

Retraction

Retracted: Global Exponential Stability of Both Continuous-Time and Discrete-Time Switched Positive Time-Varying Delay Systems with Interval Uncertainties and All Unstable Subsystems

Journal of Function Spaces

Received 23 January 2024; Accepted 23 January 2024; Published 24 January 2024

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.


The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] T. Mouktonglang and S. Yimnet, "Global Exponential Stability of Both Continuous-Time and Discrete-Time Switched Positive Time-Varying Delay Systems with Interval Uncertainties and All Unstable Subsystems," *Journal of Function Spaces*, vol. 2022, Article ID 3968850, 17 pages, 2022.

Research Article

Global Exponential Stability of Both Continuous-Time and Discrete-Time Switched Positive Time-Varying Delay Systems with Interval Uncertainties and All Unstable Subsystems

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Received 13 November 2021; Revised 23 December 2021; Accepted 16 April 2022; Published 28 May 2022

Academic Editor: Muhammad Gulzar

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The global stability problem for a class of linear switched positive time-varying delay systems (LSPTDSs) with interval uncertainties by means of a fast average dwell time (FADT) switching is analyzed in this paper. A distinctive feature of this research is that all subsystems are considered to be unstable. Both the continuous-time and the discrete-time cases of LSPTDSs with interval uncertainties and all unstable subsystems (AUSs) are investigated. By constructing a time-scheduled multiple copositive Lyapunov-Krasovskii functional (MCLKF), novel sufficient conditions are derived within the framework of the FADT switching to guarantee such systems in the case of continuous-time to be globally uniformly exponentially stable. Based on the above approach, the corresponding result is extended to the discrete-time LSPTDSs including both interval uncertainties and AUSs. In addition, new stability criteria in an exponential sense are formulated for the studied systems without interval uncertainties. The efficiency and validity of the theoretical results are shown through simulation examples.

1. Introduction

One of the crucial topics for studying the behavior of trajectories of dynamical systems under small perturbations of initial conditions is stability analysis [1–18]. The fundamental concept of stability analysis has been widely investigated in various types, such as asymptotic stability, exponential stability, robust stability, practical stability, and instability. Generally, the behavior of the considered dynamical systems depends only on the present state. Nevertheless, many phenomena cannot be explained under the specific constraints arising from the only present state, for instance, fluid and mechanical transmissions, metallurgical processes, and networked communications. Therefore, it is better to consider that the system's behavior also includes information on the former states. This characteristic is called a time

delay [1–3]. On the other hand, the time delay involved in both the continuous-time and discrete-time systems may lead to chaos and instability of the systems. Consequently, the stability problem on the systems with time delay as well as time-varying delay has been intensively analyzed, see [4–11].

In real-world systems, there exists a class of dynamical systems that compose of a family of subsystems and a switching rule orchestrating the switching among subsystems. The systems under this mechanism are well known as switched systems [19–21]. The switched systems can be described by the hybrid behavior, which is discovered in the following situations. For instance, a thermostat turning the heat on and off, a server switching between buffers in a queueing network, and the dynamics of a car changing abruptly because of wheels locking and unlocking. Among the many problems of switched systems studied both in

theory and practice, stability analysis of switched systems with an appropriate switching law is the primary concern, which has drawn great attention, see [22–29].

As a special class of switched systems, switched positive systems (SPSs) can be described by a collection of positive subsystems and a switching signal specifying the switching laws. Because of the existence of positive constraints in the switched systems, numerous results from the study on normal switched systems may not be applicable to SPSs. Besides, several phenomena can be modeled by SPSs, such as compartmental model [30], water-quality model [31], formation flying [32], congestion control [33], wireless power control [34], and network communication using transmission control protocol [35]. Due to the complex dynamics of SPSs and their numerous applications, stability analysis on SPSs has been a significant investigation, and some relevant researches have been reported in [36–44]. In addition, most practical systems often contain the term uncertainties, which refer to the differences or errors between models and reality. However, the existence of even the slight uncertainties in the considered systems can lead to the instability of those systems. Thus, it is essential to study the robust stability and stabilization problems of the systems including uncertainties. A great number of useful results on the uncertainties have been found in many switched systems, such as switched continuous-time systems [45, 46], switched discrete-time systems [47–49], stochastic switched discrete-time systems [50–52], SPSs [35, 53–55], and switched positive delay systems [56, 57].

In most of the existing works of literature, the study on the dynamic behavior of each subsystem of the overall switched system can be divided into three cases; (i) all subsystems in the first case are stable, (ii) the considered switched system in the second case composes of stable and unstable subsystems, and (iii) all subsystems interested in the last case are unstable. In general, the stability analysis of switched systems can be accomplished when either all subsystems are stable or there exists at least one stable subsystem for recompensing the state divergence caused by unstable subsystems. Nonetheless, the mentioned idea is inoperative in the case that all subsystems are unstable. Thus, an arisen natural question is how to design a suitable switching law to stabilize the switched systems without a stable subsystem? So far, this issue has been discussed, and it is still a challenging problem. By using suitable switching laws, the (robust) stability of switched systems with AUs has been investigated and reviewed briefly in the following. In [28], Xiang and Xiao studied the asymptotic stability of switched systems with AUs by using a discretized Lyapunov function and a dwell time (DT) switching law. Also, the results on exponential stability of switched systems with AUs were addressed in both the continuous-time case [23] and the discrete-time case [24]. Nevertheless, the problems of time delay, interval uncertainties, and positivity for switched systems with AUs were not taken into account in mentioned researches. Later, Feng et al. [35] dealt with the problems of stability and robust stability for linear SPSs with AUs by employing a discretized copositive Lyapunov function and a mode-dependent dwell time (MDDT)

switching rule. More recently, Zhang and Sun [32] investigated the practical exponential stability of discrete-time linear SPSs with impulse and AUs by establishing a switched time-varying vector function and applying a mode-dependent interval dwell time switching law. From the above two results, it should be noted that the existence of the time delay was not considered on the systems. On the other hand, several research articles about switched positive delay systems in the case of all subsystems are unstable were reported in [43, 44, 57]. As mentioned in [43], Liu et al. utilized the multiple discretized copositive Lyapunov-Krasovskii functionals and the DT switching rule to derive the delay-dependent sufficient criteria (DDSC) of the continuous-time and discrete-time LSPTDSs with AUs. Next, a sufficient criterion guaranteeing the global uniform exponential stability of the only continuous-time LSPTDSs with AUs based on the time-scheduled MCLKF method combining with the FADT switching law was provided in [44]. However, the interval uncertainties were not regarded in [43, 44]. Furthermore, Rojsiraphisal et al. [57] formulated the robust stability criteria within the framework of the time-scheduled MCLKF tactic and the MDDT switching strategy to ensure the continuous-time LSPTDSs including both interval uncertainties and AUs to be globally uniformly asymptotically stable.

To the best of our knowledge, there is no result on the robust exponential stability of both continuous-time and discrete-time LSPTDSs with interval uncertainties and AUs in the literature. This observed idea is the motivation of this paper. The main contributions of this study are highlighted as follows:

- (1) The global stability problem of continuous-time LSPTDSs including both interval uncertainties and AUs by adopting the time-scheduled MCLKF technique together with the FADT switching law is studied
- (2) Novel DDSC for global uniform exponential stability of the systems is derived
- (3) The corresponding results for discrete-time LSPTDSs including both interval uncertainties and AUs are also provided
- (4) Unlike the existing results in [28, 35, 43, 57], the type of stability analysis in this paper is pointed out as the exponential stability analysis instead of the asymptotic stability analysis for the underlying systems. Different from the above results, the FADT switching law, which is less conservative than the DT switching law, is applied to investigate the problem of robust exponential stability for LSPTDSs including both interval uncertainties and AUs. Moreover, it should be noted that the discrete-time case and interval uncertainties were not studied in [44]

The component of this paper is arranged as follows. In the next section, the system descriptions and preliminaries are proposed. In Section 3, the main results are presented.

In Section 4, the numerical examples are shown to support and validate our theoretical results. Lastly, the conclusions are reported in Section 5.

Notations: the following notations are exploited throughout this article. The sets of integers, nonnegative integers, and positive integers are denoted by \mathbb{Z} , \mathbb{N}_0 , and \mathbb{N} , respectively. Set $\underline{L}_0 = \{0, 1, 2, \dots, L\}$ and $\underline{N} = \{1, 2, \dots, N\}$ for any $L, N \in \mathbb{N}$. \mathbb{R}^n and \mathbb{R}_+^n refer to the vectors of n -tuples of real and positive real numbers, respectively. The set of all $m \times n$ real matrices is represented by $\mathbb{R}^{m \times n}$. I_n and A^T are the $n \times n$ dimensional identity matrix and the transpose of matrix A , respectively. The matrix A is called nonnegative if all entries are nonnegative and defined by $A \geq 0$. For given vector $v \in \mathbb{R}^n$, $v_i (1 \leq i \leq n)$ is the i th component of v . The notation $v \geq 0 (v > 0)$ stands for nonnegative (positive) vector, namely, all components of v are nonnegative (positive) for vector $v \in \mathbb{R}^n$. Let $\|v\|_1 = \sum_{i=1}^n |v_i|$ and $\|v\|_2 = (\sum_{i=1}^n v_i^2)^{1/2}$ be the 1-norm and the Euclidean norm of $v \in \mathbb{R}^n$, respectively. $\underline{\omega}(v)$ symbolizes the minimal elements of $v \in \mathbb{R}^n$. The floor function $\lfloor x \rfloor = \max \{n \in \mathbb{Z} | n \leq x, x \in \mathbb{R}\}$. In addition, $t^- = \lim_{\varepsilon \rightarrow 0^+} (t - \varepsilon)$ and $t^+ = \lim_{\varepsilon \rightarrow 0^+} (t + \varepsilon)$.

2. System Descriptions and Preliminaries

In this section, we consider the linear switched positive time-varying delay system (LSPTDS) with interval uncertainties and AUSs for both the continuous-time and the discrete-time systems.

2.1. The Continuous-Time LSPTDS including Both Interval Uncertainties and AUSs. A class of continuous-time linear switched system with time-varying delay can be stated as

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}x(t - \alpha(t)), \\ x(t_0 + \theta) = \psi(\theta), \theta \in [-\hat{\alpha}, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$. The switching signal $\sigma(t)$ is a piecewise constant function of time t , which takes values in the finite set $\underline{N} = \{1, 2, \dots, N\}$, $N > 1$ is the number of subsystems or modes of the switched system. Without loss of generality, we presume that $\sigma(t)$ is continuous from the right everywhere: $\sigma(t) = \lim_{\chi \rightarrow t^+} \sigma(\chi)$. The switching instants can be defined by a sequence $0 \leq t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} < \dots < +\infty$, where t_0 is the initial time and t_m stands for the m th switching instant, $m \in \mathbb{N}_0$. We impose that $\sigma(t) = \sigma(t_m) = i$, $i \in \underline{N}$ and the i th subsystem is activated when $t \in [t_m, t_{m+1})$. Based on the logical rule of $\sigma(t)$ at the switching instant t_m , system (1) switches from the j th subsystem to the i th subsystem, where $\sigma(t_{m-1}) = j, j \in \underline{N}$. $\alpha(t)$ is the time-varying delay satisfying $0 \leq \alpha(t) \leq \hat{\alpha}$ and $\dot{\alpha}(t) \leq \Delta < 1$, where $\hat{\alpha}$ and Δ are known constants. As mentioned in [35, 57], the system matrices A_i and D_i are supposed to be interval uncertain; namely,

$$\underline{A}_i \leq A_i \leq \bar{A}_i, \quad (2)$$

$$\underline{D}_i \leq D_i \leq \bar{D}_i, \quad (3)$$

where $\underline{A}_i, \underline{D}_i, \bar{A}_i, \bar{D}_i$ are the given constant system matrices with appropriate dimensions for all $i \in \underline{N}$. In addition, $\psi(\cdot): [-\hat{\alpha}, 0] \rightarrow \mathbb{R}^n$ is a vector-valued initial state with $\|\psi\|_{\hat{\alpha}} = \sup_{-\hat{\alpha} \leq \theta \leq 0} \|\psi(\theta)\|_2$.

First, we introduce some definitions, lemma, and assumptions helpful in obtaining the main results in the continuous-time system.

Definition 1 (see [43]). System (1) is said to be positive if for any initial condition $\psi(\theta) \geq 0, \theta \in [-\hat{\alpha}, 0]$ and any switching signal $\sigma(t)$, the corresponding trajectory $x(t) \geq 0$ holds for all $t \geq t_0$.

Definition 2 (see [35]). A matrix A is said to be a Metzler matrix if all off-diagonal elements are nonnegative.

Lemma 3 (see [43]). System (1) is positive if and only if A_i is Metzler matrices and $D_i \geq 0$ for all $i \in \underline{N}$.

Definition 4 (see [44]). System (1) is said to be globally uniformly exponentially stable (GUES) with switching signal $\sigma(t)$ if there exist two constants $\varepsilon > 0$ and $\omega > 0$ such that $\|x(t)\|_2 \leq \varepsilon e^{-\omega(t-t_0)} \|\psi\|_{\hat{\alpha}}$ for all $t \geq t_0$.

In general, actual systems can be modeled by systems in the form of interval uncertainties. Therefore, the assumption of the interval uncertainties for studying the robust exponential stability of system (1) is stated as follows.

Assumption 5 (see [35, 57]). For each A_i and D_i in system (1), there are the known Metzler matrices \underline{A}_i and the matrices $\underline{D}_i \geq 0$ such that $A_i \in [\underline{A}_i, \bar{A}_i]$ and $D_i \in [\underline{D}_i, \bar{D}_i]$, where $\underline{A}_i, \underline{D}_i, \bar{A}_i, \bar{D}_i$ are the given constant system matrices with appropriate dimensions for all $i \in \underline{N}$.

The following assumption is necessary and reasonable to analyze the problem of robust exponential stability for system (1) including interval uncertainties in the case of all subsystems are unstable.

Assumption 6 (see [35, 43, 44, 57]). All subsystems of system (1) are unstable.

Besides, it is well-known that the ADT switching law is less conservative and more general than the DT switching law. Especially, the FADT switching law can be utilized to certify the (robust) stability of the switched systems that all subsystems are unstable. Therefore, the definitions of both the DT and FADT switching laws are stated as follows.

Definition 7 (see [41, 44]). For two switching instants t_m and t_{m+1} , $m \in \mathbb{N}_0$ of system (1), if there exists a constant $\tau_m > 0$ such that $\tau_m = t_{m+1} - t_m$ holds for any $m \in \mathbb{N}_0$, then, τ_m is called DT of system (1). Moreover, if there exists a constant $\tau^* > 0$ such that $\tau^* \leq \inf_{m \in \mathbb{N}_0} \tau_m$ holds for any $m \in \mathbb{N}_0$, then, τ^* is called minimum DT of system (1).

Definition 8 (see [41, 44]). For any $T > t \geq 0$ and a switching signal $\sigma(t)$, let $N_\sigma(T, t)$ be the switching numbers over the interval $[t, T]$, if there exist two constants $N_0 \geq 0, \tau_a > 0$ such that

$$N_\sigma(T, t) \geq N_0 + \frac{T-t}{\tau_a}, \forall T > t \geq 0, \quad (4)$$

then, the constants N_0 and τ_a are called the chattering bound and FADT of system (1), respectively.

Remark 9. Already discussed in [44], the mechanism of both the FADT switching law and the ADT switching law is different. Namely, the FADT switching law imposes that the activation average time of AUSs is neither too large nor too small to stabilize the switched system that all subsystems are unstable, while the ADT switching law can be described that some stable subsystems of the switched system remain sufficiently long ADT to ensure the stability of the considered system. Moreover, from (4), it implies that the FADT switching law satisfies $\tau_a \geq (T-t)/(N_\sigma(T, t) - N_0)$, which is different from the ADT switching law satisfying $\tau_a \leq (T-t)/(N_\sigma(T, t) - N_0)$.

2.2. The Discrete-Time LSPTDS including Both Interval Uncertainties and AUSs. A class of discrete-time linear switched system with time-varying delay can be described in the form of

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + D_{\sigma(k)}x(k-d(k)), \\ x(k_0 + \zeta) = \varphi(\zeta), \zeta = -\tilde{d}, -\tilde{d} + 1, \dots, -1, 0, \end{cases} \quad (5)$$

where $x(k) \in \mathbb{R}^n$. $\sigma(k): \mathbb{N}_0 \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is the switching signal, and $N > 1$ is the number of subsystems or modes of the switched system. Given the switching signal $\sigma(k)$, we denote the set of switching moments by $\{k_m : k_m \in \mathbb{N}_0\}$ where k_0 is the initial time and $k_m < k_{m+1}$ for $m \in \mathbb{N}_0$. For two successive switching moments k_m and k_{m+1} , we impose that $\sigma(k-1) = \sigma(k_m-1) = j$ and $\sigma(k) = \sigma(k_m) = i$, where $j, i \in \underline{N}$, and the $\sigma(k_m)$ th subsystem is activated when $k \in [k_m, k_{m+1})$. Similarly, with the continuous-time system, the system matrices A_i and D_i are presumed to be interval uncertain for all $i \in \underline{N}$. The time-varying delay $d(k)$ satisfies $d_1 \leq d(k) \leq d_2$ where d_1, d_2 are known positive integers, and $\tilde{d} = \max\{d_1, d_2\}$. Moreover, $\varphi(\cdot): \{-\tilde{d}, -\tilde{d} + 1, \dots, -1, 0\} \rightarrow \mathbb{R}^n$ is a given discrete vector-valued initial state with $\|\varphi\|_{\tilde{d}} = \max_{\zeta \in \{-\tilde{d}, -\tilde{d} + 1, \dots, -1, 0\}} \|\varphi(\zeta)\|_2$.

The essential definitions, lemma, and assumptions utilized in the discrete-time system are presented in the following.

Definition 10 (see[43]). System (5) is said to be positive if for any initial condition $\varphi(\zeta) \geq 0, \zeta = -\tilde{d}, -\tilde{d} + 1, \dots, -1, 0$ and any switching signal $\sigma(k)$, the corresponding trajectory $x(k) \geq 0$ holds for all $k \in \mathbb{N}_0$.

Lemma 11 (see [43]). System (5) is positive if and only if $A_i \geq 0$ and $D_i \geq 0$ hold for all $i \in \underline{N}$.

Definition 12 (see [9]). System (5) is said to be globally uniformly exponentially stable (GUES) with switching signal $\sigma(k)$ if there exist two constants $\varepsilon > 0$ and $0 < \rho < 1$ such that $\|x(k)\|_2 \leq \varepsilon \rho^{k-k_0} \|\varphi\|_{\tilde{d}}$ for all $k \geq k_0$.

Assumption 13 (see [35]). For each A_i and D_i in system (5), there are the known constant matrices $\underline{A}_i \geq 0$ and $\underline{D}_i \geq 0$ such that $A_i \in [\underline{A}_i, \bar{A}_i]$ and $D_i \in [\underline{D}_i, \bar{D}_i]$, where $\underline{A}_i, \underline{D}_i, \bar{A}_i, \bar{D}_i$ are the given constant system matrices with appropriate dimensions for all $i \in \underline{N}$.

Assumption 14 (see [35, 43]). All subsystems of system (5) are unstable.

Definition 15 (see [43, 44]). For two switching moments k_m and $k_{m+1}, m \in \mathbb{N}_0$ of system (5), if there exists a constant $\kappa_m > 0$ such that $\kappa_m = k_{m+1} - k_m$ holds for any $m \in \mathbb{N}_0$, then, κ_m is called DT of system (5). Moreover, if there exists a constant $\kappa^* > 0$ such that $\kappa^* \leq \min_{m \in \mathbb{N}_0} \kappa_m$ holds for any $m \in \mathbb{N}_0$, then, κ^* is called minimum DT of system (5).

Definition 16 (see [44]). For any $K > k \geq 0$ and a switching signal $\sigma(k)$, let $N_\sigma(K, k)$ be the switching numbers over the interval $[k, K]$, if there exist two constants $N_0 \geq 0, \kappa_a > 0$ satisfying

$$N_\sigma(K, k) \geq N_0 + \frac{K-k}{\kappa_a}, \forall K > k \geq 0, \quad (6)$$

then, the constants N_0 and κ_a are called the chattering bound and FADT of system (5), respectively.

The main purpose of this research is to propose the global stability criteria that ensure the continuous-time system (1) with interval uncertainties and the discrete-time system (5) with interval uncertainties are positive and GUES with respect to the FADT switching law when all modes of the systems are unstable.

3. Main Results

In this section, we apply the time-scheduled MCLKF tactic and the FADT switching strategy to derive novel DDSC guaranteeing the positivity and the robust exponential stability of both the continuous-time system (1) and the discrete-time system (5) with interval uncertainties and AUSs. Besides, we propose the positivity and the exponential stability criteria of both system (1) and system (5) without interval uncertainties.

For convenience, we first define important symbols used in our main theorem as follows:

$$\tilde{D} = (\bar{d}_{kl}) \in \mathbb{R}^{n \times n}, \bar{d}_{kl} = \max_{i \in \underline{N}} \left\{ \bar{D}_i^{(kl)} \right\}, \quad (7)$$

where $\bar{D}_i^{(kl)}$ refers to the k th row and l th column element of system matrices \bar{D}_i , $i \in \underline{N}$. And

$$D = (d_{kl}) \in \mathbb{R}^{n \times n}, d_{kl} = \max_{i \in \underline{N}} \{D_i^{(kl)}\}, \quad (8)$$

where $D_i^{(kl)}$ denotes the k th row and l th column element of system matrices D_i , $i \in \underline{N}$.

3.1. Global Uniform Exponential Stability of Continuous-Time LSPTDS including Both Interval Uncertainties and AUs

Theorem 17. Consider the continuous-time system (1) satisfying Assumption 5 and Assumption 6. Given constants $0 < \mu < 1$, $\lambda > 0$, $\tau^* > 0$, and $L \in \mathbb{N}$. If there exist positive vectors $v_{i,q}$, $i \in \underline{N}$, $q \in \underline{L}$ and constants $\tau_a \geq \tau^*$ such that

$$(1 - \Delta)\Phi_{i,q} + \left[(1 - \Delta)\bar{A}_i^T + (1 + \hat{\alpha})\tilde{D}^T - \lambda(1 - \Delta)I_n \right] v_{i,q} < 0, \quad (9)$$

$$(1 - \Delta)\Phi_{i,q} + \left[(1 - \Delta)\bar{A}_i^T + (1 + \hat{\alpha})\tilde{D}^T - \lambda(1 - \Delta)I_n \right] v_{i,q+1} < 0, \quad (10)$$

$$\tilde{D}^T (\Phi_{i,q} - v_{i,q}) < 0, \quad (11)$$

$$\tilde{D}^T (\Phi_{i,q} - v_{i,q+1}) < 0, \quad (12)$$

$$\left[(1 - \Delta)\bar{A}_i^T + (1 + \hat{\alpha})\tilde{D}^T - \lambda(1 - \Delta)I_n \right] v_{i,L} < 0, \quad (13)$$

$$v_{i,0} - \mu v_{j,L} \leq 0, \quad (14)$$

hold for any $q = 0, 1, \dots, L-1$, and for any $i, j \in \underline{N}$, $i \neq j$, then, the continuous-time system (1) is positive and GUES under the switching signal with the FADT satisfying

$$\tau^* \leq \tau_a < -\frac{\ln \mu}{\lambda}, \quad (15)$$

where

$$\Phi_{i,q} = \frac{(v_{i,q+1} - v_{i,q})L}{\tau^*}, \quad (16)$$

and \tilde{D} is mentioned in (7).

Proof. We divide the proof process into the following two steps. \square

Step 1. We will prove that system (1) is positive.

Using Assumption 5, we obtain that A_i is also Metzler matrices and $D_i \geq 0$ for all $i \in \underline{N}$. According to Lemma 3, system (1) is positive.

Step 2. We will prove that system (1) is GUES under the switching signal under the FADT satisfying condition (15).

For any $t > 0$, we suppose that $t \in [t_m, t_{m+1}) = [t_m, t_m + \tau^*) \cup [t_m + \tau^*, t_{m+1})$, $m \in \mathbb{N}_0$. And we divide the interval $[t_m, t_m + \tau^*)$ into L segments with equal length $h = \tau^*/L$. In addition, we define $Y_{m,q} = [t_m + qh, t_m + (q+1)h)$, $q = 0, 1, \dots, L-1$, then $[t_m, t_m + \tau^*) = \bigcup_{q=0}^{L-1} Y_{m,q}$.

Due to each stable subsystem has been completely replaced by AUs, some previous researches about the switched systems are no longer applicable. Motivated by the concept utilized in [28, 43, 44, 57], we employ the discretized Lyapunov function and the FADT switching law to stabilize system (1) with interval uncertainties for the case that all subsystems are unstable. First, we establish the following vector function

$$v_i(t) = v_i(t_m + qh + \eta(t)h) = \begin{cases} (1 - \eta(t))v_{i,q} + \eta(t)v_{i,q+1}, & t \in Y_{m,q}, q = 0, 1, \dots, L-1, \\ v_{i,L}, & t \in [t_m + \tau^*, t_{m+1}), \end{cases} \quad (17)$$

where $i \in \underline{N}$, $m \in \mathbb{N}_0$, $\eta(t) = (t - t_m - qh)/h$ with $0 \leq \eta(t) \leq 1$, and $v_{i,q}$ are positive vectors for $i \in \underline{N}$, $q \in \underline{L}$. For $t \in Y_{m,q}$, we can get

$$\dot{v}_i(t) = \dot{\eta}(t)v_{i,q+1} - \dot{\eta}(t)v_{i,q} = \frac{v_{i,q+1} - v_{i,q}}{h}, \quad (18)$$

which yields

$$\dot{v}_i(t) = \Phi_{i,q}, \quad (19)$$

where $\Phi_{i,q}$ is defined as in (16). For any $i \in \underline{N}$, we establish the time-scheduled MCLKF:

$$V_i(t, x(t)) = (1 - \Delta)x^T(t)v_i(t) + \int_{t-\alpha(t)}^t x^T(s)\tilde{D}^T v_i(s)ds + \int_{-\hat{\alpha}}^0 \int_{t+w}^t x^T(s)\tilde{D}^T v_i(s)dsdw. \quad (20)$$

Differentiating $V_i(t, x(t))$ in (20) along the trajectories of system (1), we obtain

$$\begin{aligned} \dot{V}_i(t, x(t)) &= (1 - \Delta)[x^T(t)\dot{v}_i(t) + x^T(t)A_i^T v_i(t) \\ &\quad + x^T(t - \alpha(t))D_i^T v_i(t)] + \int_{t-\alpha(t)}^t x^T(s)\tilde{D}^T \dot{v}_i(s)ds \\ &\quad + x^T(t)\tilde{D}^T v_i(t) - x^T(t - \alpha(t))\tilde{D}^T v_i(t)(1 - \dot{\alpha}(t)) \\ &\quad + \hat{\alpha}x^T(t)\tilde{D}^T v_i(t) - \int_{t-\hat{\alpha}}^t x^T(s)\tilde{D}^T v_i(s)ds, \end{aligned} \quad (21)$$

then

$$\begin{aligned}
\dot{V}_i(t, x(t)) &\leq (1 - \Delta) \left[x^T(t) \dot{v}_i(t) + x^T(t) \bar{A}_i^T v_i(t) \right. \\
&\quad \left. + x^T(t - \alpha(t)) \bar{D}_i^T v_i(t) \right] + \int_{t-\alpha(t)}^t x^T(s) \tilde{D}^T \dot{v}_i(t) ds \\
&\quad + x^T(t) \tilde{D}^T v_i(t) - x^T(t - \alpha(t)) \tilde{D}^T v_i(t) (1 - \dot{\alpha}(t)) \\
&\quad + \hat{\alpha} x^T(t) \tilde{D}^T v_i(t) - \int_{t-\hat{\alpha}}^t x^T(s) \tilde{D}^T v_i(s) ds.
\end{aligned} \tag{22}$$

We observe that

$$\begin{aligned}
\dot{V}_i(t, x(t)) - \lambda V_i(t, x(t)) &\leq (1 - \Delta) \left[x^T(t) \dot{v}_i(t) + x^T(t) \bar{A}_i^T v_i(t) + x^T(t - \alpha(t)) \bar{D}_i^T v_i(t) \right] \\
&\quad + \int_{t-\alpha(t)}^t x^T(s) \tilde{D}^T \dot{v}_i(t) ds + x^T(t) \tilde{D}^T v_i(t) \\
&\quad - x^T(t - \alpha(t)) \tilde{D}^T v_i(t) (1 - \dot{\alpha}(t)) + \hat{\alpha} x^T(t) \tilde{D}^T v_i(t) \\
&\quad - \int_{t-\hat{\alpha}}^t x^T(s) \tilde{D}^T v_i(s) ds - \lambda(1 - \Delta) x^T(t) v_i(t) \\
&\quad - \lambda \int_{t-\alpha(t)}^t x^T(s) \tilde{D}^T v_i(t) ds - \lambda \int_{-\hat{\alpha}}^0 \int_{t+w}^t x^T(s) \tilde{D}^T v_i(s) ds dw.
\end{aligned} \tag{23}$$

Along with $\dot{\alpha}(t) \leq \Delta$, $0 < \lambda$ and $\bar{D}_i \leq \tilde{D}$ for all $i \in \underline{N}$, one has

$$\begin{aligned}
\dot{V}_i(t, x(t)) - \lambda V_i(t, x(t)) &\leq x^T(t) \left[(1 - \Delta) \dot{v}_i(t) + (1 - \Delta) \bar{A}_i^T v_i(t) \right. \\
&\quad \left. + (1 + \hat{\alpha}) \tilde{D}^T v_i(t) - \lambda(1 - \Delta) v_i(t) \right] \\
&\quad + \int_{t-\alpha(t)}^t x^T(s) \tilde{D}^T [\dot{v}_i(t) - v_i(s)] ds.
\end{aligned} \tag{24}$$

When $t \in Y_{m,q} \subset [t_m, t_m + \tau^*]$, it leads to

$$\begin{aligned}
(1 - \Delta) \dot{v}_i(t) + (1 - \Delta) \bar{A}_i^T v_i(t) + (1 + \hat{\alpha}) \tilde{D}^T v_i(t) - \lambda(1 - \Delta) v_i(t) \\
= (1 - \eta(t)) \left[(1 - \Delta) \Phi_{i,q} + \left((1 - \Delta) \bar{A}_i^T + (1 + \hat{\alpha}) \tilde{D}^T - \lambda(1 - \Delta) I_n \right) v_{i,q} \right] \\
+ \eta(t) \left[(1 - \Delta) \Phi_{i,q+1} + \left((1 - \Delta) \bar{A}_i^T + (1 + \hat{\alpha}) \tilde{D}^T - \lambda(1 - \Delta) I_n \right) v_{i,q+1} \right],
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\tilde{D}^T [\dot{v}_i(t) - v_i(s)] &= (1 - \eta(s)) \tilde{D}^T (\Phi_{i,q} - v_{i,q}) \\
&\quad + \eta(s) \tilde{D}^T (\Phi_{i,q} - v_{i,q+1}).
\end{aligned} \tag{26}$$

According to conditions (9)–(12), we obtain

$$\dot{V}_i(t, x(t)) - \lambda V_i(t, x(t)) < 0, t \in Y_{m,q}, \tag{27}$$

which implies

$$\dot{V}_i(t, x(t)) < \lambda V_i(t, x(t)), t \in \bigcup_{q=0}^{L-1} Y_{m,q} = [t_m, t_m + \tau^*]. \tag{28}$$

When $t \in [t_m + \tau^*, t_{m+1}]$, it yields that

$$\begin{aligned}
(1 - \Delta) \dot{v}_i(t) + (1 - \Delta) \bar{A}_i^T v_i(t) + (1 + \hat{\alpha}) \tilde{D}^T v_i(t) - \lambda(1 - \Delta) v_i(t) \\
= \left((1 - \Delta) \bar{A}_i^T + (1 + \hat{\alpha}) \tilde{D}^T - \lambda(1 - \Delta) I_n \right) v_{i,L},
\end{aligned} \tag{29}$$

$$\tilde{D}^T [\dot{v}_i(t) - v_i(s)] = -\tilde{D}^T v_{i,L} < 0. \tag{30}$$

Applying condition (13), we get

$$\dot{V}_i(t, x(t)) < \lambda V_i(t, x(t)), t \in [t_m + \tau^*, t_{m+1}]. \tag{31}$$

Combining (28) with (31), it is obvious that

$$\dot{V}_i(t, x(t)) < \lambda V_i(t, x(t)), t \in [t_m, t_{m+1}], m \in \mathbb{N}_0. \tag{32}$$

Integrating both sides of (32) over $[t_m, t]$ for $t \in [t_m, t_{m+1}]$, $m \in \mathbb{N}_0$, it is immediate that

$$V_i(t, x(t)) < e^{\lambda(t-t_m)} V_i(t_m, x(t_m)). \tag{33}$$

Using condition (14), we have

$$v_i(t_m) \leq \mu v_j(t_m^-) \quad \text{for all } i, j \in \underline{N}, i \neq j. \tag{34}$$

From (17) and (20), it can be seen that

$$\begin{aligned}
V_i(t_m, x(t_m)) &= (1 - \Delta) x^T(t_m) v_i(t_m) \\
&\quad + \int_{t_m - \alpha(t_m)}^{t_m} x^T(s) \tilde{D}^T v_i(t_m) ds \\
&\quad + \int_{-\hat{\alpha}}^0 \int_{t_m+w}^{t_m} x^T(s) \tilde{D}^T v_i(s) ds dw \\
&\leq (1 - \Delta) x^T(t_m^-) \mu v_j(t_m^-) \\
&\quad + \int_{t_m^- - \alpha(t_m^-)}^{t_m^-} x^T(s) \tilde{D}^T \mu v_j(t_m^-) ds \\
&\quad + \int_{-\hat{\alpha}}^0 \int_{t_m^-+w}^{t_m^-} x^T(s) \tilde{D}^T \mu v_j(s) ds dw, \\
&= \mu V_j(t_m^-, x(t_m^-)) \quad \text{for all } i, j \in \underline{N}, i \neq j.
\end{aligned} \tag{35}$$

$$= \mu V_j(t_m^-, x(t_m^-)) \quad \text{for all } i, j \in \underline{N}, i \neq j. \tag{36}$$

Based on the relationship between (33) and (36), we can derive

$$\begin{aligned}
V_{\sigma(t_m)}(t, x(t)) &< e^{\lambda(t-t_m)} V_{\sigma(t_m)}(t_m, x(t_m)) \\
&\leq \mu e^{\lambda(t-t_m)} V_{\sigma(t_{m-1})}(t_m^-, x(t_m^-)) \\
&< \mu e^{\lambda(t-t_m)} e^{\lambda(t_m-t_{m-1})} V_{\sigma(t_{m-1})}(t_{m-1}, x(t_{m-1})) : \\
&< \mu \mu \dots \mu e^{\lambda(t-t_m)} e^{\lambda(t_m-t_{m-1})} \dots e^{\lambda(t_1-t_0)} V_{\sigma(t_0)}(t_0, x(t_0)) \\
&= \mu^{N_{\sigma}(t,t_0)} e^{\lambda(t-t_0)} V_{\sigma(t_0)}(t_0, x(t_0)).
\end{aligned} \tag{37}$$

It follows from Definition 8 and $0 < \mu < 1$ by employing (4), then, we obtain

$$\begin{aligned}
V_{\sigma(t_m)}(t, x(t)) &\leq \mu^{(N_0+t-t_0/\tau_a)} e^{\lambda(t-t_0)} V_{\sigma(t_0)}(t_0, x(t_0)) \\
&= e^{N_0 \ln \mu} e^{(\lambda + \ln \mu/\tau_a)(t-t_0)} V_{\sigma(t_0)}(t_0, x(t_0)).
\end{aligned} \tag{38}$$

Without loss of generality, we impose that $V_{\sigma(t_0)}(t_0, x(t_0)) = V_{\sigma(0)}(0, x(0))$. From (17) and (20), we have

$$\begin{aligned}
V_{\sigma(0)}(0, x(0)) &\leq (1-\Delta) \|x^T(0)\|_2 \|v_{\sigma(0)}(0)\|_2 \\
&\quad + \int_{-\hat{\alpha}}^0 ds \sup_{-\hat{\alpha} \leq s \leq 0} \|x^T(s)\|_2 \|\tilde{D}^T\|_2 \|v_{\sigma(0)}(0)\|_2 \\
&\quad + \int_{-\hat{\alpha}}^0 \int_w^0 \|v_{\sigma(0)}(s)\|_2 ds dw \sup_{-\hat{\alpha} \leq s \leq 0} \|x^T(s)\|_2 \|\tilde{D}^T\|_2 \\
&\leq (1-\Delta) \|x^T(0)\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \hat{\alpha} \sup_{-\hat{\alpha} \leq s \leq 0} \|x^T(s)\|_2 \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + 2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \int_{-\hat{\alpha}}^0 \int_w^0 ds dw \sup_{-\hat{\alpha} \leq s \leq 0} \|x^T(s)\|_2 \|\tilde{D}^T\|_2 \\
&\leq (1-\Delta) \sqrt{n} \|\psi\|_{\hat{\alpha}} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \hat{\alpha} \sqrt{n} \|\psi\|_{\hat{\alpha}} \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \hat{\alpha}^2 \sqrt{n} \|\psi\|_{\hat{\alpha}} \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&= \left[(1-\Delta) + \hat{\alpha}(1+\hat{\alpha}) \|\tilde{D}^T\|_2 \right] \sqrt{n} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \|\psi\|_{\hat{\alpha}}.
\end{aligned} \tag{39}$$

Moreover, we get

$$(1-\Delta) \xi \|x(t)\|_2 \leq V_{\sigma(t_m)}(t, x(t)), \tag{40}$$

where $\xi = \min_{(a,b) \in \underline{N} \times \underline{L}_0} \{\underline{\omega}(v_{a,b})\}$. Substituting (39) and (40) into (38), we obtain

$$\begin{aligned}
\|x(t)\|_2 &\leq \frac{e^{N_0 \ln \mu}}{(1-\Delta) \xi} e^{(\lambda + \ln \mu/\tau_a)(t-t_0)} \\
&\quad \cdot \left[(1-\Delta) + \hat{\alpha}(1+\hat{\alpha}) \|\tilde{D}^T\|_2 \right] \sqrt{n} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \|\psi\|_{\hat{\alpha}},
\end{aligned} \tag{41}$$

for $t \geq t_0$. If the FADT switching signal satisfies condition (15), then

$$-\left(\lambda + \frac{\ln \mu}{\tau_a} \right) > 0. \tag{42}$$

From (41), for $t \geq t_0$, we arrive at

$$\|x(t)\|_2 \leq \varepsilon e^{-\hat{\omega}(t-t_0)} \|\psi\|_{\hat{\alpha}}, \tag{43}$$

where $\varepsilon = (e^{N_0 \ln \mu} / (1-\Delta) \xi) [(1-\Delta) + \hat{\alpha}(1+\hat{\alpha}) \|\tilde{D}^T\|_2] \sqrt{n} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2$, and $\hat{\omega} = -(\lambda + \ln \mu/\tau_a)$.

By Definition 4, we can conclude that system (1) is GUES under the switching signal with the FADT satisfying condition (15).

Remark 18. Inspired by the idea in [35, 43–44, 28, 57], the time-scheduled positive vector function defined in (17) is time-varying rather than a constant, and the time-scheduled MCLKF proposed in (20) is validly established to investigate the problem of robust stability of continuous-time system (1) when each subsystem is unstable. However, our time-scheduled MCLKF shown in Theorem 17 is different from the discretized copositive Lyapunov function, the discretized copositive Lyapunov-Krasovskii functional, the time-scheduled MCLKF, and the discretized quadratic Lyapunov function used in [28, 35, 43, 44], respectively. Although our time-scheduled MCLKF is similar to the time-scheduled MCLKF utilized in [57], our robust stability criteria under the FADT switching technique for the considered system can guarantee for being the global uniform exponential stability rather than for being the global uniform asymptotic stability derived in [57].

Remark 19. According to the result in [44], the FADT switching law is applied to deal with the stabilization problem system (1) with AUs but without interval uncertainties. This switching law can compensate for the state divergence caused by the unstable subsystem under the appropriate average time of each unstable subsystem; namely, the ADT of AUs is neither too long nor too short. However, the model in this paper, which includes the interval uncertainties, can also apply this switching law effectively. Therefore, our result obtained from Theorem 17 is more applicable than that of Theorem 1 in [44].

Next, another stability result of the continuous-time system (1) without its interval uncertainty will be presented as follows:

Corollary 20. Consider the continuous-time system (1) without its interval uncertainty satisfying Assumption 6. Let A_i be the Metzler matrices and $D_i \geq 0, \forall i \in \underline{N}$. Given constants $0 < \mu < 1, \lambda > 0, \tau^* > 0$, and $L \in \underline{N}$. If there exist positive vectors $v_{i,q}, i \in \underline{N}, q \in \underline{L}_0$, and constants $\tau_a \geq \tau^*$ such that

$$(1 - \Delta)\Phi_{i,q} + [(1 - \Delta)A_i^T + (1 + \hat{\alpha})D^T - \lambda(1 - \Delta)I_n]v_{i,q} < 0, \quad (44)$$

$$(1 - \Delta)\Phi_{i,q} + [(1 - \Delta)A_i^T + (1 + \hat{\alpha})D^T - \lambda(1 - \Delta)I_n]v_{i,q+1} < 0, \quad (45)$$

$$D^T(\Phi_{i,q} - v_{i,q}) < 0, \quad (46)$$

$$D^T(\Phi_{i,q} - v_{i,q+1}) < 0, \quad (47)$$

$$[(1 - \Delta)A_i^T + (1 + \hat{\alpha})D^T - \lambda(1 - \Delta)I_n]v_{i,L} < 0, \quad (48)$$

$$v_{i,0} - \mu v_{j,L} \leq 0, \quad (49)$$

hold for any $q = 0, 1, \dots, L - 1$, and for any $i, j \in \underline{N}$, $i \neq j$, then, the continuous-time system (1) without its interval uncertainty is positive and GUES with the same FADT satisfying (15) where $\Phi_{i,q}$ and D are defined as in (16) and (8), respectively.

Proof. With the same symbols and vector function (17) in Theorem 17, this corollary can be proved by utilizing the following time-scheduled MCLKF:

$$V_i(t, x(t)) = (1 - \Delta)x^T(t)v_i(t) + \int_{t-\alpha(t)}^t x^T(s)D^T v_i(t)ds + \int_{-\hat{\alpha}}^0 \int_{t+w}^t x^T(s)D^T v_i(s)dsdw. \quad (50)$$

The remainder of the proof is similar to that of Theorem 17. Hence, the detail is omitted. \square

Remark 21. It should be noted that our time-scheduled MCLKF proposed in (50) is similar to the time-scheduled MCLKF used in Corollary 1 of [57]. However, the DDSC obtained from this corollary are derived to ensure the global uniform exponential stability of system (1) without its interval uncertainty under the FADT switching technique rather than the global uniform asymptotic stability of the same system with the MDDT approach derived in [57]. In addition, as mentioned in [44], the ADT switching law includes the DT switching law. Consequently, the result of Theorem 3.1 in [43] is a special case of our theoretical result.

In this subsection, we extend the results of the continuous-time LSPTDS including both interval uncertainties and AUSs to the discrete-time case.

3.2. Global Uniform Exponential Stability of Discrete-Time LSPTDS including Both Interval Uncertainties and AUSs

Theorem 22. Consider the discrete-time system (5) satisfying Assumption 13 and Assumption 14. Given constants $0 < \mu < 1$, $\gamma > 1$, $\kappa^* > 0$, and $L \in \mathbb{N}$. If there exist positive vectors $v_{i,q}$, $i \in \underline{N}$, $q \in \underline{L}_0$, and constants $\kappa_a \geq \kappa^*$ such that

$$\begin{aligned} & (\bar{A}_i^T + (d_2 - d_1 + 1)\tilde{D}^T)\Pi_{i,q} \\ & + (\bar{A}_i^T + (d_2 - d_1 + 1)\tilde{D}^T - \gamma I_n)v_{i,q} < 0, \end{aligned} \quad (51)$$

$$\begin{aligned} & (\bar{A}_i^T + (d_2 - d_1 + 1)\tilde{D}^T)\Pi_{i,q} \\ & + (\bar{A}_i^T + (d_2 - d_1 + 1)\tilde{D}^T - \gamma I_n)v_{i,q+1} < 0, \end{aligned} \quad (52)$$

$$\tilde{D}^T(\Pi_{i,q} + (1 - \gamma)v_{i,q}) < 0, \quad (53)$$

$$\tilde{D}^T(\Pi_{i,q} + (1 - \gamma)v_{i,q+1}) < 0, \quad (54)$$

$$(\bar{A}_i^T + (d_2 - d_1 + 1)\tilde{D}^T - \gamma I_n)v_{i,L} < 0, \quad (55)$$

$$v_{i,0} - \mu v_{j,L} \leq 0, \quad (56)$$

hold for any $q = 0, 1, \dots, L - 1$, and for any $i, j \in \underline{N}$, $i \neq j$, then, the discrete-time system (5) is positive and GUES under the switching signal with the FADT satisfying

$$\kappa^* \leq \kappa_a < -\log_{\gamma}\mu, \quad (57)$$

where

$$\Pi_{i,q} = \frac{v_{i,q+1} - v_{i,q}}{\hbar}, \quad (58)$$

$\hbar = \lfloor \kappa^* / L \rfloor$ and \tilde{D} is defined as in (7).

Proof. The proof process of this theorem is similar to the continuous-time system; namely, it can be separated into the following two steps. \square

Step 1. We will prove that system (5) is positive.

Using Assumption 13, we obtain that $A_i \geq 0$ and $D_i \geq 0$ for all $i \in \underline{N}$. According to Lemma 11, system (5) is positive.

Step 2. We will prove that system (5) is GUES under the switching signal under the FADT satisfying condition (57).

For a given $k \in \mathbb{N}$, we suppose that $k \in [k_m, k_{m+1}) = [k_m, k_m + \hat{\kappa}) \cup [k_m + \hat{\kappa}, k_{m+1})$, $m \in \mathbb{N}_0$, $\hat{\kappa} = L \lfloor \kappa^* / L \rfloor$, and the interval $[k_m, k_m + \hat{\kappa})$ is split into L segments with equal length $\hbar = \lfloor \kappa^* / L \rfloor$. We define $\Xi_{m,q} = [k_m + q\hbar, k_m + (q + 1)\hbar)$, $q = 0, 1, \dots, L - 1$, and let $[k_m, k_m + \hat{\kappa}) = \bigcup_{q=0}^{L-1} \Xi_{m,q}$. Next, we construct the vector function:

$$v_i(k) = v_i(k_m + q\hbar + r) = \begin{cases} \left(1 - \frac{r}{\hbar}\right)v_{i,q} + \frac{r}{\hbar}v_{i,q+1}, & k \in \Xi_{m,q}, r \in \underline{\hbar}_0, q = 0, 1, \dots, L - 1, \\ v_{i,L}, & k \in [k_m + \hat{\kappa}, k_{m+1}), \end{cases} \quad (59)$$

where $i \in \underline{N}$, $m \in \mathbb{N}_0$, and $v_{i,q}$ are positive vectors for $i \in \underline{N}$, $q \in \underline{L}_0$. For $k \in \Xi_{m,q}$, we obtain

$$\begin{aligned} v_i(k+1) - v_i(k) &= v_i(k_m + q\hbar + r + 1) - v_i(k_m + q\hbar + r) \\ &= \left[\left(1 - \frac{(r+1)}{\hbar}\right)v_{i,q} + \frac{(r+1)}{\hbar}v_{i,q+1} \right] \\ &\quad - \left[\left(1 - \frac{r}{\hbar}\right)v_{i,q} + \frac{r}{\hbar}v_{i,q+1} \right], \end{aligned} \tag{60}$$

which yields

$$v_i(k+1) - v_i(k) = \Pi_{i,q}, \tag{61}$$

where $\Pi_{i,q}$ is defined as in (58). For each subsystem, we establish the time-scheduled MCLKF:

$$\begin{aligned} V_i(k, x(k)) &= x^T(k)v_i(k) \\ &\quad + \sum_{h=k-d(k)}^{k-1} x^T(h)s\tilde{D}^T v_i(k) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1}^{k-1} x^T(h)\tilde{D}^T v_i(k). \end{aligned} \tag{62}$$

Considering $V_i(k+1, x(k+1))$ in (62) along the trajectories of system (5), we have

$$\begin{aligned} V_i(k+1, x(k+1)) &= x^T(k)A_i^T v_i(k+1) \\ &\quad + x^T(k-d(k))\tilde{D}_i^T v_i(k+1) \\ &\quad + \sum_{h=k+1-d(k+1)}^k x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^k x^T(h)\tilde{D}^T v_i(k+1), \end{aligned} \tag{63}$$

then

$$\begin{aligned} V_i(k+1, x(k+1)) &\leq x^T(k)\bar{A}_i^T v_i(k+1) \\ &\quad + x^T(k-d(k))\bar{D}_i^T v_i(k+1) \\ &\quad + \sum_{h=k+1-d(k+1)}^k x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^k x^T(h)\tilde{D}^T v_i(k+1). \end{aligned} \tag{64}$$

It follows that

$$\begin{aligned} &V_i(k+1, x(k+1)) - \gamma V_i(k, x(k)) \\ &\leq x^T(k)\bar{A}_i^T v_i(k+1) + x^T(k-d(k))\bar{D}_i^T v_i(k+1) \\ &\quad + \sum_{h=k+1-d(k+1)}^k x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^k x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad - \gamma x^T(k)v_i(k) - \gamma \sum_{h=k-d(k)}^{k-1} x^T(h)\tilde{D}^T v_i(k) \\ &\quad - \gamma \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1}^{k-1} x^T(h)\tilde{D}^T v_i(k). \end{aligned} \tag{65}$$

In fact $\bar{D}_i \preceq \tilde{D}$ for all $i \in \underline{N}$. Then

$$\begin{aligned} &V_i(k+1, x(k+1)) - \gamma V_i(k, x(k)) \\ &\leq x^T(k)\bar{A}_i^T v_i(k+1) + x^T(k-d(k))\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{h=k+1-d(k+1)}^k x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^k x^T(h)\tilde{D}^T v_i(k+1) - \gamma x^T(k)v_i(k) \\ &\quad - \gamma \sum_{h=k-d(k)}^{k-1} x^T(h)\tilde{D}^T v_i(k) - \gamma \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1}^{k-1} x^T(h)\tilde{D}^T v_i(k) \\ &\leq x^T(k)\bar{A}_i^T v_i(k+1) + x^T(k-d(k))\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{h=k-d_2+1}^{k-1} x^T(h)\tilde{D}^T v_i(k+1) + x^T(k)\tilde{D}^T v_i(k+1) \\ &\quad - \gamma x^T(k)v_i(k) - \gamma \sum_{h=k-d_1+1}^{k-1} x^T(h)\tilde{D}^T v_i(k) \\ &\quad - \gamma x^T(k-d(k))\tilde{D}^T v_i(k) + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^{k-1} x^T(h)\tilde{D}^T v_i(k+1) \\ &\quad + \sum_{l=-d_2+1}^{-d_1} x^T(k)\tilde{D}^T v_i(k+1) - \gamma \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^{k-1} x^T(h)\tilde{D}^T v_i(k) \\ &\quad - \gamma \sum_{l=-d_2+1}^{-d_1} x^T(k+l)\tilde{D}^T v_i(k). \end{aligned} \tag{66}$$

We observe that

$$\sum_{l=-d_2+1}^{-d_1} x^T(k)\tilde{D}^T v_i(k+1) = (d_2 - d_1)x^T(k)\tilde{D}^T v_i(k+1), \tag{67}$$

$$\gamma \sum_{l=-d_2+1}^{-d_1} x^T(k+l) \tilde{D}^T v_i(k) = \gamma \sum_{h=k-d_2+1}^{k-d_1} x^T(h) \tilde{D}^T v_i(k). \quad (68)$$

From (66), we arrive at

$$\begin{aligned} & V_i(k+1, x(k+1)) - \gamma V_i(k, x(k)) \\ & \leq x^T(k) \left[\bar{A}_i^T v_i(k+1) + (d_2 - d_1 + 1) \tilde{D}^T v_i(k+1) - \gamma v_i(k) \right] \\ & \quad + x^T(k-d(k)) \tilde{D}^T [v_i(k+1) - \gamma v_i(k)] \\ & \quad + \sum_{h=k-d_2+1}^{k-1} x^T(h) \tilde{D}^T [v_i(k+1) - \gamma v_i(k)] \\ & \quad + \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+1+l}^{k-1} x^T(h) \tilde{D}^T [v_i(k+1) - \gamma v_i(k)]. \end{aligned} \quad (69)$$

When $k \in \Xi_{m,q} \subset [k_m, k_m + \hat{\kappa})$, it can be seen that

$$\begin{aligned} & \bar{A}_i^T v_i(k+1) + (d_2 - d_1 + 1) \tilde{D}^T v_i(k+1) - \gamma v_i(k) \\ & = \left(1 - \frac{r}{\bar{h}}\right) \left[\left(\bar{A}_i^T + (d_2 - d_1 + 1) \tilde{D}^T \right) \Pi_{i,q} \right. \\ & \quad \left. + \left(\bar{A}_i^T + (d_2 - d_1 + 1) \tilde{D}^T - \gamma I_n \right) v_{i,q} \right] \\ & \quad + \frac{r}{\bar{h}} \left[\left(\bar{A}_i^T + (d_2 - d_1 + 1) \tilde{D}^T \right) \Pi_{i,q} \right. \\ & \quad \left. + \left(\bar{A}_i^T + (d_2 - d_1 + 1) \tilde{D}^T - \gamma I_n \right) v_{i,q+1} \right], \end{aligned} \quad (70)$$

$$\begin{aligned} & \tilde{D}^T [v_i(k+1) - \gamma v_i(k)] \\ & = \left(1 - \frac{r}{\bar{h}}\right) \tilde{D}^T (\Pi_{i,q} + (1 - \gamma) v_{i,q}) \\ & \quad + \frac{r}{\bar{h}} \tilde{D}^T (\Pi_{i,q} + (1 - \gamma) v_{i,q+1}). \end{aligned} \quad (71)$$

According to conditions (51)–(54), we obtain

$$V_i(k+1, x(k+1)) - \gamma V_i(k, x(k)) < 0, \quad k \in \Xi_{m,q}, \quad (72)$$

which implies

$$V_i(k+1, x(k+1)) < \gamma V_i(k, x(k)), \quad k \in \bigcup_{q=0}^{L-1} \Xi_{m,q} = [k_m, k_m + \hat{\kappa}). \quad (73)$$

For $k \in [k_m + \hat{\kappa}, k_{m+1})$, it yields that

$$\begin{aligned} & \bar{A}_i^T v_i(k+1) + (d_2 - d_1 + 1) \tilde{D}^T v_i(k+1) - \gamma v_i(k) \\ & = \left(\bar{A}_i^T + (d_2 - d_1 + 1) \tilde{D}^T - \gamma I_n \right) v_{i,L}, \end{aligned} \quad (74)$$

$$\tilde{D}^T [v_i(k+1) - \gamma v_i(k)] = \tilde{D}^T (1 - \gamma) v_{i,L} < 0. \quad (75)$$

Utilizing condition (55), we get

$$V_i(k+1, x(k+1)) < \gamma V_i(k, x(k)), \quad k \in [k_m + \hat{\kappa}, k_{m+1}). \quad (76)$$

Combining (73) with (76), we can easily see that

$$V_i(k+1, x(k+1)) < \gamma V_i(k, x(k)), \quad k \in [k_m, k_{m+1}), \quad m \in \mathbb{N}_0. \quad (77)$$

This implies

$$V_i(k, x(k)) < \gamma^{k-k_m} V_i(k_m, x(k_m)), \quad k \in [k_m, k_{m+1}), \quad m \in \mathbb{N}_0. \quad (78)$$

Using (59), (62) and condition (56), we have

$$\begin{aligned} & V_i(k_m, x(k_m)) \leq \mu V_j(k_m - 1, x(k_m - 1)), \quad k_m - 1 \\ & \quad \in [k_{m-1} + \hat{\kappa}, k_m). \end{aligned} \quad (79)$$

Since $[k_{m-1} + \hat{\kappa}, k_m) \subset [k_{m-1}, k_m)$, one can claim from (78) that

$$\begin{aligned} & V_j(k_m - 1, x(k_m - 1)) \\ & < \gamma^{(k_m-1)-k_{m-1}} V_j(k_{m-1}, x(k_{m-1})), \quad k_m - 1 \\ & \quad \in [k_{m-1}, k_m), \end{aligned} \quad (80)$$

which implies

$$\begin{aligned} & V_j(k_m - 1, x(k_m - 1)) \\ & < \gamma^{k_m - k_{m-1}} V_j(k_{m-1}, x(k_{m-1})), \quad k_m - 1 \\ & \quad \in [k_{m-1}, k_m), \quad m \in \mathbb{N}_0. \end{aligned} \quad (81)$$

From (78), (79), and (81), we can derive

$$V_{\sigma(k_m)}(k, x(k)) < \gamma^{k-k_m} \mu \gamma^{k_m - k_{m-1}} V_j(k_{m-1}, x(k_{m-1})), \quad (82)$$

then

$$\begin{aligned} & V_{\sigma(k_m)}(k, x(k)) < \mu \mu \cdots \mu \gamma^{k-k_m} \gamma^{k_m - k_{m-1}} \cdots \gamma^{k_1 - k_0} V_{\sigma(k_0)}(k_0, x(k_0)) \\ & \quad = \mu^{N_{\sigma}(k, k_0)} \gamma^{k-k_0} V_{\sigma(k_0)}(k_0, x(k_0)). \end{aligned} \quad (83)$$

It follows from Definition 16 and $0 < \mu < 1$ by employing (6), then, we obtain

$$V_{\sigma(k_m)}(k, x(k)) \leq \mu^{(N_0 + k - k_0/\kappa_a)} \gamma^{k-k_0} V_{\sigma(k_0)}(k_0, x(k_0)), \quad (84)$$

$$= \gamma^{N_0 \log_{\gamma} \mu} \gamma^{(1 + \log_{\gamma} \mu / \kappa_a)(k-k_0)} V_{\sigma(k_0)}(k_0, x(k_0)). \quad (85)$$

Without loss of generality, we impose that $V_{\sigma(k_0)}(k_0, x(k_0)) = V_{\sigma(0)}(0, x(0))$. From (59) and (62), we have

$$\begin{aligned}
V_{\sigma(0)}(0, x(0)) &\leq \|x^T(0)\|_2 \|v_{\sigma(0)}(0)\|_2 + \sum_{h=-\hat{d}}^{-1} \max_{h \in \{-\hat{d}, -\hat{d}+1, \dots, -1, 0\}} \\
&\quad \cdot \|x^T(h)\|_2 \|\tilde{D}^T\|_2 \|v_{\sigma(0)}(0)\|_2 \\
&\quad + \sum_{l=-\hat{d}_2+1}^{-\hat{d}_1} \sum_{h=l}^{-1} \max_{h \in \{-\hat{d}, -\hat{d}+1, \dots, -1, 0\}} \\
&\quad \cdot \|x^T(h)\|_2 \|\tilde{D}^T\|_2 \|v_{\sigma(0)}(0)\|_2 \\
&\leq \|x^T(0)\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 + \hat{d} \max_{h \in \{-\hat{d}, -\hat{d}+1, \dots, -1, 0\}} \\
&\quad \cdot \|x^T(h)\|_2 \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \left(\frac{d_2-d_1}{2}\right) (d_2+d_1-1) \max_{h \in \{-\hat{d}, -\hat{d}+1, \dots, -1, 0\}} \\
&\quad \cdot \|x^T(h)\|_2 \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\leq \sqrt{n} \|\varphi\|_{\hat{d}} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \hat{d} \sqrt{n} \|\varphi\|_{\hat{d}} \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \\
&\quad + \left(\frac{d_2-d_1}{2}\right) (d_2+d_1-1) \sqrt{n} \|\varphi\|_{\hat{d}} \|\tilde{D}^T\|_2 \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2,
\end{aligned} \tag{86}$$

$$= \left[1 + \left(\hat{d} + \left(\frac{d_2-d_1}{2} \right) (d_2+d_1-1) \right) \|\tilde{D}^T\|_2 \right] \sqrt{n} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2 \|\varphi\|_{\hat{d}}. \tag{87}$$

Furthermore, we get

$$\varsigma \|x(k)\|_2 \leq V_{\sigma(k_m)}(k, x(k)), \tag{88}$$

where $\varsigma = \min_{(a,b) \in \underline{N} \times \underline{L}_0} \{\underline{\omega}(v_{a,b})\}$. Substituting (87) and (88) into (85), we obtain

$$\begin{aligned}
\|x(k)\|_2 &\leq \frac{\gamma^{N_0 \log_\gamma \mu}}{\varsigma} \gamma^{(1+\log_\gamma \mu/\kappa_a)(k-k_0)} \\
&\quad \cdot \left[1 + \left(\hat{d} + \left(\frac{d_2-d_1}{2} \right) (d_2+d_1-1) \right) \|\tilde{D}^T\|_2 \right] \sqrt{n} \sum_{b \in \underline{L}_0} \\
&\quad \cdot \|v_{\sigma(0),b}\|_2 \|\varphi\|_{\hat{d}},
\end{aligned} \tag{89}$$

for $k \geq k_0$. According to the FADT (57), it is immediate that

$$\left(1 + \frac{\log_\gamma \mu}{\kappa_a} \right) < 0, \tag{90}$$

which yields

$$0 < \gamma^{(1+\log_\gamma \mu/\kappa_a)} < 1. \tag{91}$$

From (89), for $k \geq k_0$, we arrive at

$$\|x(k)\|_2 \leq \varepsilon \rho^{k-k_0} \|\varphi\|_{\hat{d}}, \tag{92}$$

where $\varepsilon = (\gamma^{N_0 \log_\gamma \mu}/\varsigma)[1 + (\hat{d} + (d_2 - d_1/2)(d_2 + d_1 - 1))\|\tilde{D}^T\|_2] \sqrt{n} \sum_{b \in \underline{L}_0} \|v_{\sigma(0),b}\|_2$, and $\rho = \gamma^{(1+\log_\gamma \mu/\kappa_a)}$.

By Definition 12, we can conclude that system (5) is GUES under the switching signal with the FADT satisfying condition (57).

Remark 23. Unlike the discretized copositive Lyapunov function utilized in [35], the time-scheduled MCLKF defined in (62) is constructed specifically for system (3) including both interval uncertainties and AUs. Furthermore, our time-scheduled MCLKF, which is different from the discretized copositive Lyapunov-Krasovskii functional in [43], and the FADT switching method are applied together to guarantee the global uniform exponential stability of the studied system. Therefore, it is worth noting that here we consider the more general systems, and our theoretical result is more general than those of Theorem 3.4 and Theorem 3.6 in [43].

Remark 24. Apart from the studies in [32] that proposed practical exponential stability criteria for discrete-time linear SPSs with impulse, disturbance, and all modes unstable, our result shown in Theorem 22 focuses on the robust exponential stability of system (5) including both interval uncertainties and AUs. In addition, it should be noted that the conditions of Theorem 3 and Theorem 4 in [35] ensure the global asymptotic stability of discrete-time linear SPSs with AUs. Still, the existence of the time-varying delay has not been taken into account yet. Hence, our theoretical result is more general and applicable than those of the results in [32, 35].

The last result of the discrete-time system (5) without its interval uncertainty will be shown in the following:

Corollary 25. Consider the discrete-time system (5) without its interval uncertainty satisfying Assumption 14. Let $A_i \geq 0$ and $D_i \geq 0, \forall i \in \underline{N}$. Given constants $0 < \mu < 1, \gamma > 1, \kappa^* > 0$, and $L \in \underline{N}$. If there exist positive vectors $v_{i,q}, i \in \underline{N}, q \in \underline{L}_0$, and constants $\kappa_a \geq \kappa^*$ such that

$$\begin{aligned}
&(A_i^T + (d_2 - d_1 + 1)D^T)\Pi_{i,q} \\
&\quad + (A_i^T + (d_2 - d_1 + 1)D^T - \gamma I_n)v_{i,q} < 0,
\end{aligned} \tag{93}$$

$$\begin{aligned}
&(A_i^T + (d_2 - d_1 + 1)D^T)\Pi_{i,q} \\
&\quad + (A_i^T + (d_2 - d_1 + 1)D^T - \gamma I_n)v_{i,q+1} < 0,
\end{aligned} \tag{94}$$

$$D^T(\Pi_{i,q} + (1 - \gamma)v_{i,q}) < 0, \tag{95}$$

$$D^T(\Pi_{i,q} + (1 - \gamma)v_{i,q+1}) < 0, \tag{96}$$

$$(A_i^T + (d_2 - d_1 + 1)D^T - \gamma I_n)v_{i,L} < 0, \tag{97}$$

$$v_{i,0} - \mu v_{j,L} \leq 0, \quad (98)$$

hold for any $q = 0, 1, \dots, L-1$, and for any $i, j \in \underline{N}$, $i \neq j$, then, the discrete-time system (5) without its interval uncertainty is positive and GUES with the same FADT satisfying (57) where $\Pi_{i,q}$ and D are defined as in (58) and (8), respectively.

Proof. With the same symbols and vector function (59) in Theorem 22, the proof of this corollary can be achieved by using the time-scheduled MCLKF in the form of

$$\begin{aligned} V_i(k, x(k)) &= x^T(k)v_i(k) \\ &+ \sum_{h=k-d(k)}^{k-1} x^T(h)D^T v_i(k) \\ &+ \sum_{l=-d_2+1}^{-d_1} \sum_{h=k+l}^{k-1} x^T(h)D^T v_i(k). \end{aligned} \quad (99)$$

The proof is very similar to that of Theorem 22, and the methodology can easily derive it as above. Therefore, the rest of the proof of this corollary is omitted here. \square

Remark 26. Compared with the results of Theorem 3.4 and Theorem 3.6 in [43], in this corollary, we use the FADT switching technique to verify the exponential stability criteria for system (3) with AUs. Thus, our theoretical result is more general than those of the results in [43].

4. Numerical Simulations

In this section, we intensively provide two numerical examples along with the simulation results to demonstrate the correctness and effectiveness of our theoretical analysis presented in the previous section.

Example 1. The robust stability problem for the continuous-time system (1) comprising of two subsystems is studied in this example. The system data are given as follows:

$$\begin{aligned} \underline{A}_1 &= \begin{bmatrix} -2.62 & 0.66 \\ 0.14 & 0.005 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} -2.6 & 0.7 \\ 0.15 & 0.01 \end{bmatrix}, \\ \underline{D}_1 &= \begin{bmatrix} 0.004 & 0.002 \\ 0.001 & 0.008 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 0.005 & 0.003 \\ 0.002 & 0.01 \end{bmatrix}, \end{aligned} \quad (100)$$

$$\begin{aligned} \underline{A}_2 &= \begin{bmatrix} 0.035 & 0.05 \\ 0.04 & -1.92 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0.036 & 0.06 \\ 0.05 & -1.9 \end{bmatrix}, \\ \underline{D}_2 &= \begin{bmatrix} 0.001 & 0.001 \\ 0.001 & 0.006 \end{bmatrix}, \bar{D}_2 = \begin{bmatrix} 0.003 & 0.002 \\ 0.002 & 0.008 \end{bmatrix}, \end{aligned} \quad (101)$$

$$\alpha(t) = 0.15 + 0.05 \sin(t). \quad (102)$$

Under the given time-varying delay above, we select $\hat{\alpha} = 0.2$ and $\Delta = 0.05$. According to Definition 2, one can see

that \underline{A}_1 and \underline{A}_2 are Metzler matrices. Furthermore, it is obvious that $\underline{D}_1 \geq 0$ and $\underline{D}_2 \geq 0$. By Assumption 5 and Lemma 3, the studied system is positive. We set the initial state $\psi(\theta) = [510]^T$, $\theta \in [-\hat{\alpha}, 0]$, and let the system matrices be

$$\begin{aligned} A_1 &= \frac{\underline{A}_1 + \bar{A}_1}{2} = \begin{bmatrix} -2.61 & 0.68 \\ 0.145 & 0.0075 \end{bmatrix}, \\ D_1 &= \frac{\underline{D}_1 + \bar{D}_1}{2} = \begin{bmatrix} 0.0045 & 0.0025 \\ 0.0015 & 0.009 \end{bmatrix}, \end{aligned} \quad (103)$$

$$\begin{aligned} A_2 &= \frac{\underline{A}_2 + \bar{A}_2}{2} = \begin{bmatrix} 0.0355 & 0.055 \\ 0.045 & -1.91 \end{bmatrix}, \\ D_2 &= \frac{\underline{D}_2 + \bar{D}_2}{2} = \begin{bmatrix} 0.002 & 0.0015 \\ 0.0015 & 0.007 \end{bmatrix}. \end{aligned} \quad (104)$$

We first present two figures for the corresponding state responses of two subsystems. From Figures 1 and 2, it can be seen that two subsystems are positive and unstable.

As defined in (7), it is obvious that

$$\bar{D} = \begin{bmatrix} 0.005 & 0.003 \\ 0.002 & 0.01 \end{bmatrix}. \quad (105)$$

For given scalars $L = 1$, $\lambda = 0.4$, $\mu = 0.657$, and $\tau^* = 1$, we can get a set of feasible solution for Theorem 17:

$$\begin{aligned} v_{1,0} &= \begin{bmatrix} 12.8185 \\ 88.4196 \end{bmatrix}, v_{1,1} = \begin{bmatrix} 31.7095 \\ 105.1495 \end{bmatrix}, \\ v_{2,0} &= \begin{bmatrix} 20.6072 \\ 68.8325 \end{bmatrix}, v_{2,1} = \begin{bmatrix} 20.7160 \\ 135.1355 \end{bmatrix}, \end{aligned} \quad (106)$$

and $\tau_a = 1.0100$. Thus, the continuous-time system (1) is GUES with the FADT switching signal satisfying $1 \leq \tau_a < 1.0502$. The corresponding switching signal $\sigma(t)$ and the state response of the system are illustrated in Figure 3. This shows that our designed switching signal can guarantee the positivity and robust stability of the considered system effectively.

Example 2. To show some advantages of our result, the comparison between the previous result studied in [35] and our result are presented in this example. We consider the two modes of the discrete-time system (5) without time delay and take the same system matrices as given in Example 2 of [35]; namely,

$$A_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 1.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.2 & 0 \\ 0.05 & 0.7 \end{bmatrix}. \quad (107)$$

It is easy to see that $A_1 \geq 0$ and $A_2 \geq 0$. Thus, by Lemma 11, the considered system is positive. For comparison in the view of the numerical simulations, we impose the same

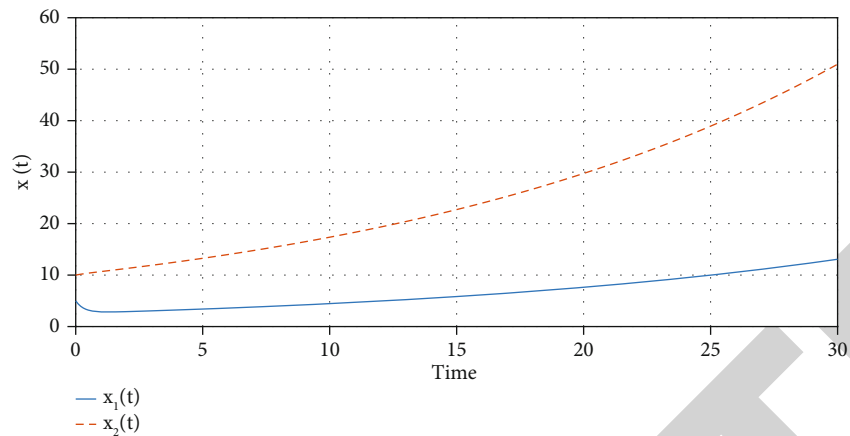


FIGURE 1: State response of the first subsystem.

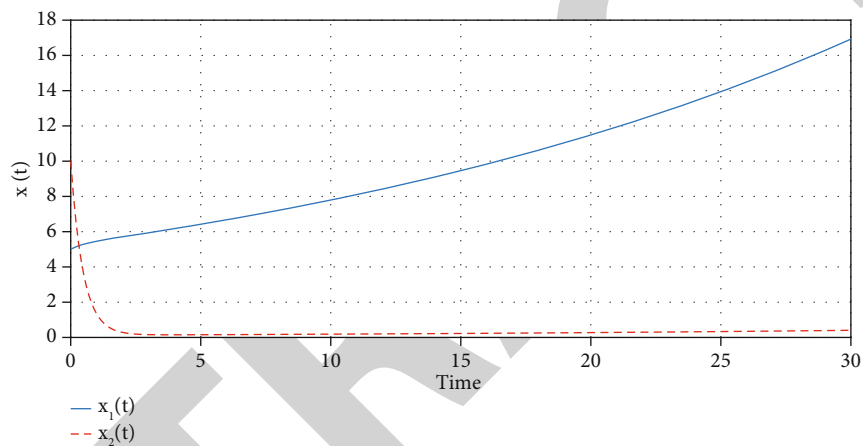


FIGURE 2: State response of the second subsystem.

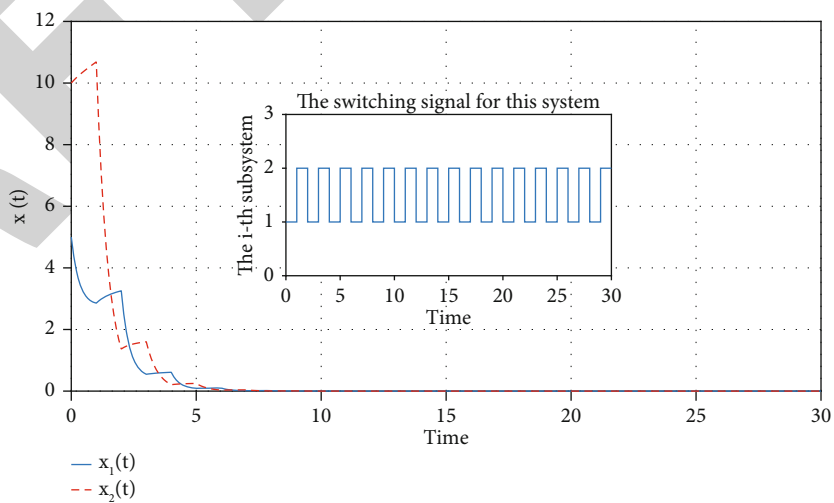


FIGURE 3: State response of the continuous-time system.

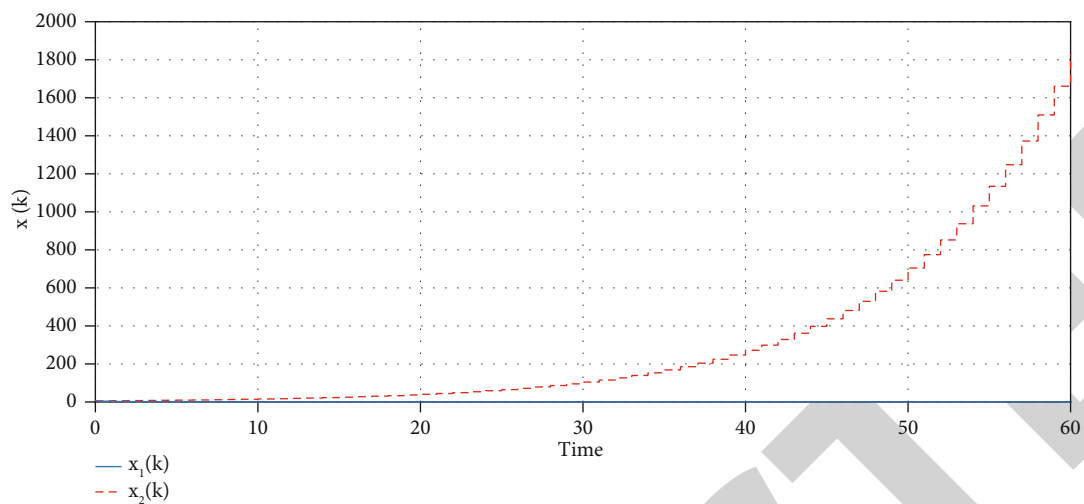


FIGURE 4: State response of the first subsystem.

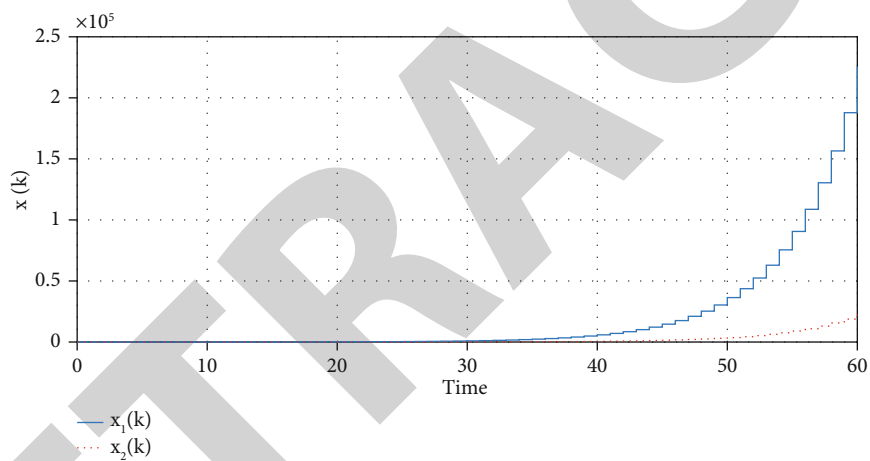


FIGURE 5: State response of the second subsystem.

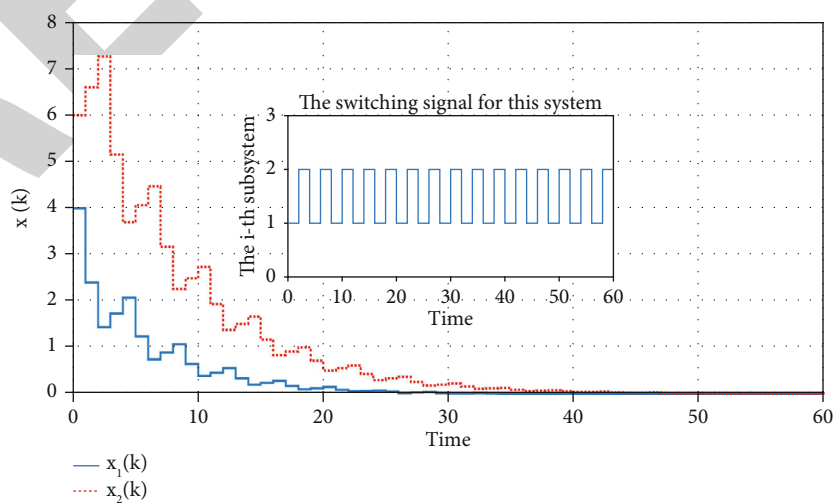


FIGURE 6: State response of the discrete-time system.

initial condition as given in Example 2 of [35]; namely, $x(0) = [46]^T$. From the system matrices above, it is obvious that the eigenvalues of A_1 are $\lambda_1 = 0.6$ and $\lambda_2 = 1.1$ and the eigenvalues of A_2 are $\lambda_1 = 1.2$ and $\lambda_2 = 0.7$. Therefore, both modes are positive and unstable which can be seen from Figures 4 and 5.

To stabilize the considered system via the FADT approach, we can choose $L = 1$, $\gamma = 1.3$, $\mu = 0.532$, and the positive integer $\kappa^* = 2$. By solving conditions in Corollary 25 with $D = 0$, we get the following feasible solution $\kappa_a = 2.2100$,

$$\begin{aligned} v_{1,0} &= \begin{bmatrix} 56.9212 \\ 35.3838 \end{bmatrix}, v_{1,1} = \begin{bmatrix} 186.2338 \\ 48.0901 \end{bmatrix}, \\ v_{2,0} &= \begin{bmatrix} 98.4268 \\ 25.5626 \end{bmatrix}, v_{2,1} = \begin{bmatrix} 107.6403 \\ 67.0016 \end{bmatrix}. \end{aligned} \quad (108)$$

Consequently, the considered system is GUES with the FADT switching signal satisfying $2 \leq \kappa_a < 2.4055$. It should be pointed out that the proportion between the maximum possible time and the minimum DT of this obtained FADT interval is 1.20275. Meanwhile, the proportion between the upper bound and the lower bound of the obtained mode-dependent dwell time for the first subsystem and the second subsystem in Example 2 of [35] is 1.2 and 1.1, respectively. Moreover, it should be noted that the average of the proportion between the upper bound and the lower bound of the obtained mode-dependent dwell time for both subsystems is 1.15. Hence, we can say that our dwell time interval is wider than that of Example 2 in [35]. This shows that our result does not need to switch too often in order to stabilize the considered system. Finally, the state response of the system and the corresponding switching signal $\sigma(k)$ are plotted mutually in Figure 6. The considered system can converge to zero under the FADT switching signal.

5. Conclusions

The global stability problem for both continuous-time and discrete-time LSPDTSs with interval uncertainties in the case of all subsystems are unstable has been intensively studied. By establishing the time-scheduled MCLKFs and applying the FADT switching law, new DDSC under the reasonable assumptions to ensure the global uniform exponential stability of both systems have been derived in the main theorems. Furthermore, novel DDSC of both systems without the interval uncertainties have also been acquired in the corollaries. Finally, two numerical examples have been displayed to validate the effectiveness along with some advantages of obtained theoretical results. In the future, it is interesting to study the global stability of nonlinear switched positive time-varying delay systems with interval uncertainties and all unstable subsystems.

Data Availability

No data were used to support this study. The authors only used MATLAB for simulation. Therefore, simulation programming can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research was supported by Chiang Mai University. This work contains a number of improvements based on comments and suggestions provided by Assoc. Prof. Dr. Thaned Rojsiraphisal, Asst. Prof. Dr. Kanyuta Pochinapan, Asst. Prof. Dr. Ben Wongsajjai, Dr. Nuttawoot Nupo, and Mr. Attapol Praleah.

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