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Research Article

Spectral Radius Formulas Involving Generalized Aluthge Transform

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In this paper, we aim to develop formulas of spectral radius for an operator S in terms of generalized Aluthge transform, numerical radius, iterated generalized Aluthge transform, and asymptotic behavior of powers of S. These formulas generalize some of the formulas of spectral radius existing in literature. As an application, these formulas are used to obtain several characterizations of normaloid operators.

1. Introduction

Generally, in mathematical analysis and particularly in functional analysis, the spectral analysis of operators is an essential research topic. It is useful to study the properties of operators, including spectrum and the spectral radius of operators (see [1]). The spectrum of an operator is connected with an invariant subspace problem on a complex Hilbert space (see [2]), and the important property of spectrum is the expression of spectral radius in various formulas (see [3–5]). These formulas help to obtain several characterizations of operators, including normaloid and spectraloid operators (see [6]). Since the advent of various transformations of bounded linear operators, including Aluthge transform and its generalizations, the study of spectral properties of operators has become the center point for many researchers (see [7–9]).

An operator can be decomposed into two Hermitian operators being its real and imaginary parts, and this decomposition is known as Cartesian decomposition. Clearly, Hermitian operators are self-adjoint and hence symmetric operators. The symmetric operators involved in Cartesian decomposition are helpful to develop the spectral radius formulas and numerical radius inequalities involving Aluthge transform [10–12].

This paper is aimed at studying the generalization of spectral radius formulas involving generalized Aluthge transform. Henceforward, we will give the notions to proceed with the results of this paper.

Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on complex Hilbert space H. Let $S = U|S|$ be the polar decomposition of $S \in \mathscr{B}(\mathscr{H})$, where $|S|$ is the square root of an operator defined as $|S| = \sqrt{S^* S}$ and U is a partial isometry.

In [13], Aluthge introduced a transform to study the properties of hyponormal operators that were connected with the invariant subspace problem in operator theory. This transform is called Aluthge transform, which is defined as

$$\Delta_{1/2} S = |S|^{1/2} U |S|^{1/2}, \tag{1}$$

and its nth iterated Aluthge transform is defined as

$$\begin{aligned} \Delta_{1/2}^{n}(S) &= \Delta\left(\Delta_{1/2}^{n-1}(S)\right), \\ \Delta_{1/2}^{1}(S) &= \Delta(S), \forall n \in \mathbb{N}. \end{aligned} \tag{2}$$

Yamazaki, in [3], gave the formula of spectral radius for bounded linear operator involving iterated Aluthge transform, i.e.,

$$r(S) = \lim_n \|\Delta^n_{1/2}(S)\|. \quad (3)$$

In [14], a generalization of Aluthge transform was introduced that is called λ-Aluthge transform which is defined as

$$\Delta_\lambda S = |S|^\lambda U |S|^{1-\lambda}, \lambda \in [0, 1]. \quad (4)$$

Tam [4] gave a formula of spectral radius involving iterated λ-Aluthge transform for invertible operators using unitarily invariant norm, i.e.,

$$r(S) = \lim_n \|\Delta^n_\lambda(S)\|, \lambda \in (0, 1). \quad (5)$$

Chabbabi and Mbekhta [12] gave various expressions for spectral radius formulas involving λ-Aluthge transform, iterated λ-Aluthge transform, asymptotic behavior of powers of an operator, and numerical radius. The expression of spectral radius involving λ-Aluthge transform is given by

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta_\lambda\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{selfadjoint}}} \left\|\Delta_\lambda\left(e^A S e^{-A}\right)\right\|, \quad (6)$$

and the expressions of spectral radius involving iterated λ-Aluthge transform and the asymptotic behavior of powers of S are given by

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta^n_\lambda\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta^n_\lambda\left(e^A S e^{-A}\right)\right\|, \quad (7)$$

$$r(S) = \lim_k \left\|\Delta^n_\lambda\left(S^k\right)\right\|^{1/k} = \lim_k \left\|\Delta_\lambda\left(S^k\right)\right\|^{1/k}, \quad (8)$$

for each $n \geq 0$.

The expressions of spectral radius involving iterated λ-Aluthge transform, numerical radius, and the asymptotic behavior of powers of S are given by

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|w\left(\Delta^n_\lambda\left(YSY^{-1}\right)\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|w\left(\Delta^n_\lambda\left(e^A S e^{-A}\right)\right)\right\|, \quad (9)$$

$$r(S) = \lim_k \left\|w\left(\Delta^n_\lambda\left(S^k\right)\right)\right\|^{1/k} = \lim_k \left\|w\left(\Delta_\lambda\left(S^k\right)\right)\right\|^{1/k}, \quad (10)$$

for each $n \geq 0$. With the help of the above formulas, the author [14] gave a characterization of normaloid operators.

In [15], Shebrawi and Bakherad introduced a new generalization of Aluthge transform, called generalized Aluthge transform. This transform is defined as

$$\Delta_{f,g} S = f(|S|)Ug(|S|), \quad (11)$$

where f and g both are continuous functions such that $g(x)f(x) = x, x \geq 0$. The iterated generalized Aluthge transform is defined as

$$\Delta^n_{f,g}(S) = \Delta\left(\Delta^{n-1}_{f,g}(S)\right), \forall n \in \mathbb{N}. \quad (12)$$

In this paper, we establish the formulas of spectral radius for operator S by assuming that $\|\Delta_{f,g}(S)\| \leq \|S\|$. These formulas generalize the spectral radius formulas (6)–(10).

The paper is organized as follows. In Section 2, we give the properties of the generalized Aluthge transform. In Section 3, spectral radius formulas involving generalized Aluthge transform and asymptotic behavior of powers of the bounded operator S are given. In Section 4, we develop spectral radius formulas of bounded linear operators involving numerical radius of generalized Aluthge transform. Furthermore, some characterizations of normaloid operators are established.

2. Preliminaries and Some Auxiliary Results

We start this section with some basic definitions and properties of generalized Aluthge transform which will be useful in establishing the main results of this paper. An operator T is *similar* to S if there exists an invertible operator Y such that $S = Y^{-1}TY$ (see [16]). If $r(S) = \|S\|$, then the operator is said to be *normaloid*. An operator S is said to be a *contraction* if $\|S\| \leq 1$. The *spectral radius* of an operator S is defined as

$$r(S) = \sup\{|\lambda| : \lambda \in \sigma(S)\}, \quad (13)$$

where $\sigma(S)$ is the spectrum of the operator S.

To prove spectral radius formulas, we recall some properties of generalized Aluthge transform.

Proposition 1 [7]. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

(i) $\sigma(S) = \sigma(\Delta_{f,g}(S))$

(ii) $r(S) = r(\Delta_{f,g}(S))$

Proposition 2. *Let $T, S \in \mathscr{B}(\mathscr{H})$. If T is similar to S, then*

(i) $\sigma(S) = \sigma(T)$

(ii) $\sigma(\Delta_{f,g}(S)) = \sigma(\Delta_{f,g}(T))$

(iii) $r(\Delta_{f,g}(S)) = r(\Delta_{f,g}(T))$

Proof. The proofs of parts (i) and (iii) are trivial. The proof of part (ii) follows from part (i) and Proposition 1 (i). □

Proposition 3. *Let $S \in \mathscr{B}(\mathscr{H})$ and f be any continuous function on $\sigma(S)$. Then,*

$$f(U|S|U^*) = Uf(|S|)U^*, \tag{14}$$

for any unitary $U \in \mathscr{B}(\mathscr{H})$.

Proof. Since $U^*U|S| = |S|$, we have

$$(U|S|U^*)^n = U|S|^n U^*, \tag{15}$$

for each $n \in \mathbb{N}$, which implies

$$P(U|S|U^*) = UP(|S|)U^*, \tag{16}$$

for any polynomial $P(t)$. Since f is a continuous, so there exist a sequence of polynomial $\{P_n(t)\}_{n=1}^{\infty}$ such that $P_n(0) = 0$ for each $n \in \mathbb{N}$, and $\{P_n(t)\}_{n=1}^{\infty}$ converges uniformly to $f(t)$ on the interval $[0, \||T|\|]$. Then, from Equation (16), we have

$$\begin{aligned} f(U|S|U^*) &= \lim_{n\longrightarrow\infty} P_n(U|S|U^*) = \lim_{n\longrightarrow\infty} (UP_n(|S|)U^*) \\ &= U \lim_{n\longrightarrow\infty} P_n(|S|)U^* = Uf(|S|)U^*, \end{aligned} \tag{17}$$

as required. □

Proposition 4. *Let $S, U \in \mathscr{B}(\mathscr{H})$ such that U is unitary. Then, we have*

$$\Delta_{f,g}(USU^*) = U\Delta_{f,g}(S)U^*. \tag{18}$$

Proof. Let $S = V|S|$ be the polar decomposition of S. Then, we have

$$|USU^*| = U|S|U^*. \tag{19}$$

Now by using Proposition 3, we have

$$f(|USU^*|) = Uf(|S|)U^*. \tag{20}$$

The polar decomposition of operator USU^* is as follows:

$$\begin{aligned} USU^* &= UV|S|U^*, \\ USU^* &= (UVU^*)(U|S|U^*), \end{aligned} \tag{21}$$

where UVU^* is partial isometry. Therefore,

$$\begin{aligned} \Delta_{f,g}(USU^*) &= f((USU^*))UVU^*g((USU^*)) \\ &= Uf((S))Vg((S))U^* = U\Delta_{f,g}(S)U^*. \end{aligned} \tag{22}$$

The second equality holds by Proposition 3 and by the fact that $U^*U = I$. □

Proposition 5. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the sequence $\{\|\Delta_{f,g}^n(S)\|\}_{n=1}^{\infty}$ is nonincreasing.*

Proof. The proof follows from the repeated application of the inequality

$$\left\|\Delta_{f,g}(S)\right\| \le \|S\|. \tag{23}$$

□

3. Formulas of Spectral Radius Involving Generalized Aluthge Transform

In this section, we give formulas of the spectral radius by using Rota's theorem [16] and the properties of generalized Aluthge transform.

Theorem 6. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H}) \\ invertible}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| = \inf_{\substack{A\in\mathscr{B}(\mathscr{H}) \\ self\text{-}adjoint}} \left\|\Delta_{f,g}\left(e^A S e^{-A}\right)\right\|. \tag{24}$$

Proof. From Propositions 1 and 2, we have

$$r(S) = r\left(\Delta_{f,g}\left(YSY^{-1}\right)\right). \tag{25}$$

It follows that

$$r(S) = r\left(\Delta_{f,g}\left(YSY^{-1}\right)\right) \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}). \tag{26}$$

Hence,

$$r(S) \le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\|. \tag{27}$$

Let $Y = U|Y|$ be the polar decomposition of Y. Since Y is an invertible operator, then U is unitary and $|Y|$ invertible. Therefore, there exists $\beta > 0$ such that $\sigma(|Y|) \subseteq [\beta,\infty)$. Consequently, $A = \ln(|Y|)$ exists and self-adjoint; then, we have

$$\begin{aligned} |Y| &= e^A, \\ |Y|^{-1} &= e^{-A}. \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| &= \left\|\Delta_{f,g}(U|Y|)S(U|Y|)^{-1})\right\| \\ &= \left\|U\left(\Delta_{f,g}|Y|S|Y|^{-1}\right)U^*\right\| \\ &= \left\|U\left(\Delta_{f,g}\left(e^A S e^{-A}\right)U^*\right\| \\ &= \left\|\Delta_{f,g}\left(e^A S e^{-A}\right)\right\|. \end{aligned} \tag{29}$$

The second equality holds by Proposition 4. Hence,

$$r(S) \le \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| \Delta_{f,g}\left(YSY^{-1}\right) \right\| \le \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}\left(e^{A}Se^{-A}\right) \right\|. \tag{30}$$

To prove above inequality in other direction, for an arbitrary $\varepsilon > 0$, we define an operator

$$S_\varepsilon = \frac{S}{r(S) + \varepsilon}. \tag{31}$$

For operator S_ε, we have

$$r(S_\varepsilon) = r\left(\frac{S}{r(S)+\varepsilon}\right) = \frac{r(S)}{r(S)+\varepsilon} < 1. \tag{32}$$

From [16], Theorem 2, the spectrum of operator S_ε lies in the unit disk; thus, the operator S_ε is similar to contraction for which there exists an invertible operator $Y_\varepsilon \in \mathscr{B}(\mathscr{H})$ such that

$$\left\| \frac{Y_\varepsilon S Y_\varepsilon^{-1}}{r(S)+\varepsilon} \right\| < 1, \tag{33}$$

and this implies that

$$\left\| \Delta_{f,g}\left(e^{A_\varepsilon} S e^{-A_\varepsilon}\right) \right\| \le \left\| Y_\varepsilon S Y_\varepsilon^{-1} \right\| < r(S) + \varepsilon. \tag{34}$$

For $\varepsilon > 0$, we obtain

$$\begin{aligned} r(S) &\le \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| \Delta_{f,g}\left(YSY^{-1}\right) \right\| \le \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}\left(e^{A}Se^{-A}\right) \right\| \\ &\le \inf_{\substack{A_\varepsilon \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}\left(e^{A_\varepsilon}Se^{-A_\varepsilon}\right) \right\| \le \inf_{\substack{Y_\varepsilon \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| Y_\varepsilon S Y_\varepsilon^{-1} \right\| \le r(S) + \varepsilon. \end{aligned} \tag{35}$$

Since $\varepsilon > 0$ is arbitrary, therefore

$$r(S) = \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| \Delta_{f,g}\left(YSY^{-1}\right) \right\| = \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}\left(e^{A}Se^{-A}\right) \right\|. \tag{36}$$

□

The next Corollary is the direct result of Theorem 6 involving iterated generalized Aluthge transform.

Corollary 7. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for each $n \in \mathbb{N}$, we have*

$$r(S) = \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| \Delta_{f,g}^{n}\left(YSY^{-1}\right) \right\| = \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}^{n}\left(e^{A}Se^{-A}\right) \right\|. \tag{37}$$

Proof. From Propositions 1 and 2, we can easily obtain

$$r\left(\Delta_{f,g}^{n}\left(YSY^{-1}\right)\right) = r(S), \forall n \in \mathbb{N}. \tag{38}$$

From above equality and by using Proposition 5, we have

$$r(S) \le \left\| \Delta_{f,g}^{n}\left(YSY^{-1}\right) \right\| \le \left\| \Delta_{f,g}\left(YSY^{-1}\right) \right\|, \tag{39}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. Therefore,

$$\begin{aligned} r(S) &\le \inf_{\substack{Y \in \mathscr{B}(\mathscr{H}) \\ \text{invertible}}} \left\| \Delta_{f,g}^{n}\left(YSY^{-1}\right) \right\| \le \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}^{n}\left(e^{A}Se^{-A}\right) \right\| \\ &\le \inf_{\substack{A \in \mathscr{B}(\mathscr{H}) \\ \text{self-adjoint}}} \left\| \Delta_{f,g}\left(e^{A}Se^{-A}\right) \right\| = r(S). \end{aligned} \tag{40}$$

The third inequality holds by Proposition 5, and the last equality holds by Theorem 6, which completes the proof. □

The next Corollary is the direct result of Corollary 7 that is the characterization of normaloid operators.

Corollary 8. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the following assertions are equivalent*

(i) S is normaloid

(ii) $\|S\| \le \|YSY^{-1}\|$, for invertible $Y \in \mathscr{B}(\mathscr{H})$

Proof. Assume that S is normaloid. Then,

$$\|S\| = r\left(YSY^{-1}\right) \le \left\| \Delta_{f,g}\left(YSY^{-1}\right) \right\| \le \left\| YSY^{-1} \right\|, \tag{41}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The first equality holds by Proposition 2. The first inequality holds because the spectral radius is less than the operator norm, and the second inequality holds by Proposition 5.

Assume that assertion (ii) holds. Then, we have

$$r(S) \le \|S\| \le \left\| YSY^{-1} \right\| \le \left\| Y\varepsilon SY\varepsilon^{-1} \right\| \le r(S) + \varepsilon, \tag{42}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The last inequality holds by inequality (33) in Theorem 6. Since $\varepsilon > 0$ is arbitrary, hence S is normaloid. □

Corollary 9. *Let $S \in \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent.*

(i) S is normaloid;

(ii) $\|S\| \le \|\Delta_{f,g}(YSY^{-1})\|$ for invertible $Y \in \mathscr{B}(\mathscr{H})$;

(iii) $\|S\| \le \|\Delta_{f,g}^{n}(YSY^{-1})\|$ for invertible $Y \in \mathscr{B}(\mathscr{H})$ and every $n \in \mathbb{N}$.

Proof. (i)⇒(iii)⇒(ii). Since S is normaloid, therefore

$$\|S\| = r\left(\Delta^n_{f,g} YSY^{-1}\right) \le \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\| \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\|, \tag{43}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The first inequality holds because the spectral radius is less than the operator norm, and the second inequality holds by Proposition 5. Hence,

$$\begin{aligned} \|S\| &\le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}), \\ \|S\| &\le \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\| \text{ for invertible } Y \in \mathscr{B}(\mathscr{H}). \end{aligned} \tag{44}$$

(ii)⇒(i)

Since spectral radius is less than operator norm and by assertion (ii), we have

$$r(S) \le \|S\| \le \left\|\Delta_{f,g}\left(YSY^{-1}\right)\right\| \le \left\|\Delta_{f,g}\left(Y_\varepsilon S Y_\varepsilon^{-1}\right)\right\| \le \left\|Y_\varepsilon S Y_\varepsilon^{-1}\right\| \le r(S) + \varepsilon, \tag{45}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. The third inequality holds by inequality (34) of Theorem 6. Since $\varepsilon > 0$ is arbitrary, therefore S is normaloid. □

Now, we will give a formula of spectral radius involving iterated generalized Aluthge transform and asymptotic behavior of powers of S.

Theorem 10. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \lim_k \left\|\Delta^n_{f,g}\left(S^k\right)\right\|^{1/k}, \forall n \in \mathbb{N} = \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k}. \tag{46}$$

Proof.

$$r(S) = r\left(\Delta^n_{f,g}(S)\right) \le \left\|\Delta^n_{f,g}(S)\right\| \le \left\|\Delta_{f,g}(S)\right\| \le \|S\|, \forall n \in \mathbb{N}. \tag{47}$$

The first equality holds by Proposition 1, second inequality holds by $r(S) \le \|S\|$, and third inequality holds by Proposition 5. Thus, for kth power of an operator, we have

$$\begin{aligned} r(S)^k &= r\left(S^k\right) = r\left(\Delta^n_{f,g}\left(S^k\right)\right) \le \left\|\Delta^n_{f,g}\left(S^k\right)\right\| \\ &\le \left\|\Delta_{f,g}\left(S^k\right)\right\| \le \left\|S^k\right\|, \forall n, k \in \mathbb{N}, \\ r(S) &\le \left\|\Delta^n_{f,g}\left(S^k\right)\right\|^{1/k} \le \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} \le \left\|S^k\right\|^{1/k}, \forall n, k \in \mathbb{N}, \\ r(S) &\le \lim_k \left\|\Delta^n_{f,g}\left(S^k\right)\right\|^{1/k} \le \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} \le \lim_k \left\|S^k\right\|^{1/k}, \forall n \in \mathbb{N}. \end{aligned} \tag{48}$$

Since

$$r(S) = \lim_k \left\|S^k\right\|^{1/k}. \tag{49}$$

Thus,

$$\begin{aligned} r(S) &\le \lim_k \left\|\Delta^n_{f,g}\left(S^k\right)\right\|^{1/k} \le \lim_k \left\|\Delta_{f,g}\left(S^k\right)\right\|^{1/k} \\ &\le \lim_k \left\|S^k\right\|^{1/k} = r(S), \forall n \in \mathbb{N}, \end{aligned} \tag{50}$$

which completes the proof. □

The next Corollary is obtain in the consequence of Theorem 10.

Corollary 11. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, the following assertions are equivalent.*

(i) S is normaloid

(ii) $\|S\|^k = \|\Delta_{f,g}(S^k)\|, \forall k \in \mathbb{N}$

(iii) $\|S\|^k = \|\Delta^n_{f,g}(S^k)\|, \forall n, k \in \mathbb{N}$

Proof. (i)⇒(ii).

$$\begin{aligned} \|S\| &= \lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \\ \|S\|^k &= \left(\lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}\right)^k, \\ \|S\|^k &= \left\|\Delta_{f,g}(S)^k\right\|, \forall k \in \mathbb{N}. \end{aligned} \tag{51}$$

The first equality holds by assertion (i) and Theorem 10.

(i)⇒(iii)

$$\begin{aligned} \|S\| &= \lim_k \left\|\Delta^n_{f,g}(S)^k\right\|^{1/k}, \forall n \in \mathbb{N}, \\ \|S\|^k &= \left\|\Delta^n_{f,g}(S)^k\right\|, \forall n, k \in \mathbb{N}. \end{aligned} \tag{52}$$

The first equality holds by assertion (i) and Theorem 10.

(ii)⇒(i)

$$\begin{aligned} \|S\|^k &= \left\|\Delta_{f,g}(S)^k\right\|, \forall k \in \mathbb{N}, \\ \left(\|S\|^k\right)^{1/k} &= \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \forall k \in \mathbb{N}, \\ \lim_k \|S\| &= \lim_k \left\|\Delta_{f,g}(S)^k\right\|^{1/k}, \\ r(S) &= \|S\|. \end{aligned} \tag{53}$$

The last equality holds by Theorem 10.

(iii)⇒(i)

$$\|S\|^k = \left\|\Delta^n_{f,g}(S)^k\right\|, \forall n, k \in \mathbb{N},$$
$$\lim_k \|S\| = \lim_k \left\|\Delta^n_{f,g}(S)^k\right\|^{1/k}, \forall n \in \mathbb{N}, \quad (54)$$
$$\|S\| = r(S).$$

The last equality holds by Theorem 10. Hence, S is normaloid. □

4. Formulas of Spectral Radius Involving Generalized Aluthge Transform and Numerical Radius

This section gives spectral radius formulas for the bounded linear operator in terms of numerical radius and iterated generalized Aluthge transform. The numerical radius is defined as

$$w(S) = \sup \{|\lambda|: \lambda \in W(S)\}, \quad (55)$$

where $W(S)$ is the numerical range.

Theorem 12. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for all $n \in \mathbb{N}$, we have*

$$r(S) = \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) = \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right). \quad (56)$$

Proof. As we know that

$$r(S) \le w(S) \le \|S\|. \quad (57)$$

Thus, for every invertible operator $Y \in \mathscr{B}(\mathscr{H})$, we have

$$r(S) = r\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \le w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \le \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\|, \forall n \in \mathbb{N}. \quad (58)$$

Let Y be any bounded linear invertible operator with polar decomposition $Y = U|Y|$. Since Y is an invertible operator, then U is unitary and $|Y|$ is also invertible and positive. Thus, there exists $\beta > 0$ such that $\sigma(|Y|) \subseteq [\beta, \infty)$. So, $A = \ln(|Y|)$ exists and self-adjoint. Thus, we have

$$|Y| = e^A,$$
$$|Y|^{-1} = e^{-A}. \quad (59)$$

Therefore,

$$\begin{aligned} W\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) &= \left\langle \Delta^n_{f,g}\left(YSY^{-1}\right)x, x\right\rangle \\ &= \left\langle \Delta^n_{f,g}\left((U|Y|)S(U|Y|)^{-1}\right)x, x\right\rangle \\ &= \left\langle \Delta^n_{f,g}\left((U|Y|)S|Y|^{-1}U^*\right)x, x\right\rangle \\ &= \left\langle \Delta^n_{f,g}\left(|Y|S|Y|^{-1}\right)U^*x, U^*x\right\rangle \\ &= \left\langle \Delta^n_{f,g}\left(e^A S e^{-A}\right)\frac{U^*x}{\|U^*x\|}, \frac{U^*x}{\|U^*x\|}\right\rangle \cdot \langle UU^*x, x\rangle. \end{aligned} \quad (60)$$

The second equality holds by $Y = U|Y|$, third equality holds because U is unitary, and fourth equality holds by Proposition 4. Thus,

$$W\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \subseteq W\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right) W(UU^*). \quad (61)$$

In the above equation, U is unitary. This implies that

$$w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \le w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right). \quad (62)$$

It follows that

$$\begin{aligned} r(S) = r\left(\Delta^n_{f,g} YSY^{-1}\right) &\le w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right), \text{for invertible } Y \in \mathscr{B}(\mathscr{H}) \\ &\le w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right), \text{for self-adjoint } A \in \mathscr{B}(\mathscr{H}) \\ &\le \left\|\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right\|, \text{for self-adjoint } A \in \mathscr{B}(\mathscr{H}). \end{aligned} \quad (63)$$

For every invertible $Y \in \mathscr{B}(\mathscr{H})$, all above inequalities are satisfied; thus, we have

$$\begin{aligned} r(S) &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} w\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right) \\ &\le \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} w\left(\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right) \\ &\le \inf_{\substack{A\in\mathscr{B}(\mathscr{H})\\ \text{self-adjoint}}} \left\|\Delta^n_{f,g}\left(e^A S e^{-A}\right)\right\| = r(S). \end{aligned} \quad (64)$$

The last equality holds by Corollary 7, which completes the proof. □

Let A be any bounded linear operator with cartesian decomposition

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2i}. \quad (65)$$

In this decomposition $1/2(A + A^*)$ is the real part and $1/2i(A - A^*)$ is the imaginary part.

In [17], the spectrum of a bounded linear operator is contained in the closure of the numerical range.

Theorem 13. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for all $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, we have*

$$\begin{aligned} r(S) &= \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ invertible}} w\left(\operatorname{Re}\left(e^{i\theta}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right)\right) \\ &= \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ invertible}} \left\|\operatorname{Re}\left(e^{i\theta}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right)\right\|. \end{aligned} \tag{66}$$

Proof. Let $r(S) \in \sigma(\mathcal{S})$. Then,

$$r(S) \in \operatorname{Re}\left(\sigma(S)\right) = \operatorname{Re}\left(\sigma\left(\Delta^n_{f,g} YSY^{-1}\right)\right), \text{for invertible operator } Y \in \mathscr{B}(\mathscr{H}). \tag{67}$$

Thus,

$$\begin{aligned} r(S) &\in \operatorname{Re}\left(\sigma\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right) \subseteq \operatorname{Re}\left(\bar{W}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right) \\ &= \bar{W}\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right), \end{aligned} \tag{68}$$

which implies

$$r(S) \le w\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right) \le \left\|\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right\| \le \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\|, \tag{69}$$

for all invertible $Y \in \mathscr{B}(\mathscr{H})$. Thus, we have

$$\begin{aligned} r(S) &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right) \\ &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right\| \\ &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta^n_{f,g}\left(YSY^{-1}\right)\right\| = r(S). \end{aligned} \tag{70}$$

The last equality holds by Corollary 7. For $r(S) \in \sigma(\mathcal{S})$, we have proved

$$\begin{aligned} r(S) &= \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right) \\ &= \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta^n_{f,g}\left(YSY^{-1}\right)\right)\right\|. \end{aligned} \tag{71}$$

If S is an arbitrary operator, then there exists $z \in \sigma(S)$ such that $|z| = r(S)$. Put $\theta = -\arg(z)$. Then, $r(S) = ze^{i\theta} \in \sigma(e^{i\theta}S)$. Hence, by the first part of the proof, we conclude that

$$\begin{aligned} r(S) = r\left(e^{i\theta}S\right) &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} w\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right)\right) \\ &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\operatorname{Re}\left(\Delta^n_{f,g}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right)\right\| \\ &\le \inf_{\substack{Y\in\mathscr{B}(\mathscr{H})\\ \text{invertible}}} \left\|\Delta^n_{f,g}\left(e^{i\theta}\left(YSY^{-1}\right)\right)\right\| = r\left(e^{i\theta}S\right). \end{aligned} \tag{72}$$

The last inequality holds by Corollary 7, which completes the proof. □

The next Corollary is the characterization of normaloid operators.

Corollary 14. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, for each $n \in \mathbb{N}$, the following assertions are equivalent:*

(i) S is normaloid

(ii) There exists $\theta \in \mathbb{R}$ such that for any invertible $Y \in \mathscr{B}(\mathscr{H})$

$$\|S\| \le w\left(\operatorname{Re}\left(\Delta^n_{f,g}\left(e^{i\theta}YSY^{-1}\right)\right)\right) \tag{73}$$

(iii) There exists $\theta \in \mathbb{R}$ such that for any invertible $Y \in \mathscr{B}(\mathscr{H})$

$$\|S\| \le \left\|\operatorname{Re}\left(\Delta^n_{f,g}\left(e^{i\theta}YSY^{-1}\right)\right)\right\| \tag{74}$$

Theorem 15. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$r(S) = \lim_k w\left(\Delta^n_{f,g}\left(S^k\right)\right)^{1/k}, \forall n \in \mathbb{N}. \tag{75}$$

Proof. Since $r(S) \le w(S) \le \|S\|$, therefore

$$\begin{aligned} r(S)^k = r\left(S^k\right) = r\left(\Delta^n_{f,g}\left(S^k\right)\right) &\le w\left(\Delta^n_{f,g}\left(S^k\right)\right) \\ &\le \left\|\Delta^n_{f,g}\left(S^k\right)\right\|, \forall n, k \in \mathbb{N}. \end{aligned}$$

$$r(S) \le \left(w\left(\Delta^n_{f,g}\left(S^k\right)\right)\right)^{1/k} \le \left\|\Delta^n_{f,g}\left(S^k\right)\right\|^{1/k}, \forall n, k \in \mathbb{N}. \tag{76}$$

By Theorem 10, we obtain

$$r(S) \le \lim_k \left(w\left(\Delta^n_{f,g}\left(S^k\right)\right)\right)^{1/k} \le \lim_k \left\| \Delta^n_{f,g}\left(S^k\right)\right\|^{1/k} = r(S), \forall n \in \mathbb{N}, \tag{77}$$

which completes the proof. □

Data Availability

There is no any data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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