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Research Article

Extragradient Methods for Solving Split Feasibility Problem and General Equilibrium Problem and Resolvent Operators in Banach Spaces

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In this paper, we introduce a new extragradient algorithm by using generalized metric projection. We prove a strong convergence theorem for finding a common element of the solution set of split feasibility problem and the set of fixed points of relatively nonexpansive mapping and a finite family of resolvent operator and the set of solutions of an equilibrium problem.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X with norm $\|.\|$ and X^* be the dual of X. We consider the following variational inequality problem (VI), which consists in finding a point $x \in C$ such that

$$\langle x^*, y - x \rangle \geq 0 \quad \forall y \in C, \quad \forall x^* \in Ax, \tag{1}$$

where $A : C \longrightarrow 2^{X^*}$ is a mapping and $\langle .,. \rangle$ denotes the duality pairing. The solution set of the variational inequality problem is denoted by $VI(C, A)$.

The operator $A : X \longrightarrow 2^{X^*}$ is called

(i) Monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \quad \forall x, y \in X, \quad \forall x^* \in Ax, \quad y^* \in Ay. \tag{2}$$

(ii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y, x^* - y^* \rangle \geq \alpha \|x^* - y^*\|^2, \quad \forall x, y \in X, \quad \forall x^* \in Ax, \quad y^* \in Ay. \tag{3}$$

(iii) Demiclosed if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ in X, and $y_n \in Ax_n$ with $y_n \longrightarrow y$ in X^*, we have $x \in X$ and $y \in Ax$

A monotone mapping B is said to be maximal if its graph $G(B) = \{(x, Bx): x \in D(B)\}$ is not properly contained in the graph of any other monotone mapping. Obviously, the monotone mapping B is maximal if and only if for $(x, x^*) \in X \times X^*$, $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(y, y^*) \in G(B)$, then it is implied that $x^* \in Bx$.

Assume that $A : C \longrightarrow 2^{X^*}$ is a nonlinear mapping and $f : C \times C \longrightarrow \mathbb{R}$ is a bifunction. The equilibrium problem (EP) is as follows: find $x \in C$ such that

$$f(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall \quad y \in C. \tag{4}$$

The solution set of (4) is denoted by $\mathrm{EP}(f)$. The equilibrium problem is very general because it includes many well-known problems such as variational inequality problems and saddle point problems (see [1–4]). Several methods have been proposed to solve the equilibrium problem in Hilbert space (see [5]), and some authors obtained weak and strong convergence algorithms for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space (see [6–9]). Then, the authors proved the strong convergence of the algorithms in a uniformly convex and uniformly smooth Banach space (see [10]).

Suppose that C and D are nonempty, closed, and convex subsets of real Banach spaces X_1 and X_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \quad \text{such that} \quad x \in A^{-1}D, \tag{5}$$

which $A : X_1 \longrightarrow X_2$ is a bounded linear operator. The solution set of (5) is denoted by Ω.

In 1994, the split feasibility problem was first studied by Censor and Elfving [11] in finite dimensional Hilbert spaces. In solving (SFP), Schöpfer et al. [12] proposed the next algorithm in p-uniformly convex real Banach spaces: $x_1 \in X_1$ is chosen arbitrarily and for $n \geq 1$,

$$x_{n+1} = \Pi_C J_{X_1}^{*}\left(J_{X_1}x_n - t_n A^{*} J_{X_2}\left(Ax_n - P_D Ax_n\right)\right), \tag{6}$$

where J is the duality mapping, Π_C denotes the Bregman projection, A is a bounded linear operator, and A^{*} is the adjoint of A. Also, they have proven the generated sequence $\{x_n\}$ by algorithm (6) is weakly convergent under suitable conditions. The split feasibility problems were studied extensively by many authors [13, 14].

In this paper, motivated by Schöpfer et al. [12], we present a new hybrid algorithm using the inverse strongly monotone operation and a finite family of resolvent operator. Then, we show that our generated sequence is strongly converges to a common point, the set of solution set of split feasibility problem, and the fixed point of relatively nonexpansive mapping and the fixed point of resolvent operator.

2. Preliminaries

Let X be a real smooth Banach space with norm $\|.\|$ and let X^{*} be the dual space of X. We denote the strong convergence and the weak convergence $\{x_n\}$ to x in X by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. A function $\delta : [0,2] \longrightarrow [0,1]$ is said to be the modulus of convexity of X as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}, \tag{7}$$

for every $\varepsilon \in [0,2]$. A Banach space X is said to be uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. It is known that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \longrightarrow \|u\|$ imply that $x_n \longrightarrow u$ (see [15]). Let p be a fixed real number with $p \geq 2$. A Banach space X is called p-uniformly convex [16], if there exists a constant $c > 0$ such that $\delta \geq c\epsilon^p$ for all $\epsilon \in [0,2]$. Let $S(E) = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be smooth if for all $x \in S(X)$, there exists a unique functional $j_x \in X^{*}$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$ (see [17]).

The norm of X is said to be *Gâteaux* differentiable if for all $x, y \in S(X)$, the limit

$$\lim_{t \longrightarrow 0} \frac{\|x+ty\| - \|x\|}{t}, \tag{8}$$

exists. In this case, X is said to be smooth, and X is called uniformly smooth if the limit (8) is attained uniformly for all $x, y \in S(X)$ [18]. If a Banach space X is uniformly convex, then X is reflexive and strictly convex, and X^{*} is uniformly smooth [17]. The duality mapping J_X^p on X is defined by

$$J_X^p(x) = \left\{f \in X^{*} : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\right\}, \tag{9}$$

for every $x \in X$. If X is a p-uniformly convex and uniformly smooth, then J_X^p is single valued, one-to-one and satisfies $J_X^p = (J_X^{*})^{-1} = (J_X^q)^{-1}$, where $J_X^{*} = J_X^q$ is the duality mapping of X (see [19]). If $p = 2$, then $J_X = J_2 = J$ is the normalized duality mapping. It is well known that if X is a reflexive, strictly convex, and smooth Banach space and $J_X^{*} : X^{*} \longrightarrow 2^X$ is the duality mapping on X^{*}, then $J_X^{-1} = J_X^{*}$. If X is a uniformly smooth and uniformly convex Banach space, then J_X is uniformly norm to norm continuous on bounded sets of X , and $J_X^{-1} = J_X^{*}$ is also uniformly norm to norm continuous on bounded sets of X^{*}. Let X be a smooth Banach space and let J_X be the duality mapping on X. The function $\phi : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, J_X y\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{10}$$

Clearly, from (10), we can conclude that

$$\left(\|x\| - \|y\|\right)^2 \leq \phi(x,y) \leq \left(\|x\| + \|y\|\right)^2. \tag{11}$$

If X is a reflexive, strictly convex, and smooth Banach space, then for all $x, y \in X$

$$\phi(x,y) = 0 \Leftrightarrow x = y. \tag{12}$$

Also, it is clear from the definition of the function ϕ that the following condition holds for all $x, y \in X$,

$$\begin{aligned}\phi(x,y) &= \langle x, J_X x - J_X y\rangle + \langle y - x, J_X y\rangle \\ &\leq \|x\|\|J_X x - J_X y\| + \|y - x\|\|y\|.\end{aligned} \tag{13}$$

Now, the function $V : X \times X^{*} \longrightarrow \mathbb{R}$ is defined as follows:

$$V(x, x^{*}) = \|x\|^2 - 2\langle x, x^{*}\rangle + \|x^{*}\|^2, \tag{14}$$

for all $x \in X$ and $x^{*} \in X^{*}$. Moreover, $V(x, x^{*}) = \phi(x, J_X^{-1}x^{*})$ for all $x \in X$ and $x^{*} \in X^{*}$. If X is a reflexive strictly convex and smooth Banach space with X^* as its dual, then

$$V(x,x^*)+2\langle J_X^{-1}x^*-x,y^*\rangle \le V(x,x^*+y^*), \tag{15}$$

for all $x\in X$ and all $x^*,y^*\in X^*$ [20].

An operator $A:C\longrightarrow X^*$ is hemicontinuous at $x_0\in C$, if for any sequence $\{x_n\}$ converging to x_0 along a line implies that the sequence $\{Ax_n\}$ is weakly convergent to Ax_0, i.e., $Ax_n=A(x_0+t_nx)\rightharpoonup Ax_0$ as $t_n\longrightarrow 0$ for all $x\in C$.

The generalized projection $\Pi_C:X\longrightarrow C$ is a mapping that assigns to an arbitrary point $x\in X$, the minimum point of the functional $\phi(y,x)$; that is, $\Pi_C x=x_0$, where x_0 is the solution of the minimization problem

$$\phi(x_0,x)=\min_{y\in C}\phi(y,x). \tag{16}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J [21]. Suppose that C is a nonempty closed convex subset of X and T is a self mapping on C. We denote the set of fixed points of T by $F(T)$, that is $F(T)=\{x\in C: x\in Tx\}$. A point $p\in C$ is called an asymptotically fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Tx_n-x_n\longrightarrow 0$ [17]. The set of asymptotical fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if $\widehat{F}(T)=F(T)$ and $\phi(p,Tx)\le\phi(p,x)$ for all $x\in C$ and $p\in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [22, 23].

We need the following lemmas for proving our main results.

Lemma 1. (see [24]). *Let X be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If $\phi(x_n,y_n)\longrightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n-y_n\longrightarrow 0$.*

Lemma 2. (see [21]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X and let $y\in X$. Then,*

$$\phi(x,\Pi_C y)+\phi(\Pi_C y,y)\le\phi(x,y),\quad \forall x\in C. \tag{17}$$

Lemma 3. (see [21]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, let $x\in X$, and let $z\in C$. Then,*

$$z=\Pi_C x \Leftrightarrow \langle y-z, J_Xx-J_Xz\rangle\le 0,\quad \text{for all } y\in C. \tag{18}$$

Lemma 4. (see [25]). *Let X be a 2-uniformly convex and smooth Banach space. Then, for all x, $y\in X$, we have that*

$$\|x-y\|\le\frac{2}{c^2}\|J_Xx-J_Xy\|, \tag{19}$$

where $1/c(0\le c\le 1)$ is the 2-uniformly convex constant of X.

Lemma 5. (see [25]). *Let X be a uniformly convex Banach space and $r>0$. Then, there exists a continuous strictly increasing convex function $g:[0,2r]\longrightarrow[0,\infty)$ such that $g(0)=0$ and*

$$\|tx+(1-t)y\|^2\le t\|x\|^2+(1-t)\|y\|^2-t(1-t)g(\|x-y\|), \tag{20}$$

for all $x,y\in B_r(0)=\{z\in X:\|z\|\le r\}$ and $t\in[0,1]$.

Lemma 6. (see [24]). *Let X be a uniformly convex Banach space and $r>0$. Then, there exists a continuous strictly increasing convex function $g:[0,2r]\longrightarrow[0,\infty)$ such that $g(0)=0$ and*

$$g(\|x-y\|)\le\phi(x,y), \tag{21}$$

for all $x,y\in B_r(0)=\{z\in X:\|z\|\le r\}$.

Lemma 7. (see [25]). *Let $x,y\in X$. If X is p-uniformly smooth, then there is a $c>0$ so that*

$$\|x-y\|^p\le\|x\|^p-p\langle y,J_X^p(x)\rangle+c\|y\|^p. \tag{22}$$

Throughout this paper, we assume that $f:C\times C\longrightarrow\mathbb{R}$ is a bifunction satisfying the following conditions

(A1) $f(x,x)=0$ for all $x\in C$

(A2) f is monotone, i.e., $f(x,y)+f(y,x)\le 0$, for all $x,y\in C$

(A3) $\lim_{t\downarrow 0}f(tz+(1-t)x,y)\le f(x,y)$, for all $x,y,z\in C$

(A4) For each $x\in C, y\mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 8. (see [26]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X. Let $A:C\longrightarrow X^*$ be an α-inverse-strongly monotone operator and f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A_1)-(A_4)$. Then, for all $r>0$ the following hold*

(i) For $x\in X$, there exists $u\in C$ such that

$$f(u,x)+\langle Au,y-u\rangle+\frac{1}{r}\langle y-u,J_Xu-J_Xx\rangle\ge 0,\quad \forall y\in C, \tag{23}$$

(ii) If X is additionally uniformly smooth and $K_r:X\longrightarrow C$ is defined as

$$K_r(x)=\left\{u\in C: f(u,y)+\langle Au,y-u\rangle+\frac{1}{r}\langle y-u,J_Xu-J_Xx\rangle \ge 0,\quad \forall y\in C\right\}, \tag{24}$$

then, the following conditions hold:

K_r is single-valued

K_r is firmly nonexpansive, i.e., for all $x, y \in X$,

$$\langle K_r x - K_r y, J_X K_r x - J_X K_r y\rangle \le \langle K_r x - K_r y, J_X x - J_X y\rangle,$$
$$F(K_r) = \widehat{F}(K_r) = EP(f). \tag{25}$$

EP is a closed convex subset of C.

$$\phi(p, K_r x) + \phi(K_r x, x) \le \phi(p, x), \quad \forall \quad p \in F(K_r). \tag{26}$$

Definition 9.

Let X be a real smooth and uniformly convex Banach space and let $M : X \longrightarrow 2^{X^*}$ be a maximal monotone operator. For all $\iota > 0$, define the operator $Q_\iota^M : X \longrightarrow X$ by $Q_\iota^M = (J_X + \iota M)^{-1} J_X x$ for all $x \in X$.

Lemma 10. (see [18]). *Let X be a real smooth and uniformly convex Banach space, and let $M : X \longrightarrow 2^{X^*}$ be a maximal monotone operator. Then, $M^{-1}0$ is a closed and convex subset of X, and the graph $G(M)$ of M is demiclosed.*

Lemma 11. *Let X be a real reflexive, strictly convex, and let smooth Banach space and $M : X \longrightarrow 2^{X^*}$ be a maximal monotone operator with $M^{-1}0 \ne \emptyset$. Then, for all $x \in X, y \in M^{-1}0$ and $\iota > 0$, then $\phi(y, Q_\iota^M x) + \phi(Q_\iota^M x, x) \le \phi(y, x)$.*

3. Main Results

In this section, we introduce our new extragradient algorithm.

Theorem 12. *Let X_1 and X_2 are real 2-uniformly convex and uniformly smooth Banach spaces. Suppose that C and D are nonempty closed and convex subsets of X_1 and X_2, respectively. Suppose that g is a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies the conditions A1-A4, $A : X_1 \longrightarrow X_2$ is a bounded linear operator and $A^* : X_2^* \longrightarrow X_1^*$ is the adjoint of A. Let $M_i : X_1 \longrightarrow 2^{X_1^*}$ be a maximal monotone operator with $M_i^{-1}0 \ne \emptyset$ for all $i = 1, 2, \cdots, k$. Assume that $B : C \longrightarrow X^*$ is an α-inverse strongly monotone operator, and f is a relatively nonexpansive mappings from C into itself and $\Gamma = \Omega \cap F(f) \cap (\cap_{i=1}^k F(Q_\iota^{M_i})) \cap EP(g) \ne \emptyset$. Let $\{x_n\}$ is a sequence generated by $v_1 \in C$ and*

$$\begin{cases} x_n \in C \quad s.t \quad g(x_n, y) + \langle Bx_n, y - x_n\rangle + \dfrac{1}{r_n}\langle y - x_n, J_{X_1}x_n - J_{X_1}v_n\rangle \ge 0, \\ u_n = \Pi_C J_{X_1}^{-1}\left(s_n J_{X_1}x_n + (1 - s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right), \\ z_n = \Pi_C J_{X_1}^{-1}\left(J_{X_1}u_n - \tau A^* J_{X_2}(Au_n - P_D A u_n)\right), \\ y_n = \Pi_C J_{X_1}^{-1}\left(J_{X_1}z_n - \tau A^* J_{X_2}(Az_n - P_D A z_n)\right), \\ w_n = \Pi_C J_{X_1}^{-1}\left(\beta_n J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n + (1 - \beta_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right), \\ v_{n+1} = \Pi_C J_{X_1}^{-1}\left[\alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right], \end{cases} \tag{27}$$

where $r_n \in [a, \infty)$ for some $a > 0$, $\{s_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$, and τ and $\{\alpha_{n,i}\}_{i=1}^3$ satisfy the following conditions:

(i) $\{\alpha_{n,i}\}_{i=1}^3 \subset (0, 1)$, $\sum_{i=1}^3 \alpha_{n,i} = 1$, $\liminf_{n\longrightarrow\infty}\alpha_{n,1}\alpha_{n,2} > 0$, and $\liminf_{n\longrightarrow\infty}\alpha_{n,3} > 0$

(ii) τ is real number such that $0 < \tau < 2/c\|A\|^2$, where c depends on 2-uniformly smoothness of X_1^*

Then, $\{x_n\}$ converges strongly to $q = \Pi_{\Omega\cap(\cap_{i=1}^k F(Q_\iota^{M_i}))\cap EP(g)} \circ f(q)$.

Proof. Let $\widehat{u} \in \Gamma$. By (10), Lemma 2 and the convexity of $\|.\|^2$, we have

$$\begin{aligned} \phi(\widehat{u}, u_n) &\le \phi\left(\widehat{u}, J_{X_1}^{-1}\left(s_n J_{X_1}x_n + (1 - s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right)\right) \\ &= \|\widehat{u}\|^2 - 2\left\langle \widehat{u}, s_n J_{X_1}x_n + (1 - s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\rangle \\ &\quad + \left\|s_n J_{X_1}x_n + (1 - s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\|^2 \\ &\le \|\widehat{u}\|^2 - 2s_n\left\langle \widehat{u}, J_{X_1}x_n\right\rangle - 2(1 - s_n)\left\langle \widehat{u}, J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\rangle \\ &\quad + s_n\|x_n\|^2 + (1 - s_n)\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\|^2 = s_n\phi(\widehat{u}, x_n) \\ &\quad + (1 - s_n)\phi\left(\widehat{u}, Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right). \end{aligned} \tag{28}$$

□

Now, it follows from Lemma 11 and the above that

$$\begin{aligned} \phi(\widehat{u}, u_n) &\le s_n\phi(\widehat{u}, x_n) + (1 - s_n)\phi\left(\widehat{u}, Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right) \\ &\le s_n\phi(\widehat{u}, x_n) + (1 - s_n)\phi\left(\widehat{u}, Q_\iota^{M_3}\cdots Q_\iota^{M_k}x_n\right), \end{aligned} \tag{29}$$

$$\vdots \tag{30}$$

$$\le s_n\phi(\widehat{u}, x_n) + (1 - s_n)\phi(\widehat{u}, x_n) = \phi(\widehat{u}, x_n). \tag{31}$$

Let $k_n = Au_n - P_D A u_n$. From (10) and Lemmas 2 and 7, we have that

$$\begin{aligned} \phi(\widehat{u}, z_n) &\le \phi\left(\widehat{u}, J_{X_1}^{-1}\left(J_{X_1}u_n - \tau A^* J_{X_2}k_n\right)\right) = \|\widehat{u}\|^2 \\ &\quad - 2\left\langle \widehat{u}, J_{X_1}u_n - \tau A^* J_{X_2}k_n\right\rangle + \left\|J_{X_1}u_n - \tau A^* J_{X_2}k_n\right\|^2 \\ &= \|\widehat{u}\|^2 - 2\left\langle \widehat{u}, J_{X_1}u_n\right\rangle + 2\tau\left\langle \widehat{u}, A^* J_{X_2}k_n\right\rangle \\ &\quad + \left\|J_{X_1}u_n - \tau A^* J_{X_2}k_n\right\|^2 \le \|\widehat{u}\|^2 - 2\left\langle \widehat{u}, J_{X_1}u_n\right\rangle \\ &\quad + 2\tau\left\langle \widehat{u}, A^* J_{X_2}k_n\right\rangle + \left\|J_{X_1}u_n\right\|^2 \\ &\quad - 2\tau\left\langle A^* J_{X_2}k_n, J_{X_1}^* J_{X_1}u_n\right\rangle + c\tau^2\left\|A^* J_{X_2}k_n\right\|^2 \\ &= \phi(\widehat{u}, u_n) + 2\tau\left\langle A\widehat{u}, J_{X_2}k_n\right\rangle - 2\tau\left\langle J_{X_2}k_n, Au_n\right\rangle \\ &\quad + c\tau^2\|A\|^2\left\|J_{X_2}k_n\right\|^2 = \phi(\widehat{u}, u_n) \\ &\quad + 2\tau\left\langle A\widehat{u} - Au_n, J_{X_2}k_n\right\rangle + c\tau^2\|A\|^2\|k_n)\|^2. \end{aligned} \tag{32}$$

Since $\langle J_{X_2}(x - P_D x), y - P_D x\rangle \le 0$ for each $y \in D$ and for each $x \in X_2$, we have that

$$\begin{aligned}\left\langle J_{X_2}k_n, Au_n - A\hat{u}\right\rangle &= \left\langle J_{X_2}k_n, P_D Au_n - A\hat{u}\right\rangle + \left\langle J_{X_2}k_n, Au_n - P_D Au_n\right\rangle \\ &= \left\langle J_{X_2}k_n, P_D Au_n - A\hat{u}\right\rangle + \|P_D Au_n - Au_n\|^2 \\ &\ge \|P_D Au_n - Au_n\|^2.\end{aligned} \tag{33}$$

From (32), our assumptions, and the above, we conclude that

$$\begin{aligned}\phi(\hat{u}, z_n) &\le \phi(\hat{u}, u_n) - 2\tau\|P_D Au_n - Au_n\|^2 + c\tau^2\|A\|^2\|k_n\|^2 \\ &= \phi(\hat{u}, u_n) - \tau\left(2 - c\tau\|A\|^2\right)\|P_D Au_n - Au_n\|^2 \\ &\le \phi(\hat{u}, u_n).\end{aligned} \tag{34}$$

In a similar way as above, we obtain that

$$\phi(\hat{u}, y_n) \le \phi(\hat{u}, z_n) - \tau\left(2 - c\tau\|A\|^2\right)\|P_D Az_n - Az_n\|^2 \le \phi(\hat{u}, z_n). \tag{35}$$

It follows from (10), (29), (34), (35), Lemma 11, and the convexity of $\|.\|^2$ that

$$\begin{aligned}\phi(\hat{u}, w_n) &\le \phi\Big(\hat{u}, J_{X_1}^{-1}\big(\beta_n J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n \\ &\quad + (1-\beta_n) J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big)\Big) = \|\hat{u}\|^2 \\ &\quad - 2\big\langle \hat{u}, \beta_n J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n \\ &\quad + (1-\beta_n) J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big\rangle \\ &\quad + \big\|\beta_n J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n \\ &\quad + (1-\beta_n) J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big\|^2 \le \|\hat{u}\|^2 \\ &\quad - 2\beta_n \big\langle \hat{u}, J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n\big\rangle \\ &\quad - 2(1-\beta_n)\big\langle \hat{u}, J_{X_1} Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big\rangle \\ &\quad + \beta_n \big\|Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n\big\|^2 \\ &\quad + (1-\beta_n)\big\|Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big\|^2 \\ &= \beta_n \phi\big(\hat{u}, Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} z_n\big) \\ &\quad + (1-\beta_n)\, \phi\big(\hat{u}, Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} y_n\big),\end{aligned} \tag{36}$$

$$\le \beta_n \phi\big(\hat{u}, Q_\iota^{M_2} Q_\iota^{M_3} \cdots Q_\iota^{M_k} z_n\big) + (1-\beta_n)\phi\big(\hat{u}, Q_\iota^{M_2} Q_\iota^{M_3} \cdots Q_\iota^{M_k} y_n\big), \tag{37}$$

$$\vdots \tag{38}$$

$$\le \beta_n \phi(\hat{u}, z_n) + (1-\beta_n)\phi(\hat{u}, y_n) \le \phi(\hat{u}, x_n). \tag{39}$$

By (10), (29), (37), Lemmas 2, 8, the condition (i), the convexity of $\|.\|^2$, and the relatively nonexpansiveness of f, we have that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &= \phi\big(\hat{u}, K_{r_n} v_{n+1}\big) \le \phi(\hat{u}, v_{n+1}) \\ &\le \phi\Big(\hat{u}, J_{X_1}^{-1}\big[\alpha_{n,1} J_{X_1} f(x_n) + \alpha_{n,2} J_{X_1} u_n + \alpha_{n,3} J_{X_1} w_n\big]\Big) \\ &= \|\hat{u}\|^2 - 2\big\langle \hat{u}, \alpha_{n,1} J_{X_1} f(x_n) + \alpha_{n,2} J_{X_1} u_n + \alpha_{n,3} J_{X_1} w_n\big\rangle \\ &\quad + \big\|\alpha_{n,1} J_{X_1} f(x_n) + \alpha_{n,2} J_{X_1} u_n + \alpha_{n,3} J_{X_1} w_n\big\|^2 \\ &\le \|\hat{u}\|^2 - 2\alpha_{n,1}\big\langle \hat{u}, J_{X_1} f(x_n)\big\rangle - 2\alpha_{n,2}\big\langle \hat{u}, J_{X_1} u_n\big\rangle \\ &\quad - 2\alpha_{n,3}\big\langle \hat{u}, J_{X_1} w_n\big\rangle + \alpha_{n,1}\|f(x_n)\|^2 + \alpha_{n,2}\|u_n\|^2 \\ &\quad + \alpha_{n,3}\|w_n\|^2 = \alpha_{n,1}\phi(\hat{u}, f(x_n)) + \alpha_{n,2}\phi(\hat{u}, u_n) \\ &\quad + \alpha_{n,3}\phi(\hat{u}, w_n) \le \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) \\ &\quad + \alpha_{n,3}\phi(\hat{u}, w_n) \le \phi(\hat{u}, x_n).\end{aligned} \tag{40}$$

Therefore, $\{\phi(\hat{u}, x_n)\}$ is bounded, and $\lim_{n\longrightarrow\infty}\phi(\hat{u}, x_n)$ exists. Now, by (11), we conclude that $\{x_n\}$ is bounded. It follows from (29), (34), (35), (37), (40), and relatively nonexpansiveness of f that the sequences $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{w_n\}$, $\{v_n\}$, and $\{f(x_n)\}$ are bounded.

Next, by (28), (37), (40), and Lemma 11, we conclude that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\le \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, w_n) \\ &\le (1-\alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) \\ &\le (1-\alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}(s_n\phi(\hat{u}, x_n) \\ &\quad + (1-s_n)\phi\big(\hat{u}, Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big)) \\ &\le (1-\alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}(s_n\phi(\hat{u}, x_n) \\ &\quad + (1-s_n)\big[\phi\big(\hat{u}, Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big) \\ &\quad - \phi\big(Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n, Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big)\big]) \\ &\le (1-\alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}(s_n\phi(\hat{u}, x_n) \\ &\quad + (1-s_n)\big[\phi\big(\hat{u}, Q_\iota^{M_3} \cdots Q_\iota^{M_k} x_n\big) \\ &\quad - \phi\big(Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n, Q_\iota^{M_3} \cdots Q_\iota^{M_k} x_n\big) \\ &\quad - \phi\big(Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n, Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big)\big]),\end{aligned} \tag{41}$$

$$\vdots \tag{42}$$

$$\begin{aligned}&\le (1-\alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}(s_n\phi(\hat{u}, x_n) + (1-s_n)[\phi(\hat{u}, x_n) \\ &\quad - \phi\big(Q_\iota^{M_k} x_n, x_n\big) - \cdots - \phi\big(Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n, Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big)]) \\ &\quad = \phi(\hat{u}, x_n) - \alpha_{n,2}(1-s_n)\big[\phi\big(Q_\iota^{M_k} x_n, x_n\big) \\ &\quad - \cdots - \phi\big(Q_\iota^{M_1} Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n, Q_\iota^{M_2} \cdots Q_\iota^{M_k} x_n\big)\big].\end{aligned} \tag{43}$$

Now, from (11) and (41), we have the following inequalities:

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,2}(1-s_n)\phi\left(Q_\iota^{M_k}x_n, x_n\right),\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,2}(1-s_n)\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}x_n, Q_\iota^{M_k}x_n\right),\\
&\vdots\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,2}(1-s_n)\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right).
\end{aligned} \tag{44}$$

Now, since $\{\phi(\hat{u}, x_n)\}$ is convergent, it follows from (44), the conditions (i), and our assumptions that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_k}x_n, x_n\right) = 0,\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}x_n, Q_\iota^{M_k}x_n\right) = 0,\\
&\vdots\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right) = 0.
\end{aligned} \tag{45}$$

Therefore, from Lemma 1, we have that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\left\|Q_\iota^{M_k}x_n - x_n\right\| = 0,\\
&\lim_{n\longrightarrow\infty}\left\|Q_\iota^{M_{k-1}}Q_\iota^{M_k}x_n - Q_\iota^{M_k}x_n\right\| = 0,\\
&\vdots\\
&\lim_{n\longrightarrow\infty}\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n - Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\| = 0.
\end{aligned} \tag{46}$$

Then,

$$\lim_{n\longrightarrow\infty}\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n - x_n\right\| = 0. \tag{47}$$

From (13), (47), the boundedness of the sequences $\{x_n\}$, $\{Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\}$, and using uniformly norm-to-norm continuity of J on bounded sets, it is clear that

$$\lim_{n\longrightarrow\infty}\phi\left(x_n, Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right) = 0. \tag{48}$$

By (29), (34), (35), (36), (40), and Lemma 11, we conclude that

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\le \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, w_n)\\
&\le (1-\alpha_{n,3})\phi(\hat{u}, x_n) + \alpha_{n,3}\phi(\hat{u}, w_n)\\
&\le (1-\alpha_{n,3})\phi(\hat{u}, x_n) + \alpha_{n,3}\left(\beta_n\phi\left(\hat{u}, Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\right.\\
&\quad \left.+ (1-\beta_n)\phi\left(\hat{u}, Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right)\\
&\le (1-\alpha_{n,3})\phi(\hat{u}, x_n) + \alpha_{n,3}\left(\beta_n\left[\phi\left(\hat{u}, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\right.\right.\\
&\quad \left. - \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\right]\\
&\quad + (1-\beta_n)\left[\phi\left(\hat{u}, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right.\\
&\quad \left.\left. - \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right]\right),\\
&\vdots\\
&\le (1-\alpha_{n,3})\phi(\hat{u}, x_n) + \alpha_{n,3}\left(\beta_n\left[\phi(\hat{u}, z_n) - \phi\left(Q_\iota^{M_k}z_n, z_n\right)\right.\right.\\
&\quad \left. - \cdots - \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\right]\\
&\quad + (1-\beta_n)\left[\phi(\hat{u}, y_n) - \phi\left(Q_\iota^{M_k}y_n, y_n\right)\right.\\
&\quad \left.\left. - \cdots - \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right]\right)\\
&\le \phi(\hat{u}, x_n) - \alpha_{n,3}\beta_n\left[\phi\left(Q_\iota^{M_k}z_n, z_n\right)\right.\\
&\quad \left. + \cdots + \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\right]\\
&\quad - \alpha_{n,3}(1-\beta_n)\left[\phi\left(Q_\iota^{M_k}y_n, y_n\right)\right.\\
&\quad \left. + \cdots + \phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right].
\end{aligned} \tag{49}$$

Hence, from (11), the above, and our assumptions, we obtain the following results

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}\beta_n\phi\left(Q_\iota^{M_k}z_n, z_n\right),\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}\beta_n\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}z_n, Q_\iota^{M_k}z_n\right),\\
&\vdots\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}\beta_n\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right),
\end{aligned} \tag{50}$$

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}(1-\beta_n)\phi\left(Q_\iota^{M_k}y_n, y_n\right),\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}(1-\beta_n)\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}y_n, Q_\iota^{M_k}y_n\right),\\
&\vdots\\
\phi(\hat{u}, x_{n+1}) &\le \phi(\hat{u}, x_n) - \alpha_{n,3}(1-\beta_n)\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right).
\end{aligned} \tag{51}$$

Since $\{\phi(\hat{u}, x_n)\}$ is convergent, we conclude from (i), (50), (51), and our assumptions that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_k}z_n, z_n\right) = 0,\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}z_n, Q_\iota^{M_k}z_n\right) = 0,\\
&\vdots\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right) = 0.
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_k}y_n, y_n\right) = 0,\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_{k-1}}Q_\iota^{M_k}y_n, Q_\iota^{M_k}y_n\right) = 0,\\
&\vdots\\
&\lim_{n\longrightarrow\infty}\phi\left(Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n, Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right) = 0.
\end{aligned} \tag{53}$$

Now, from (29), (34), (35), (37), and (40), we have

$$\begin{aligned}\phi(\widehat{u}, x_{n+1}) &\leq \alpha_{n,1}\phi(\widehat{u}, x_n) + \alpha_{n,2}\phi(\widehat{u}, u_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}(\beta_n\phi(\widehat{u}, z_n) + (1-\beta)\phi(\widehat{u}, y_n))\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}\phi(\widehat{u}, z_n)\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}(\phi(\widehat{u}, u_n)\\ &\quad - \tau(2 - c\tau\|A\|^2)\|P_D A u_n - A u_n\|^2)\\ &\leq \phi(\widehat{u}, x_n) - \alpha_{n,3}\tau(2 - c\tau\|A\|^2)\|P_D A u_n - A u_n\|^2.\end{aligned} \tag{54}$$

Hence, it follows from (54) that

$$\alpha_{n,3}\tau(2 - c\tau\|A\|^2)\|P_D A u_n - A u_n\|^2 \leq \phi(\widehat{u}, x_n) - \phi(\widehat{u}, x_{n+1}). \tag{55}$$

Then, it follows from (i) and our assumptions that

$$\lim_{n\longrightarrow\infty}\|P_D A u_n - A u_n\|^2 = 0. \tag{56}$$

From (29), (34), (35), (37), and (40), we have

$$\begin{aligned}\phi(\widehat{u}, x_{n+1}) &\leq \alpha_{n,1}\phi(\widehat{u}, x_n) + \alpha_{n,2}\phi(\widehat{u}, u_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}(\beta_n\phi(\widehat{u}, z_n) + (1-\beta_n)\phi(\widehat{u}, y_n))\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}(\beta_n\phi(\widehat{u}, z_n) + (1-\beta_n)[\phi(\widehat{u}, z_n)\\ &\quad - \tau(2 - c\tau\|A\|^2)\|P_D A z_n - A z_n\|^2])\\ &\leq (1-\alpha_{n,3})\phi(\widehat{u}, x_n) + \alpha_{n,3}\phi(\widehat{u}, z_n)\\ &\quad - \alpha_{n,3}(1-\beta_n)\tau(2 - c\tau\|A\|^2)\|P_D A z_n - A z_n\|^2\\ &\leq \phi(\widehat{u}, x_n) - \alpha_{n,3}(1-\beta_n)\tau(2 - c\tau\|A\|^2)\|P_D A z_n - A z_n\|^2.\end{aligned} \tag{57}$$

So,

$$\alpha_{n,3}(1-\beta_n)\tau(2 - c\tau\|A\|^2)\|P_D A z_n - A z_n\|^2 \leq \phi(\widehat{u}, x_n) - \phi(\widehat{u}, x_{n+1}). \tag{58}$$

Therefore, it follows from (i) and our assumptions that

$$\lim_{n\longrightarrow\infty}\|P_D A z_n - A z_n\|^2 = 0. \tag{59}$$

Suppose that $r_1 = \sup_n\{\|f(x_n)\|, \|u_n\|\}$. Therefore, from Lemma 5, there exists a continuous strictly increasing convex function $g_1 : [0, 2r_1] \longrightarrow [0,\infty)$ such that $g_1(0) = 0$ and using (29), (37), Lemmas 2 and 8, the convexity of $\|.\|^2$, and the condition relatively nonexpansiveness of f, we have that

$$\begin{aligned}\phi(\widehat{u}, x_{n+1}) &= \phi(\widehat{u}, K_{r_n} v_{n+1}) \leq \phi(\widehat{u}, v_{n+1})\\ &\leq \phi\left(\widehat{u}, J_{X_1}^{-1}\left[\alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right]\right)\\ &= \|\widehat{u}\|^2 - 2\langle\widehat{u}, \alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\rangle\\ &\quad + \left\|\alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right\|^2\\ &\leq \|\widehat{u}\|^2 - 2\alpha_{n,1}\langle\widehat{u}, J_{X_1}f(x_n)\rangle - 2\alpha_{n,2}\langle\widehat{u}, J_{X_1}u_n\rangle\\ &\quad - 2\alpha_{n,3}\langle\widehat{u}, J_{X_1}w_n\rangle + \alpha_{n,1}\|f(x_n)\|^2 + \alpha_{n,2}\|u_n\|^2\\ &\quad + \alpha_{n,3}\|w_n\|^2 - \alpha_{n,1}\alpha_{n,2}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right)\\ &\leq \alpha_{n,1}\phi(\widehat{u}, f(x_n)) + \alpha_{n,2}\phi(\widehat{u}, u_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\quad - \alpha_{n,1}\alpha_{n,2}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right) \leq \alpha_{n,1}\phi(\widehat{u}, x_n)\\ &\quad + \alpha_{n,2}\phi(\widehat{u}, u_n) + \alpha_{n,3}\phi(\widehat{u}, w_n)\\ &\quad - \alpha_{n,1}\alpha_{n,2}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right) \leq \phi(\widehat{u}, x_n)\\ &\quad - \alpha_{n,1}\alpha_{n,2}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right).\end{aligned} \tag{60}$$

So,

$$\alpha_{n,1}\alpha_{n,2}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right) \leq \phi(\widehat{u}, x_n) - \phi(\widehat{u}, x_{n+1}). \tag{61}$$

Since $\lim_{n\longrightarrow\infty}\phi(\widehat{u}, x_n)$ exists. Therefore, it follows from the condition (i) that

$$\lim_{n\longrightarrow\infty}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right) = 0. \tag{62}$$

Because g_1 is continues function, we conclude that

$$\begin{aligned}g_1\left(\lim_{n\longrightarrow\infty}\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right) &= \lim_{n\longrightarrow\infty}g_1\left(\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\|\right)\\ &= 0 = g_1(0).\end{aligned} \tag{63}$$

Therefore,

$$\lim_{n\longrightarrow\infty}\left\|J_{X_1}f(x_n) - J_{X_1}u_n\right\| = 0. \tag{64}$$

Since $J_{X_1}^{-1}$ is uniformly norm-to-norm continuous on bounded sets, it imply that

$$\lim_{n\longrightarrow\infty}\|f(x_n) - u_n\| = 0. \tag{65}$$

Using (13), (65), the uniformly norm-to-norm continuity of J_{X_1} on bounded sets, and the boundedness of the sequences $\{f(x_n)\}$ and $\{u_n\}$, we conclude that

$$\lim_{n\longrightarrow\infty}\phi(u_n, f(x_n)) = 0. \tag{66}$$

By (48) and using our assumptions, we obtain that

$$\begin{aligned}\phi(x_n,u_n)&\le\phi\left(x_n,J_{X_1}^{-1}\left(s_nJ_{X_1}x_n+(1-s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right)\right)\\&=\|x_n\|^2-2\langle x_n,s_nJ_{X_1}x_n+(1-s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\rangle\\&\quad+\left\|s_nJ_{X_1}x_n+(1-s_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\|^2\\&\le\|x_n\|^2-2s_n\langle x_n,J_{X_1}x_n\rangle-2(1-s_n)\langle x_n,J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\rangle\\&\quad+s_n\|x_n\|^2+(1-s_n)\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right\|^2\\&=s_n\phi(x_n,x_n)+(1-s_n)\phi\left(x_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right)\\&=(1-s_n)\phi\left(x_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n\right)\longrightarrow0\quad as\quad n\longrightarrow\infty.\end{aligned}\tag{67}$$

Then, it follows from Lemma 1 that

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0.\tag{68}$$

Now, by (15), (56), and Lemmas 2 and 4, we conclude that

$$\begin{aligned}\phi(u_n,z_n)&\le\phi\left(u_n,J_{X_1}^{-1}\left(J_{X_1}u_n-\tau A^*J_{X_2}k_n\right)=V\left(u_n,J_{X_1}u_n-\tau A^*J_{X_2}k_n\right)\right.\\&\le V\left(u_n,J_{X_1}u_n\right)-2\left\langle J_{X_1}^{-1}\left(J_{X_1}u_n-\tau A^*J_{X_2}k_n\right)-u_n,\tau A^*J_{X_2}k_n\right\rangle\\&=\phi(u_n,u_n)-2\left\langle J_{X_1}^{-1}\left(J_{X_1}u_n-\tau A^*J_{X_2}k_n\right)-J_{X_1}^{-1}\left(J_{X_1}u_n\right),\tau A^*J_{X_2}k_n\right\rangle\\&\le2\left\|J_{X_1}^{-1}\left(J_{X_1}u_n-\tau A^*J_{X_2}k_n\right)-J_{X_1}^{-1}\left(J_{X_1}u_n\right)\right\|\left\|\tau A^*J_{X_2}k_n\right\|\\&\le\frac{4\tau^2}{c^2}\left\|A^*J_{X_2}k_n\right\|^2\le\frac{4\tau^2}{c^2}\|A\|^2\|Au_n-P_DAu_n\|^2\\&\longrightarrow0\quad as\quad n\longrightarrow\infty.\end{aligned}\tag{69}$$

Then, using Lemma 1, we obtain

$$\lim_{n\longrightarrow\infty}\|u_n-z_n\|=0.\tag{70}$$

Also, from (15), (59), and Lemma 2, and the same way used for proving (70), we can conclude that

$$\begin{aligned}\phi(z_n,y_n)&\le\phi\left(z_n,J_{X_1}^{-1}\left(J_{X_1}z_n-\tau A^*J_{X_2}(Az_n-P_DAz_n)\right)\right)\\&\le\frac{4\tau^2}{c^2}\|A\|^2\|Az_n-P_DAz_n\|^2\longrightarrow0\quad as\quad n\longrightarrow\infty.\end{aligned}\tag{71}$$

Then, using Lemma 1, we get

$$\lim_{n\longrightarrow\infty}\|z_n-y_n\|=0.\tag{72}$$

Now, it follows from (13), (52), and Lemma 1 that

$$\begin{aligned}\left\|z_n-Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right\|&\le\left\|z_n-Q_\iota^{M_k}z_n\right\|+\left\|Q_\iota^{M_k}z_n-Q_\iota^{M_{k-1}}Q_\iota^{M_k}z_n\right\|\\&\quad+\cdots+\left\|Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n-Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right\|\\&\longrightarrow0\quad as\quad n\longrightarrow\infty.\end{aligned}\tag{73}$$

Similarly, from (13), (53), and Lemma 1, we have

$$\lim_{n\longrightarrow\infty}\left\|y_n-Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right\|\longrightarrow0,\tag{74}$$

then, by (72), we obtain that

$$\lim_{n\longrightarrow\infty}\left\|z_n-Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right\|=0.\tag{75}$$

Now, by (13), (73), (75), and using uniformly norm-to-norm continuity of J_{X_1} on bounded sets, it is implied that

$$\lim_{n\longrightarrow\infty}\phi\left(z_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)=0,\quad\lim_{n\longrightarrow\infty}\phi\left(z_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)=0.\tag{76}$$

It follows from (10), (76), Lemma 2, and the convexity of $\|.\|^2$ that

$$\begin{aligned}\phi(z_n,w_n)&\le\phi\left(z_n,J_{X_1}^{-1}\left(\beta_nJ_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right.\right.\\&\qquad\left.\left.+(1-\beta_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\right)\\&=\|z_n\|^2-2\langle z_n,\beta_nJ_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\\&\quad+(1-\beta_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\rangle\\&\quad+\left\|\beta_nJ_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right.\\&\quad\left.+(1-\beta_n)J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n)\right\|^2\\&\le\|z_n\|^2-2\beta_n\langle z_n,J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\rangle\\&\quad-2(1-\beta_n)\langle z_n,J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n)\rangle\\&\quad+\beta_n\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right\|^2\\&\quad+(1-\beta_n)\left\|Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n)\right\|^2\\&=\beta_n\phi\left(z_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}z_n\right)\\&\quad+(1-\beta_n)\phi\left(z_n,Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}y_n\right)\\&\longrightarrow0\quad as\quad n\longrightarrow\infty.\end{aligned}\tag{77}$$

Now, by Lemma 1, we have $\lim_{n\longrightarrow\infty}\|z_n-w_n\|=0$. Therefore, we obtain from (70) that $\lim_{n\longrightarrow\infty}\|u_n-w_n\|=0$, then by (13), we conclude that

$$\lim_{n\longrightarrow\infty}\phi(u_n,w_n)=0.\tag{78}$$

From (66), (78), Lemma 2, and our assumptions, it implied that

$$\begin{aligned}\phi(u_n, v_{n+1}) &\le \phi\left(u_n, J_{X_1}^{-1}\left(\alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right)\right)\\ &= \|u_n\|^2 - 2\left\langle u_n, \alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right\rangle\\ &\quad + \left\|\alpha_{n,1}J_{X_1}f(x_n) + \alpha_{n,2}J_{X_1}u_n + \alpha_{n,3}J_{X_1}w_n\right\|^2\\ &\le \|u_n\|^2 - 2\alpha_{n,1}\left\langle u_n, J_{X_1}f(x_n)\right\rangle - 2\alpha_{n,2}\left\langle u_n, J_{X_1}u_n\right\rangle\\ &\quad - 2\alpha_{n,3}\left\langle u_n, J_{X_1}w_n\right\rangle + \alpha_{n,1}\|f(x_n)\|^2 + \alpha_{n,2}\|u_n\|^2\\ &\quad + \alpha_{n,3}\|w_n\|^2 = \alpha_{n,1}\phi(u_n, f(x_n)) + \alpha_{n,2}\phi(u_n, u_n)\\ &\quad + \alpha_{n,3}\phi(u_n, w_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty.\end{aligned} \tag{79}$$

Therefore, by Lemma 1, we have

$$\lim_{n\longrightarrow\infty}\|v_{n+1} - u_n\| = 0. \tag{80}$$

Let $r_2 = \sup_n\{\|v_n\|, \|x_n\|\}$. Therefore, by Lemma 6, there exists a continuous, convex, and strictly increasing function $g_2 : [0, 2r_2] \longrightarrow [0,\infty)$ such that $g_2(0) = 0$ and

$$g_2(\|x_n - v_n\|) \le \phi(x_n, v_n). \tag{81}$$

It follows from (40), (81), Lemma 8, and the fact that $x_n = K_{r_n}v_n$, we conclude that

$$\begin{aligned}g_2(\|x_n - v_n\|) &\le \phi(x_n, v_n) \le \phi(\widehat{u}, v_n) - \phi(\widehat{u}, x_n) \le \phi(\widehat{u}, x_{n-1})\\ &\quad - \phi(\widehat{u}, x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty.\end{aligned} \tag{82}$$

Therefore,

$$\lim_{n\longrightarrow\infty}\|x_n - v_n\| = 0, \tag{83}$$

because g_2 is a continuous strictly increasing convex function. Now, by (80) and (83), we have

$$\lim_{n\longrightarrow\infty}\|u_n - x_{n+1}\| = 0. \tag{84}$$

From (68) and (84), we obtain that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0. \tag{85}$$

This shows that $\{x_n\}$ is a Cauchy sequence, so $\{x_n\}$ converges strongly to a point $q \in C$. Therefore, by (68), (70), and (72), we imply that $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to q.

Next, we prove that $q \in \cap_{i=1}^k F(Q_\iota^{M_i})$. It follows from (46) and uniformly continuity of J_{X_1} on bounded subset of X_1 that $J_{X_1}Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n - J_{X_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n \longrightarrow 0$ as $n \longrightarrow \infty$. Get $\eta_n = Q_\iota^{M_1}Q_\iota^{M_2}\cdots Q_\iota^{M_k}x_n$; hence, by Definition 9, we have $J_{X_1}\eta_n + \iota M_1\eta_n = J_{X_1}Q_\iota^{M_2}Q_\iota^{M_3}\cdots Q_\iota^{M_k}x_n$. Therefore, there exists $h_n \in M_1\eta_n$ such that

$$h_n = \frac{J_{X_1}Q_\iota^{M_2}Q_\iota^{M_3}\cdots Q_\iota^{M_k}x_n - J_{X_1}\eta_n}{\iota}. \tag{86}$$

So, by the above observation, $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, since $x_n \rightharpoonup q$, we can conclude from (47) that $\eta_n \rightharpoonup q$. Then, from Lemma 10, $0 \in M_1q$, i.e., $q \in M_1^{-1}0 = F(Q_\iota^{M_1})$. Similar to the above, by using (46), we can also prove that $q \in M_i^{-1}0 = F(Q_\iota^{M_i})$ for all $i = 2, 3, \cdots k$. Hence, $q \in \cap_{i=1}^k F(Q_\iota^{M_i})$.

Next, we show that $q \in F(f)$. From (65), (68), and the triangle inequality, we conclude that

$$\lim_{n\longrightarrow\infty}\|f(x_n) - x_n\| = 0. \tag{87}$$

Hence, q is an asymptotic fixed point of f. Then, $\widehat{F}(f) = F(f)$ because f is a relatively nonexpansive mapping. Hence, $q \in F(f)$.

Now, we prove that $q \in EP(g)$. Since J_{X_1} is uniformly norm-to-norm continuous on bounded sets, it follows from (83) that

$$\lim_{n\longrightarrow\infty}\left\|J_{X_1}x_n - J_{X_1}v_n\right\| = 0. \tag{88}$$

By $x_n = K_{r_n}v_n$, we conclude that $g(x_n, y) + \langle Bx_n, y - x_n\rangle + (1/r_n)\langle y - x_n, J_{X_1}x_n - J_{X_1}v_n\rangle \ge 0$ for all $y \in C$. Moreover, by the condition A2, $g(y, x_n) \le -g(x_n, y)$ for all $y \in C$. Therefore,

$$g(y, x_n) \le \langle Bx_n, y - x_n\rangle + \frac{1}{r_n}\left\langle y - x_n, J_{X_1}x_n - J_{X_1}v_n\right\rangle, \tag{89}$$

for all $y \in C$. Using (88), the condition A4, and letting $n \longrightarrow \infty$, we have that

$$g(y, q) \le \langle Bq, y - q\rangle, \tag{90}$$

for all $y \in C$. Let $y_\lambda = \lambda y + (1 - \lambda)q$ for all $y \in C$ and $\lambda \in (0, 1)$. It follows from (90), the conditions A1, A4, and the monotonicity of B that

$$\begin{aligned}0 &= g(y_\lambda, y_\lambda) + \langle By_\lambda, y_\lambda - y_\lambda\rangle \le \lambda g(y_\lambda, y) + (1 - \lambda)g(y_\lambda, q)\\ &\quad + \langle By_\lambda, \lambda y + (1 - \lambda)q - y_\lambda\rangle = \lambda g(y_\lambda, y) + (1 - \lambda)g(y_\lambda, q)\\ &\quad + \lambda\langle By_\lambda, y - y_\lambda\rangle + (1 - \lambda)\langle By_\lambda, q - y_\lambda\rangle = \lambda g(y_\lambda, y)\\ &\quad + (1 - \lambda)g(y_\lambda, q) + \lambda\langle By_\lambda, y - y_\lambda\rangle\\ &\quad + (1 - \lambda)\langle By_\lambda - Bq, q - y_\lambda\rangle + (1 - \lambda)\langle Bq, q - y_\lambda\rangle\\ &\le \lambda g(y_\lambda, y) + \lambda\langle By_\lambda, y - y_\lambda\rangle,\end{aligned} \tag{91}$$

for all $y \in C$. Therefore, $0 \le g(y_\lambda, y) + \langle By_\lambda, y - y_\lambda\rangle$. Using the condition A3 and letting $\lambda \longrightarrow 0$, we obtain that $0 \le g(q, y) + \langle Bq, y - q\rangle$ for all $y \in C$. Then, $q \in EP(g)$.

Finally, we prove that $q \in \Omega$. From (56), we have that $\| P_D Aq - Aq \| = \lim_{n \longrightarrow \infty} \| P_D Au_n - Au_n \| = 0$. Therefore, $Aq \in D$, i.e., $q \in \Omega$. Hence, $q = \Pi_{\Omega \cap (\cap_{i=1}^{k} F(Q_i^{M_i})) \cap EP(g)} \circ f(q)$, and this completed the proof.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

References

[1] M. J. Bin and Z. H. Liu, "Relaxation in nonconvex optimal control for nonlinear evolution hemivariational inequalities," *Nonlinear Analysis: Real World Applications*, vol. 50, pp. 613–632, 2019.

[2] M. Ghadampour, E. Soori, R. P. Agarwal, and D. O'Regan, "Two generalized strong convergence algorithms for variational inequality problems in Banach spaces," http://arxiv.org/abs/2101.08047.

[3] Z. H. Liu, D. Motreanu, and S. D. Zeng, "Generalized penalty and regularization method for differential variational-hemivariational inequalities," *SIAM Journal on Optimization*, vol. 31, no. 2, pp. 1158–1183, 2021.

[4] A. Tada and W. Takahashi, *Strong Convergence Theorem for an Equilibrium Problem and a Nonexpansive Mapping, Nonlinear Analysis and Convex Analysis*, W. Takahashi and T. Tanaka, Eds., Yokohama Publishers, Yokohama, Japan, 2007.

[5] G. Cai, Y. Shehu, and O. S. Iyiola, "Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules," *Numerical Algorithms*, vol. 77, pp. 535–558, 2017.

[6] M. Ghadampour, D. O'Regan, E. Soori, and R. P. Agarwal, "A generalized strong convergence algorithm in the presence of errors for variational inequality problems in Hilbert spaces," *Journal of Function Spaces*, vol. 2021, Article ID 9911241, 9 pages, 2021.

[7] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[8] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[9] L. J. Zhu, Y. Yao, and M. Postolache, "Projection methods with linesearch tech- nique for pseudomonotone equilibrium problems and fixed point problems," *UPB Scientific Bulletin, Series A*, vol. 83, no. 1, pp. 3–14, 2021.

[10] G. Cai and S. Bu, "Weak convergence theorems for general equilibrium problems and variational inequality problems and fixed point problems in Banach spaces," *Acta Mathematica Scientia*, vol. 33, no. 1, pp. 303–320, 2013.

[11] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.

[12] F. Schöpfer, T. Schuster, and A. K. Louis, "An iterative regularization method for the solution of the split feasibility problem in Banach spaces," *Inverse Problems*, vol. 24, no. 5, pp. 055008–055020, 2008.

[13] Q. L. Dong, L. Liu, and Y. Yao, "Self-adaptive projection and contraction methods with alternated inertial terms for solving the split feasibility problem," *Journal of Nonlinear and Convex Analysis*, vol. 23, no. 3, pp. 591–605, 2022.

[14] X. Zhao, J. C. Yao, and Y. Yao, "A proximal algorithm for solving split monotone variational inclusions," *UPB Scientific Bulletin, Series A*, vol. 82, no. 3, pp. 43–52, 2020.

[15] S. Reich, "Book review: geometry of Banach spaces, duality mappings and nonlinear problems," *Bulletin of the American Mathematical Society*, vol. 26, no. 2, pp. 367–371, 1992.

[16] Y. Takahashi, K. Hashimoto, and M. Kato, "On sharp uniform convexity, smoothness, and strong type, cotype inegualities," *Journal of Nonlinear and Convex Analysis*, vol. 3, pp. 267–281, 2002.

[17] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, vol. 6, Springer, New York, 2009.

[18] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.

[19] Y. Shehu and O. S. Iyiola, "A cyclic iterative method for solving multiple sets split feasibility problems in Banach spaces," *Quaestiones Mathematicae*, vol. 39, no. 7, pp. 959–975, 2016.

[20] F. Kohsaka and W. Takahashi, "Strong convergence of an iterative sequence for maximal monotone operators in a Banach space," *Abstract and Applied Analysis*, vol. 2004, no. 3, p. 249, 2004.

[21] Y. I. Alber, *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications*, A. G. Kartsatos, Ed., vol. 178, Theory Appl Nonlinear Operat Accretive Monotone Type. Marcel Dekker, New York, 1996.

[22] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banach spaces," *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.

[23] D. Butanriu, S. Reich, and A. J. Zas Iavaski, "Weak convergence of orbits of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.

[24] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[25] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis*, vol. 16, no. 12, pp. 1127–1138, 1991.

[26] Y. C. Liou, "Shrinking projection method of proximal-type for a generalized equilibrium problem, a maximal monotone operator and a pair of relatively nonexpansive mappings," *Taiwanese Journal of Mathematics*, vol. 14, no. 2, pp. 517–540, 2010.