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Research Article

BVP for Hadamard Sequential Fractional Hybrid Differential Inclusions

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The study is concerned with the Hadamard sequential fractional hybrid differential inclusions with two-point hybrid integral boundary conditions. With the help of the Dhage fixed-point theorem for the product of two operators and the Covitz-Nadler fixed-point theorem in the case of fractional inclusions, we obtain the existence results of solutions for Hadamard sequential fractional hybrid differential inclusions. Finally, two examples are presented to illustrate the main results.

1. Introduction

Nowadays, with the increasing demand of researchers for the study of natural phenomena, the use of fractional differential operators and fractional differential equations become an effective means to achieve this goal. Compared with integer order operators, fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. This kind of operator has been expanded and widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission, and control [1–6]. Also, some studies in the biological models with fractional-order derivative have been conducted in recent years [7–9]. In the past year, fractional differential operators and fractional differential equations have been used in modeling the spread of some viruses, such as Zika virus and mumps virus [10, 11]. All of these have enabled researchers to discover the structure of fractional boundary value problems (BVP) and the hereditary nature of their solutions from various aspects. In this regard, many researchers investigated advanced fractional-order modelings and related theoretical results and qualitative behaviors of such fractional-order boundary value problems, see [12–20] and the references therein.

There have been appeared different versions of fractional operators during these years. Much of the work on fractional differential equations only involves either Riemann-Liouville derivative or Caputo derivative [21–29]. Guo et al. ([30, 31]) discussed the existence and Hyers–Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1<\beta<2$ and the existence and Hyers–Ulam stability of the almost periodic solution to the fractional differential equation with impulse and fractional Brownian motion under nonlocal condition. Ma et al. [32] investigated the existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay in Hilbert space.

However, there is another concept of fractional derivative in the literature which was introduced by Hadamard in 1892 [33]. This derivative is known as Hadamard fractional derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. Many researchers have studied and obtained some results on the existence of solutions of Hadamard fractional differential equations in recent years. Yang ([34, 35]) studied the extremal solutions for a coupled system of nonlinear Hadamard fractional differential equations with Cauchy initial value conditions and the existence and nonexistence of positive solutions for the eigenvalue problems of nonlinear Hadamard fractional differential equations with p-Laplacian operator. Tomar et al. [36] established certain generalized Hermite-Hadamard inequalities for generalized convex functions via local fractional integral.

In 1993, Miller and Ross also defined another type of fractional derivative called sequential derivative, which is a combination of the existing derivative operators. From then on, the attention of some researchers was attracted to finding a connection between the Hadamard fractional derivative and the sequential fractional derivative [37–40]. In [41], by using the topological degree theory and Leray–Schauder fixed-point theory, Rezapour and Etemad studied the existence of solutions for the following Caputo–Hadamard fractional boundary value problem via mixed multiorder integroderivative conditions:

$$\begin{cases} \left({}^{\mathrm{CH}}D_{1+}^{\alpha} + \lambda {}^{\mathrm{CH}}D_{1+}^{\varsigma}\right)u(t) = f(t, u(t)), \quad t \in [1, M], \\ u(1) = 0, \\ \mu_1 {}^{\mathrm{CH}}D_{1+}^{\gamma_1}u(M) + {}^{\mathrm{CH}}D_{1+}^{\gamma_2}u(\eta) = \delta_1, \\ \mu_2 {}^{H}I_{1+}^{q_1}u(M) + {}^{H}I_{1+}^{q_2}u(\eta) = \delta_2, \end{cases} \tag{1}$$

where $\lambda, \mu_1 \mu_2, \in (0, 1], \gamma_1, \gamma_2 \in (0, \varsigma - \alpha)$ with $2 < \alpha < \varsigma < 3$, $q_1, q_2 > 0$, $\delta_1, \delta_2 \in R$. The symbol ${}^{\mathrm{CH}}D_{1+}^{\xi}$ points out the Caputo–Hadamard fractional derivative of order $\xi \in \{\alpha, \varsigma, \gamma_1, \gamma_2\}$, with the notation ${}^{H}I_{1+}^{q}$ standing for the Hadamard fractional integrals of order $q \in \{q_1, q_2\}$. The function f formulated by $f\colon [1, M] \times R \longrightarrow R$ is assumed to be continuous on $[1, M] \times R$ with respect to its both components.

As a generalization of fractional boundary value problems, hybrid differential problems with different kinds of boundary conditions have received a lot of attention in recent years [42–44]. The research in this field started from Dhage and Lakshmikantham in 2010 [45]. There is a new concept of differential equation in the literature which was introduced by Dhage and Lakshmikantham. They described this novel differential equation as a hybrid differential equation and investigated the extremal solutions of this new BVP by using some useful fundamental differential inequalities [45]. So far, there are few studies about the existence and various properties of solutions for hybrid boundary value problems of fractional order. In [46], by using a fixed-point theorem due to Dhage, the authors developed some existence theorem for Hadamard-type fractional hybrid differential inclusions problem:

$$\begin{cases} {}^{H}D^{\alpha}\left(\dfrac{x(t)}{f(t, x(t))}\right) \in F(t, x(t)), \quad t \in (1, e), \alpha \in (1, 2], \\ x(1) = x(e) = 0, \end{cases} \tag{2}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1, e] \times R, R \setminus \{0\})$, F: $[1, e] \times R \longrightarrow \mathscr{P}(R)$ is a multivalued map, and $\mathscr{P}(R)$ is the family of all nonempty subsets of R. In [47], by using a hybrid fixed-point theorem of Schaefer type for a sum of three operators due to Dhage, the authors investigated the existence of solutions for the nonlocal fractional BVP of hybrid inclusion problem given by

$$\begin{cases} {}^{C}D_{0+}^{\varrho}\left[\dfrac{w(s) - \sum_{j=1}^{m} {}^{R}I_{0^+}^{\beta_j}h_j(s, w(s))}{f(s, w(s))}\right] \in F(s, w(s)), \quad s \in J \coloneqq [0, 1], \\ w(0) = \mu(w), w(1) = \alpha, \end{cases} \tag{3}$$

where $\mu\colon C(J, R) \longrightarrow R$, $\alpha \in R, \varrho \in (1, 2]$, ${}^{C}D_{0+}^{\varrho}$ is the Caputo derivative, and ${}^{R}I_{0^+}^{\beta_j}$ is the Riemann-Liouville integral of order $\varphi > 0$, such that $\varphi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$. In [48], by using the well-known Dhage fixed-point theorems for single-valued and set-valued maps, Baleanu and Etemad studied a new fractional hybrid model of thermostat in which the thermostat controls an amount of heat based on the temperature detected by sensors. This hybrid differential inclusions of Caputo type are illustrated by

$$\begin{cases} -{}^{C}D_{0+}^{\varrho}\left[\dfrac{x(s)}{\varphi(s,w(s))}\right] \in \Phi(s,x(s)), \quad \varrho \in (1,2], s \in [0,1], \\ D\left[\dfrac{x(s)}{\varphi(s,x(s))}\right]\Bigg|_{s=0} = 0, \\ \lambda {}^{C}D_{0+}^{\varrho-1}\left[\dfrac{x(s)}{\varphi(s,x(s))}\right]\Bigg|_{s=1} + \left[\dfrac{x(s)}{\varphi(s,x(s))}\right]\Bigg|_{s=\eta} = 0, \end{cases} \tag{4}$$

where $\lambda > 0, \eta \in [0,1], \varrho - 1 \in (0,1], D = d/ds, {}^{C}D_{0+}^{\alpha}$ is the Caputo derivative of fractional order $\alpha \in \{\varrho, \varrho - 1\}$, the function Φ: $[0,1] \times R \longrightarrow \mathscr{P}(R)$ is a multivalued map, and $\varphi \in C([0,1] \times R, R\setminus\{0\})$. In [49], the authors investigated the following fractional three-point hybrid problem:

$$\begin{cases} {}^{C}D_{0+}^{\varrho}\left(\dfrac{w(s)}{g(s,w(s))}\right) = G(s,w(s)), \quad s \in [0,1], \\ w(0) = 0, \\ \left[\dfrac{w(s)}{g(s,w(s))}\right]\Bigg|_{s=0} + {}^{R}I_{0^+}^{\varrho^*}\left[\dfrac{w(s)}{g(s,w(s))}\right]\Bigg|_{s=\eta} = 0, \\ \left[\dfrac{w(s)}{g(s,w(s))}\right]\Bigg|_{s=0} + {}^{R}I_{0^+}^{\varrho^*}\left[\dfrac{w(s)}{g(s,w(s))}\right]\Bigg|_{s=1} = 0, \end{cases} \tag{5}$$

where $\varrho \in (2,3]\, \varrho^* > 0, \eta \in (0,1)$. The function G: $[0,1] \times R \longrightarrow R$ is continuous, and $g \in C([0,1] \times R, R\setminus\{0\})$. In [50], by using various novel analytical techniques based on $\alpha - \psi-$contractive mappings, endpoints, and the fixed points of the product operators, the authors investigated a new category of the sequential hybrid inclusion problem with three-point integroderivative boundary conditions:

$$\begin{cases} p_1\left({}^{C}D_{0+}^{\varrho} + p_2 {}^{C}D_{0+}^{\varrho-1}\right)\left[\dfrac{w(s)}{\zeta\left(s,w(s),{}^{R}I_{0+}^{\gamma}w(s)\right)}\right] \in \mathcal{S}(s,w(s)), \quad s \in [0,1], \\ \left[\dfrac{w(s)}{\zeta(s,w(s),{}^{R}I_{0+}^{\gamma}w(s))}\right]\Bigg|_{s=0} = 0, \\ {}^{C}D_{0+}^{1}\left[\dfrac{w(s)}{\zeta(s,w(s),{}^{R}I_{0+}^{\gamma}w(s))}\right]\Bigg|_{s=0} + {}^{C}D_{0+}^{2}\left[\dfrac{w(s)}{\zeta(s,w(s),{}^{R}I_{0+}^{\gamma}w(s))}\right]\Bigg|_{s=0} = 0, \\ \left[\dfrac{w(s)}{\zeta(s,w(s),{}^{R}I_{0+}^{\gamma}w(s))}\right]\Bigg|_{s=1} + {}^{R}I_{0+}^{\xi}\left[\dfrac{w(s)}{\zeta(s,w(s),{}^{R}I_{0+}^{\gamma}w(s))}\right]\Bigg|_{s=p} = 0, \end{cases} \tag{6}$$

where $\varrho \in (2,3]$, $p \in (0,1)$, $p_1, p_2, \gamma, \xi > 0$, ${}^{C}D_{0+}^{(\cdot)}$ and ${}^{R}I_{0+}^{(\cdot)}$ denote the Caputo-fractional derivative and the Riemann-Liouville fractional integral, respectively. Note that ${}^{C}D_{0+}^{1} = \mathrm{d}/\mathrm{d}s$ and ${}^{C}D_{0+}^{2} = \mathrm{d}^2/\mathrm{d}s^2$. The nonzero continuous real-valued function ζ is supposed to be defined on $[0,1] \times R \times R$. $\mathcal{S}\colon [0,1] \times R \longrightarrow \mathcal{P}(R)$ is a set-valued map equipped via some properties.

Motivated by these problems, in this study, we will study the following Hadamard sequential fractional hybrid differential inclusion with two-point hybrid Hadamard integroboundary conditions:

$$\begin{cases} \left({}^{H}D^{\alpha} + \lambda {}^{H}D^{\alpha-1}\right)\left(\dfrac{x(t)}{\rho\left(t, x(t), {}^{H}I^{p}x(t)\right)}\right) \in G\left(t, x(t), {}^{H}I^{p}x(t)\right), \quad t \in (1,e), \\ \alpha_1\left(\dfrac{x(\xi)}{\rho\left(\xi, x(\xi), {}^{H}I^{p}x(\xi)\right)}\right) = \alpha_2 {}^{H}I^{r}\left(\dfrac{x(e)}{\rho\left(e, x(e), {}^{H}I^{p}x(e)\right)}\right), \\ \beta_1\left(\dfrac{x(e)}{\rho\left(e, x(e), {}^{H}I^{p}x(e)\right)}\right) = \beta_2 {}^{H}I^{r}\left(\dfrac{x(\xi)}{\rho\left(\xi, x(\xi), {}^{H}I^{p}x(\xi)\right)}\right), \end{cases} \tag{7}$$

where $\alpha \in (1,2]$, $\xi \in (1,e)$, $p \in (0,1)$, $\lambda, r > 0$, $\alpha_i, \beta_i \in R$, $i = 1, 2$, ${}^{H}D^{(\cdot)}$ and ${}^{H}I^{(\cdot)}$ denote the Hadamard fractional derivative and the Hadamard fractional integral, respectively. The nonzero continuous real-valued function ρ is supposed to be defined $[1,e] \times R \times R$, and $G\colon [1,e] \times R \times R \longrightarrow \mathcal{P}(R)$ is a set-valued map equipped via some properties.

The Hadamard sequential fractional hybrid differential inclusion BVP (7) is modeled with respect to the generalized operators with kernels, including logarithmic functions. In other words, the presented formulation for the given Hadamard sequential fractional hybrid differential inclusion BVP (7) involves two different derivatives in the format of the Hadamard. The supposed abstract fractional hybrid differential inclusion problem (7) with given hybrid integral boundary conditions can describe some mathematical models of real and physical processes in which some parameters are often adjusted to suitable situations. The value of these parameters can change the effects of fractional derivatives and integrals. Moreover, we express that such a Hadamard sequential fractional hybrid differential inclusion BVP is new and enriches the literature on boundary value problems for nonlinear Hadamard fractional differential inclusions. In this way, with the help of Dhage fixed-point theorem and Covitz-Nadler fixed-point theorem in the case of multivalued mapping, we try to find the existence criteria of solutions for the proposed problem (7).

The rest of this study is organized as follows. In Section 2, some preliminary facts that we need in the sequel are given. In Section 3, the existence results of solution for system (7) are discussed. In Section 4, two examples are given to prove validity of the results we obtained.

2. Preliminaries

Definition 1 (see [2]). The Hadamard derivative of fractional-order α for a function $f\colon [1,\infty) \longrightarrow R$ is defined as

$$ {}^{H}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{f(s)}{s}\mathrm{d}s, \quad n-1 < \alpha < n, n = [\alpha]+1, \tag{8}$$

provided the right side is pointwise defined on $[1,\infty)$, where $\Gamma(\cdot)$ is the gamma function and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 (see [2]). The Hadamard fractional integral of order β for a function g is defined as

$$ {}^{H}I^{\beta}g(t) = \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{g(s)}{s}\mathrm{d}s, \quad \beta > 0, \tag{9}$$

provided the integral exists.

Definition 3 (see [15]). Let $w\colon [1,+\infty) \longrightarrow R$ is a sufficiently smooth function; then, the sequential fractional derivative is defined by

$$ {}^{H}D^{\varrho}w(s) = \left({}^{H}D^{\varrho_1}{}^{H}D^{\varrho_2}\ldots{}^{H}D^{\varrho_n}\right)w(s), \tag{10}$$

where $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_n)$ is a multiindex.

Lemma 1. *For any $h \in C([1,e], R)$. A function $x \in AC([1,e], R)$ is a solution of the Hadamard sequential fractional hybrid differential equations:*

$$\left({}^{H}D^{\alpha} + \lambda^{H}D^{\alpha-1}\right)\left(\frac{x(t)}{\rho\left(t, x(t), {}^{H}I^{p}x(t)\right)}\right) = h(t), \quad t \in [1, e], \alpha \in (1, 2], \tag{11}$$

supplemented with the boundary conditions in (7) if and only if it satisfies the following integral equation:

$$\begin{aligned} x(t) = {} & \rho\left(t, x(t), {}^{H}I^{p}x(t)\right)\Bigg(\frac{t^{-\lambda}}{\Delta}\Bigg\{B_2\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg] \\ & -A_2\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right) \\ & \cdot\Bigg\{A_1\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg] \\ & -B_1\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Delta &= A_1B_2 - A_2B_1 \neq 0, \\ A_1 &= \alpha_1\xi^{-\lambda} - \frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\mathrm{d}s, \end{aligned} \tag{13}$$

$$\begin{aligned} A_2 &= \alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s - \frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1}(\log\tau)^{\alpha-2}\mathrm{d}\tau\right)\mathrm{d}s, \\ B_1 &= \beta_1 e^{-\lambda} - \frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\mathrm{d}s, \\ B_2 &= \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s - \frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1}(\log\tau)^{\alpha-2}\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \tag{14}$$

Proof. As argued in [2], the solution of Hadamard differential equation in (11) can be written as

$$\begin{aligned} x(t) = {} & \rho\left(t, x(t), {}^{H}I^{p}x(t)\right)\Bigg(c_0t^{-\lambda} + c_1t^{-\lambda} \\ & \cdot\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s + t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}h(s)\mathrm{d}s\Bigg), \end{aligned} \tag{15}$$

where c_i, $(i = 0, 1)$ are the unknown arbitrary constants. Making use of the integral boundary conditions given by (7) in (15), we obtain

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \tag{16}$$

where A_i and B_i $(i = 1, 2)$ are, respectively, given by (13) and (14), and

$$J_1 = \frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1 \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s, \quad (17)$$

$$J_2 = \frac{\beta_2}{\Gamma(r)} \int_1^{\xi} \left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda} \int_1^{e} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s. \quad (18)$$

Solving (16) for c_0 and c_1 and using notation (13), we find that

$$\begin{aligned} c_0 = \frac{1}{\Delta}\Bigg\{ & B_2\left[\frac{\alpha_2}{\Gamma(r)} \int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1 \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s\right] \\ & - A_2\left[\frac{\beta_2}{\Gamma(r)} \int_1^{\xi} \left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda} \int_1^{e} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s\right]\Bigg\}, \end{aligned} \quad (19)$$

$$\begin{aligned} c_1 = \frac{1}{\Delta}\Bigg\{ & A_1\left[\frac{\beta_2}{\Gamma(r)} \int_1^{\xi} \left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda} \int_1^{e} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s\right] \\ & - B_1\left[\frac{\alpha_2}{\Gamma(r)} \int_1^{e} \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1H} I^{\alpha-1} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1 \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1H} I^{\alpha-1} h(s)\mathrm{d}s\right]\Bigg\}. \end{aligned} \quad (20)$$

Substituting the values of c_0 and c_1 in (15), we get the desired solution (12). This completes the proof.

For a normed space $(X, \|\cdot\|)$, let $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X)\colon Y \text{ is bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X)\colon Y \text{ is compact}\}$, $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X)\colon Y \text{ is closed}\}$, and $\mathscr{P}_{cp,cv}(X) = \{Y \in \mathscr{P}(X)\colon Y \text{ is compact and convex}\}$. □

Definition 4 (see [51]). A multivalued map $G\colon X \longrightarrow \mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.

Definition 5 (see [51]). The multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in \mathscr{P}_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y|\colon y \in G(x)\}\} < \infty$).

Definition 6 (see [51]). A multivalued map G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0, such that $G(\mathscr{N}_0) \subseteq N$.

Definition 7 (see [51]). A multivalued map G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathscr{P}_b(X)$.

Definition 8 (see [51]). A multivalued map G has a fixed point if there is $x \in X$, such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

Definition 9 (see [51]). A multivalued map $G\colon [0,1] \longrightarrow \mathscr{P}_{cl}(R)$ is said to be measurable if for every $y \in R$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z|\colon z \in G(t)\}, \quad (21)$$

is measurable.

Lemma 2 (see [51]). *If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, and $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.*

Let $C([1,e], R)$ denote a Banach space of continuous functions from $[1,e]$ into R with the norm $\|x\| = \sup_{t\in[1,e]} |x(t)|$. Let $L^p([1,e], R)$ be the Banach space of measurable functions $x\colon [1,e] \longrightarrow R$ which are p-th Lebesgue integrable and normed by $\|x\|_{L^p} = (\int_1^e |x(t)|^p \mathrm{d}t)^{1/p}$.

Definition 10 (see [51]). A collection of selections of multivalued map G at point $x \in C[1,e]$ is defined by

$$\begin{aligned} S_{G,x} = \Big\{ & v(s) \in L^1([1,e])\colon v(s) \in \\ & \cdot G\left(t, x(t), {}^H I^p x(t)\right) \quad \text{for a.e. } t \in [1,e]\Big\}. \end{aligned} \quad (22)$$

Definition 11 (see [51]). A multivalued map $G\colon [1,e] \times R^2 \longrightarrow \mathscr{P}(R)$ is said to be Caratheodory if

(i) $t \longmapsto G(t, x, y)$ is measurable for each $x, y \in R$
(ii) $(x, y) \longmapsto G(t, x, y)$ is upper semicontinuous for almost all $t \in [1,e]$

Definition 12 (see [37]). A function $x \in \mathrm{AC}([1,e], R)$ is called a solution of problem (7) if there exists a function $v \in L^1([1,e], R)$ with $v(t) \in G(t, x(t), {}^H I^p x(t))$, a.e. on $[1,e]$, such that

$$\begin{cases} \left({}^{H}D^{\alpha} + \lambda^{H}D^{\alpha-1}\right)\left(\dfrac{x(t)}{\rho\left(t, x(t), {}^{H}I^{p}x(t)\right)}\right) = v(t), \quad \text{a.e. } t \in (1,e), \\ \alpha_1\left(\dfrac{x(\xi)}{\rho\left(\xi, x(\xi), {}^{H}I^{p}x(\xi)\right)}\right) = \alpha_2{}^{H}I^{r}\left(\dfrac{x(e)}{\rho\left(e, x(e), {}^{H}I^{p}x(e)\right)}\right), \\ \beta_1\left(\dfrac{x(e)}{\rho\left(e, x(e), {}^{H}I^{p}x(e)\right)}\right) = \beta_2{}^{H}I^{r}\left(\dfrac{x(\xi)}{\rho\left(\xi, x(\xi), {}^{H}I^{p}x(\xi)\right)}\right). \end{cases} \tag{23}$$

Lemma 3 (see [52]). *Let X be a Banach space. Let G: $[1,e]\times R^2 \longrightarrow \mathscr{P}_{cp,cv}(X)$ be an L^1-Caratheodory multivalued map, and let Θ be a linear continuous mapping from $L^1([1,e],X)$ to $C([1,e],X)$. Then, the operator*

$$\Theta\circ S_G\colon C([1,e],X) \longrightarrow \mathscr{P}_{cp,cv}(C([1,e],X)), x \longmapsto (\theta^{\circ} S_G)(x) = \Theta\left(S_{G,x}\right), \tag{24}$$

is a closed graph operator in $C([1,e],X)\times C([[1,e],X)$.

Lemma 4 (see [53]). *Let X be a Banach algebra and A: $X \longrightarrow X$ be a single-valued and B: $X \longrightarrow \mathscr{P}_{cp,cv}(X)$ be a multivalued operator satisfying the following:*

(i) A is single-valued Lipschitz with a Lipschitz constant k

(ii) B is compact and upper semicontinuous operator

(iii) $2MK<1$, where $M = \|B(X)\|$

Then, either

(i) The operator inclusion $x \in AxBx$ has a solution or

(ii) The set $\mathscr{E} = \{u \in X\colon \mu u \in AuBu, \mu>1\}$ is unbounded.

Definition 13 (see [51]). A multivalued map G: $X \longrightarrow \mathscr{P}(X)$ is said to be a contraction mapping if there is a constant $0<\lambda<1$, such that

$$H_d(G(x),G(y)) \le \lambda\|x-y\|_X, \tag{25}$$

for every $x, y \in X$, where H_d is the Hausdorff metric.

Lemma 5 (see [54]). *Let (X,d) be a complete metric space. If N: $X \longrightarrow \mathscr{P}_{cl}(X)$ is a contraction, then Fix $N \neq \varnothing$.*

3. Main Results

In this section, we will study the existence results of solutions for problem (7). First of all, we fix our terminology.

Let $X = C([1,e],R)$ denote the space equipped with the norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$. Observe that $(X,\|\cdot\|)$ is a Banach space, and $(X,\|\cdot\|)$ with multiplication given by $(x\cdot x')(s) = x(s)x'(s)$ is a Banach algebra.

Now, we enlist the assumptions that we need in the sequel.

(H_1) The function ρ: $[1,e]\times R\times R \longrightarrow R\diagdown\{0\}$ is continuous, and there exists a bounded function Ψ, with bound $\|\Psi\|$, such that $\Psi(t)>0$, a.e. $t \in [1,e]$, and

$$\begin{aligned} &\left|\rho(t,x_1,y_1) - \rho(t,x_2,y_2)\right| \\ &\quad \le \Psi(t)\left(\left|x_1-x_2\right| + \left|y_1-y_2\right|\right), \\ &\quad \text{a.e. } t \in [1,e], \quad \forall x_1,y_1,x_2,y_2 \in R. \end{aligned} \tag{26}$$

(H_2) G: $[1,e]\times R\times R \longrightarrow \mathscr{R}$ is Caratheodory and has nonempty compact and convex values

(H_3) There exists a constant $p_1 \in (0,p)$ and a function $g \in L^{1/p_1}([1,e],R^+)$, such that

$$\|G(t,x,y)\| = \sup\{|v|\colon v \in G(t,x,y)\} \le g(t), \tag{27}$$

For all $x, y \in R$ and for a.e. $t \in [[1,e]$.

(H_4) There exists a positive real number $\mathscr{R}$, such that

$$\mathscr{R} > \frac{\rho_0 M_1\|g\|_{L^{1/p_1}}}{1 - M_0 M_1\|g\|_{L^{1/p_1}}}, \tag{28}$$

where $M_0 M_1\|g\|_{L^{1/p_1}} < 1/2$, $\rho_0 = \sup_{t\in[1,e]}|\rho(t,0,0)|$.

(H_5) There exists a continuous nondecreasing, sub-homogeneous function Φ: $R^+ \longrightarrow R^+$ (that is, $\Phi(\mu x) \le \mu\ \Phi(x)$ for all $\mu\ge1$ and $x \in R^+$) and a function $\varsigma \in L^{1/p_1}([1,e],R^+)$, such that

$$\begin{aligned} \|G(t,x,y)\|_{\mathscr{P}} &\coloneqq \sup\{|y|\colon y \in G(t,x,y)\} \\ &\le \varsigma(t)\Phi(|x|+|y|), \end{aligned} \tag{29}$$

For each $(t,x,y) \in [1,e]\times R^2$.

(H_6) There exists a constant $r>0$, such that

$$r > \frac{\rho_0 M_1(1+1/\Gamma(p+1))\|\varsigma\|_{L^{1/p_1}}\Phi(r)}{1 - M_0 M_1(1+1/\Gamma(p+1))\|\varsigma\|_{L^{1/p_1}}\Phi(r)}, \tag{30}$$

where

$$M_0M_1\left(1+\frac{1}{\Gamma(p+1)}\right)\|\varsigma\|_{L^{1/p_1}}\Phi(r)<\frac{1}{2},\ \rho_0=\sup_{t\in[1,e]}|\rho(t,0,0)|. \tag{31}$$

(H_7) G: $[1,e]\times R^2\longrightarrow \mathscr{P}_{cp}(R)$ satisfying the condition $G(\cdot,x,y)$: $[1,e]\longrightarrow \mathscr{P}_{cp}(R)$ is measurable for each $(x,y)\in R^2$

(H_8) $H_d(G(t,x,y),G(t,\overline{x},\overline{y}))\le\zeta(t)(|x-\overline{x}|+|y-\overline{y}|)$ for a.e. $t\in[1,e]$ and for all $x,y,\overline{x},\overline{y}\in R$ with $\zeta\in L^{1/p_1}([1,e],R^+)$ and $d(0,G(t,0,0))\le\zeta(t)$ for a.e. $t\in[1,e]$.

(H_9) The function ρ: $[1,e]\times R\times R\longrightarrow R\setminus\{0\}$ is continuous, and there exists a function $\eta\in C([1,e],R^+)$, such that

$$|\rho(t,x,y)|\le\eta(t),\ \forall(t,x,y)\in[1,e]\times R^2. \tag{32}$$

In assumption (H_8), H_d is the Hausdorff metric, where d is the Euclidean metric in R defined by $d(x,y)=|x-y|$ for x, $y\in R$.

Furthermore, we set the notations:

$$M_1=\frac{1}{|\Delta|(\alpha-1)}\left\{\frac{|\alpha_2|\left[(\alpha-1)|B_2|+|B_1|\right]}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right.$$
$$+\frac{|\beta_2|\left[(\alpha-1)|A_2|+|A_1|\right](\log\xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}$$
$$+\frac{|\beta_1|\left[(\alpha-1)|A_2|+|A_1|\right]}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}+\frac{|\alpha_1|\left[(\alpha-1)|B_2|+|B_1|\right](\log\xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}$$
$$\left.+\frac{|\Delta|(\alpha-1)}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right\},$$

$$M_2=\frac{1}{|\Delta|}\left[\frac{|\alpha_2||B_2|}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}+\frac{|\beta_2||A_2|(\log\xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right.$$
$$\left.+\frac{|\alpha_1||B_2|(\log\xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}+\frac{|\beta_1||A_2|}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right],$$

$$M_3=\frac{1}{|\Delta|(\alpha-1)}\left[\frac{|\alpha_2||B_1|}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}+\frac{|\beta_2||A_1|(\log\xi)^{(1+a)(1-p_1)+r+1}}{\Gamma(\alpha-1)\Gamma(r+1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right.$$
$$\left.+\frac{|\alpha_1||B_1|(\log\xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}+\frac{|\beta_1||A_1|}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right],$$

$$\|\Psi\|=\sup_{t\in[1,e]}|\Psi(t)|,\ \|\eta\|=\sup_{t\in[1,e]}|\eta(t)|,\ a=\frac{\alpha-2}{1-p_1},\ \|\zeta\|_{L^{1/p_1}}=\left(\int_1^e|\zeta(t)|^{1/p_1}\mathrm{d}t\right)^{p_1}. \tag{33}$$

Theorem 1. *Let the hypotheses (H_1)–(H_4) be satisfied. Then, inclusion problem (7) has at least one mild solution on $C([1,e],R)$.*

Proof. Consider the operator $\mathcal{N}: X \longrightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned}\mathcal{N}x(t) = \Bigg\{ & w \in C([1,e],R): w(t) = \rho\left(t, x(t), {}^{H}I^{p}x(t)\right)\Bigg(\frac{t^{-\lambda}}{\Delta}\Bigg\{B_2\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1} \\ & \times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg] \\ & -A_2\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right) \\ & \cdot\Bigg\{A_1\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg] \\ & -B_1\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg), v \in S_{G,x}\Bigg\},\end{aligned} \tag{34}$$

and we define two operators $\mathcal{A}: X \longrightarrow X$ by

$$\mathcal{A}x(t) = \rho\left(t, x(t), {}^{H}I^{p}x(t)\right), \quad t \in [1,e], \tag{35}$$

and $\mathcal{B}: X \longrightarrow \mathcal{P}(X)$ by

$$\begin{aligned}\mathcal{B}x(t) = \Bigg\{ & w \in C([1,e],R): w(t) = \frac{t^{-\lambda}}{\Delta}\Bigg\{B_2\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda} \\ & \cdot\int_1^{\xi} s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg] - A_2\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\Bigg\{A_1\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1} \\ & \times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg] \\ & -B_1\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\Bigg]\Bigg\} \\ & +t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s, \quad v \in S_{G,x}\Bigg\}.\end{aligned} \tag{36}$$

Observe that $\mathcal{N}(x) = \mathcal{A}x\mathcal{B}x$. We will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 4. For the sake of convenience, we split the proof into several steps. □

Step 1. $\mathcal{A}$ is a Lipschitz on X, that is, (i) of Lemma 4 holds.

Let $x, y \in X$. By H_1, we have

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left|\rho\left(t, x(t), {}^{H}I^{p}x(t)\right) - \rho\left(t, y(t), {}^{H}I^{p}y(t)\right)\right| \\
&\le |\Psi(t)|\left(|x(t) - y(t)| + \left|{}^{H}I^{p}x(t) - {}^{H}I^{p}y(t)\right|\right) \\
&\le |\Psi(t)|\left(|x(t) - y(t)| + \frac{1}{\Gamma(p)}\int_1^t \left(\log\frac{t}{s}\right)^{p-1}\frac{1}{s}|x(s) - y(s)|ds\right) \\
&\le |\Psi|\left(\|x - y\| + \frac{\|x - y\|}{\Gamma(p)}\int_1^t (\log t - \log s)^{p-1} d \log s\right) \\
&\le |\Psi|\left(\|x - y\| + \frac{\|x - y\|}{\Gamma(p)}\cdot\frac{(\log t)^{p}}{p}\right) \\
&\le |\Psi|\left(\|x - y\| + \frac{1}{\Gamma(1 + p)}\|x - y\|\right) \\
&\le |\Psi|\left(1 + \frac{1}{\Gamma(1 + p)}\right)\|x - y\|.
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
\|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t\in[1,e]} |\mathcal{A}x(t) - \mathcal{A}y(t)| \\
&\le \|\Psi\|\left(1 + \frac{1}{\Gamma(1 + p)}\right)\|x - y\|,
\end{aligned} \tag{38}$$

for all $x, y \in X$. So, $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $M_0 = \|\Psi\|(1 + 1/\Gamma(1 + p))$.

Step 2. The multivalued operator $\mathcal{B}$ is compact and upper semicontinuous on X, that is, (ii) of Lemma 4 holds.

First, we show that $\mathcal{B}$ has convex values. Let $w_1, w_2 \in \mathcal{B}x$, and then, there are $v_1, v_2 \in S_{G,x}$, such that

$$\begin{aligned}
w_i = \frac{t^{-\lambda}}{\Delta}&\Bigg\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v_i(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1 H}I^{\alpha-1}v_i(s)ds\right] \\
&-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi \left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v_i(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1 H}I^{\alpha-1}v_i(s)ds\right]\Bigg\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\Bigg\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi \left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v_i(\tau)d\tau\right)ds\right. \\
&\left.-\beta_1 e^{-\lambda}\int_1^e s^{\lambda-1 H}I^{\alpha-1}v_i(s)ds\right] - B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v_i(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1 H}I^{\alpha-1}v_i(s)ds\right]\Bigg\} \\
&+ t^{-\lambda}\int_1^t s^{\lambda-1 H}I^{\alpha-1}v_i(s)ds,
\end{aligned} \tag{39}$$

$i = 1, 2$, $t \in [1, e]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned}
&\theta w_1(t) + (1-\theta) w_2(t) \\
&= \frac{t^{-\lambda}}{\Delta}\Bigg\{B_2\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}(\theta v_1(\tau) + (1-\theta)v_2(\tau))d\tau\right)ds \\
&-\alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1 H}I^{\alpha-1}(\theta v_1(s) + (1-\theta)v_2(s))ds\Bigg] \\
&-A_2\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}(\theta v_1(\tau) + (1-\theta)v_2(\tau))d\tau\right)ds \\
&-\beta_1 e^{-\lambda}\int_1^{e} s^{\lambda-1 H}I^{\alpha-1}(\theta v_1(s) + (1-\theta)v_2(s))ds\Bigg]\Bigg\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\Bigg\{A_1\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1} \\
&\times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}(\theta v_1(\tau) + (1-\theta)v_2(\tau))d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^{e} s^{\lambda-1 H}I^{\alpha-1}(\theta v_1(s) + (1-\theta)v_2(s))ds\Bigg] \\
&-B_1\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}(\theta v_1(\tau) + (1-\theta)v_2(\tau))d\tau\right)ds \\
&-\alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1 H}I^{\alpha-1}(\theta v_1(s) + ((1-\theta)v_2(s))ds\Bigg]\Bigg\} \\
&+t^{-\lambda}\int_1^t s^{\lambda-1 H}I^{\alpha-1}(\theta v_1(s) + (1-\theta)v_2(s))ds,
\end{aligned} \tag{40}$$

where $\bar{v}(t) = (\theta v_1(t) + (1-\theta)v_2(t)) \in G(t, x(t), {}^H I^q x(t))$ for all $t \in [1, e]$. Hence, $\theta u_1(t) + (1-\theta)u_2(t) \in \mathscr{B}x$, and consequently, $\mathscr{B}x$ is convex for each $x \in X$. As a result, $\mathscr{B}$ defines a multivalued operator $\mathscr{B}: X \longrightarrow \mathscr{P}_{cv}(X)$.

Next, we show that $\mathscr{B}$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X, and then, there exists a real number $r > 0$, such that $\|x\| \le r$, $\forall x \in Q$. Now, for each $h \in \mathscr{B}x$, there exist $v \in S_{G,x}$, such that

$$\begin{aligned}
h(t) &= \frac{t^{-\lambda}}{\Delta}\Bigg\{B_2\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1 H}I^{\alpha-1}v(s)ds\Bigg] \\
&-A_2\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^{e} s^{\lambda-1 H}I^{\alpha-1}v(s)ds\Bigg]\Bigg\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\Bigg\{A_1\Bigg[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v(\tau)d\tau\right)ds \\
&-\beta_1 e^{-\lambda}\int_1^{e} s^{\lambda-1 H}I^{\alpha-1}v(s)ds\Bigg] - B_1\Bigg[\frac{\alpha_2}{\Gamma(r)}\int_1^e \left(\log\frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H}I^{\alpha-1}v(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^{\xi} s^{\lambda-1 H}I^{\alpha-1}v(s)ds\Bigg]\Bigg\} \\
&+t^{-\lambda}\int_1^t s^{\lambda-1 H}I^{\alpha-1}v(s)ds.
\end{aligned} \tag{41}$$

Then, for each $t \in [1,e]$, using (H_2), we have

$$
\begin{aligned}
|h(t)| \le{}& \frac{t^{-\lambda}}{|\Delta|}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|d\tau\right)ds+|\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1}\times{}^H I^{\alpha-1}|v(s)|ds\right]\right.\\
&\left.+|A_2|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|d\tau\right)ds+|\beta_1|e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}|v(s)|ds\right]\right\}\\
&+\frac{1}{|\Delta|}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\\
&\times\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|d\tau\right)ds+|\beta_1|e^{-\lambda}\int_1^e\int_1^e s^{\lambda-1H}I^{\alpha-1}|v(s)|ds\right]\right.\\
&\left.+|B_1|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|d\tau\right)ds+|\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}|v(s)|ds\right]\right\}\\
&+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}|v(s)|ds\\
\le{}& \frac{t^{-\lambda}}{|\Delta|}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}g(\tau)d\tau\right)ds+|\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1}\times{}^H I^{\alpha-1}g(s)ds\right]\right.\\
&\left.+|A_2|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}g(\tau)d\tau\right)ds+|\beta_1|e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}g(s)ds\right]\right\}\\
&+\frac{1}{|\Delta|}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\\
&\times\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}g(\tau)d\tau\right)ds+|\beta_1|e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}g(s)ds\right]\right.\\
&\left.+|B_1|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}g(\tau)d\tau\right)ds+|\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}g(s)ds\right]\right\}\\
&+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}g(s)ds\\
\le{}& \frac{\|g\|_{L^{p_1/1}}}{|\Delta|}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-1}\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\sigma^{-1/1-p_1}d\sigma\right)^{1-p_1}d\tau\right)ds\right.\right.\\
&\left.+\frac{|\alpha_1|}{\Gamma(\alpha-1)}\int_1^\xi s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\tau^{-1/1-p_1}d\tau\right)^{1-p_1}ds\right]\\
&+|A_2|\left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\sigma^{-1/1-p_1}d\sigma\right)^{1-p_1}d\tau\right)ds\right.\\
&\left.\left.+\frac{|\beta_1|}{\Gamma(\alpha-1)}\int_1^e s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\tau^{-1/1-p_1}d\tau\right)^{1-p_1}ds\right]\right\}
\end{aligned}
$$

$$+\frac{\|g\|_{L^{1/p_1}}}{|\Delta|}\left(\int_1^t s^{-1}(\log s)^{\alpha-2}ds\right)\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}\right.\right.$$

$$\times\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\sigma^{-1/1-p_1}d\sigma\right)^{1-p_1}d\tau\right)ds+\frac{|\beta_1|}{\Gamma(\alpha-1)}\int_1^e s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\tau^{-1/1-p_1}d\tau\right)^{1-p_1}ds\right]$$

$$+|B_1|\left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-1}\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\sigma^{-1/1-p_1}d\sigma\right)^{1-p_1}d\tau\right)ds\right.$$

$$\left.\left.+\frac{|\alpha_1|}{\Gamma(\alpha-1)}\int_1^\xi s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\tau^{-1/1-p_1}d\tau\right)^{1-p_1}ds\right]\right\}$$

$$+\frac{\|g\|_{L^{1/p_1}}}{\Gamma(\alpha-1)}\int_1^t s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\tau^{-1/1-p_1}d\tau\right)^{1-p_1}ds$$

$$\leq\frac{\|g\|_{L^{1/p_1}}}{|\Delta|}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-1}\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\frac{1}{\sigma}d\sigma\right)^{1-p_1}d\tau\right)ds\right.\right.$$

$$\left.+\frac{|\alpha_1|}{\Gamma(\alpha-1)}\int_1^{e\xi}s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\frac{1}{\tau}d\tau\right)^{1-p_1}ds\right]$$

$$+|A_2|\left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}\times\left(\int_1^s\tau^{-1}\left(\int_1^e\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\frac{1}{\sigma}d\sigma\right)^{1-p_1}d\tau\right)ds\right.$$

$$\left.\left.+\frac{|\beta_1|}{\Gamma(\alpha-1)}\int_1^e s^{-1}\left(\int_1^e\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\frac{1}{\tau}d\tau\right)^{1-p_1}ds\right]\right\}$$

$$+\frac{\|g\|_{L^{1/p_1}}}{|\Delta|}\left(\int_1^t s^{-1}(\log s)^{\alpha-2}ds\right)\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}\right.\right.$$

$$\times\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\frac{1}{\sigma}d\sigma\right)^{1-p_1}d\tau\right)ds+\frac{|\beta_1|}{\Gamma(\alpha-1)}\int_1^e s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\frac{1}{\tau}d\tau\right)^{1-p_1}ds\right]$$

$$+|B_1|\left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-1}\left(\int_1^s\tau^{-1}\left(\int_1^\tau\left(\log\frac{\tau}{\sigma}\right)^{\alpha-2/1-p_1}\frac{1}{\sigma}d\sigma\right)^{1-p_1}d\tau\right)ds\right.$$

$$\left.\left.+\frac{|\alpha_1|}{\Gamma(\alpha-1)}\int_1^\xi s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\frac{1}{\tau}d\tau\right)^{1-p_1}ds\right]\right\}+\frac{|g|_{L^{1/p_1}}}{\Gamma(\alpha-1)}\int_1^t s^{-1}\left(\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha-2/1-p_1}\frac{1}{\tau}d\tau\right)^{1-p_1}ds$$

$$\leq\frac{|g|_{L^{1/p_1}}}{|\Delta|}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(\alpha-1)\Gamma(r)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-1}(\log s)^{(1+a)(1-p_1)+1}ds\right.\right.$$

$$\left.+\frac{|\alpha_1|(\log\xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right]$$

$$
\begin{aligned}
&+\left|A_2\right|\left[\left[\frac{\left|\beta_2\right|}{\Gamma(\alpha-1)\Gamma(r)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\times\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}(\log s)^{(1+a)(1-p_1)+1}ds\right.\right.\\
&\left.\left.+\frac{\left|\beta_1\right|}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right]\right]\Bigg\}\\
&+\frac{\|g\|_{L^{1/p_1}}}{|\Delta|(\alpha-1)}\left\{\left|A_1\right|\left[\left[\frac{\left|\beta_2\right|}{\Gamma(\alpha-1)\Gamma(r)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right.\right.\right.\\
&\left.\times\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-1}(\log s)^{(1+a)(1-p_1)+1}ds+\frac{\left|\beta_1\right|}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right]\\
&+\left|B_1\right|\left[\left[\frac{\left|\alpha_2\right|}{\Gamma(\alpha-1)\Gamma(r)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\times\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-1}(\log s)^{(1+a)(1-p_1)+1}ds\right.\right.\\
&\left.\left.\left.+\frac{\left|\alpha_1\right|(\log\xi)^{(1+a)(1-p_1)+1}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\right]\right]\right\}\\
&+\frac{\|g\|_{L^{1/p_1}}}{\Gamma(\alpha-1)\left[(1+a)^{2-p_1}(1-p_1)+(1+a)^{1-p_1}\right]}\leq M_1\|g\|_{L^{1/p_1}},
\end{aligned}
\tag{42}
$$

hence,

$$
\|h\|\leq M_1\|g\|_{L^{1/p_1}}. \tag{43}
$$

Therefore, $\mathscr{B}(Q)$ is uniformly bounded.

Next, we show that $\mathscr{B}$ maps bounded sets into equicontinuous sets. For this purpose, we assume that Q be, as above, a bounded set and $h\in\mathscr{B}x$ for some $x\in Q$, and then, there exists a $v\in S_{G,x}$, such that

$$
\begin{aligned}
h(t)=&\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^{s}\tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right.\\
&\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^{s}\tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds-\beta_1e^{-\lambda}\int_1^{e}s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right\}\\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^{t}s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\\
&\cdot\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^{s}\tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds-\beta_1e^{-\lambda}\int_1^{e}s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right.\\
&\left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^{s}\tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right\}\\
&+t^{-\lambda}\int_1^{t}s^{\lambda-1H}I^{\alpha-1}v(s)ds.
\end{aligned}
\tag{44}
$$

Thus, for any $t_1, t_2 \in [1, e]$, $t_2 > t_1$, we have

$$\begin{aligned}
&\left|h(t_2) - h(t_1)\right| \\
&\leq \frac{M_2\|g\|_{L^{1/p_1}}}{|\Delta|}\left|t_2^{-\lambda} - t_1^{-\lambda}\right| + \frac{M_3\|g\|_{L^{1/p_1}}}{|\Delta|}\left|t_2^{-\lambda}\int_1^{t_2} s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s - t_1^{-\lambda}\int_1^{t_1} s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right| \\
&\quad + \frac{\|g\|_{L^{1/p_1}}}{(1+a)^{1-p_1}\Gamma(\alpha-1)}\left|t_2^{-\lambda}\int_1^{t_2} s^{\lambda-1}(\log s)^{(1+a)(1-p_1)}\mathrm{d}s - t_1^{-\lambda}\int_1^{t_1} s^{\lambda-1}(\log s)^{(1+a)(1-p_1)}\mathrm{d}s\right| \\
&\leq \frac{M_2\|g\|_{L^{1/p_1}}}{|\Delta|}\left|t_2^{-\lambda} - t_1^{-\lambda}\right| + \frac{M_3\|g\|_{L^{1/p_1}}}{|\Delta|}\left(\left|t_2^{-\lambda} - t_1^{-\lambda}\right| \cdot \left|\int_1^{t_2}\frac{(\log s)^{\alpha-2}}{s^{1-\lambda}}\mathrm{d}s\right| + \left|t_1^{-\lambda}\right|\int_{t_1}^{t_2}\frac{(\log s)^{\alpha-2}}{s^{1-\lambda}}\mathrm{d}s|\right) \\
&\quad + \frac{\|g\|_{L^{1/p_1}}}{(1+a)^{1-p_1}\Gamma(\alpha-1)}\left(\left|t_2^{-\lambda} - t_1^{-\lambda}\right| \cdot \left|\int_1^{t_2}\frac{(\log s)^{(1+a)(1-p_1)}}{s^{1-\lambda}}\mathrm{d}s\right| + \left|t_1^{-\lambda}\right|\int_{t_1}^{t_2}\frac{(\log s)^{(1+a)(1-p_1)}}{s^{1-\lambda}}\mathrm{d}s|\right) \\
&\quad \longrightarrow 0,
\end{aligned} \tag{45}$$

independent of $x \in Q$ as $t_1 - t_2 \longrightarrow 0$.

Therefore, $\mathscr{B}(Q)$ is an equicontinuous set in X. Now, an application of the Arzela-Ascoli theorem yields that $\mathscr{B}(Q)$ is relatively compact.

In our next step, we show that $\mathscr{B}$ is upper semicontinuous. By Lemma 2, $\mathscr{B}$ will be upper semicontinuous if we prove that it has a closed graph. Let $x_n \longrightarrow x_*, h_n \in \mathscr{B}x_n$, and $h_n \longrightarrow h_*$. Then, we need to show that $h_* \in \mathscr{B}x_*$. Associated with $h_n \in \mathscr{B}x_n$, there exists $v_n \in S_{G,x_n}$, such that for each $t \in [1, e]$,

$$\begin{aligned}
h_n(t) = \frac{t^{-\lambda}}{\Delta}&\Bigg\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right] \\
&-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\Bigg\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right) \\
&\cdot\Bigg\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right] \\
&-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\Bigg\} \\
&+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s.
\end{aligned} \tag{46}$$

Thus, it suffices to show that there exists $v_* \in S_{G,x_*}$, such that for each $t \in [1,e]$,

$$\begin{aligned}
h_*(t) = {} & \frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right.\\
& \left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right\}\\
& +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\\
& \cdot\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right.\\
& \left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right\}\\
& + t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_*(s)ds.
\end{aligned} \tag{47}$$

Let us consider the linear operator $\Theta: L^1([1,e],R) \longrightarrow C([1,e],R)$ given by

$$\begin{aligned}
v(t) \longmapsto {} & \Theta(v)(t)\\
= {} & \frac{t^{-\lambda}}{\Delta}\left\{B_2\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v(s)ds\right.\\
& \left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right\}\\
& +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\\
& \cdot\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds - \beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right.\\
& \left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)d\tau\right)ds - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v(s)ds\right]\right\}\\
& + t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v(s)ds.
\end{aligned} \tag{48}$$

Notice that the operator Θ is continuous. Indeed, for $v_n, v_* \in L^1([1,e],R)$ with $v_n \longrightarrow v_*$ in $L^1([1,e],R)$, we obtain

$$\begin{aligned}
&\|\Theta(v_n)(t)-\Theta(v_*)(t)\| \\
&\leq \frac{1}{|\Delta|}\left\|\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}(v_n(\tau)-v_*(\tau))d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}(v_n(s)-v_*(s))ds\right]\right.\right. \\
&\left.\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}(v_n(\tau)-v_*(\tau))d\tau\right)ds-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}(v_n(s)-v_*(s))ds\right]\right\}\right\| \\
&+\frac{1}{|\Delta|(\alpha-1)}\left\|\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}(v_n(\tau)-v_*(\tau))d\tau\right)ds\right.\right.\right. \\
&\left.-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}(v_n(s)-v_*(s))ds\right] \\
&\left.\left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}(v_n(\tau)-v_*(\tau))d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}(v_n(s)-v_*(s))ds\right]\right\}\right\| \\
&+\left\|\int_1^t s^{-1H}I^{\alpha-1}(v_n(s)-v_*(s))ds\right\|, \quad \forall t\in[1,e],
\end{aligned} \tag{49}$$

which implies that $\Theta(v_n) \longrightarrow \Theta(v_*)$ in $C([1,e],R)$.

Thus, it follows by Lemma 3 that $\Theta^\circ S_G$ is a closed graph operator. Furthermore, we have $h_n(t) \in \Theta(S_{G,x_n})$. Since $x_n \longrightarrow x_*$, therefore, we have

$$\begin{aligned}
h_*(t)=&\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right. \\
&\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}ds\right)\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right. \\
&\left.\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right] \\
&\left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_*(\tau)d\tau\right)ds-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_*(s)ds\right]\right\} \\
&+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_*(s)ds,
\end{aligned} \tag{50}$$

for some $v_* \in S_{G,x_*}$.

As a result, we have that the operator $\mathscr{B}$ is compact and upper semicontinuous.

Step 3. Now, we show that $2MK<1$, that is, (iii) of Lemma 4 holds.

This is obvious by (H_4) since we have

$$M = \|B(X)\| = \sup\{|\mathscr{B}x|: x \in X\} \leq M_1 \|g\|_{L^{1/p_1}}, \tag{51}$$

and $K = M_0$.

Thus, all the conditions of Lemma 4 are satisfied, and a direct application of it yields that either conclusion (i) or conclusion (ii) holds. We show that conclusion (ii) is not possible.

Supposed the conclusion (ii) is true. Let $u \in \mathscr{E}$ be arbitrary. Then, we have, for $\lambda > 1, \lambda u \in \mathscr{A}u\mathscr{B}u$, and then, there exists $v \in S_{G,x}$ such that

$$\begin{aligned} u(t) = {} & \lambda^{-1}\rho\left(t, x(t), {}^{H}I^{p}x(t)\right)\left(\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\right.\right.\right. \\ & \cdot\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1p}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1p}v(s)\mathrm{d}s\Big] \\ & -A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1p}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1p}v(s)\mathrm{d}s\right]\Bigg\} \\ & +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right. \\ & \times\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1p}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1p}v(s)\mathrm{d}s\Big] \\ & -B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1p}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s - \alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1p}v(s)\mathrm{d}s\right]\Bigg\} \\ & +t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1p}v(s)\mathrm{d}s\Bigg), \end{aligned} \tag{52}$$

and so, for all $t \in [1, e]$, we have

$$\begin{aligned} |u(t)| \leq {} & \lambda^{-1}\left|\rho\left(t, u(t), {}^{H}I^{p}u(t)\right)\right| \\ & \times\left(\frac{t^{-\lambda}}{\Delta}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s + |\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\right.\right. \\ & +|A_2|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s + |\beta_1|e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\Bigg\} \\ & +\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^\xi\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right. \\ & \times\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s + |\beta_1|e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\Big] \\ & +|B_1|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s + |\alpha_1|\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\Bigg\} \\ & +t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\left|\rho\left(t, u(t), {}^{H}I^{p} u(t)\right)-\rho(t,0,0)\right|+|\rho(t,0,0)|\right] \\
&\times\left(\frac{t^{-\lambda}}{\Delta}\left\{|B_2|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s+|\alpha_1|\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\right.\right. \\
&\left.+|A_2|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s+|\beta_1|e^{-\lambda}\int_1^{e}s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\right\} \\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\left\{|A_1|\left[\frac{|\beta_2|}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right. \\
&\left.\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s+|\beta_1|e^{-\lambda}\int_1^{e}s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right] \\
&\left.+|B_1|\left[\frac{|\alpha_2|}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}|v(\tau)|\mathrm{d}\tau\right)\mathrm{d}s+|\alpha_1|\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right]\right\} \\
&\left.+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}|v(s)|\mathrm{d}s\right)\leq M_1\|g\|_{L^{1/p_1}}\left[\|\Psi\|\left(1+\frac{1}{\Gamma(1+p)}\right)\|u\|+\rho_0\right].
\end{aligned} \tag{53}
$$

Therefore,

$$
\begin{aligned}
\|u\| &= \sup_{t\in[1,e]}|u(t)| \\
&\leq M_1\|g\|_{L^{1/p_1}}\left[\|\Psi\|\left(1+\frac{1}{\Gamma(1+p)}\right)\|u\|+\rho_0\right],
\end{aligned} \tag{54}
$$

where we have put $\rho_0=\sup_{t\in[1,e]}|\rho(t,0,0)|$. Then, with $\|u\|=\mathscr{R}$, we have

$$
\mathscr{R}\leq\frac{\rho_0 M_1\|g\|_{L^{1/p_1}}}{1-M_0M_1\|g\|_{L^{1/p_1}}}. \tag{55}
$$

This is contradictory. Thus, conclusion (ii) of Lemma 4 does not hold by (28). Therefore, the operator equation $\mathscr{A}x\mathscr{B}x$ and consequent problem (7) have a solution on $[1,e]$. This completes the proof.

Theorem 2. *Suppose that the conditions (H_1), (H_2), (H_5), and (H_6) hold. Then, inclusion problem (7) has at least one mild solution on $C([1,e],R)$.*

Proof. The proof is similar to that of Theorem 1 and is omitted. □

Theorem 3. *Suppose that the conditions (H_1), (H_3), and (H_7)–(H_9) hold. If*

$$
\lambda_0 := M_1\left(1+\frac{1}{\Gamma(1+p)}\right)\left(\|g\|_{L^{1/p_1}}\|\Psi\|+\|\eta\|\|\zeta\|_{L^{1/p_1}}\right)<1, \tag{56}
$$

where M_1, $\|\Psi\|$, $\|\eta\|$, $\|\zeta\|_{L^{1/p_1}}$ are given by (33); then, inclusion problem (7) has at least one mild solution on $C([1,e],R)$.

Proof. Observe that the set $S_{G,x}$ is nonempty for each $x\in C[1,e]$ by assumption (H_7), and thus, G has a measurable selection. We now show that the operator $\mathscr{N}: C[1,e]\longrightarrow\mathscr{P}(C[1,e])$ satisfies the assumptions of Lemma 5. To establish that $\mathscr{N}x\in\mathscr{P}_{cl}(C[1,e])$, for each $x\in C[1,e]$, let $\{w_n\}_{n\geq1}\subset\mathscr{N}x$ be such that $w_n\longrightarrow w$ as $n\longrightarrow\infty$ in $C[1,e]$. Then, $w\in C[1,e]$, and there exists $v_n\in S_{G,x}$, such that for each $t\in[1,e]$, we have

$$
\begin{aligned}
w_n(t)=\rho\left(t,x(t),{}^{H}I^{p}x(t)\right)&\left(\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\right.\right. \\
&\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s\tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1e^{-\lambda}\int_1^{e}s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\right\}
\end{aligned}
$$

$$+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)$$
$$\times\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\right.$$
$$\left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_n(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right]\right\}$$
$$\left.+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_n(s)\mathrm{d}s\right), \tag{57}$$

with $v_n(t)\in G(t,x(t),{}^HI^qx(t))$, $t\in[1,e]$.

Since G has compact values, therefore, we can pass onto a subsequence (denoted in a same way) to obtain that v_n converges to v in $L^1[1,e]$. Thus, $v\in S_{G,x}$, and for each $t\in[1,e]$, we have $w_n(t)\longrightarrow w(t)$, where

$$w(t)=\rho\left(t,x(t),{}^HI^px(t)\right)\left(\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\right]\right.\right.$$
$$\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\right]\right\}$$
$$+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)$$
$$\times\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\right]\right. \tag{58}$$
$$\left.-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\right]\right\}$$
$$\left.+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v(s)\mathrm{d}s\right).$$

Hence, $w\in\mathcal{N}(x)$.

Next, we show that $\mathcal{N}$ is a contraction, that is,

$$H_{d_1}(\mathcal{N}x,\mathcal{N}\overline{x})\le\lambda_0\|x-\overline{x}\|,\quad \forall x,\overline{x}\in X, \tag{59}$$

where λ_0 is defined in (56), and d_1 is the metric induced by the norm $\|\cdot\|$ in $C[1,e]$.

For this, let $x,\overline{x}\in[1,e]$ and $w_1\in\mathcal{N}x$. Then, there exists $v_1\in S_{G,x}$, such that for all $t\in[1,e]$, we obtain

$$w_1(t)=\rho\left(t,x(t),{}^HI^px(t)\right)\left(\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\right.\right.\right.$$
$$\left.\cdot\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^\xi s^{\lambda-1H}I^{\alpha-1}v_1(s)\mathrm{d}s\right]$$
$$\left.-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_1(s)\mathrm{d}s\right]\right\}$$

$$
\begin{aligned}
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right.\\
&\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_1(s)\mathrm{d}s\Bigg]\\
&-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v_1(s)\mathrm{d}s\right]\Bigg\}\\
&\left.+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_1(s)\mathrm{d}s\right).
\end{aligned}
\tag{60}
$$

By (H_7), we have

$$
\begin{aligned}
&H_{d_1}\left(G\left(t,x(t),{}^HI^px(t)\right),G\left(t,\overline{x}(t),{}^HI^p\overline{x}(t)\right)\right)\\
&\quad\leq\zeta(t)\left(|x(t)-\overline{x}(t)|+\left|{}^HI^px(t)-{}^HI^p\overline{x}(t)\right|\right),
\end{aligned}
\tag{61}
$$

for a.e. $t\in[1,e]$, and then, there exists $\psi\in G(t,\overline{x}(t),{}^HD^q\overline{x}(t))$, such that

$$
\begin{aligned}
&|v_1(t)-\psi|\leq\zeta(t)(|x(t)-\overline{x}(t)|\\
&\quad+\left|{}^HI^px(t)-{}^HI^p\overline{x}(t)\right|\right),\quad \text{a.e. } t\in[1,e].
\end{aligned}
\tag{62}
$$

We define $\widetilde{U}$: $[1,e]\longrightarrow\mathscr{P}(R)$ by $\widetilde{U}(t)=\{\psi\in R: |v_1(t)-\psi|\leq\zeta(t)(|x(t)-\overline{x}(t)|+|{}^HI^px(t)-{}^HI^p\overline{x}(t)|)\}$. As the multivalued operator $V(t)=\widetilde{U}(t)\cap G(t,\overline{x}(t),{}^HI^q\overline{x}(t))$ is measurable (proposition III.4, [55]), there exists a function $v_2(t)$ which is a measurable selection for $V(t)$. Hence, $v_2(t)\in G(t,\overline{x}(t),{}^HI^q\overline{x}(t))$ for a.e. $t\in[1,e]$ and

$$
\begin{aligned}
&|v_1(t)-v_2(t)|\leq\zeta(t)(|x(t)-\overline{x}(t)|\\
&\quad+\left|{}^HI^px(t)-{}^HI^p\overline{x}(t)\right|\right),\quad \text{a.e. } t\in[1,e].
\end{aligned}
\tag{63}
$$

Let us define the function $w_2(t), t\in[1,e]$ by

$$
\begin{aligned}
w_2(t)=\rho\left(t,\overline{x}(t),{}^HI^p\overline{x}(t)\right)&\left(\frac{t^{-\lambda}}{\Delta}\left\{B_2\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\right.\right.\right.\\
&\cdot\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v_2(s)\mathrm{d}s\Bigg]\\
&-A_2\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_2(s)\mathrm{d}s\right]\Bigg\}\\
&+\frac{1}{\Delta}\left(t^{-\lambda}\int_1^t s^{\lambda-1}(\log s)^{\alpha-2}\mathrm{d}s\right)\left\{A_1\left[\frac{\beta_2}{\Gamma(r)}\int_1^{\xi}\left(\log\frac{\xi}{s}\right)^{r-1}s^{-\lambda-1}\right.\right.\\
&\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\beta_1 e^{-\lambda}\int_1^e s^{\lambda-1H}I^{\alpha-1}v_2(s)\mathrm{d}s\Bigg]\\
&-B_1\left[\frac{\alpha_2}{\Gamma(r)}\int_1^e\left(\log\frac{e}{s}\right)^{r-1}s^{-\lambda-1}\times\left(\int_1^s \tau^{\lambda-1H}I^{\alpha-1}v_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s-\alpha_1\xi^{-\lambda}\int_1^{\xi}s^{\lambda-1H}I^{\alpha-1}v_2(s)\mathrm{d}s\right]\Bigg)\\
&\left.+t^{-\lambda}\int_1^t s^{\lambda-1H}I^{\alpha-1}v_2(s)\mathrm{d}s\right).
\end{aligned}
\tag{64}
$$

Then, we conclude that

$$
\begin{aligned}
\left|w_1(t)-w_2(t)\right| \leq & \left|\rho\left(t, x(t), {}^H I^p x(t)\right)-\rho\left(t, \bar{x}(t), {}^H I^p \bar{x}(t)\right)\right| \\
& \left(\frac{t^{-\lambda}}{\Delta}\left\{\left|B_2\right|\left[\frac{\left|\alpha_2\right|}{\Gamma(r)} \int_1^e\left(\log \frac{e}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s+\left|\alpha_1\right| \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)\right| \mathrm{d} s\right]\right.\right. \\
& \left.+\left|A_2\right|\left[\frac{\left|\beta_2\right|}{\Gamma(r)} \int_1^{\xi}\left(\log \frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s+\left|\beta_1\right| e^{-\lambda} \int_1^e s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)\right| \mathrm{d} s\right]\right\} \\
& +\frac{1}{\Delta}\left(t^{-\lambda} \int_1^t s^{\lambda-1}(\log s)^{\alpha-2} \mathrm{d} s\right)\left\{\left|A_1\right|\left[\frac{\left|\beta_2\right|}{\Gamma(r)} \int_1^{\xi}\left(\log \frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\right.\right. \\
& \left.\times\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s+\left|\beta_1\right| e^{-\lambda} \int_1^e s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)\right| \mathrm{d} s\right] \\
& \left.+\left|B_1\right|\left[\frac{\left|\alpha_2\right|}{\Gamma(r)} \int_1^e\left(\log \frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s+\left|\alpha_1\right| \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)\right| \mathrm{d} s\right]\right\} \\
& \left.+t^{-\lambda} \int_1^t s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)\right| \mathrm{d} s\right) \\
& +\left|\rho\left(t, \bar{x}(t), {}^H I^p \bar{x}(t)\right)\right|\left(\frac{t^{-\lambda}}{\Delta}\left\{\left|B_2\right|\left[\frac{\left|\alpha_2\right|}{\Gamma(r)} \int_1^e\left(\log \frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)-v_2(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s\right.\right.\right. \\
& \left.+\left|\alpha_1\right| \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)-v_2(s)\right| \mathrm{d} s\right] \\
& +\left|A_2\right|\left[\frac{\left|\beta_2\right|}{\Gamma(r)} \int_1^{\xi}\left(\log \frac{\xi}{s}\right)^{r-1} s^{-\lambda-1} \times\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)-v_2(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s\right. \\
& \left.\left.+\left|\beta_1\right| e^{-\lambda} \int_1^e s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)-v_2(s)\right| \mathrm{d} s\right]\right\} \\
& +\frac{1}{\Delta}\left(t^{-\lambda} \int_1^t s^{\lambda-1}(\log s)^{\alpha-2} \mathrm{d} s\right)\left\{\left|A_1\right|\left[\frac{\left|\beta_2\right|}{\Gamma(r)} \int_1^{\xi}\left(\log \frac{\xi}{s}\right)^{r-1} s^{-\lambda-1}\right.\right. \\
& \left.\times\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)-v_2(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s+\left|\beta_1\right| e^{-\lambda} \int_1^e s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)-v_2(s)\right| \mathrm{d} s\right] \\
& +\left|B_1\right|\left[\frac{\left|\alpha_2\right|}{\Gamma(r)} \int_1^e\left(\log \frac{e}{s}\right)^{r-1} s^{-\lambda-1}\left(\int_1^s \tau^{\lambda-1 H} I^{\alpha-1}\left|v_1(\tau)-v_2(\tau)\right| \mathrm{d} \tau\right) \mathrm{d} s\right. \\
& \left.\left.+\left|\alpha_1\right| \xi^{-\lambda} \int_1^{\xi} s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)-v_2(s)\right| \mathrm{d} s\right]\right\} \\
& \left.+t^{-\lambda} \int_1^t s^{\lambda-1 H} I^{\alpha-1}\left|v_1(s)-v_2(s)\right| \mathrm{d} s\right) \\
\leq & M_1\|g\|_{L^{1/p_1}}\left|\rho\left(t, x(t), {}^H I^p x(t)\right)-\rho\left(t, \bar{x}(t), {}^H I^p \bar{x}(t)\right)\right|+M_1\|\zeta\|_{L^{1/p_1}}\left(1+\frac{1}{\Gamma(1+p)}\right)\|x-\bar{x}\| \\
& \times\left|\rho\left(t, \bar{x}(t), {}^H I^p \bar{x}(t)\right)\right| \\
\leq & M_1\left(1+\frac{1}{\Gamma(1+p)}\right)\left(\|g\|_{L^{1/p_1}}\|\Psi\|+\|\eta\|\|\zeta\|_{L^{1/p_1}}\right)\|x-\bar{x}\|, \quad \forall t \in[1, e],
\end{aligned}
\tag{65}
$$

which yield

$$\begin{aligned}\|w_1 - w_2\| &= \sup_{t\in[1,e]} |w_1(t) - w_2(t)| \\ &\le M_1\left(1 + \frac{1}{\Gamma(1+p)}\right) \\ &\quad \cdot \left(\|g\|_{L^{1/p_1}}\|\Psi\| + \|\eta\|\|\zeta\|_{L^{1/p_1}}\right)\|x - \overline{x}\| \\ &= \lambda_0\|x - \overline{x}\|.\end{aligned} \tag{66}$$

By interchanging the roles of x and $\overline{x}$, we obtain a similar relation, and thus, we get

$$H_{d_1}(\mathcal{N}x, \mathcal{N}\overline{x}) \le \lambda_0\|x - \overline{x}\|. \tag{67}$$

In view of the condition $\lambda_0 < 1$ (given by (56)), it follows that $\mathcal{N}$ is a contraction, and therefore, by Lemma 5, $\mathcal{N}$ has a fixed point x, which is a solution of problem (7). This completes the proof. □

4. Examples

(a) Consider the following equation:

$$\begin{cases} \left({}^{H}D^{3/2} + {}^{H}D^{1/2}\right)\left(\dfrac{x(t)}{e^{1-t}/460\tan^{-1}\left(x(t) + {}^{H}I^{1/2}x(t) + \pi/4\right) + 2}\right) \in G\left(t, x(t), {}^{H}I^{1/2}x(t)\right), \quad t \in (1,e), \\ 3\left(\dfrac{x\left(e^{1/2}\right)}{e^{1-e^{1/2}}/460\tan^{-1}\left(x\left(e^{1/2}\right) + {}^{H}I^{1/2}x\left(e^{1/2}\right) + \pi/4\right) + 2}\right) = {}^{H}I^{1/2}\left(\dfrac{x(e)}{e^{1-e}/460\tan^{-1}\left(x(e) + {}^{H}I^{1/2}x(e) + \pi/4\right) + 2}\right), \\ 3\left(\dfrac{x(e)}{e^{1-e}/460\tan^{-1}\left(x(e) + {}^{H}I^{1/2}x(e) + \pi/4\right) + 2}\right) = {}^{H}I^{1/2}\left(\dfrac{x\left(e^{1/2}\right)}{e^{1-e^{1/2}}/460\tan^{-1}\left(x\left(e^{1/2}\right) + {}^{H}I^{1/2}x\left(e^{1/2}\right) + \pi/4\right) + 2}\right), \end{cases} \tag{68}$$

where G: $[1,e] \times R \times R \longrightarrow \mathcal{P}(R)$ is a multivalued map given by

$$t \longrightarrow G\left(t, x(t), {}^{H}I^{1/2}x(t)\right) = \left[\frac{|x|^3 + \left|{}^{H}I^{1/2}x\right|^3}{20\left(|x|^3 + \left|{}^{H}I^{1/2}x\right|^3 + 4\right)}, \frac{\left|\sin\left(x + {}^{H}I^{1/2}x\right)\right|}{9\left(\left|\sin\left(x + {}^{H}I^{1/2}x\right)\right| + 1\right)} + \frac{8}{9}\right]. \tag{69}$$

By condition (H_1), $\Psi(t) = e^{1-t}/460$ with $\|\Psi\| = 1/460$. For $\tilde{g} \in G$, we have

$$\begin{aligned}|\tilde{g}| &\le \max\left(\frac{|x|^3 + \left|{}^{H}I^{1/2}x\right|^3}{20\left(|x|^3 + \left|{}^{H}I^{1/2}x\right|^3 + 4\right)}, \frac{\left|\sin\left(x + {}^{H}I^{1/2}x\right)\right|}{9\left(\left|\sin\left(x + {}^{H}I^{1/2}x\right)\right| + 1\right)} + \frac{8}{9}\right) \le 1, \quad \forall x \in R, \\ \|G(t,x,y)\| &= \sup\{|y|: \ y \in G(t,x,y)\} \le 1 = g(t), \quad \forall x, y \in R.\end{aligned} \tag{70}$$

Let $p_1 = 1/4$; then, $g(t) \in L^4([1,e], R^+)$. Using the given data, we find that $|A_1| \le 1.8196, |A_2| \le 4.2428, |B_1| \le 1.1041, |B_2| \le 6, |\Delta| \le 15.6018, M_0 \le 0.0048, M_1 \le 84.6585, \rho_0 = 1/460 + 2$. Furthermore,

$$M_0 M_1 \|g\|_{L^4} \le 0.4485 < \frac{1}{2}, \tag{71}$$

and $\mathscr{R} > 160(e-1)^{1/4}(1/460+2) \ge \rho_0 M_1 \|g\|_{L^4}/(1 - M_0 M_1 \|g\|_{L^4})$. Hence, all the conditions of Theorem 1 are satisfied, and accordingly, problem (68) has a solution on $[1, e]$.

(b) Let us consider the following inclusion problem:

$$\begin{cases} \left({}^H D^{3/2} + {}^H D^{1/2}\right)\left(\dfrac{x(t)}{3/800e^{t-1} + 20\left(\sin(x(t)) + \left|{}^H I^{1/2} x(t)\right|/1 + \left|{}^H I^{1/2} x(t)\right|\right) + 1/10}\right) \in G\left(t, x(t), {}^H I^{1/2} x(t)\right), \quad t \in (1, e), \\ 3\left(\dfrac{x\left(e^{1/2}\right)}{3/800e^{e^{1/2}-1} + 20\left(\sin\left(x\left(e^{1/2}\right)\right) + \left|{}^H I^{1/2} x\left(e^{1/2}\right)\right|/1 + \left|{}^H I^{1/2} x\left(e^{1/2}\right)\right|\right) + 1/10}\right) \\ = {}^H I^{1/2}\left(\dfrac{x(e)}{3/800e^{e-1} + 20\left(\sin(x(e)) + \left|{}^H I^{1/2} x(e)\right|/1 + \left|{}^H I^{1/2} x(e)\right|\right) + 1/10}\right), \\ 3\left(\dfrac{x(e)}{3/800e^{e-1} + 20\left(\sin(x(e)) + \left|{}^H I^{1/2} x(e)\right|/1 + \left|{}^H I^{1/2} x(e)\right|\right) + 1/10}\right) \\ = {}^H I^{1/2}\left(\dfrac{x\left(e^{1/2}\right)}{3/800e^{e-1} + 20\left(\sin\left(x\left(e^{1/2}\right)\right) + \left|{}^H I^{1/2} x\left(e^{1/2}\right)\right|/1 + \left|{}^H I^{1/2} x\left(e^{1/2}\right)\right|\right) + 1/10}\right). \end{cases} \tag{72}$$

In order to demonstrate the application of Theorem 3, we consider

$$t \longrightarrow G\left(t, x(t), {}^H I^{1/2} x(t)\right) = \left[0, \frac{1}{512\sqrt{[4]}\,t}\left(\frac{|x(t)|}{12(8 + |x(t)|)} + \frac{\tan^{-1}\left({}^H I^{1/2}\, x(t)\right)}{1 + \tan^{-1}\left({}^H I^{1/2}\, x(t)\right)}\right) + \frac{1}{300 + t}\right]. \tag{73}$$

Clearly,

$$\begin{gathered} H_{d_1}\left(G\left(t, x, {}^H I^{1/2} x\right), G\left(t, \overline{x}, {}^H I^{1/2} \overline{x}\right)\right) \le \frac{3}{512\sqrt[4]{t}} \|x - \overline{x}\|, \\ \|G(t, x, y)\| = \sup\{|\nu|: \nu \in G(t, x, y)\} \le 1 = g(t), \quad \forall x, y \in R. \end{gathered} \tag{74}$$

Letting $\zeta(t) = 3/512\sqrt[4]{t}$, it is easy to check that $d(0, G(t, 0, 0)) \le \zeta(t)$ holds for almost $t \in [1, e]$ and that $\zeta(t) \in L^4([1, e], R^+)$ $(p_1 = 1/4)$, $\|\zeta\|_{L^4} = 3/512$. From the following inequalities, we get $\eta(t) = 6/(800e^{t-1} + 20) + 1/10$ and $\|\eta\| = 88/820$:

$$|\rho(t, x, y)| \le \frac{6}{800e^{t-1} + 20} + \frac{1}{10}, \quad (t, x, y) \in [1, e] \times R^2. \tag{75}$$

In addition, by condition (H_1), we obtain $\Psi(t) = 3/(800e^{t-1} + 20)$ with $\|\Psi\| = 3/820$. Furthermore, using the given data, we find that $|A_1| \le 1.8196$, $|A_2| \le 4.2428$, $|B_1| \le 1.1040$, $|B_2| \le 6$, $|\Delta| \le 15.6018$, $M_1 \le 84.6585$. Furthermore,

$$M_1\left(1 + \frac{1}{\Gamma(1 + p)}\right)\left(\|g\|_{L^4}\|\Psi\| + \|\eta\|\|\zeta\|_{L^4}\right) \le 0.8682 < 1. \tag{76}$$

Thus, all the conditions of Theorem 3 are satisfied. Hence, it follows by the conclusion of Theorem 3 that there exists a solution for problem (72) on $[1, e]$.

5. Conclusion

Nowadays, we need to study more natural phenomena to gain more abilities for modeling. Therefore, fractional calculus came into being, and today, their importance has become more and more apparent to researchers. In this way, it is necessary to design different and complicated modelings by utilizing the fractional differential problems. This is useful in making modern software which helps us to allow for more cost-free testing and less material consumption. In this work, we have developed the existence theory for a class of Hadamard sequential fractional hybrid differential inclusions equipped with two-point hybrid Hadamard integral boundary value conditions. The nonlinearities in the given problems implicitly depend on the unknown function together with its Hadamard fractional integral of order $p \in (0, 1)$. We apply fixed-point theorem due to Dhage and Covitz-Nadler fixed-point theorem to establish the desired results. Eventually, we give two numerical examples to support the applicability of our findings.

The work accomplished in this study is new and enriches the literature on boundary value problems for nonlinear Hadamard fractional differential inclusions. For future works, one can extend the given fractional boundary value problem to more general structures, such as finitely point multistrip integral boundary value conditions given by newly introduced generalized fractional operators with nonsingular kernels.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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