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Research Article

The Strong Convex Functions and Related Inequalities

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The study of convex functions is one of the most researched of the classical fields. Analysis of the geometric characteristics of these functions is a core area of research in this field; however, a paradigm shift in this research is the application of convexity in optimization theory. The Jensen-Mercer type inequalities are studied extensively in recent years. In the present paper, we extend Jensen-Mercer type inequalities for strong convex function. Some improved inequalities in Hölder sense are also derived. The previously established results are generalized and strengthened by our results.

1. Introduction and Preliminary Results

Convex functions and their consequences are useful in the establishment of different kinds of inequalities; therefore, they are considered the base of theory of inequalities in mathematical analysis. A real valued function $\psi : I \longrightarrow \mathbb{R}$ is said to be convex on the interval $I \subset \mathbb{R}$, if

$$\psi(zx + (1 - z)y) \leq z\psi(x) + (1 - z)\psi(y) \tag{1}$$

holds for all $x, y \in I$ and $z \in [0, 1]$. The function ψ is said to be concave if reverse of inequality (1) holds.

Convex functions are also very important in the fields of mathematical analysis, mathematical statistics, and optimization theory. These functions motivate towards a nice theory named convex analysis (see [1–3]). Convex functions have been defined in various ways by using different techniques, for example, by support function, by chords joining two points, and Jensen's inequality. Inequality (1) represents the convex function analytically and provides encouragement to define further general notions.

The study of convex functions [4–14] began with Jensen's thought-provoking concepts and interesting work over the period from 1905 to 1906. It is used in the analysis as an efficient tool for solving optimization issues. Additionally, inequalities involving convex functions are very stimulating in the development of different sections of mathematics, such as mathematical finance, economics, management sciences, and optimization theory.

If the function $\psi : I \subset \mathbb{R}$ is convex, then the inequality

$$\psi\left(\frac{c + d}{2}\right) \leq \frac{1}{d - c}\int_c^d \psi(\alpha_1)d\alpha_1 \leq \frac{\psi(c) + \psi(d)}{2} \tag{2}$$

is called the Hermite-Hadamard inequality [15, 16].

Definition 1 (Convex set) (see [17]). A set I is considered to be convex if the line segment between any two points in I lies in I; i.e., $\forall \alpha_1, \alpha_2 \in I, \forall \in [0, 1]$

$$l\alpha_1 + (1 - l)\alpha_2 \in I. \tag{3}$$

Authors [18–20] expanded on the idea of a strongly convex function by replacing the nonnegative term with a real-valued nonnegative function and defined it as follows:

Definition 2 (Strongly convex function (see [18])). A function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is strongly convex, if

$$\psi(l\alpha_1 + (1-l)\alpha_2) \le l\psi(\alpha_1) + 1(1-l)\psi(\alpha_2) - l(1-l)M(\alpha_1 - \alpha_2) \tag{4}$$

holds for all $\alpha_1, \alpha_2 \in I$ and $l \in [0, 1]$.

Definition 3 (Riemann-Liouville fractional integral). For a function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, the Riemann-Liouville fractional integral operator of order $\xi \le 0$ with $c \le 0$ is defined as

$$\begin{aligned} J_c^{\xi}\psi(\alpha_1) &= \frac{1}{\Gamma_\xi}\int_c^{\alpha_1}(\alpha_1 - l)^{\zeta-1}\psi(l)dl, \\ J_c^0\psi(\alpha_1) &= \psi(\alpha_1). \end{aligned} \tag{5}$$

Many scholars have recently analyzed a variety of inequalities by using the Riemann-Liouville fractional integrals (see [21–24]).

Definition 4 (Hadamard fractional integral). For a function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, the Hadamard fractional integral of order $\zeta \le \mathbb{R}^+$ for all $\alpha_1 > 1$ is defined as

$$_H J_{1,\alpha_1}^{-\zeta}\psi(\alpha_1) = \frac{1}{\Gamma_\xi}\int_1^{\alpha_1}\ln\left(\frac{\alpha_1}{l}\right)^{\zeta-1}\psi(l)dl, \tag{6}$$

where $\Gamma_\zeta = \int_0^\infty e^{-l}l^{\zeta-1}dl.$

Definition 5 (Conformable fractional integral). Let $\zeta \in (0, 1)$ and $0 \le c < d$. A function $\psi : [c, d] \longrightarrow \mathbb{R}$ is ζ-fractional integrable on $[c, d]$ if the integral

$$\int_c^d \psi(\alpha_1)d_\zeta\alpha_1 = \int_c^d \psi(\alpha_1)\alpha_1^{\zeta-1} \tag{7}$$

exists and is finite.

This paper is aimed at establishing Hermite-Jensen-Mercer type inequalities and some other inequalities including improved Hölder inequality for strong convex function.

2. New Hermite-Jensen-Mercer Type Inequalities

Theorem 6. *Let $\zeta, \xi > 0$ and $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ be a strong convex function. Then, the inequality*

$$\begin{aligned} \psi\left(\phi + \varphi - \frac{\alpha_1 + \alpha_2}{2}\right) &\le \frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi + k)}{(\alpha_2 - \alpha_1)^{\zeta\xi/k}}\left[{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}\psi(\phi \right. \\ &\quad \left. + \varphi - \alpha_2) + {}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}\psi(\phi + \varphi - \alpha_2) - \frac{a}{2}M(\alpha_2 - \alpha_1)\right] \\ &\le \psi(\phi) + \psi(\varphi) - \left(\frac{\psi(\alpha_2) + \psi(\alpha_2)}{2}\right) - \frac{a}{2}\psi\left(\phi + \varphi - \frac{\alpha_1 - \alpha_2}{2}\right) \end{aligned} \tag{8}$$

holds for all $\alpha_1, \alpha_2 \in [\phi, \varphi]$.

Proof. To prove that the first inequality holds, take

$$\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) = \psi\left(\frac{2\phi + 2\varphi - \alpha_{11} - \alpha_{21}}{2}\right). \tag{9}$$

Since ψ is a strong convex function, so

$$\begin{aligned} &\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right)\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) \\ &\le \left(\frac{1}{2}\right)\psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \left(\frac{1}{2}\right)\psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) \\ &\quad - a\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2 \\ &\le \left(\frac{1}{2}\right)\left[\psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) - \left(\frac{a}{2}\right)\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2\right]. \end{aligned} \tag{10}$$

Suppose $\alpha_{11} = l/2\alpha_1 + (2 - l)/2\alpha_2$ and $\alpha_{21} = (2 - l)/2\alpha_1 + l/2\alpha_2$; then, for $\alpha_1, \alpha_2 \in [\phi, \varphi]$ and $l \in [0, 1]$, we have

$$\begin{aligned} &\psi\left(\frac{\phi + \varphi - l/2\alpha_1 - 2 - l/2\alpha_2}{2}\right) + \psi\left(\frac{\phi + \varphi - 2 - l/2\alpha_1 - l/2\alpha_2}{2}\right) \\ &\quad - \frac{a}{2}\left(\frac{\alpha_1 - \alpha_2}{2}\right)^2(1 - l)^2 2\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right) \le \psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) \\ &\quad + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) - \frac{a}{2}\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2. \end{aligned} \tag{11}$$

Multiplying both sides of Equation (11) with $(1 - (1 - l)^\zeta/\zeta)^{\xi/k-1}(1 - l)^{\zeta-1}$, we get

$$\begin{aligned} &\left(2\psi\left(\phi + \varphi - \frac{\alpha_{11} + \alpha_{21}}{2}\right)\right)\left(\frac{1 - (1 - l)^\zeta}{\zeta}\right)^{\xi/k-1}(1 - l)^{\zeta-1} \\ &\le \psi\left(\frac{\phi + \varphi - \alpha_{11}}{2}\right) + \psi\left(\frac{\phi + \varphi - \alpha_{21}}{2}\right) \\ &\quad - \frac{a}{2}\left(\frac{\alpha_{21} - \alpha_{11}}{2}\right)^2\left(\frac{1 - (1 - l)^\zeta}{\zeta}\right)^{\xi/k-1}(1 - l)^{\zeta-1}. \end{aligned} \tag{12}$$

Integrating the above inequality with respect to l over the range $[0, 1]$ and then combining the result with the integral operator yield

$$2\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\alpha-1}$$
$$\leq\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)+\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right)-\frac{c}{2}\left(\frac{\alpha_{21}-\alpha_{11}}{2}\right)^2\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl$$
$$=\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl+\int_0^1\left[\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right)\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl-\int_0^1\left[\frac{a}{2}\left(\frac{\alpha_{21}-\alpha_{11}}{2}\right)^2\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\right]dl. \tag{13}$$

Now, by altering the variables, we can obtain

$$\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-(\alpha_1+\alpha_2/2)}\left[\frac{1-(2\phi+2\varphi-(\alpha_1+\alpha_2)-2l_1/(\alpha_2-\alpha_2))^{\zeta}}{\zeta}\right]^{\xi/k-1}\times\left(\frac{2\phi+2\varphi-(\alpha_1+\alpha_2)-2l_1}{(\alpha_2-\alpha_1)}\right)^{\zeta-1}\psi(l_1)\frac{\alpha_2-\alpha_1}{2}dl+\int_{\phi+\varphi-\alpha_1}^{\phi+\varphi-(\alpha_1+\alpha_2/2)}\left[\frac{1-(2l_2-2\phi+2\varphi-(\alpha_1+\alpha_2)/(\alpha_2-\alpha_2))^{\zeta}}{\zeta}\right]^{\xi/k-1}\times\left(\frac{2l^2-2\phi+2\varphi-(\alpha_1+\alpha_2)}{(\alpha_2-\alpha_1)}\right)^{\zeta-1}\psi(l_2)\frac{\alpha_2-\alpha_1}{2}dl+\int_{\alpha_1-\alpha_2}^{\alpha_1+\alpha_2}\left[\frac{1-(2l_3/(\alpha_2-\alpha_1))^{\zeta}}{\zeta}\right]^{\xi/k-1}\cdot\left(\frac{l_3}{\alpha_1-\alpha_2}\right)a\psi(l_3)(\alpha_2-\alpha_1)dl. \tag{14}$$

So,

$$\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}dl=\frac{1}{\xi/k\zeta^{\xi/k}},$$
$${}_{\kappa}^{\xi}J_{\phi^+}^{\zeta}\psi(\alpha_2)=\frac{1}{\kappa\Gamma_{\kappa}(\xi)}\int_{\phi}^{\alpha_2}\left(\frac{(\alpha_2-\phi)^{\zeta}-(l-\phi)^{\zeta}}{\zeta}\right)^{\xi/k-1}\left(\frac{\psi(l)}{(l-\phi)^{1-\zeta}}\right)dl,$$
$${}_{\kappa}^{\xi}J_{\varphi^-}^{\zeta}\psi(\alpha_2)=\frac{1}{\kappa\Gamma_{\kappa}(\xi)}\int_{\phi}^{\varphi}\left(\frac{(\varphi-\alpha_2)^{\zeta}-(\varphi-l)^{\zeta}}{\zeta}\right)^{\xi/k-1}\left(\frac{\psi(l)}{(\varphi-l)^{1-\zeta}}\right)dl. \tag{15}$$

Therefore,

$$2\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\frac{1}{\xi/k\zeta^{\xi/k}}\leq\left(\frac{2}{\alpha_2-\alpha_1}\right)^{\zeta\xi/k}\cdot\left\{\Gamma_k(\xi)_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}S(\phi+\varphi-\alpha_2)+\Gamma_k(\xi)_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_1+\alpha_2/2)^-}\psi(\phi+\varphi-\alpha_1)-\frac{a}{2}M(\alpha_{21}-\alpha_{11})\right\}. \tag{16}$$

As a result, the first inequality of (8) is proved.

We can prove the second inequality of (8) by using strong convexity of ψ for l over $[0,1]$.

$$\psi(\phi+\varphi-\alpha_1)=\psi(\phi+\varphi)+\psi(\alpha_1)-\frac{1}{2}aM(\phi+\varphi-\alpha_1), \tag{17}$$

$$\psi(\phi+\varphi-\alpha_2)=\psi(\phi+\varphi)+\psi(\alpha_1)-\frac{1}{2}aM(\phi+\varphi-\alpha_2), \tag{18}$$

$$\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)=\psi(\phi+\varphi)+\left(\frac{l}{2}\psi\alpha_1+\frac{2-l}{2}\psi\alpha_2\right)-\frac{a}{2}M\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right), \tag{19}$$

$$\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{5l}{2}\alpha_2\right)\right)=\psi(\phi+\varphi)+\left(\frac{2-l}{2}\psi\alpha_1+\frac{l}{2}\psi\alpha_2\right)-\frac{a}{2}M\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right). \tag{20}$$

Adding (17) and (20) leads to

$$\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)+\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\leq2\psi(\phi+\varphi)+\left[\psi\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)+\psi\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right]-\frac{a}{2}(2\phi+2\varphi-(\alpha_2-\alpha_1)). \tag{21}$$

Multiply 9 with $(1-(1-l)^{\zeta}/\zeta)^{\xi/k-1}(1-l)^{\zeta-1}$, and integrating the obtained inequality w.r.t to l over [0,1] gives

$$\int_0^1 \left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}(1-l)^{\zeta-1}\left[\psi\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right.$$
$$\left.+\psi\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right]dl \le 2\psi(\phi+\varphi)$$
$$+\left[\psi\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)+\psi\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right]$$
$$-\frac{a}{2}[2\phi+2\varphi-(\alpha_2-\alpha_1)]\int_0^1$$
$$\times\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k1}(1-l)^{\zeta-1}-\left(\frac{2}{\alpha_2-\alpha_1}\right)^{\zeta\xi/k}$$
$$\times\left\{\Gamma_k(\xi){}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)^+}\psi(\phi+\varphi-\alpha_2)\right.$$
$$\left.+\Gamma_k(\xi){}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)^-}\psi(\phi+\varphi-\alpha_1)-\frac{a}{2}M(\alpha_{21}-\alpha_{11})\right\}dl$$
$$\le \frac{1}{\xi/k\zeta^{\xi/k}}2\psi(\phi+\varphi)-[\psi\alpha_1+\psi\alpha_2]-\frac{a}{2}[2\phi+2\varphi-(\alpha_2-\alpha_1)]. \tag{22}$$

This completes the proof. □

Remark 7. It is obvious from Theorem 6 that

(1) Theorem 2.1 of [25] is obtained if we take $a=0$, $\alpha_1=x$, and $\alpha_2=y$ in Theorem 6

(2) Theorem 2.1 of [26] is obtained if we take $a=0$, $k=1$, $\alpha_1=\theta$, and $\alpha_2=\vartheta$ in Theorem 6

(3) Theorem 2 of [27] is obtained by taking $a=0$, $\alpha=k=1$, $\alpha_1=\theta$, and $\alpha_2=\vartheta$ in Theorem 6

Theorem 8. *Let $\zeta,\xi>0$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a strong convex function. Then, the inequalities*

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^\xi J^\zeta_{\left(\phi+\varphi-\alpha_1^+\right)}\psi(\alpha_2)+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2)^-}\psi(\alpha_1)-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}$$
$$\le\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{a}{2}(\alpha_2-\alpha_1)^2 \tag{23}$$

and

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_1^+)}\psi(\alpha_2)+{}_k^\xi J^\zeta_{(\phi+\varphi-\alpha_2)}-\psi(\alpha_1)-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}$$
$$\le\psi(\phi)+\psi(\varphi)-\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right)-\frac{a}{2}(\alpha_2-\alpha_1)^2 \tag{24}$$

hold $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. The Jensen-Mercer inequality dictates that

$$\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\psi(\alpha_{11})+\psi(\alpha_{21})}{2}$$
$$-\frac{a}{2}M(\alpha_2-\alpha_1), \tag{25}$$

$\forall\alpha_{11},\alpha_{21}\in[\phi,\varphi]$. □

Taking $\alpha_{11}=l\alpha_1+(1-l)\alpha_2$ and $\alpha_{21}=(1-l)\alpha_1+(l)\alpha_2$ for $\alpha_1,\alpha_2\in[\phi,\varphi]$ and $l\in[0,1]$ in (25), we get

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)$$
$$-\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+(l)\alpha_2)}{2}$$
$$-\frac{a}{2}M(\alpha_2-\alpha_1). \tag{26}$$

Multiplying the above inequality by $(1-(1-l)^\zeta/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and integrating the obtained inequality with respect to l over [0,1] give

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-la)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl$$
$$\le\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}$$
$$\times\left\{\psi(\phi)+\psi(\varphi)-\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+(l)\alpha_2)}{2}\right.$$
$$\left.-\frac{a}{2}M(\alpha_2-\alpha_1)\right\}dl, \tag{27}$$

that is,

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\le\psi(\phi)+\psi(\varphi)-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}_k^\xi J^\zeta_{\alpha_1^+}\psi(\alpha_2)+{}_k^\xi J^\zeta_{\alpha_2^-}\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}. \tag{28}$$

That proves the first inequality of (24).

To prove the second inequality of (24), from the definition of strong convexity of ψ, for $l\in[0,1]$, we get

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)=\psi\left(\frac{l\alpha_1+(1-l)\alpha_2+(1-l)\alpha_1+l\alpha_2}{2}\right)$$
$$\le\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+l\alpha_2)}{2}$$
$$-\frac{a}{2}M(\alpha_1-\alpha_2). \tag{29}$$

Multiplying the above inequality by $(1-(1-l)^\zeta/\zeta)^{\xi/k}$

$(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1} \leq \int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1} \times\left\{\frac{\psi(l\alpha_1+(1-l)\alpha_2)+\psi((1-l)\alpha_1+l\alpha_2)}{2}-\frac{a}{2}M(\alpha_1-\alpha_2)\right\}dl, \quad (30)$$

where

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\leq\left\{{}_k^{\xi}J_{\alpha_1^+}^{\zeta}\psi(\alpha_2)+k^{\xi}J_{\alpha_2^-}^{\zeta}\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2))\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}} \quad (31)$$

implies

$$-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\geq-\left\{{}_k^{\xi}J_{\alpha_1^+}^{\zeta}\psi(\alpha_2)+k^{\xi}J_{\alpha_2^-}^{\zeta}\psi(\alpha_1)\frac{a}{2}M(\alpha_1+\alpha_2)\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}. \quad (32)$$

Adding $\psi(\phi)+\psi(\varphi)$ on both side of above inequality,

$$\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)\geq\psi(\phi)+\psi(\varphi)-\left\{{}_k^{\xi}J_{\alpha_1^+}^{\zeta}\psi(\alpha_2)+k^{\xi}J_{\alpha_2^-}^{\zeta}\psi(\alpha_1)\right\}\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}, \quad (33)$$

which gives (23).

To prove (24), using strong convexity of ψ, we get

$$\psi\left(\phi+\varphi-\frac{\alpha_{11}+\alpha_{21}}{2}\right)=\psi\left(\frac{\phi+\varphi-\alpha_{11}+\phi+\varphi-\alpha_{21}}{2}\right)\leq\psi\left(\frac{\phi+\varphi-\alpha_{11}}{2}\right)+\psi\left(\frac{\phi+\varphi-\alpha_{21}}{2}\right)-\frac{a}{2}M(\alpha_2-\alpha_1), \quad (34)$$

$$\forall\alpha_{11},\alpha_{21}\in[\phi,\varphi]. \quad (35)$$

Let $\alpha_{11}=l\alpha_1+(1-l)\alpha_2$ and $\alpha_{21}=(1-l)\alpha_1+l\alpha_2$; then, (34) leads

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\psi\left(\frac{\phi+\varphi-(l\alpha_1+(1-l)\alpha_2)}{2}\right)+\psi\left(\frac{\phi+\varphi-((1-l)\alpha_1+l\alpha_2)}{2}\right)-\frac{a}{2}M(\alpha_1+\alpha_2). \quad (36)$$

Multiplying the above inequality by $(1-(1-l)^{\zeta}/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl \leq\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}\left\{\psi\left(\frac{\phi+\varphi-(l\alpha_1+(1-l)\alpha_2)}{2}\right)+\psi\left(\frac{\phi+\varphi-((1-l)\alpha_1+l\alpha_2)}{2}\right)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}dl, \quad (37)$$

which can be written as

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J_{\alpha_1^+}^{\zeta}\psi(\alpha_2)+k^{\xi}J_{\alpha_2^-}^{\zeta}\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}. \quad (38)$$

If follows from the definition of strong convexity of ψ that

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))\leq l\psi(\phi+\varphi-\alpha_1)+(1-l)\psi(\phi+\varphi-\alpha_2)-l\frac{a}{2}(1-l)M(\alpha_1-\alpha_2), \quad (39)$$

$$\psi((1-l)(\phi+\varphi-\alpha_1)+l(\phi+\varphi-\alpha_2))\leq(1-l)\psi(\phi+\varphi-\alpha_1)+l\psi(\phi+\varphi-\alpha_2)-al(1-l)M(\alpha_1-\alpha_2). \quad (40)$$

Adding the above two inequalities and with the help of Jensen-Mercer inequality, we have

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1)+l(\phi+\varphi-\alpha_2))\leq l\psi(\phi+\varphi-\alpha_1)+(1-l)\psi(\phi+\varphi-\alpha_2)-l(1-l)M(\alpha_1-\alpha_2)+(1-l)\psi(\phi+\varphi-\alpha_1)+l\psi(\phi+\varphi-\alpha_2)-l(1-l)M(\alpha_1-\alpha_2),$$

$$\psi(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2))\leq\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)-2al(1-l)M(\alpha_2-\alpha_1)\leq 2(\psi(\phi)+\psi(\varphi))-(\psi(\alpha_1)+\psi(\alpha_2))-2aM(\alpha_2-\alpha_1). \tag{41}$$

Multiplying the above inequality by $(1-(1-l)^{\zeta}/\zeta)^{\xi/k}(1-l)^{\zeta-1}$ and then integrating with respect to l over [0,1], we have

$$\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}[S(l(\phi+\varphi-\alpha_1)+(1-l)(\phi+\varphi-\alpha_2))+\psi((1-l)(\phi+\varphi-\alpha_1)+l(\phi+\varphi-\alpha_2))]dl\leq 2(\psi(\phi)+\psi(\varphi))-(\psi(\alpha_1)+\psi(\alpha_2))-\frac{a}{2}M(\alpha_2-\alpha_1)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}(1-l)^{\zeta-1}dl, \tag{42}$$

that is,

$$\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{\alpha_1^+}\psi(\alpha_2)+{}_k^{\xi}J^{\zeta}_{\alpha_2^-}\psi(\alpha_1)-\frac{a}{2}M(\alpha_1+\alpha_2)\right\}\leq\psi(\phi)+\psi(\varphi)-\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right)-\frac{a}{2}M(\alpha_2-\alpha_1)^2. \tag{43}$$

Combining (38) and (43) leads to (24).

Remark 9. Let a = 0, $\zeta=\xi=k=1$; then, Theorem 8 leads to

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\psi(\phi)+\psi(\varphi)-\int_0^1\psi(l\alpha_1+(1-l)\alpha_2)dl\leq\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\alpha_1+\alpha_2}{2}\right), \tag{44}$$

$$\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{1}{\alpha_2-\alpha_2}\int_{\alpha_1}^{\alpha_2}\psi(\phi+\varphi_l)dl\leq\psi(\phi)+\psi(\varphi)-\psi\left(\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}\right), \tag{45}$$

which was proved in Theorem 2.1 of [28].

Lemma 10. *Let $\zeta,\xi>0$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow R$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$. Then, the inequality*

$$\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)\zeta\xi/k}\left\{{}_k^{\xi}J_{(\phi+\varphi-\alpha_1+\alpha_2/2)}\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J_{(\phi+\varphi-\alpha_1+\alpha_2/2)}\psi(\phi+\varphi-\alpha_2)x\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right\}=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\times\left\{\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)-\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right\} \tag{46}$$

holds $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. Suppose

$$P=(P_1-P_2)\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}, \tag{47}$$

where

$$P_1=\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)dl,$$
$$P_2=\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)dl. \tag{48}$$

Using integration by parts, we get

$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\frac{2\zeta\xi/k}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\int_0^1\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)dl$$
$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\frac{2\zeta\xi/k}{\zeta^{\xi/k}(\alpha_2-\alpha_1)^{\zeta\xi/k}+1}\int_{\phi+\varphi-\frac{\alpha_1-\alpha_2}{2}}^{\phi+\varphi-\alpha_1}\times\left(\left(\frac{\alpha_2-\alpha_2}{2}\right)^{\zeta}-\left(l_1-\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)^{\zeta}\right)^{\xi/k-1}\times\frac{\psi(l_1)}{(\alpha_1-\phi+\varphi-\alpha_1+\alpha_2/2)^{1-\zeta}}\right)dl_1$$
$$=-\frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)+\left(\frac{2}{y-x}\right)^{\zeta\xi/k+1}\frac{\Gamma_k(\xi+k)^{\xi}}{\zeta^{\xi/k-1}}{}_kJ^{\zeta}_{\phi+\varphi-\alpha_1+\alpha_2/2^+}\psi(\phi+\varphi-\alpha_1). \tag{49}$$

Similarly, using integration by parts for P_2, we get

$$P_2 = \int_0^1 \left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k} \psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)dl$$
$$= \frac{2}{\zeta^{\xi/k}(\alpha_2-\alpha_1)}\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)$$
$$-\left(\frac{2}{y-x}\right)^{\zeta\xi/k+1}\frac{\Gamma_k(\xi+k)^\xi}{\zeta^{\xi/k-1}}\,{}_kJ^\zeta_{(\phi+\varphi-\alpha_1+\alpha_2/2)}-\psi(\phi+\varphi-\alpha_2). \quad (50)$$

Therefore, inequality (43) follows from (47)–(50). □

Remark 11.

(1) If we take $K=1$, $\alpha_1=\theta$, and $\alpha_2=\upsilon$ in Lemma 10, then we can get Lemma 3.1 of [26]

(2) If we take $\zeta=K=1$, $\alpha_1=\theta$, and $\alpha_2=\upsilon$, then Lemma 10 reduces to Lemma 1.1 of [29]

Lemma 12. *Let $\zeta,\xi>0$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in\psi[\phi,\varphi]$. Then, the identity*

$$\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k\Gamma_k(\xi+k)}}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}$$
$$\times\left\{{}^\xi_kJ^\zeta_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}^\xi_kJ^\zeta_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)\right\}$$
$$=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl \quad (51)$$

holds $\forall\alpha_1,\alpha_2\in[\phi,\varphi]$.

Proof. Suppose

$$P=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\{P_1-P_2\}. \quad (52)$$

Then, we clearly see that

$$P_1=\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dla$$
$$=\frac{1}{\alpha^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_1)}{\alpha_2-\alpha_1}-\frac{\xi/k}{\alpha_2-\alpha_1}\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k-1}$$
$$\times(1-l)^{\zeta-1}\psi(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=\frac{1}{\alpha^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_1)}{\alpha_2-\alpha_1}-\frac{\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k+1}}$$
$$\cdot\left\{{}^\zeta_kJ^\zeta_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_2)\right\}, \quad (53)$$

and

$$P_2=\int_0^1\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\times\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))dl$$
$$=-\frac{1}{\zeta^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_2)}{\alpha_2-\alpha_1}+\frac{\xi/k}{\alpha_2-\alpha_1}\int_0^1\left(\frac{1-l^\zeta}{\zeta}\right)^{\xi/k-1}$$
$$\times l^{\zeta-1}\psi(\phi+\varphi(l\alpha_1+(1-l)y))dl$$
$$=-\frac{1}{\zeta^{\xi/k}}\frac{\psi(\phi+\varphi-\alpha_2)}{\alpha_2-\alpha_1}+\frac{\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k+1}}$$
$$\cdot\left\{{}^\xi_kJ^\zeta_{(\phi+\varphi-\alpha_2)}-\psi(\phi+\varphi-\alpha_1)\right\}. \quad (54)$$

Therefore, identity (51) follows from (52)–(54). □

Corollary 13. *If we take $\zeta=\xi=k=1$, then Lemma 12 leads to the equality*

$$\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-\alpha_1}\psi(la)dl$$
$$=\frac{\alpha_2-\alpha_1}{2}\int_0^1(2l-1)S'(\phi+\varphi-(l\alpha_1+(1-l)y))dl. \quad (55)$$

Remark 14. If we take $\alpha_1=\phi=\theta$ and $\alpha_2=\varphi=\upsilon$ in Corollary 13, then (55) becomes

$$\frac{\psi(\phi)+\psi(\varphi)}{2}-\frac{1}{\varphi-\phi}\int_0^1\psi(l)dl=\frac{\phi-\varphi}{2}\int_0^1(2l-1)\psi'((1-l)\phi+l\varphi)dl, \quad (56)$$

which was proved in Lemma 2.1 of [30].

Theorem 15. *Let $\zeta,\xi>0$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|^q$ is a* *convex mapping on* $[\phi, \varphi]$. *Then, the inequality*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\cdot\left[\left(\left|\psi'(\phi)+\psi'(\varphi)\right|\right)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)-\left\{\left|\psi'(\alpha_1)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}+\left\{\left|\psi'(\alpha_2)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl+\left(\left|\psi'(\phi)+\psi'(\varphi)\right|\right)\cdot\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right)-\left\{\left|\psi'(\alpha_1)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}+\left\{\left|\psi'(\alpha_2)\right|\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl\right] \tag{57}$$

holds for all $\alpha_2, \alpha_1 \in [\phi, \varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power mean inequality, and the convexity of function $|\psi'|^q$ that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|dl\right\}. \tag{58}$$

Using the definition of strong convexity

$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|-\left(\frac{2-l}{2}\left|\psi'(\alpha_1|)+\frac{l}{2}\right|\psi'(\alpha_2|)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right\}dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|-\left(\frac{l}{2}\left|\psi'(\alpha_1|)\right|+\frac{2-l}{2}\left|\psi'(\alpha_2|)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right\}dl\right\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\cdot\left\{\left(\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi\xi/k/k}dl-\left(\left|\psi'(\alpha_1)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+\left|\psi'(\alpha_2)\right|\int_0^1\cdot\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl\right)-(\alpha_2-\alpha_1)^2\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\cdot\left(\frac{2-l}{2}\right)dl\right\}+\left\{\left(\left|\psi'(\phi)\right|+\left|\psi'(\varphi)\right|\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl-\left(\left|\psi'(\alpha_1)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl+\left|\psi'(\alpha_2)\right|\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl\right)-(\alpha_2-\alpha_1)^2\int_0^1\cdot\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}. \tag{59}$$

Therefore, inequality (57) can be derived after some simple calculation. □

Remark 16. From Theorem 15, we clearly see that

(1) If we take $a = 0$, $\alpha_1 = x$, and $\alpha_2 = y$ in Theorem 15, then we get Theorem 3.1 of [25]

(2) If we take $a = 0$, $\zeta = k = 1$, $\alpha_1 = \theta$, and $\alpha_2 = \vartheta$ in Theorem 15, then we get Theorem 3.1 of [26]

(3) If we take $a = 0$, $\zeta = k = 1$, $\alpha_1 = \theta$, and $\alpha_2 = \vartheta$ in Theorem 15, then we get Theorem 5 of [29] in the case of $q = 1$

Theorem 17. *Let* $q > 1, \zeta, \xi > 0, \phi < 0$ *and* $\psi : [\phi, \varphi] \longrightarrow \mathbb{R}$ *be a differentiable mapping such that* $\psi' \in L[\phi, \varphi]$ *and* $|\psi'|^q$ *is a convex mapping on* $[\phi, \varphi]$. *Then, the inequality*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}+\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|$$
$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\Bigg)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)^{1-1/q}$$
$$\times\left\{\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{1}{\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right)\right.$$
$$-\left\{|\psi'(\alpha_1)|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}\right.$$
$$+\left\{|\psi'(\alpha_2)|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}-\frac{a}{4}\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl+\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)$$
$$\times\left(\frac{1}{\zeta^{\xi/k+1}}\times B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right)-\left\{|\psi'(\alpha_1)|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right.\right.$$
$$\left.-B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)\right\}+\left\{|\psi'(\alpha_2)|^q\left(\frac{1}{2\zeta^{\xi/k+1}}B\left(\frac{\xi}{k}+1,\frac{1}{\zeta}\right)+B\left(\frac{\xi}{k}+1,\frac{2}{\zeta}\right)\right\}\right.$$
$$\left.-\frac{a}{4}\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}l(2-l)dl\right]^{1/q} \tag{60}$$

holds for all $\alpha_2,\alpha_1\in[\phi,\varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power-mean inequality, and the convexity of function $|\psi'|^q$ that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)-\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right\}\right|$$
$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\left(\left(\int_0^1\left(\frac{\left(1-(1-l)^{\zeta}\right.}{\zeta}\right)^{\xi/k}dl\right)^{1-1/q}\right.\right.$$
$$\times\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\zeta+\frac{2-l}{2}y\right)\right)\right|^q dl\right)^{1/q}$$
$$+\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right)^{1-1/q}\times\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\right.$$
$$\left.\left.\times\left|\psi'\left(\phi+\varphi-\left(\frac{2-la}{2}\zeta+\frac{l}{2}y\right)\right)\right|^q dl\right)^{1/q}\right\}. \tag{61}$$

Using definition of strong convexity, we have

$$\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\left\{|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right.\right.$$
$$-\left(\frac{2-l}{2}|\psi'(\alpha_1)|^q+\frac{l}{2}|\psi'(\alpha_2)|^q\right)-\frac{al}{2}$$
$$\times\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl+\left\{\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\right.$$
$$\times\left\{|\psi'(\phi)|^q+|\psi'(\varphi)|^q-\left(\frac{l}{2}|\psi'(\alpha_1)|^q+\frac{2-l}{2}|\psi'(\alpha_2)|^q\right)\right.$$
$$-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}$$
$$\times\left\{\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\int_0^1\left(\frac{1-(1-la)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.$$
$$-\left(|\psi'(\alpha_1)|^q\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+|\psi'(\alpha_2)|^q\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1$$
$$\left.\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}$$
$$+\left\{\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}dl\right.$$
$$-\left(|\psi'(\alpha_1)|^q\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{l}{2}dl+|\psi'(\alpha_2)|^q\int_0^1\right.$$
$$\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1$$
$$\left.\times\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\right\}. \tag{62}$$

Making simple simplification, we get (60) from (61). □

Remark 18. Theorem 17 leads to

(1) If we take $a=0$, $\alpha_1=x$, and $\alpha_2=y$ in Theorem 17, then we get Theorem 2.12 of [25]

(2) If we take $a=0$, $k=1$, $\alpha_1=x=\phi$, and $\alpha_2=y=\varphi$ in Theorem 17, then we get Theorem 3.1 of [26]

(3) If we take $a=0$, $\zeta=k=1$, $\alpha_1=x=\phi$, and $\alpha_2=y=\varphi$ in Theorem 17, then we get Theorem 5 of [29] in the case of $q=1$

Theorem 19. *Let $q>1,\zeta,\xi>0,\phi<0$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|^q$ is a convex mapping on $[\phi,\varphi]$. Then, the inequality*

$$\begin{aligned}&\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.\\&\left.\left.+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\\&\le\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\frac{1}{\zeta^{\xi/kp+1}}B\left(\frac{\xi}{k}p+1,\frac{1}{\zeta}\right)\right)^{1/p}\\&\times\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{3\left|\psi'\alpha_1\right|^q+\left|\psi'\alpha_2\right|^q}{4}\right)-\frac{al}{2}\right.\right.\\&\left.\times\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)\Bigg)^{1/q}+\Big(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\\&\left.\times\left(\frac{\left|\psi^1(\alpha^1)\right|+3\left|\psi'(\alpha_2)\right|}{\quad}\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)^{1/q}\Bigg\}.\end{aligned}\tag{63}$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, power mean inequality, and the convexity of function $|\psi'|^q$ that

$$\begin{aligned}&\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.\\&\left.\left.+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)-\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right\}\right|\\&\le\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\left(\left(\int_0^1\left(\frac{\left(1-(1-l)^{\zeta}\right)}{\zeta}\right)^{\xi/kp}dl\right)^{1/p}\right.\right.\\&\times\left(\int_0^1\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\zeta+\frac{l}{2}y\right)\right)\right|^q dl\right)^{1/q}+\\&\left.\times\left(\int_0^1\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\zeta+\frac{2-l}{2}y\right)\right)\right|^q dl\right)^{1/q}\right\}.\end{aligned}\tag{64}$$

It follows from the strong convexity of $|\psi'|^q$

$$\begin{aligned}&\le\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\int_0^1\left(\frac{1-(1-l)^{\zeta}}{\zeta}\right)^{\xi/kp}dl\right)^{1/p}\times\left\{\left(\int_0^1\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right.\right.\right.\\&-\left(\frac{2-l}{2}\left|\psi'(\alpha_1)\right|^q+\frac{l}{2}\left|\psi'(\alpha_2)\right|^q\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\Big)\Big)^{1/q}dl\\&+\left(\int_0^1\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{l}{2}\left|\psi'(\alpha_1)\right|^q+\frac{2-l}{2}\left|\psi'(\alpha_2)\right|^q\right)\right.\right.\\&\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\Big)\Big)^{1/q}dl\right\}\le\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left(\frac{1}{\zeta^{\xi/kp+1}}B\left(\frac{\xi}{k}p+1,\frac{1}{\zeta}\right)\right)^{1/p}\\&\cdot\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{3\left|\psi'\alpha_1\right|^q+\left|\psi'\alpha_2\right|^q}{4}\right)-\frac{al}{2}\right.\right.\\&\cdot\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\Big)\Big)^{1/q}+\Big(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\\&\left.\cdot\left(\frac{\left|\psi'(\alpha_1)\right|+3\left|\psi'(\alpha_2)\right|}{\quad}\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)^{1/q}\Bigg\},\end{aligned}\tag{65}$$

which completes the proof. □

Corollary 20. *Let $a=0$ and $\xi=k=1$. Then, Theorem 19 leads to*

$$\begin{aligned}&\left|\frac{1}{\alpha_2-\alpha_1}\int_{\phi+\varphi-\alpha_2}^{\phi+\varphi-\alpha_1}\psi(l)dl-\psi\left(\phi+\varphi-\frac{\alpha_1+\alpha_2}{2}\right)\right|\\&\le\frac{1}{2^{1/p}}\left\{\left(\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{3\left|\psi'\alpha_1\right|^q+\left|\psi'\alpha_2\right|^q}{4}\right)\right.\right.\right.\\&\left.\times-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)\Bigg)^{1/q}\\&+\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q-\left(\frac{\left|\psi'(\alpha_1)\right|+3\left|\psi'(\alpha_2)\right|}{\quad}\right)\right.\\&\left.\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)^{1/q}\right\}.\end{aligned}\tag{66}$$

Theorem 21. *Let $\zeta,\xi>0,p,q>1$ with $1/p+1/q=1,\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|^q$ is a convex mapping on $[\phi,\varphi]$. Then, the inequality*

$$\begin{aligned}&\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}_k^{\xi}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\right.\right.\\&\left.+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)|\le\frac{\left(\alpha_2-\alpha_1\zeta^{\xi/k}\right)}{4}\\&\cdot\left\{\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)\left(\frac{B(\xi/k+1,\xi/k)}{\xi^{\xi/k+1}}\right)\right.\\&-\left(\left(\frac{B(\xi/k+1,2/\xi)+B(\xi/k+1,1/\xi)}{2\zeta^{\xi/k+1}}\right)\left|\psi'(\alpha_1)\right|^q\right.\\&+\left(\frac{B(\xi/k+1,1/\xi)+B(\xi/k+1,2/\xi)}{2\zeta^{\xi/k+1}}\right)\left|\psi'(\alpha_1)\right|^q\\&\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)\left|\alpha_2-\alpha_1\right|^2\right)^{1/q}+\left(\left|\psi'(\phi)\right|^q+\left|\psi'(\varphi)\right|^q\right)\\&\cdot\left(\frac{B(\xi/k+1,1/\xi)}{\xi^{\xi/k+1}}\right)-\left(\left(\frac{B(\xi/k+1,1/\xi)-B(\xi/k+1,2/\xi)}{2\zeta^{\xi/k+1}}\right)\left|S'(\alpha_1)\right|^q\right.\\&+\left(\frac{B(\xi/k+1,2/\xi)+B(\xi/k+1,1/\xi)}{2\zeta^{\xi/k+1}}\right)\left|\psi'(\alpha_2)\right|^q\\&\left.\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)\left|\alpha_2-\alpha_1\right|^2\right)^{1/q}\right\}\end{aligned}\tag{67}$$

holds $\forall\alpha_1,\alpha_1\in[\phi,\varphi]$.

Proof. It follows from Lemma 10, Jensen-Mercer inequality, strong convexity of $|\psi'|^q$, and Holder integral inequality that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\right.\right.$$
$$\left.\left.+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}$$
$$\times\left\{\left(\int_0^1 1dl\right)^{1/p}\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'(\phi+\varphi-\right.\right.\right.$$
$$\left.\left.\times\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|^q dl\Bigg)^{1/q}+\left(\int_0^1 1dl\right)^{1/p}$$
$$\left.\times\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|^q dl\right)^{1/q}\right\}$$
$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left\{\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left[|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right.\right.\right.$$
$$\left.\left.-\left(\frac{2-l}{2}|\psi'(\alpha_1)|^q+\frac{l}{2}|\psi'(\alpha_2)|^q\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right]^q\right)^{1/q}$$
$$+\left(\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left[|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right.\right.$$
$$\left.\left.-\left(\frac{l}{2}|\psi'(\alpha_1)|^q+\frac{2-l}{2}|\psi'(\alpha_2)|^q\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right]dl\right)^{1/q}\right\}. \tag{68}$$

By making necessary changes, we get (67). □

Theorem 22. *Let $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in L[\phi,\varphi]$ and $|\psi'|$ is a convex mapping on $[\phi,\varphi]$. Then, one has*

$$\left|\frac{(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\right.$$
$$\left.\times\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_1)}+\psi(\phi+\varphi-\alpha_2)\right\}\right|$$
$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\left\{\left(\left||\psi'(\phi)+\psi'(\varphi)\right|\right)\frac{B((1/\xi),\xi/k+1)}{\zeta^{\xi/k+1}}\right.\right.$$
$$-\frac{|\psi'(\alpha_1)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)\right\}$$
$$\left.-\frac{|\psi'(\alpha_2)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}\right\}$$
$$+\left\{\left(\left||\psi'(\phi)+\psi'(\varphi)\right|\right)\frac{B((1/\xi),\xi/k+1)}{\zeta^{\xi/k+1}}-\frac{|\psi'(\alpha_1)|}{\zeta^{\xi/k+1}}\right.$$
$$\times\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)\right\}-\frac{|\psi'(\alpha_2)|}{\zeta^{\xi/k+1}}\left\{B_{1/2^\zeta}\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)\right.$$
$$\left.\left.\left.+B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}\right\}\right],\forall\alpha_2,\alpha_1\in[\phi,\varphi]. \tag{69}$$

Proof. By using Lemma 12 and similar arguments as in the proofs of previous theorem, we have

$$\left|\frac{\psi(\phi+\varphi-\alpha_1)+\psi(\phi+\varphi-\alpha_2)}{2}-\frac{\zeta^{\xi/k}\Gamma_k(\xi+k)}{2(\alpha_2-\alpha_1)^{\zeta\xi/k}}\right.$$
$$\left.\times\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_1)}\psi(\phi+\varphi-\alpha_2)\right\}\right|$$
$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left|\left(\frac{1-(1-la)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right|$$
$$\times\left|\psi'(\phi+\varphi-(l\alpha_1+(1-l)\alpha_2))\right|dl$$
$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\int_0^1\left|\left(\frac{1-(1-la)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right|$$
$$\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|+(1-l)|(\alpha_2)|\right)\right.$$
$$\left.-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\int_0^{1/2}\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right.\right.$$
$$\left.-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|\right.\right.$$
$$\left.\left.+(1-l)|(\alpha_2)|\right)-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl+\int_{1/2}^1$$
$$\times\left[\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}-\left(\frac{(1-l)^\zeta}{\zeta}\right)^{\xi/k}\right]$$
$$\times\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(l|\psi'(\alpha_1)|+(1-l)|(\alpha_2)|\right)\right.$$
$$\left.-al(1-l)|\alpha_2-\alpha_1|^2\right\}dl, \tag{70}$$

which completes the proof. □

3. New Inequalities by Improved Hölder Inequality

Theorem 23. *Let $\xi,\zeta>0,p,q>1$ with $1/p+1/q=1,\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in l[\phi,\varphi]$ and $|\psi'|^q$ is a strong convex mapping on $[\phi,\varphi]$. Then, one has*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)\right.\right.$$
$$\left.\left.+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|$$
$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{2}\left[\left\{\left(\frac{B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.\right.\right.$$
$$-\left(\frac{5}{12}|\psi'(\alpha_1)|^q+\frac{1}{12}|\psi'(\alpha_2)|^q\right)-a\frac{5}{144}(\alpha_2-\alpha_1)^2)+$$
$$\times\left(\frac{B(1/\zeta,\xi/kp+1)-B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}$$
$$\times\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}-\left(\frac{1}{3}|\psi'(\alpha_1)|^q+\frac{1}{6}|\psi'(\alpha_2)|^q\right)\right.$$

$$-a\frac{1}{18}(\alpha_2-\alpha_1)^2)\}+\left(\frac{B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.$$
$$-\left(\frac{1}{12}|\psi'(\alpha_1)|^q+\frac{5}{12}|\psi'(\alpha_2)|^q\right)-a\frac{5}{144}(\alpha_2-\alpha_1)^2)$$
$$+\left(\frac{B(1/\zeta,\xi/kp+1)-B(2/\zeta,\xi/kp+1)}{\zeta^{\xi/kp+1}}\right)^{1/p}\times\left(\frac{|\psi'(\phi)|^q+|\psi'(\varphi)|^q}{2}\right.$$
$$\left.\left.-\left(\frac{1}{6}|\psi'(\alpha_1)|^q+\frac{1}{3}|\psi'(\alpha_2)|^q\right)-a\frac{1}{18}(\alpha_2-\alpha_1)^2)\right\}\right]. \quad (71)$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, the convexity of $|\psi'|^q$, and Holder-Iscan integral inequality given in Theorem 1.4 of [31] that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_2)\right\}\right.$$
$$\left.-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\right.\right.\right.$$
$$\times\left(\int_0^1(1-l)\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\right)\right)\right|^q dl\right)^{1/q}\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}$$
$$\left.\times\left(\int_0^1(l)\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\right)\right)\right|^q dl\right)^{1/q}\right\}$$
$$+\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)|\psi'(\phi\right.\right.\right.$$
$$\left.+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\right))|^q dl\right)^{1/q}\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}$$
$$\left.\left.\times\left(\int_0^1(l)\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\right)\right)\right|^q dl\right)^{1/q}\right\}\right]. \quad (72)$$

Applying definition of strong convexity,

$$\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)\right.\right.\right.$$
$$\times\left[|\psi'\phi|^q+|\psi'\varphi|^q-\left(\frac{2-l}{2}|\psi'\alpha_1|^q+\left|\psi'\frac{l}{2}\right|^q\right)\right.$$
$$\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2]dl\right)^{1/q}$$
$$+\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(l)\left[|\psi'\phi|^q+|\psi'\varphi|^q\right.\right.$$
$$\left.\left.-\left(\frac{2-l}{2}|\psi'\alpha_1|^q+\left|\psi'\frac{l}{2}\right|^q-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\right)\right]^q dl]^{1/q}\right\}$$
$$+\left\{\left(\left(\int_0^1(1-l)\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(1-l)\left[|\psi'\phi|^q\right.\right.\right.$$
$$+|\psi'\varphi|^q-\left(\frac{l}{2}|\psi'\alpha_1|^q+\frac{2-l}{2}|\psi'\alpha_2|^q\right.$$
$$\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2)\right]^q dl\right)^{1/q}$$
$$\times\left(\int_0^1 l\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/kP}dl\right)^{1/P}\left(\int_0^1(l)\left[|\psi'\phi|^q\right.\right.$$
$$+|\psi'\varphi|^q-\left(\frac{l}{2}|\psi'\alpha_1|^q+\frac{2-l}{2}|\psi'\alpha_2|^q\right)$$
$$\left.\left.\left.-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}]^q dl\right)^{1/q}\right\}\right]. \quad (73)$$

By some computations, one can get the required result. □

Theorem 24. *Let $\xi,\zeta>0$, $p,q>1$ with $1/p+1q=1$, $\phi<\varphi$ and $\psi:[\phi,\varphi]\longrightarrow\mathbb{R}$ be a differentiable mapping such that $\psi'\in l[\phi,\varphi]$ and $|\psi'|^q$ is a strong convex mapping on $[\phi,\varphi]$. Then, one has*

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\psi(\phi+\varphi-\alpha_2)\right\}\right.$$
$$\left.-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\le\frac{(\alpha_2-\alpha_1)\zeta^{\xi/k}}{4}\left[\left\{\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}\right.\right.$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}+\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$+\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-2B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}\left.\right\}+\left\{\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}\right.$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_2)|^q\left(B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right.$$
$$-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}+\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)^{1-1/q}$$
$$\times\left(\left(|\psi'(\phi)|^q+|\psi'(\varphi)|^q\right)\left(\frac{B(1/\zeta,\xi/k+1)-B(2/\zeta,\xi/k+1)}{\zeta^{\xi/k+1}}\right)\right.$$
$$-\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-2B\left(\frac{2}{\zeta},\frac{\xi}{k}+1\right)+B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)$$
$$+\left(\frac{1}{2\zeta^{\xi/k+1}}|\psi'(\alpha_1)|^q\left(B\left(\frac{1}{\zeta},\frac{\xi}{k}+1\right)-B\left(\frac{3}{\zeta},\frac{\xi}{k}+1\right)\right)\right)$$
$$\left.\left.-\frac{al}{4\zeta^{\xi/k+1}}\left(\frac{2-l}{2}\right)|\alpha_2-\alpha_1|^2)))^{1/q}\right\}\right]. \quad (74)$$

Proof. It follows from Lemma 10, Jensen-Mercer inequality, the convexity of $|\psi'|^q$, and the improved power-mean integral inequality given in Theorem 1.5 of [31] that

$$\left|\frac{2^{\zeta\xi/k-1}\zeta^{\xi/k}\Gamma_k(\xi+k)}{(\alpha_2-\alpha_1)^{\zeta\xi/k}}\left\{{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}+\psi(\phi+\varphi-\alpha_1)+{}^{\xi}_{k}J^{\zeta}_{(\phi+\varphi-\alpha_2+\alpha_1/2)}\right.\right.$$
$$\left.\left.+\psi(\phi+\varphi-\alpha_2)\right\}-\psi\left(\phi+\varphi-\frac{\alpha_2+\alpha_1}{2}\right)\right|\le\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}$$
$$\times\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{2-l}{2}\alpha_1+\frac{l}{2}\alpha_2\right)\right)\right|dl\right.$$
$$\left.+\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left|\psi'\left(\phi+\varphi-\left(\frac{l}{2}\alpha_1+\frac{2-l}{2}\alpha_2\right)\right)\right|dl\right\}.\right. \quad (75)$$

Using definition of strong convexity,

$$\begin{aligned}
&\leq \frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left\{|\psi'(\phi)|+|\psi'(\varphi)|\right.\right.\\
&-\left(\frac{2-l}{2}\left|\psi'(\alpha_1|\right)+\frac{l}{2}\left|\psi'(\alpha_2|\right)\right)-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\\
&+\left\{\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\left\{|\psi'(\phi)|+|\psi'(\varphi)|-\left(\frac{l}{2}|\psi'(\alpha_1|\right)\right.\right.\\
&+\frac{2-l}{2}|\psi'(\alpha_2|))-\frac{al}{2}\left(\frac{2-l}{2}\right)(\alpha_2-\alpha_1)^2\}dl\}\leq\frac{\alpha_2-\alpha_1}{4}\zeta^{\xi/k}\\
&\times\left\{\left(|\psi'(\phi)|+|\psi'(\varphi)|\right)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}dl\right.\\
&-\left(|\psi'(\alpha_1)|\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl+|\psi'(\alpha_2)|\int_0^1\right.\\
&\times\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{al}{2}\\
&\times\left(\frac{2-l}{2}\right)dl\}+\left\{\left(|\psi'(\phi)|+|\psi'(\varphi)|\right)\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}dl\right.\\
&-\left(|\psi'(\alpha_1)\times|\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{l}{2}dl\right.\\
&+|\psi'(\alpha_2)|\int_0^1\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{2-l}{2}dl)-(\alpha_2-\alpha_1)^2\int_0^1\\
&\times\left(\frac{1-(1-l)^\zeta}{\zeta}\right)^{\xi/k}\frac{al}{2}\left(\frac{2-l}{2}\right)dl\}.
\end{aligned} \tag{76}$$

□

4. Applications to Special Means

Means are important in applied and pure mathematics, especially they are used frequently in numerical approximation. In literature, they are order in the following way:

$$H\leq G\leq L\leq I\leq A. \tag{77}$$

The arithmetic mean of two numbers a, b such that a $a \neq b$ is defined as

$$A=A(a,b)=\frac{a+b}{2}, a,b\in\mathbb{R}. \tag{78}$$

The generalized logarithmic mean is defined as follows:

$$\begin{aligned}
L=L_r^r(a,b)&=\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}, r\in\mathbb{R}\,[-1,0], a,b\in\mathbb{R}, a\neq b,\\
L_p(a,b)&=\left[\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^p\right], p\in\mathbb{R}\,[-1,0]
\end{aligned} \tag{79}$$

Proposition 25. *Assume $a, b > 0$ and $a < b$; then,*

$$M_p(a,b)\leq L_{1-p}^{p-1}(a,b).L_p^p(a,b)\leq A \tag{80}$$

holds for $p\in(-\infty,1)/0$ where

$$M_p(a,b)=\left[\left(\frac{a^p+b^p}{2}\right)^{1/p}\right]. \tag{81}$$

Proof. From Theorem 23, we have

$$\begin{aligned}
\left[\left(\frac{a^p+b^p}{2}\right)^{1/p}\right]&\leq\frac{pB(\alpha)}{\alpha(b^p-a^p)}\left[\left({}_a^{CF}I^\alpha\psi\right)(k)+\left({}_a^{CF}I^\alpha\psi\right)(k)\right.\\
&\left.-\frac{2(1-\alpha)}{B(\alpha)}\psi(k)\right]\leq\frac{\phi(a)+\phi(b)}{2},
\end{aligned} \tag{82}$$

with $\psi(x)=\phi(x)/x^{1-p}$ holds. Setting $f(x)=x$, $\alpha=1$ and $B(\alpha)=B(1)=1$ in the above theorem, we obtain

$$\left(\frac{a^p+b^p}{2}\right)^{1/p}\leq\frac{p(b-a)}{b^p-a^p}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]\leq\frac{a+b}{2}. \tag{83}$$

Now use the following:

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}. \tag{84}$$

For $p=p-1$, we have

$$L_{p-1}(a,b)=\left(\frac{b^p-a^p}{p(b-a)}\right)^{1/p-1}. \tag{85}$$

This implies that

$$L_{p-1}^{p-1}(a,b)=\frac{b^p-a^p}{p(b-a)}. \tag{86}$$

By using these means, we get

$$M_p(a,b)\leq L_{p-1}^{1-p}(a,b).L_p^p(a,b)\leq A. \tag{87}$$

By using the results of Section 3, we get some application to special means. □

Proposition 26. *Let $a,b\in\mathbb{R}^+$, $a<b$; then,*

$$\left|A(a^2,b^2)-pL_{p+1}^{1+p}(a^p,b^p)\right|\leq\frac{b^p-a^p}{p}[|a|C_1(a,b)+|b|C_2(a,b)]. \tag{88}$$

Proof. We obtain the result immediately from Theorem 23. □

Proposition 27. *Let $a, b \in \mathbb{R}^+$, $a < b$, then*

$$\left|A(a^n, b^n) - pL_{n-p+1}^{n-1+p}(a^p, b^p)\right| \leq \frac{b^p - a^p}{2p}\left[\left|a^{n-1}\right|C_1(a, b) + \left|b^{n-1}\right|C_2(a, b)\right]. \tag{89}$$

Proof. We obtain the result immediately from Theorem 24. □

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

Authors of this paper declare that they have no competing interests.

Authors' Contributions

X.W. analyzed and approved the results, wrote the final version of the paper, and arranged the funding. A.H. proved the main results. M.S.S. proposed the problem and supervised this work. S.U.Z. wrote the first version of the paper.

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