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Research Article

On the Janowski Starlikeness of the Coulomb Wave Functions

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In this article, we are interested in finding sufficient conditions on $A, B, L,$ and η which ensure the normalized Coulomb wave function to be Janowski starlike. Sufficient conditions are also obtained for $g_{L,\eta}/z \in \mathscr{P}[A, B]$, which readily yield conditions for $g_{L,\eta}$ to be close-to-convex.

1. Introduction

Let $\mathscr{A}$ be the class of functions f which are analytic in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathscr{U}$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathscr{U}$, then $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$. Let $\mathscr{P}[A, B]$ denote the class of analytic functions p such that $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, -1 \le B < A \le 1, z \in \mathscr{U}. \tag{1}$$

Note that for $0 \le \beta < 1$, $\mathscr{P}[1 - 2\beta, -1]$ is the class of analytic functions p with $p(0) = 1$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathscr{U}$. For $-1 \le B < A \le 1$, the class $\mathscr{S}^*[A, B]$ defined by

$$\mathscr{S}^*[A, B] := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathscr{U} \right\}, \tag{2}$$

is the class of Janowski starlike functions [13]. For $0 \le \beta < 1$, $\mathscr{S}^*[1 - 2\beta, -1] := \mathscr{S}^*(\beta)$ is the usual class of starlike functions of order β:

$$\mathscr{S}^*[1 - \beta, 0] := \mathscr{S}^*_\beta = \left\{ f \in \mathscr{A} : |zf'(z)/f(z) - 1| < 1 - \beta \right\},$$
$$\mathscr{S}^*[\beta, -\beta] := \mathscr{S}^*[\beta] = \left\{ f \in \mathscr{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1| \right\}. \tag{3}$$

These classes have been studied in [6, 8]. A function $f \in \mathscr{A}$ is said to be close-to-convex of order β with respect to a function $g \in \mathscr{S}^*$ if $\operatorname{Re}\,(zf'(z)/g(z)) > \beta$. In particular case, if $f \in \mathscr{A}$ and satisfies the condition $\operatorname{Re} f'(z) > \beta$ for all z in $\mathscr{U}$, then $f(z)$ is a close-to-convex of order β.

Let ${}_1F_1$ denote the Kummer confluent hypergeometric function. The regular Coulomb wave function is defined as

$$\begin{aligned} F_{L,\eta}(z) &= z^{L+1} e^{-iz} C_L(\eta)\, {}_1F_1(L + 1 - i\eta, 2L + 2\,; 2iz) \\ &= C_L(\eta) \sum_{n \ge 0} a_{L,n} z^{n+L+1}, L, \eta \in \mathbb{C}, z \in \mathbb{C}, \end{aligned} \tag{4}$$

where

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2} |\Gamma(L + 1 + i\eta)|}{\Gamma(2L + 2)} = \begin{cases} \dfrac{2^L}{(2L + 1)!} \sqrt{\dfrac{2\pi \prod_{k=0}^{L} (k^2 + \eta^2)}{\eta(e^{2\pi\eta} - 1)}}, & \text{if } \eta \ne 0, \\ \dfrac{2^L L!}{(2L + 1)!}, & \text{if } \eta = 0, \end{cases}$$
$$a_{L,0} = 1, a_{L,1} = \frac{\eta}{L + 1}, a_{L,n} = \frac{2\eta a_{L,n-1} - a_{L,n-2}}{n(n + 2L + 1)}, n \in \{2, 3, \cdots\}, \tag{5}$$

which is the solution of following differential equation:

$$z^2\omega''(z) + \left[z^2 - 2\eta z - L(L+1)\right]\omega(z) = 0. \tag{6}$$

In this paper, we focus on the following normalized form:

$$g_{L,\eta}(z) = C_L(\eta)^{-1} z^{-L} F_{L,\eta}(z). \tag{7}$$

The function $g_{L,\eta}(z)$ satisfies the following homogenous second-order differential equation:

$$z^2 g_{L,\eta}''(z) + 2Lz g_{L,\eta}'(z) + \left(z^2 - 2\eta z - 2L\right) g_{L,\eta}(z) = 0. \tag{8}$$

Baricz [9, 10] studied the Turan-type inequalites of regular Coulomb wave functions and zeros of a cross-product of the Coulomb wave and Tricomi hypergeometric functions, respectively. Baricz et al. [11] also investigated the radii of starlikeness and convexity of regular Coulomb wave functions. Recently, Aktas [1] has studied lemniscate and exponential starlikeness of Coulomb wave functions. In some recent papers [2–5, 12], the authors have discussed certain geometric properties of some special functions. The relationships of generalized Bessel function, Bessel-Struve kernal function, and Struve function with the Janowski class have also been studied by various researchers, see [7, 14, 17, 18]. Motivated by the above papers in this subject, in this paper, our aim is to present some geometric results for the normalized regular Coulomb wave function.

The following lemmas are needed in the paper.

Lemma 1 (see [15, 16]). *Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^2 \times \mathcal{U} \longrightarrow \mathbb{C}$ satisfy*

$$\Psi(i\rho, \sigma; z) \notin \Omega, \tag{9}$$

whenever $z \in \mathcal{U}$, ρ real, $\sigma \le -(1+\rho^2)/2$. If p is analytic in $\mathcal{U}$ with $p(0) = 1$, and $\Psi(= (p(z), zp'(z); z) \in \Omega$ for $z \in \mathcal{U}$, then $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$. In the case $\Psi : \mathbb{C}^3 \times \mathcal{U} \longrightarrow \mathbb{C}$, then the condition in Lemma 1 generalized to

$$\Psi(i\rho, \sigma, u + iv; z) \notin \Omega, \tag{10}$$

ρ real, $\sigma + \mu \le 0$, and $\sigma \le -(1+\rho^2)/2$.

Lemma 2 (see [19]). *Let $f \in \mathcal{A}$. If*

$$\operatorname{Re} \frac{f(z)}{z} > 0, \tag{11}$$

then

$$\operatorname{Re} f'(z) > 0, \tag{12}$$

for $|z| < \sqrt{2} - 1$.

2. Inclusion of Generalized Coulomb Wave Function in the Janowski Class

Our first result is related with Janowski starlikeness of normalized Coulomb wave function.

Theorem 3. *Let $-1 \le B < A \le 1$ and $L, \eta \in \mathbb{C}$. Suppose that*

$$\begin{aligned} \operatorname{Re}(2L-1) \le \frac{(1+A)}{4} - (\operatorname{Im}(L))^2 \left(\frac{1+A}{2+A}\right) \\ - 2(1 + |\eta| + |L|) \text{for} - 1 \\ = B < A \le 1, \end{aligned} \tag{13}$$

or, for $-1 < B < A \le 1$,

$$\begin{cases} (1 + 2|\eta| + 2|L|)\dfrac{(1+B)}{(1+A)} - \dfrac{(1+A)}{(1+B)} - \dfrac{(A-B)}{(1+A)(1+B)} < \operatorname{Re}(2L-1), \\ \dfrac{(A-B)}{2(1-B)} - \dfrac{2}{(1-B)} - (1 + 2|\eta| + 2|L|)\dfrac{(1-B)}{2} > \operatorname{Re}(2L-1), \end{cases} \tag{14}$$

$$\begin{aligned} \left[2\operatorname{Im}(L)(1-AB)\right]^2 \le {} & \left[1 + \frac{(1+A)(1+B)}{(A-B)}\operatorname{Re}(2L-1) \right. \\ & \left. + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1 + 2|\eta| + 2|L|)\right] \\ & \times \left[\frac{2(1-B)}{(A-B)}\operatorname{Re}(2L-1) - 1 + \frac{4}{(A-B)} \right. \\ & \left. + \frac{(1-B)^2}{(A-B)}(1 + 2|\eta| + 2|L|)\right]. \end{aligned} \tag{15}$$

If $(1+B)z g_{L,\eta}'(z) \ne (1+A) g_{L,\eta}(z)$, $0 \ne g_{L,\eta}(z)$ and $0 \ne g_{L,\eta}'(z)$, then $g_{L,\eta} \in \mathcal{S}^*[A, B]$.

Proof. Define an analytic function $p : \mathcal{U} \longrightarrow \mathbb{C}$ by

$$p(z) := \frac{(1-A) g_{L,\eta}(z) - (1-B) z g_{L,\eta}'(z)}{(1+B) z g_{L,\eta}'(z) - (1+A) g_{L,\eta}(z)}, p(0) = 1. \tag{16}$$

Then,

$$\frac{z g_{L,\eta}'(z)}{g_{L,\eta}(z)} = \frac{(1-A) + (1+A)p(z)}{(1-B) + (1+B)p(z)}, \tag{17}$$

$$\frac{z g_{L,\eta}''(z)}{g_{L,\eta}'(z)} = \frac{2(A-B) z p'(z)}{[(1-A) + (1+A)p(z)][(1-B) + (1+B)p(z)]} - 1 + \frac{z g_{L,\eta}'(z)}{g_{L,\eta}(z)}. \tag{18}$$

A rearrangement of (18) gives.

$$\left(\frac{zg_{L,\eta}'{}'(z)}{g_{L,\eta}'(z)}\right)\left(\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}\right)=\frac{2(A-B)zp'(z)}{[(1-B)+(1+B)p(z)]^2}-\frac{[(1-A)+(1+A)p(z)]}{[(1-B)+(1+B)p(z)]}+\frac{[(1-A)+(1+A)p(z)]^2}{[(1-B)+(1+B)p(z)]^2}. \tag{19}$$

Now, define a function $q_{L,\eta}:\mathscr{U}\longrightarrow\mathbb{C}$ by

$$q_{L,\eta}(z)=\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}. \tag{20}$$

This function $q_{L,\eta}$ is analytic in $\mathscr{U}$ and $q_{L,\eta}(0)=1$. Suppose that $z\neq 0$. We know that $g_{L,\eta}(z)\neq 0$. This function satisfies the following equation:

$$\frac{z^2g_{L,\eta}'{}'(z)}{g_{L,\eta}(z)}+2L\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}+\left(z^2-2\eta z-2L\right)=0. \tag{21}$$

which yields

$$\left(\frac{zg_{L,\eta}'{}'(z)}{g_{L,\eta}'(z)}\right)\left(\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}\right)+2L\frac{zg_{L,\eta}'(z)}{g_{L,\eta}(z)}+\left(z^2-2\eta z-2L\right)=0. \tag{22}$$

Substituting (17) and (19) in (22), we get

$$\begin{aligned}&\frac{2(A-B)zp'(z)}{[(1-B)+(1+B)p(z)]^2}+\frac{[(1-A)+(1+A)p(z)]^2}{[(1-B)+(1+B)p(z)]^2}\\&+(2L-1)\frac{[(1-A)+(1+A)p(z)]}{[(1-B)+(1+B)p(z)]}\\&+\left(z^2-2\eta z-2L\right)=0,\end{aligned} \tag{23}$$

or equivalently

$$\begin{aligned}&zp'(z)+\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)p(z)\}\{(1-B)+(1+B)p(z)\}\right]\\&+\frac{[(1-A)+(1+A)p(z)]^2}{2(A-B)}\\&+\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}[(1-B)+(1+B)p(z)]^2=0.\end{aligned} \tag{24}$$

Now setting

$$\begin{aligned}\Psi\left(p(z),zp'(z);z\right)&:=zp'(z)+\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)p(z)\}\right.\\&\left.\cdot\{(1-B)+(1+B)p(z)\}\right]\\&+\frac{[(1-A)+(1+A)p(z)]^2}{2(A-B)}\\&+\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}[(1-B)+(1+B)p(z)]^2.\end{aligned} \tag{25}$$

Then, for $\rho\in\mathbb{R}$and $\sigma=-(1+\rho^2)/2$, we get

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma;z)&=\sigma+\operatorname{Re}\left[\frac{(2L-1)}{2(A-B)}\{(1-A)+(1+A)i\rho\}\right.\\&\left.\cdot\{(1-B)+(1+B)i\rho\}\right]\\&+\operatorname{Re}\frac{[(1-A)+(1+A)i\rho]^2}{2(A-B)}\\&+\operatorname{Re}\left[\frac{\left(z^2-2\eta z-2L\right)}{2(A-B)}\{(1-B)+(1+B)i\rho\}^2\right]\\&\leq-\frac{\left(1+\rho^2\right)}{2}+\frac{\operatorname{Re}(2L-1)}{2(A-B)}[(1-A)(1-B)\\&-(1+A)(1+B)\rho^2]-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho\\&+\frac{(1-A)^2-(1+A)^2\rho^2}{2(A-B)}\\&+\frac{(1+|2\eta|+|2L|)}{2(A-B)}\left|\{(1-B)+(1+B)i\rho\}^2\right|\\&=-\rho^2\left[\frac{1}{2}+\frac{(1+A)(1+B)}{2(A-B)}\operatorname{Re}(2L-1)+\frac{(1+A)^2}{2(A-B)}\right.\\&\left.-\frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho\\&+\left[\frac{(1-A)(1-B)}{2(A-B)}\operatorname{Re}(2L-1)-\frac{1}{2}+\frac{(1-A)^2}{2(A-B)}\right.\\&\left.+\frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&\leq-\rho^2\left[\frac{1}{2}+\frac{(1+A)(1+B)}{2(A-B)}\operatorname{Re}(2L-1)+\frac{(1+A)^2}{2(A-B)}\right.\\&\left.-\frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]\\&-\frac{2\operatorname{Im}(L)}{(A-B)}[1-AB]\rho+\frac{(1-B)}{(A-B)}\operatorname{Re}(2L-1)-\frac{1}{2}\\&+\frac{2}{(A-B)}+\frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|):=Q(\rho).\end{aligned} \tag{26}$$

To get the contradiction, we have to show $Q(\rho)\leq 0$ for

$\rho \in \mathbb{R}$. We split the proof into two cases. First, consider the case $B = -1 < A \leq 1$. Then, the function Q becomes

$$Q(\rho) = -\rho^2\left[\frac{2+A}{2}\right] - 2\,\mathrm{Im}\,(L)\rho + \left[\frac{2}{(1+A)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(1+A)} + \frac{2}{(1+A)}(1+|2\eta|+|2L|)\right], \tag{27}$$

that achieve its maximum at $\rho_0 = -2\,\mathrm{Im}\,(L)/(2+A)$, and

$$Q(\rho_0) = \frac{2(\mathrm{Im}\,(L))^2}{[2+A]} + \left[\frac{2}{(1+A)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(1+A)} + \frac{2}{(1+A)}(1+|2\eta|+|2L|)\right], \tag{28}$$

which is nonpositive if and only if

$$\mathrm{Re}\,(2L-1) \leq \left[\frac{(1+A)}{4} - (\mathrm{Im}\,(L))^2\left(\frac{1+A}{2+A}\right) - 2(1+|\eta|+|L|)\right]. \tag{29}$$

Now, consider the case $-1 < B < A \leq 1$. Rewriting Q in the form

$$Q(\rho) = -P\rho^2 + R\rho - S = -P\left\{\left(\rho - \frac{R}{2P}\right)^2 + \frac{4PS - R^2}{4P^2}\right\}, \tag{30}$$

where

$$\begin{aligned} P &= \left[\frac{1}{2} + \frac{(1+A)(1+B)}{2(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{2(A-B)} - \frac{(1+B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right],\\ R &= -\frac{2\,\mathrm{Im}\,(L)}{(A-B)}[1-AB],\\ S &= -\left[\frac{(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - \frac{1}{2} + \frac{2}{(A-B)} + \frac{(1-B)^2}{2(A-B)}(1+|2\eta|+|2L|)\right]. \end{aligned} \tag{31}$$

The inequality $Q(\rho) \leq 0$ holds for any real ρ, if $P > 0$, $S > 0$ and $R^2 \leq 4PS$ or

$$\begin{cases} (1+|2\eta|+|2L|)\dfrac{(1+B)}{(1+A)} - \dfrac{(1+A)}{(1+B)} - \dfrac{(A-B)}{(1+A)(1+B)} < \mathrm{Re}\,(2L-1),\\ \dfrac{(A-B)}{2(1-B)} - \dfrac{2}{(1-B)} - (1+|2\eta|+|2L|)\dfrac{(1-B)}{2} > \mathrm{Re}\,(2L-1), \end{cases} \tag{32}$$

and

$$\begin{aligned} [2\,\mathrm{Im}\,(L)(1-AB)]^2 &\leq \left[1 + \frac{(1+A)(1+B)}{(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1+|2\eta|+|2L|)\right]\\ &\quad\times\left[\frac{2(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - 1 + \frac{4}{(A-B)} + \frac{(1-B)^2}{(A-B)}(1+|2\eta|+|2L|)\right],\\ &\leq \left[1 + \frac{(1+A)(1+B)}{(A-B)}\,\mathrm{Re}\,(2L-1) + \frac{(1+A)^2}{(A-B)} - \frac{(1+B)^2}{(A-B)}(1+|2\eta|+|2L|)\right]\\ &\quad\times\left[\frac{2(1-B)}{(A-B)}\,\mathrm{Re}\,(2L-1) - 1 + \frac{4}{(A-B)} + \frac{(1-B)^2}{(A-B)}(1+|2\eta|+|2L|)\right], \end{aligned} \tag{33}$$

that holds by hypothesis (14) and (15). Thus, in both cases, the function Ψ satisfies the hypothesis of lemma (8) and hence $\mathrm{Re}\,p(z) > 0$, or

$$\frac{(1-A)g_{L,\eta}(z) - (1-B)zg'_{L,\eta}(z)}{(1+B)zg'_{L,\eta}(z) - (1+A)g_{L,\eta}(z)} \prec \frac{1+z}{1-z}. \tag{34}$$

By definition of subordination, there exist a map ω in $\mathcal{U}$with $\omega(0) = 0$, and

$$\frac{(1-A)g_{L,\eta}(z) - (1-B)zg'_{L,\eta}(z)}{(1+B)zg'_{L,\eta}(z) - (1+A)g_{L,\eta}(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \tag{35}$$

which yields

$$\frac{zg'_{L,\eta}(z)}{g_{L,\eta}(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}. \tag{36}$$

Hence,

$$\frac{zg'_{L,\eta}(z)}{g_{L,\eta}(z)} \prec \frac{1+Az}{1+Bz}. \tag{37}$$

□

If take $A = 1-2\beta$ and $B = -1$ for $0 \leq \beta < 1$ in Theorem 3, we obtain following result.

Corollary 4. *Let $0 \leq \beta < 1$ and$L, \eta \in \mathbb{C}$. If*

$$\mathrm{Re}\,(L) \leq -\frac{1+\beta}{4} - (\mathrm{Im}\,(L))^2\left(\frac{1-\beta}{3-2\beta}\right) - (|\eta|+|L|), \tag{38}$$

then the normalized Coulomb wave function $g_{L,\eta}(z)$ is starlike of order β.

Theorem 5. *Let $-1 \le B < A \le 1$ and $L, \eta \in \mathbb{C}$ satisfy*

$$\operatorname{Re}(2L+1) \ge \begin{cases} \dfrac{(1+|2\eta|)}{(1+A)}\left(\sqrt{2(1+A^2)}+(1-A)\right), & -1 = B < A \le 3-2\sqrt{2}, \\ \dfrac{(1+|2\eta|)(1+A)}{2\sqrt{A}} \text{ and } \operatorname{Re}(2L+1) \le \dfrac{(1+|2\eta|)(1+A)}{(1-A)} (A \ne 1), & B = -1, A > 3-2\sqrt{2}, \\ \dfrac{(1+|2\eta|)(1+A)(1-B)^2}{(A-B)(1+B)} - \dfrac{(1-B)}{(1+B)}, & -1 < B \le 0, \\ \dfrac{(1+|2\eta|)(1+A)(1+B)}{(A-B)} - \dfrac{(1-B)}{(1+B)}, & B \ge 0. \end{cases} \tag{39}$$

If $(1+B)s_{L,\eta} \ne (1+A)$, then $g_{L,\eta}(z)/z \in \mathscr{P}[A,B]$.

Proof. Define a function $s_{L,\eta} : \mathscr{U} \longrightarrow \mathbb{C}$ by

$$s_{L,\eta}(z) = \frac{g_{L,\eta}(z)}{z}. \tag{40}$$

The function $s_{L,\eta}$ is analytic in $\mathscr{U}$ and $s_{L,\eta}(0) = 1$. Suppose that $z \ne 0$.. This function satisfies the following equation:

$$z^2 s_{L,\eta}''(z) + 2(L+1) z s_{L,\eta}'(z) + \left(z^2 - 2\eta z\right) s_{L,\eta}(z) = 0. \tag{41}$$

Define the analytic function p::$\mathscr{U} \longrightarrow \mathbb{C}$ by

$$p(z) = -\frac{(1-A) - (1-B)s_{L,\eta}(z)}{(1+A) - (1+B)s_{L,\eta}(z)}. \tag{42}$$

Then, simple computation yields

$$s_{L,\eta}(z) = \frac{(1-A) + (1+A)p(z)}{(1-B) + (1+B)p(z)}, \tag{43}$$

$$s_{L,\eta}'(z) = \frac{2(A-B)p'(z)}{((1-B) + (1+B)p(z))^2}, \tag{44}$$

$$s_{L,\eta}''(z) = \frac{2(A-B)((1-B)+(1+B)p(z))p''(z) - 4(1+B)(A-B)p'^2(z)}{((1-B)+(1+B)p(z))^3}. \tag{45}$$

Thus, using (43)–(45), the differential equation (41) can be rewritten as

$$z^2 p''(z) - \frac{2z^2(1+B)p'^2(z)}{(1-B)+(1+B)p(z)} + 2(L+1)zp'(z) + \left[\frac{\{(1-B)+(1+B)p(z)\}\{(1-A)+(1+A)p(z)\}}{2(A-B)}\right]\left(z^2 - 2\eta z\right) = 0. \tag{46}$$

Assume $\Omega = \{0\}$ and define $\Psi(r,s,t;z)$ by

$$\Psi(r,s,t;z) := t - \frac{2(1+B)}{(1-B)+(1+B)r}s^2 + 2(L+1)s + \left[\frac{\{(1-B)+(1+B)r\}\{(1-A)+(1+A)r\}}{2(A-B)}\right]\left(z^2 - 2\eta z\right). \tag{47}$$

It follows from (47) that $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. To ensure $\operatorname{Re} p(z) > 0$ for $z \in \mathscr{U}$, from Lemma 1, it is enough to establish $\operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) \le 0$ in $\mathscr{U}$ for any real ρ, $\sigma \le -(1+\rho^2)/2$,, and $\sigma + \mu \le 0$. Let $z = x + iy \in \mathscr{U}$.. A computation yields

$$\begin{aligned} \operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) &= \mu - \frac{2(1+B)(1-B)\sigma^2}{(1-B)^2 + (1+B)^2\rho^2} + \operatorname{Re} 2(L+1)\sigma \\ &\quad + \left|\frac{\{(1-B)+(1+B)i\rho\}\{(1-A)+(1+A)i\rho\}}{2(A-B)}\right|\left(\left|z^2 - 2\eta z\right|\right). \end{aligned} \tag{48}$$

Since $\sigma \le -(1+\rho^2)/2$. Thus,

$$\begin{aligned} \operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) &\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2} - \frac{\left(1-B^2\right)\left(1+\rho^2\right)^2}{2\left[(1-B)^2 + (1+B)^2\rho^2\right]} \\ &\quad + \left|\frac{\{(1-B)+(1+B)i\rho\}\{(1-A)+(1+A)i\rho\}}{2(A-B)}\right|\left(\left|z^2 - 2\eta z\right|\right) \\ &\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2} - \frac{\left(1-B^2\right)\left(1+\rho^2\right)^2}{2\left[(1-B)^2 + (1+B)^2\rho^2\right]} \\ &\quad + \left[\frac{|(1-B)+(1+B)i\rho||(1-A)+(1+A)i\rho|}{2(A-B)}\right](1+|2\eta|). \end{aligned} \tag{49}$$

The proof will be divided into four cases. Consider first $B = -1$, $B < A \le 3 - 2\sqrt{2}$. The inequality (49) reduces to

$$\begin{aligned}
&\operatorname{Re}\Psi(i\rho,\sigma,\mu+iv;z)\\
&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}+\left[\frac{|(1-A)+(1+A)i\rho|}{(1+A)}\right](1+|2\eta|)\\
&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}+\frac{(1+|2\eta|)}{(1+A)}[(1-A)+(1+A)|\rho|]\\
&=-\frac{\operatorname{Re}(2L+1)}{2}\rho^2+(1+|2\eta|)|\rho|+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}\\
&=-\frac{\operatorname{Re}(2L+1)}{2}\left(|\rho|-\frac{(1+|2\eta|)}{\operatorname{Re}(2L+1)}\right)^2+\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}\\
&\quad+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}=:G(\rho).
\end{aligned} \tag{50}$$

A quadratic function G takes nonpositive values for any ρ, if

$$\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}+\frac{(1-A)}{(1+A)}(1+|2\eta|)-\frac{\operatorname{Re}(2L+1)}{2}\le 0. \tag{51}$$

Last inequality can be rewritten as

$$-[\operatorname{Re}(2L+1)]^2+\frac{2(1-A)}{(1+A)}(1+|2\eta|)\operatorname{Re}(2L+1)+(1+|2\eta|)^2\le 0, \tag{52}$$

that holds, if

$$\operatorname{Re}(2L+1)\ge\frac{(1+|2\eta|)}{(1+A)}\left(\sqrt{(1-A)^2+(1+A)^2}-(1-A)\right), \tag{53}$$

which reduces to the assumption. Therefore, the assertion follows.

In second case, we consider $B=-1$, $A>3-2\sqrt{2}$. According to (49), we have

$$\begin{aligned}
\operatorname{Re}\Psi(i\rho,\sigma,\mu+iv;z)&\le -\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}\\
&\quad+\left[\frac{|(1-A)+(1+A)i\rho|}{(1+A)}\right](1+|2\eta|)\\
&=-\frac{\operatorname{Re}(2L+1)\left(1+\rho^2\right)}{2}\\
&\quad+\frac{(1+|2\eta|)}{(1+A)}\left(\sqrt{(1-A)^2+(1+A)^2\rho^2}\right)=:H(\rho).
\end{aligned} \tag{54}$$

We note that the function H is even with respect to ρ, and

$$H(0)=-\frac{\operatorname{Re}(2L+1)}{2}+\frac{(1-A)}{(1+A)}(1+|2\eta|), \tag{55}$$

that satisfies $H(0)\le 0$, if

$$\operatorname{Re}(2L+1)\ge\frac{2(1-A)}{(1+A)}(1+|2\eta|). \tag{56}$$

Moreover, $\lim_{\rho\longrightarrow\infty}H(\rho)=-\infty$, and

$$H'(\rho)=-\operatorname{Re}(2L+1)\rho+\frac{(1+A)\rho(1+|2\eta|)}{\sqrt{(1-A)^2+(1+A)^2\rho^2}}, \tag{57}$$

with $H'(\rho)=0$ if and only if $\rho=0$ or

$$\rho_0^2=\frac{(1+|2\eta|)^2}{\operatorname{Re}^2(2L+1)}-\frac{(1-A)^2}{(1+A)^2}. \tag{58}$$

We observe that $\rho_0^2\ge 0$ by the inequality

$$\frac{(1+|2\eta|)^2}{\operatorname{Re}^2(2L+1)}\ge\frac{(1-A)^2}{(1+A)^2}, \tag{59}$$

$$\operatorname{Re}(2L+1)\le\frac{(1+A)}{(1-A)}(1+|2\eta|). \tag{60}$$

Additionally,

$$H''(\rho)=-\operatorname{Re}(2L+1)+\frac{(1-A)^2}{(1+A)^2}\frac{\operatorname{Re}^3(2L+1)}{(1+|2\eta|)^2}\le 0, \tag{61}$$

in view of (60). Hence, $H(\rho_0)=\max H(\rho)$, and

$$H(\rho_0)=-\frac{\operatorname{Re}(2L+1)}{2}\left[1-\left(\frac{1-A}{1+A}\right)^2\right]+\frac{(1+|2\eta|)^2}{2\operatorname{Re}(2L+1)}\le 0, \tag{62}$$

that holds if

$$\operatorname{Re}(2L+1)\ge\frac{(1+A)}{2\sqrt{A}}(1+|2\eta|). \tag{63}$$

Since,

$$\frac{(1+A)}{(1-A)}(1+|2\eta|)\ge\frac{(1+A)}{2\sqrt{A}}(1+|2\eta|)\ge\frac{(1-A)}{(1+A)}(1+|2\eta|), \tag{64}$$

holds for $3-2\sqrt{2}\le A\le 1$, then the condition (56), (60), and (63) reduce to the assumption (39). Therefore, the assertion follows. Let now $-1<B\le 0, A>B$. By the fact $(1-A)/(1+A)<(1-B)/(1+B)$, we get

$$\begin{aligned}
&|(1-A)+(1+A)i\rho||(1-B)+(1+B)i\rho|\\
&\quad=(1+A)(1+B)\sqrt{\left(\frac{1-B}{1+B}\right)^2+\rho^2}\sqrt{\left(\frac{1-A}{1+A}\right)^2+\rho^2}\\
&\quad\le(1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right].
\end{aligned} \tag{65}$$

Also, for $B \le 0$, we have $(1+B)/(1-B) \le 1$; therefore,

$$\frac{1+\rho^2}{(1-B)^2+(1+B)^2\rho^2} = \frac{1}{(1-B)^2}\frac{1+\rho^2}{1+((1+B)/(1-B))^2\rho^2} \ge \frac{1}{(1-B)^2}, \tag{66}$$

for any real ρ. Thus,

$$\begin{aligned} \operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) \le & -\frac{\operatorname{Re}(2L+1)(1+\rho^2)}{2} - \frac{(1+B)(1+\rho^2)}{2(1-B)} \\ & + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right] \\ = & \rho^2\left(-\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\right) \\ & - \frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} \\ & + \frac{(1+|2\eta|)}{2(A-B)(1+B)}(1+A)(1-B)^2.. \end{aligned} \tag{67}$$

Since for $B \le 0$,,

$$\begin{aligned} & -\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B) \\ & \le -\frac{\operatorname{Re}(2L+1)}{2} - \frac{(1+B)}{2(1-B)} + \frac{(1+|2\eta|)}{2(A-B)(1+B)}(1+A)(1-B)^2, \end{aligned} \tag{68}$$

and the last expression is nonpositive in view of (39); then, the assertion follows. Finally, consider $0 \le B < A \le 1$. In this case $\beta = (1+B)/(1-B) \le 1$. Hence, setting $t = \beta^2 + \rho^2$ with $t \ge \beta^2$ and using (65), we get from (49)

$$\begin{aligned} \operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) \le & -\frac{\operatorname{Re}(2L+1)}{2}(1-\beta^2+t) - \frac{\beta(1-\beta^2+t)^2}{2t} \\ & + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)t \\ = & t\left\{-\frac{\operatorname{Re}(2L+1)}{2} - \frac{\beta}{2} + \frac{(1+|2\eta|)}{2(A-B)}(1+A)(1+B)\right\} \\ & - \frac{\operatorname{Re}(2L+1)}{2}(1-\beta^2) - \frac{\beta(1-\beta^2)^2}{2t} - \beta(1-\beta^2)^2, \end{aligned} \tag{69}$$

that is nonpositive due to inequality

$$\operatorname{Re}(2L+1) \ge \frac{(1+|2\eta|)}{(A-B)}(1+A)(1+B) - \frac{(1-B)}{(1+B)}, \tag{70}$$

that is equivalent to the assumption (39). Evidently, Ψ satisfies the hypothesis of Lemma 1, and thus, $\operatorname{Re} p(z) > 0$, that is

$$-\frac{(1-A)-(1-B)s_{L,\eta}(z)}{(1+A)-(1+B)s_{L,\eta}(z)} \prec \frac{1+z}{1-z}. \tag{71}$$

Hence, there exists an analytic self-map ω of $\mathscr{U}$ with $\omega(0)=0$ such that

$$-\frac{(1-A)-(1-B)s_{L,\eta}(z)}{(1+A)-(1+B)s_{L,\eta}(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \tag{72}$$

which implies that

$$\frac{g_{L,\eta}(z)}{z} \prec \frac{1+Az}{1+Bz}. \tag{73}$$

□

If we take $A = 1-2\beta$ and $B=-1$ for $0 \le \beta < 1$ in Theorem 5, we obtain following result.

Corollary 6. *Let $0 \le \beta < 1$ and $L, \eta \in \mathbb{C}$. If*

$$\begin{aligned} & \frac{(1+2|\eta|)(1-\beta)}{2\sqrt{1-2\beta}} - \frac{1}{2} \le \operatorname{Re}(L) \\ & \le \frac{(1+2|\eta|)(1-\beta)}{2\beta} - \frac{1}{2}, \left(0 \le \beta < \sqrt{2}-1\right), \end{aligned} \tag{74}$$

$$\operatorname{Re}(L) \ge \frac{1}{2} + \frac{1+2|\eta|}{2(1-\beta)}\left(\sqrt{1-2\beta+2\beta^2}+\beta\right), \left(\sqrt{2}-1 \le \beta < 1\right), \tag{75}$$

then $\operatorname{Re}(g_{L,\eta}(z)/z) > \beta$, *that is,* $z + \int_0^z (g_{L,\eta}(t)/t)dt$ *is close-to-convex of order* β.

Applying Corollary 6 for $\beta = 0$ and Lemma 2, the following result for close-to-convexity of $g_{L,\eta}(z)$ immediately follows.

Corollary 7. *Let $L, \eta \in \mathbb{C}$. If* $\operatorname{Re}(L) \ge |\eta|$, *then* $g_{L,\eta}(z)$ *is close-to-convex (univalent) for* $|z| < \sqrt{2}-1$.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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