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Research Article

Existence Results of Fuzzy Delay Impulsive Fractional Differential Equation by Fixed Point Theory Approach

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The main aim of this article is to study controllability and existence of solution of fuzzy delay impulsive fractional nonlocal integro-differential equation in the sense of Caputo operator. The existence and uniqueness of the solution have been carried out with the help of the Banach fixed point theorem. Moreover, for fuzzy fractional differential equations (FFDEs) driven by the Liu process, this present work introduced a concept of stability in credibility space. Finally, efficient examples are presented to demonstrate the main theoretical findings.

1. Introduction

Fractional-order dynamical equations can be used to model a huge spectrum of physical processes in modern-world observations [1]. Due to its wide range application in various areas of sciences such as physics, chemistry, biology, electronics, thermal systems, electrical engineering, mechanics, signal processing, weapon systems, electrohydraulics, population modeling, robotics, and control, the concept of fuzzy sets continues to catch the attention of researchers [2]. As a result, in recent years, scholars have been increasingly interested in it. As a concept of describing a set with uncertain boundary, the fuzzy set was developed by Zadeh et al. [3]. The concept of possibility measure was studied by Zadeh [4] in 1978. Fuzzy set theory is a very useful technique for simulating uncertain problems. In fuzzy calculus, therefore, the concept of the fractional derivative is essential. Although the possibility measure provides the theoretical basis for the measurement of fuzzy events, it does not satisfy self-duality. Liu B. and Liu Y. [5] studied the concept of credibility measure in 2002, and a sufficient and necessary condition for credibility measure was derived by Li and Liu [6] in 2006. Fractional differential equations (FDEs) are differential equations with fractional derivatives. It is known from the research on fractional derivatives that they originate uniformly from major mathematical reasons. Different types of derivatives exist, such as Caputo and RL [7]. In 1965, Zadeh used the membership function to propose the concept of fuzzy sets for the first time. The FFDE is the most fascinating field. They are useful for understanding phenomena that have an underlying effect. Kwun et al. [8] and Lee et al. [9] investigated the solution of uniqueness-existence for FDEs. Controlled processes have been explored by several researchers. In the case of the fuzzy system, Kwun et al. [10] for the impulsive semilinear FDEs, controllability in n-dimension fuzzy vector space was demonstrated. Park et al. [11] controllability of semilinear fuzzy integro-differential equations with nonlocal conditions was investigated. Park et al. [12] established controllability of impulsive semilinear fuzzy integro-differential equations. Phu and Dung [13] studied stability analysis and controllability of fuzzy control set differential equations. According to Lee et al. [14], in the n-dimensional fuzzy space $\mathbf{E_N}^n$ of a nonlinear fuzzy control system, controllability with nonlocal initial conditions was examined.

Balasubramaniam and Dauer [15] examined the controllability of stochastic systems in Hilbert space of quasilinear stochastic evolution equations, while Feng [16] explored the controllability of stochastic with control systems associated with time-variant coefficients. Arapostathis et al. [17] analyzed the controllability of stochastic differential systems of equations with linear-controlled diffusion affected by

Lipschitz nonlinearity that is limited, smooth, and uniform. Stochastic differential equations given by Brownian motion are a well-known and well-studied area of modern mathematics. A new type of FDE was created using the Liu technique [18], which was described as follows:

$$dX_\nu = f(X_\nu, \nu)d\nu + g(X_\nu, \nu)dC_\nu, \tag{1}$$

where C_ν denotes Liu operation and f and g are functions that have been assigned to it. This class of equations is solved using a fuzzy technique. For homogeneous FDEs, Chen and Qin [19] studied solutions of existence-uniqueness of few special FDEs. Liu [20] investigated an approximate method for solving unknown differential equations. Abbas et al. [21, 22] worked on a partial differential equation. Niazi et al. [23, 24], Iqbal et al. [25], Shafqat et al. [26], Abuasbeh et al. [27], and Alnahdi [28] existence-uniqueness of the FFEE were investigated. Arjunan et al. [29–32] worked on the fractional differential inclusions.

Using conclusions of Liu [20], Jeong et al. [33] focused on exact controllability in credibility space for FDEs. Abstract FDEs' complete controllability in credibility space is as follows:

$$dx(\nu, \varpi) = Ax(\nu, \varpi)d\nu + f(\nu, x(\nu, \varpi))d\mathscr{C}_\nu + Bu(\nu), \nu \in [0, \mathfrak{J}],$$
$$x(0) = x_0. \tag{2}$$

We used the Caputo derivative to prove controllability for the fuzzy delay impulsive fractional integro-evolution equation in credibility space with nonlocal condition; as a result of the above research,

$$\begin{aligned} {}_0^{\mathscr{C}}D_\nu^\beta u(\nu, \zeta) &= \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu, \zeta) \\ &\quad + \int_0^\nu f\left((\nu, u(\nu, \zeta)), \int_0^s k(s, u(\nu, \zeta))\right) d\mathscr{C}_\nu \\ &\quad + Bx(\nu)\mathscr{C}x(\nu)d\nu, \nu \in (0, \nu_i], i = 1, 2, \cdots, N, \\ u(0) &= u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), \end{aligned} \tag{3}$$

where $U(\subset \mathbf{E_N})$ and $V(\subset \mathbf{E_N})$ are two bounded spaces. $\mathbf{E_N}$ is denoted for the set of numbers; all upper semicontinuously convex fuzzy on $\mathbf{R^m}$, and $(\Theta_1, \mathbf{P^m}, \mathscr{C}_r)$, is the credibility space.

The fuzzy coefficient is defined by the state function $u : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow U$. $f : [0, \mathfrak{J}] \times U \longrightarrow U$ is a fuzzy process. $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is regular fuzzy function, $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow V$ is control function, and $\mathscr{B}$ is linear bounded operator on V to U. The initial value is $u_0 \in \mathbf{E_N}$, and $\mathscr{C}_\nu$ denotes the Liu process.

The goal of this work is to investigate the existence and stability of results to FDEs and the exact controllability driven by the Liu process, in order to deal with a fuzzy process. Some scholars discovered FDE results in the literature, although the vast majority of them were differential equations of the first order. We discovered the results for Caputo derivatives of order $(0, 1)$ in our research. Stability, as a part of differential equation theory, is vital in both theory and application. As a result, stability is a key subject of study for researchers, and research papers on stability for FDE have been published in the last two decades, for example, essential conditions for solution stability and asymptotic stability of FDEs. We use fuzzy delay impulsive fractional integro-evolution equations with the nonlocal condition. The theory of fuzzy sets continues to gain scholars' attention because of its huge range of applications in different fields of sciences such as engineering, robotics, mechanics, control, thermal systems, electrical, and signal processing.

In Section 2, we go over some basic notions relating to Liu's processes and fuzzy sets. Section 3 demonstrates the existence of solutions of FDE and shows that FDE is precisely controllable. The concept of credibility stability for FDEs driven by the Liu process was developed in Section 4. Finally, in Section 5, several theorems for FDEs driven by the Liu process that is stable in credibility space were demonstrated.

2. Preliminary

If $M_k(\mathbf{R^m})$ be the family of all nonempty compact convex subsets of $\mathbf{R^m}$, then addition and scalar multiplication are commonly defined as $M_k(\mathbf{R^m})$. Consider two nonempty bounded subsets of $\mathbf{R^m}$, A_1 and B_1. The distance between A_1 and B_1 is measured using the Hausdorff metric as

$$d(A_i, B_i) = \max \left\{ \sup_{a_i \in A_i} \inf_{b_i \in B_i} \|a_i - b_i\|, \sup_{b_i \in B_i} \inf_{a_i \in A_i} \|a_i - b_i\| \right\}, \tag{4}$$

where $\|\cdot\|$ indicates the usual Euclidean norm in $\mathbf{R^m}$. It follows that $(M_k(\mathbf{R^m}), d)$ is a separable and complete metric space [20]. Satisfy the below condition:

$$\mathbf{E^m} = \{j : \mathbf{R^m} \longrightarrow [0, 1] | j \text{ satisfies(a)} - \text{(b)below}\}, \tag{5}$$

where

(a) j is normal; there exists an $j_0 \in \mathbf{R^m}$ such that $j(j_0) = 1$.

(b) j is fuzzy convex, such that is $j(\lambda\nu + (1 - \lambda)s) \geq 1$.

(c) j is upper semicontinuous function on $\mathbf{R^m}$, that is, $j(\nu_0) \geq \lim_{k \longrightarrow \infty} j(\bar{\nu_k})$ for any $\nu_k \in \mathbf{R^m}(k = 0, 1, 2, \cdots)$, $\nu_k \longrightarrow \nu_0$.

(d) $[j]^0 = cl\{u \in \mathbf{R^m} | j(\nu) > 0\}$ is compact.

In $\mathbf{R^m}$ [34], for $0 < \beta < 1$, denote $[j]^\beta = \{\nu \in \mathbf{R^m} | u(\nu) \geq \beta\}$ and $[u]^0$ are nonempty compact convex sets. Then from (a) to (b), it concludes that β-level set $[j]^\beta \nu \in M_k(\mathbf{R^m})$ for all $0 < \beta < 1$. Using Zadeh's extension principle, we can have scalar multiplication and addition in fuzzy number space $\mathbf{E^m}$ as follows:

$$[j \oplus \wp]^\beta = [j]^\beta \oplus [\wp]^\beta, [kj]^\beta = k[\wp]^\beta, \tag{6}$$

where $j, \wp \in \mathbf{E^m}$, $k \in \mathbf{R^m}$ and $0 < \beta < 1$. Assume $\mathbf{E_N}$ denotes a set of all numbers upper semicontinuously convex fuzzy on $\mathbf{R^m}$.

Definition 1 (see [35]). Given a complete metric D_L by

$$D_L(j, y) = \sup_{0<\beta<1} d_L\left\{[j]^\beta, [\wp]^\beta\right\} = \sup_{0<\beta<1} \max\left\{\left|j_l^\beta - \wp_l^\beta\right|, \left|j_l^\beta - \wp_r^\beta\right|\right\}, \quad (7)$$

for any $u, v \in \mathbf{E_N}$, which satisfies $D_L(j+z, \wp+z) = D_L(j, \wp)$ for each $z \in \mathbf{E_N}$ and $[j]^\alpha = [j_l^\beta, u_r^\beta]$, for each $\beta \in (j, \wp)$ where χ_l^β, $u_r^\beta \in \mathbf{R^m}$ with $j_l^\beta \leq u_r^\beta$.

Definition 2 (see [36]). The fractional derivative of RL is stated as

$${}_aD_\nu^\lambda f(\nu) = \left(\frac{d}{d\nu}\right)^{n+1} \int_a^\nu (\nu-\tau)^{n-\lambda} f(\tau) d\tau, \text{ where } (n \leq \lambda \leq n+1). \quad (8)$$

Definition 3 (see [37]). The fractional derivatives in the sense of Caputo ${}_a^{\mathscr{C}}D_\nu^\sigma f(\nu)$ of order $\alpha \in \mathbf{R}^{m^+}$ are described by

$${}_a^{\mathscr{C}}D_\nu^\sigma f(\nu) = {}_aD_\nu^\sigma\left(f(\nu) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(\nu-a)^k\right), \quad (9)$$

where $n = [\sigma] + 1$ for $\sigma \notin N_0$; $n = \sigma$ for $\sigma \in N_0$.

Definition 4 (see [37]). The Wright function ψ_σ is defined by

$$\psi_\sigma(\varpi) = \sum_{n=0}^{\infty} \frac{(-\varpi)^n}{n!\Gamma(-\sigma n + 1 - \sigma)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-\varpi)^n}{(n-1)!} \Gamma(n\sigma) \sin(n\pi\sigma), \quad (10)$$

where $\varpi \in \mathbb{C}$ with $0 < \sigma < 1$.

Definition 5 (see [38]). For any $j, \wp \in \mathscr{C}([0, T], E_N)$, metric $H_1(\chi, \wp)$ on $\mathscr{C}([0, T], E_N)$ is defined by

$$H_1(j, \wp) = \sup_{0\leq\nu\leq T} D_L(j(\nu), \wp(\nu)). \quad (11)$$

Consider that Θ_1 is a nonempty set and $\mathbf{P^m}$ denotes power set on Θ_1. A case is a label given to each element of $\mathbf{P^m}$. To present an axiomatic credibility, an idea based on the consideration of A_i will occur. To validate that the number $\mathscr{C}_r\{A_i\}$ is applied to each A_i event, representing the probability of A_i happens. We accept the four main axioms to ensure that the number $\mathscr{C}_r\{A_i\}$ has certain mathematical features that we predict:

(a) Normality property $\mathscr{C}_r\{\Theta_1\} = 1$,

(b) Monotonicity property $\mathscr{C}_r\{A_i\} \leq \mathscr{C}_r\{B_i\}$, whenever $A_i \subset B_i$,

(c) Self-duality property $\mathscr{C}_r\{A_i\} + \mathscr{C}_r\{A_i^c\} = 1$ for any event A_i,

(d) Maximality property $\mathscr{C}_r\{\cup_i A_i\} = \sup_i \mathscr{C}_r\{A_i\}$ for any events $\{A_i\}$ with $\sup_i \mathscr{C}_r\{A_i\} < 0.5$.

Definition 6 (see [39]). Take Θ be the nonempty set, P^m be the power set of Θ_1, and $\mathscr{C}_r$ be the credibility measure. After that, the triplet $(\Theta_1, P^m, \mathscr{C}_r)$ is assigned to the set of real numbers.

Definition 7 (see [39]). A fuzzy variable is a function that is generated from a set of real numbers $(\Theta_1, P^m, \mathscr{C}_r)$ to credibility space $(\Theta_1, P^m, \mathscr{C}_r)$.

Definition 8 (see [39]). If $(\Theta_1, P^m, \mathscr{C}_r)$ be credibility space and $(\Theta_1, P^m, \mathscr{C}_r)$ be an index set, a fuzzy process is a function that takes a set of real numbers and multiplies them by $T \times (\Theta_1, P^m, \mathscr{C}_r)$.

It is a fuzzy method. $u(\nu, \zeta)$ is a two-variable function in which $u(\nu, \zeta^*)$ represents a fuzzy variable for each ν^*. For each fixed ζ^*, the function $u(\nu, \zeta)$ is termed a sample path of fuzzy process. The fuzzy process $u(\nu, \zeta)$ is said to be sample continuous if sample ping is continuous for almost all ζ. Alternately of $u(\nu, \zeta)$, we frequently use the notation u_ν.

Definition 9 (see [39]). $(\Theta_1, P^m, \mathscr{C}_r)$ is the symbol of a credibility space. The β-level set is applied for the fuzzy random variable u_ν in credibility space for each $\beta \in (0, 1)$.

$$[u_\nu]^\beta = \left[(u_\nu)_l^\beta, (u_\nu)_r^\beta\right], \quad (12)$$

is defined by

$$(u_\nu)_l^\beta = \inf\,(u_\nu)^\beta = \inf\,\{a \in \mathbf{R^m}\,;\, u_\nu(a) \geq \beta\},$$
$$(u_\nu)_r^\beta = \sup\,(u_\nu)^\beta = \inf\,\{a \in \mathbf{R^m}\,;\, u_\nu(a) \geq \beta\}, \quad (13)$$

where $(u_\nu)_l^\beta, (u_\nu)_r^\beta \in \mathbf{R^m}$ with $(u_\nu)_l^\beta \leq (u_\nu)_r^\beta$ when $\beta \in (0, 1)$.

Definition 10 (see [5]). Suppose that ϖ is a fuzzy variable and that r is a real number. Then, ϖ's expected value is defined:

$$E\varpi = \int_0^{+\infty} C_r\{\varpi \geq r\} dr - \int_{-\infty}^0 \mathscr{C}_r\{\varpi \leq r\} dr, \quad (14)$$

if at least one of the integrals is finite.

Lemma 11 (see [5]). *If ϖ is a fuzzy vector, then the following are properties of expected value operator E:*

(a) If $f \leq \mathfrak{g}$, $\mathbf{E}[f(\varpi)] \leq \mathbf{E}[\mathfrak{g}(\varpi)]$

(b) $\mathbf{E}[-f(\varpi)] = -\mathbf{E}[f(\varpi)]$

(c) If f and $\mathfrak{g}$ are comonotonic, we have for any nonnegative real numbers a_i and b_i,

(a)

$$\mathbf{E}[a_i f(\varpi) + b_i \mathfrak{g}(\varpi)] = a_1 \mathbf{E}[f(\varpi)] + b_1 \mathbf{E}[\mathfrak{g}(\varpi)], \tag{15}$$

where $f(\varpi)$ and $\mathfrak{g}(\varpi)$ are fuzzy variables, respectively.

Definition 12 (see [5]). A fuzzy process $\mathscr{C}_\nu$ is Liu process, if

(a) $\mathscr{C}_0 = 0$,

(b) the $\mathscr{C}_\nu$ has independent and stationary increments,

(c) any increment $\mathscr{C}_{\nu+s} - \mathscr{C}_s$ is normally distributed fuzzy variable with expected value $e\nu$ and variance $\phi^2\nu^2$, with membership function.

$$\xi(u) = 2\left(1 + \exp\left(\frac{\pi|u - e\nu|}{\sqrt{6}\phi\nu}\right)\right)^{-1}, u \in \mathbf{R^m} \tag{16}$$

The parameters ϕ and e represent the diffusion and drift coefficients, respectively. If $e = 0$ and $\phi = 1$, the Liu process is standard.

Definition 13 (see [40]). Suppose that $\mathscr{C}_\nu$ is a standard Liu process and u_ν is a fuzzy process. The mesh is fixed as $c = \nu_0 < \cdots < \nu_n = d$ for any partition of the closed interval $[c, d]$ with $c = \nu_0 < \cdots < \nu_n = d$,

$$\Delta = \max_{1\le i\le n}(\nu_i - \nu_{i-1}). \tag{17}$$

After that, the fuzzy integral of u_ν with regard to $\mathscr{C}_\nu$ is calculated:

$$\int_c^d u_\nu d\mathscr{C}_\nu = \lim_{\Delta\longrightarrow 0}\sum_{i=1}^{n}\mu(\nu_{i-1})\left(\mathscr{C}_{\nu_i} - \mathscr{C}_{\nu_{i-1}}\right), \tag{18}$$

determined by the limit exists almost positively and is a fuzzy variable.

Lemma 14 (see [40]). *Consider that $\mathscr{C}_\nu$ represent the standard Liu process with $\mathscr{C}_r\{\zeta\} > 0$, and the direction $\mathscr{C}_\nu$ is Lipschitz continuous, employing the below inequality:*

$$\left|\mathscr{C}_{\nu_1} - \mathscr{C}_{\nu_2}\right| < \mathscr{K}(\zeta)|\nu_1 - \nu_2|, \tag{19}$$

where $\mathscr{K}(\zeta)$ is Lipschitz, which is a fuzzy variable described by

$$\mathscr{K}(\zeta) = \begin{cases} \sup\limits_{0\le s\le\nu} \dfrac{|\mathscr{C}_\nu - \mathscr{C}_s|}{\nu} - s, & \mathscr{C}_r\{\zeta\} > 1, \\ \infty, & \text{otherwise}, \end{cases} \tag{20}$$

and $E[\mathscr{K}^p] < \infty$ for all $p > 1$.

Lemma 15 (see [40]). *Assume that $h(\nu\,;c)$ is a continuously differentiable function and that $\mathscr{C}_\nu$ is a standard Liu process. The function is defined as $u_\nu = h(\nu\,;\mathscr{C}_\nu)$. Then, there is the chain rule, which is as follows:*

$$du_\nu = \frac{\partial h(\nu\,;\mathscr{C}_\nu)}{\partial \nu}d\nu + \frac{\partial h(\nu\,;\mathscr{C}_\nu)}{\partial \mathscr{C}}d\mathscr{C}_\nu. \tag{21}$$

Lemma 16 (see [40]). *The fuzzy integral inequality exists if $f(nu)$ is a continuous fuzzy process:*

$$\left|\int_c^d f(\nu)d\mathscr{C}_\nu\right| \le \mathscr{K}\int_c^d |f(\nu)|d\nu. \tag{22}$$

In Lemma 14, the term $\mathscr{K} = \mathscr{K}(\zeta)$ is defined.

3. Existence of Solutions

This part applies the symbol u_ν instead of the lengthy notation $u(\nu, \zeta)$, as defined by Definition 8. The existence-uniqueness of solutions to FDE 1 ($x \equiv 0$) has been investigated.

$$\begin{cases} {}^{\mathscr{C}}_{0}D^{\beta}_{\nu}u_\nu = \mathfrak{g}_i u_\nu + Au_\nu + \displaystyle\int_0^\nu f\left((\nu, u_\nu) + \int_0^s \mathscr{K}(s, u_\nu)\right)d\mathscr{C}_\nu, & \beta \in (0, 1), \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), & \in E_N, \end{cases} \tag{23}$$

where u_ν is state that includes values from the $U(\subset\mathbf{E_N})$ set of values. The set of all upper semicontinuously convex fuzzy numbers on $\mathbf{R^m}$ is called $\mathbf{E_N}$, credibility space is $(\Theta_1, \mathbf{P^m}, \mathscr{C}_r)$, fuzzy coefficient is A, and state function $u : [0, \mathfrak{I}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow U$ is fuzzy process, $f : [0, \mathfrak{I}] \times U \longrightarrow U$ is regular fuzzy function, $\mathscr{C}_\nu$ is standard Liu process, and $u_0 \in \mathbf{E_N}$ is initial value.

Lemma 17. *If $u(\nu)$ is the solution of equation (3) for $u(0) = u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)$, then $u(\nu)$ is given by*

$$\begin{aligned} u(\nu) = {} & \nu^{\beta-1}(u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)) + \frac{1}{\sqrt{q}}\left[\left[\int_0^\nu (\nu - s)^{\beta-1}\mathfrak{g}_i(s, x(s))ds\right.\right. \\ & + \int_0^\nu (\nu - s)^{\beta-1}\left[Au(s, \zeta) + \int_0^\nu f\right. \\ & \left.\left.\left. \cdot \left(s, u(s, \zeta), \int_0^s \mathscr{K}(s, u(s, \zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right]\right]\right] ds, \end{aligned} \tag{24}$$

holds, and then,

$$\begin{aligned} u(\nu) = {} & \nu^{\beta-1}P_\beta(\nu)(u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, x(s))ds \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)[Au(s, \zeta) \\ & + \left.\int_0^\nu f\left(s, u(s, \zeta), \int_0^s \mathscr{K}(s, u(s, \zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right] ds, \end{aligned} \tag{25}$$

where

$$P_q(\nu) = \int_0^\infty q\zeta M_q(\zeta)Q(\nu^q\zeta)d\zeta. \tag{26}$$

Suppose that the statements below are correct:

(J_1) For $u_\nu, \nu_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U), \nu \in [0, \mathfrak{T}]$. There exist positive number m that is

$$\begin{aligned} d_L\left([f(\nu, u_\nu)]^\beta, [f(\nu, \nu_\nu)]^\beta\right) &\leq m d_L\left([u_\nu]^\beta, [\nu_\nu]^\beta\right) \\ f\left(0, X_{\{0\}}(0)\right) &\equiv 0. \end{aligned} \tag{27}$$

(J_2) $2cm\mathscr{K}\mathfrak{T} \leq 1$. Because of Lemma 17, (23) has the solution u_ν. As a result, we establish in Theorem 18 that the solution to (23) is unique.

Theorem 18. *For $(u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \in E_N$, if ($J_1$) and ($J_2$) are hold, (23) has an unique solution $u_\nu \in \mathscr{C}([0, \mathfrak{T}]) \times (\Theta_1, P^m, \mathscr{C}_r), U)$.*

Proof. For all $\varpi_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U), \nu \in [0, \mathfrak{T}]$, define

$$\begin{aligned} \phi\varpi_\nu = \nu^{\beta-1}P_\beta(\nu)\big(u_0 &+ h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ &+ \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, \varpi_s)\Big)ds \\ &+ \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s) \\ &\cdot \left[A\varpi_s + \int_0^\nu f\left(s, \varpi_s, \int_0^s K(s, \varpi_s)d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right]ds. \end{aligned} \tag{28}$$

□

As a result, the $\phi\varpi : [0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r) \longrightarrow ([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$ can be established as

$$\phi : \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U) \longrightarrow \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U). \tag{29}$$

For equation (23), ϕ is a fixed point which is likewise an obvious solution. $\varpi_\nu, \mu_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$, according to hypothesis (J_1) and Lemma 16.

$$\begin{aligned} &d_L\left([\phi\varpi_\nu]^\beta, [\phi\mu_\nu]^\beta\right) \\ &= d_L\Bigg(\Bigg[\Bigg[\int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, \varpi_s) + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s) \\ &\quad\cdot \left[A(s, \varpi_s) + f\left((s, \varpi_s), \int_0^s \mathscr{K}(s, \varpi_s)d\mathscr{C}_s\right)\right]\Bigg]\Bigg]^\beta, \\ &\quad\cdot \Bigg[\int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, \mu_s) + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s) \\ &\quad\cdot \left[A\mu_s + f\left((s, \mu_s), \int_0^s \mathscr{K}(s, \mu_s)dC_s\right)\right]^\beta\Bigg) \\ &\leq cm\mathscr{K}\int_0^\nu d_L\left([\theta_s]^\beta, [\mu_s]^\beta\right)ds. \end{aligned} \tag{30}$$

Therefore, we obtain

$$\begin{aligned} D_L(\phi\varpi_\nu, \phi\mu_\nu) &= \sup_{\beta\in(0,1)} d_L\left([\phi\varpi_\nu]^\beta, [\phi\mu_\nu]^\beta\right) \\ &\leq \mathrm{cm}\mathscr{K}\int_0^\nu \sup_{\beta\in(0,1)} d_L\left([\varpi_\nu]^\beta, [\mu_\nu]^\beta\right)ds \\ &= \mathrm{cm}\mathscr{K}\int_0^\nu D_L(\varpi_s, \mu_s)ds. \end{aligned} \tag{31}$$

As a result, according to Lemma 11, for a.s. $\varpi \in \Theta_1$,

$$\begin{aligned} E(H_1(\phi\varpi, \phi\mu)) &= E\left(\sup_{\nu\in(0,T]} D_L(\phi\varpi_\nu, \phi\mu_\nu)\right) \\ &\leq E\left(cm\mathscr{K}\sup_{\nu\in(0,\mathfrak{T}]}\int_0^\nu D_L(\varpi_\nu, \mu_\nu)\right) \\ &\leq cm\mathscr{K}\mathfrak{T}E(H_1(\varpi, \mu)). \end{aligned} \tag{32}$$

A contraction mapping is ϕ according to hypothesis (J_2). The Banach fixed point theorem equation (23) has unique fixed point $x_\nu \in \mathscr{C}([0, \mathfrak{T}] \times (\Theta_1, \mathbf{P^m}, \mathscr{C}_r), U)$.

3.1. Exact Controllability. In this section, we will study exact controllability for differential equation in the context of Caputo operator (3). We investigate a solution for equation (3) x in $V(\subset \mathbf{E_N})$.

$$\begin{cases} \phi\varpi_\nu = \nu^{\beta-1}P_\beta(\nu)(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, u_s))ds + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\left[Au_s + \int_0^\nu f\left(s, u_s, \int_0^s \mathscr{K}(s, u_s)d\mathscr{C}_s\right) + Bu_s\mathscr{C}u_s\right]ds, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), \end{cases} \tag{33}$$

where $\mathbf{S}(\nu)$ continuous, such that $\mathbf{S}(0) = I = \mathbf{S}'(0)$ and $|\mathbf{S}(\nu)| \le c, c > 0, \nu \in [0, \mathfrak{T}]$. The term of controllability is defined for Caputo fuzzy differential equations.

Definition 19. Equation (3) is called a controllable on $[0, \mathfrak{T}]$, if there is control $u_\nu \in V$ for every $u_0 \in E_N$ where the solution u of (3) satisfies the condition $u_\nu = u^{-1} \in U$, a.s. ζ, that is, $[u_\nu]^\beta = [u^1]^\beta$.

Given fuzzy $\tilde{G} : \tilde{P}(\mathbf{R^m}) \longrightarrow U$ mapping such that

$$\tilde{G}^\beta(\nu) = \begin{cases} \displaystyle\int_0^T (\nu - s)^{\beta-1} P_\beta(\nu - s) B\nu_s \mathscr{C}\nu_s ds, & \wp \subset \bar{\Gamma}_j, \\ 0, & otherwise, \end{cases} \quad (34)$$

where $\bar{\Gamma}_x$ is closure of support x and a nonempty fuzzy subset $\tilde{P}(\mathbf{R^m})$ of $\mathbf{R^m}$.

After that, there is a $\tilde{G}_i^\beta (i = m, n)$,

$$\begin{aligned} \tilde{G}_m^\beta(\wp_m) &= \int_0^T (\nu - s)^{q-1} P_m^\beta(\nu - s) B(\wp_s)_m \mathscr{C}(\wp_s)_m ds, (\wp_s)_m \in \left[(\wp_s)_m^\beta, (\wp_s)^1\right], \\ \tilde{G}_n^\beta(\wp_n) &= \int_0^T (\nu - s)^{q-1} P_n^\beta(\nu - s) B(\wp_s)_n \mathscr{C}(\wp_s)_n ds, (\wp_s)_n \in \left[(\wp_s)^1, (\wp_s)_n^\beta\right]. \end{aligned} \quad (35)$$

We assume that $\tilde{G}_m^\beta, \tilde{G}_n^\beta$ are bijective functions. A β-level set of x_s can be presented as below:

$$\begin{aligned} [x_s]^\beta &= \left[(x_s)_m^\beta, (x_s)_n^\beta\right] \\ &= \Bigg[\left(\tilde{G}_m^\beta\right)^{-1} \Big\{ \left(u^1\right)_m^\beta - \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_m^\beta \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{im}^\beta(s, u_s)\Big) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s + \int_0^\nu f_m^\beta\left(s, u_s, \int_0^s \mathscr{K}_m^\beta(s, u_s) d\mathscr{C}_s\right)\right. \\ &\quad \left. + B_m^\beta(u_s) \mathscr{C}_m^\beta(u_s) ds\right] ds\Big\}, \left(\tilde{G}_n^\beta\right)^{-1} \\ &\quad \cdot \Big\{ -\left(u^1\right)_n^\beta - \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_n^\beta \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{\text{in}}^\beta(s, u_s)\Big) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \\ &\quad \cdot \left[A u_s + \int_0^\nu f_n^\beta\left(s, u_s, \int_0^s \mathscr{K}_n^\beta(s, u_s) d\mathscr{C}_s\right) + B_n^\beta(u_s) \mathscr{C}_n^\beta(u_s) ds\right]\Big\}\Bigg]. \end{aligned} \quad (36)$$

This expression is substituted into (33) to get the β-level of x_ν.

$$\begin{aligned} [u_\nu]^\beta &= \Big[\nu^{\beta-1} P_\beta(\nu) \big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right) \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_i(s, u_s)\Big) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s + f\left(s, u_s, \int_0^s \mathscr{K}(s, u_s) d\mathscr{C}_s\right)\right. \\ &\quad + B u_s \mathscr{C} u_s] ds]^\beta \\ &= \Big[\nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_m^\beta \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{mi}^\beta(s, u_s)\Big) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \\ &\quad \cdot \left[A u_s + f_m^\beta\left(s, u_s, \int_0^s \mathscr{K}_m^\beta(s, u_s) d\mathscr{C}_s\right)\right] \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) B \left(\tilde{G}_m^\beta\right)^{-1} \\ &\quad \cdot \Big\{\left(u^1\right)_m^\beta - \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + \mathfrak{g}\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_m^\beta \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{mi}^\beta(s, u_s)\Big) ds \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s - f_m^\beta\left(s, u_s, \int_0^s \mathscr{K}_m^\beta(s, u_s) d\mathscr{C}_s\right)\right. \\ &\quad \left. - B u_s \mathscr{C} u_s\right]\Big\} ds, \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_n^\beta \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{ni}^\beta(s, u_s)) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s + f_n^\beta\left(s, u_s, \int_0^s \mathscr{K}_n^\beta(s, u_s) d\mathscr{C}_s\right)\right] \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) B \left(\tilde{G}_n^\beta\right)^{-1} \\ &\quad \cdot \Big\{\left(u^1\right)_n^\beta - \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_n^\beta \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{ni}^\beta(s, u_s)) ds \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \\ &\quad \cdot \left[A u_s - f_n^\beta\left(s, u_s, \int_0^s \mathscr{K}_n^\beta(s, u_s) d\mathscr{C}_s\right) - B u_s \mathscr{C} u_s\right]\Big\} ds\Big] \\ &= \Big[\nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_m^\beta \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{mi}^\beta(s, u_s)) ds \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s + f_m^\beta\left(s, u_s, \int_0^s \mathscr{K}_m^\beta(s, u_s) d\mathscr{C}_s\right)\right] \\ &\quad + \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \tilde{G}_n^\beta \left(\tilde{G}_n^\beta\right)^{-1} \\ &\quad \cdot \Big\{\left(u^1\right)_m^\beta - \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_m^\beta \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \mathfrak{g}_{mi}^\beta(s, u_s)) ds \\ &\quad - \int_0^\nu (\nu - s)^{\beta-1} P_\beta(\nu - s) \left[A u_s - f_m^\beta\left(s, u_s, \int_0^s \mathscr{K}_m^\beta(s, u_s) d\mathscr{C}_s\right)\right. \\ &\quad - B u_s \mathscr{C} u_s]\} ds, \nu^{\beta-1} P_\beta(\nu) \Big(u_0 + h\left(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right)_n^\beta \end{aligned}$$

$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^{\beta}(s,u_s))ds$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f_n^{\beta}\left(s,u_s,\int_0^s\mathcal{K}_n^{\beta}(s,u_s)d\mathcal{C}_s\right)\right]$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\tilde{G}_n^{\beta}\left(\tilde{G}_n^{\beta}\right)^{-1}$$
$$\cdot\left\{\left(u^1\right)_n^{\beta}-\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right)_n^{\beta}\right.$$
$$-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_{ni}^{\beta}(s,u_s))ds$$
$$-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)$$
$$\left.\cdot\left[Au_s-f_n^{\beta}\left(s,u_s,\int_0^s\mathcal{K}_n^{\beta}(s,u_s)d\mathcal{C}_s\right)Bu_s\mathcal{C}u_s\right]\}ds\right]$$
$$=\left[\left(u^1\right)_m^{\beta},\left(u^1\right)_n^{\beta}\right]=\left[u^1\right]^{\alpha}. \tag{37}$$

Hence, this control x_ν satisfies $u_\nu=u^1$, a.s. ζ.
We now set

$$\psi u_\nu=\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right.$$
$$+\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds+\int_0^{\nu}(\nu-s)^{\beta-1}$$
$$\cdot P_\beta(\nu-s)\left[Au_s+f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)\right]$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}\right)^{-1}\left\{\left(u^1\right)-\nu^{\beta-1}P_\beta(\nu)\right. \tag{38}$$
$$\cdot\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\right.$$
$$\left.\cdot\,\mathfrak{g}_i(s,u_s)\right)ds-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f\right.$$
$$\left.\left.\cdot\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)-Bu_s\mathcal{C}u_s\right]\right\}ds.$$

Fuzzy mappings $\tilde{G}^{-1}$ holds the above equation.

$$d_L\left([\psi u_\nu]^{\beta},[\psi\nu_\nu]^{\beta}\right]\Big)$$
$$=d_L\left(\left[\nu^{\beta-1}P_\beta(\nu)\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)\right.\right.\right.$$
$$+\left.\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s+f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)\right]$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\left(\tilde{G}\right)^{-1}\left\{\left(u^1\right)-\nu^{\beta-1}P_\beta(\nu)\right.$$
$$\cdot\left(u_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,u(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds$$
$$\left.-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au_s-f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)-Bu_s\mathcal{C}u_s\right]\right\}ds\Big]^{\beta},$$

$$\nu^{\beta-1}P_\beta(\nu)\left(\nu_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,\nu(.)\right)+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)\right)ds$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathcal{K}(s,\nu_s)d\mathcal{C}_s\right)\right]$$
$$+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\left\{\left(\nu^1\right)-\nu^{\beta-1}P_\beta(\nu)\right.$$
$$\cdot\left(\nu_0+h\left(\nu_1,\nu_2,\cdots,\nu_p,\nu(.)\right)-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)\right)ds$$
$$\left.\left.-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s-f\left(s,\nu_s,\int_0^s\mathcal{K}(s,\nu_s)dC_s\right)-B\nu_s\mathcal{C}\nu_s\right]\right\}ds\right)$$
$$\leq d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)ds+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\right.\right.\right.$$
$$\left.\left.\cdot\left[Au_s+f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)\right]\right]^{\beta},\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds\right.\right.$$
$$\left.\left.\left.+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathcal{K}(s,\nu_s)d\mathcal{C}_\tau(s)\right)\right]\right]^{\beta}\right)$$
$$+d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\times\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\right.\right.$$
$$\left.-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_\tau(s)ds\right],$$
$$\cdot\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds+\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)$$
$$\left.\cdot\left[A\nu_s+f\left(s,\nu_s,\int_0^s\mathcal{K}(s,\nu_s)d\mathcal{C}_s\right)\right]\right]^{\beta}\Big)$$
$$+d_L\left(\left[\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)B\tilde{G}^{-1}\times\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)\right)ds\right.\right.$$
$$\left.\left.-\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_\tau(s)ds\right)\right]^{\beta}\right)$$
$$\leq cm\mathcal{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds$$
$$+d_L\left(\left[\tilde{G}\tilde{G}^{-1}\left[\int_0^{\nu}(\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s,u_s)ds\right.\right.\right.$$
$$\left.\left.+\int_0^{\nu}f\left(s,u_s,\int_0^s\mathcal{K}(s,u_s)d\mathcal{C}_s\right)\right]\right]^{\beta}\left[\tilde{G}\tilde{G}^{-1}\left[\int_0^{\nu}(\nu-s)^{\beta-1}\right.\right.$$
$$\left.\left.\left.\cdot P_\beta(\nu-s)\mathfrak{g}_i(s,\nu_s)ds+\int_0^{\nu}f\left(s,\nu_s,\int_0^s\mathcal{K}(s,\nu_s)d\mathcal{C}_s\right)\right]\right]^{\beta}\right)$$
$$\leq cm\mathcal{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds+cm\mathcal{K}\int_0^{\nu}d_L\left([f(s,u_s)]^{\beta},[f(s,\nu_s)]^{\beta}\right)ds$$
$$\leq 2cm\mathcal{K}\int_0^{\nu}d_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds. \tag{39}$$

Theorem 20. *If Lemma 16 and hypotheses (J_1) and (J_2) are hold, then equation (3) is controllable on $[0,\mathfrak{I}]$.*

Proof. From $\mathcal{C}([0,\mathfrak{I}]\times(\Theta_1,\mathbf{P^m},U)$ to $\mathcal{C}([0,\mathfrak{I}]$, we can clearly see that ψ is continuous. We have Lemma 16 and hypotheses (J_1) and (J_2) for any given ζ with $\mathcal{C}_r\{\zeta\}>0,x_\nu$, $\wp_\nu\in\mathcal{C}([0,\mathfrak{I}]\times(\Theta_1,\mathbf{P^m},\mathcal{C}_r),U)$. □

Hence, by Lemma 11,

$$E(H_1(\psi u,\psi\nu))=E\left(\sup_{\nu\in[0,\mathfrak{I}]}D_L(\psi u_\nu,\psi\nu_\nu)\right)=E\left(\sup_{\nu\in[0,\mathfrak{I}]}\sup_{0<\beta\leq1}D_L\left(|\psi u_\nu|^{\beta},|\psi\nu_\nu|^{\beta}\right)ds\right)$$
$$\leq E\left(\sup_{\nu\in[0,\mathfrak{I}]}\sup_{0<\beta\leq1}2cm\mathcal{K}\int_0^{\nu}D_L\left([u_s]^{\beta},[\nu_s]^{\beta}\right)ds\right)$$
$$\leq E\left(\sup_{\nu\in[0,\mathfrak{I}]}2cm\mathcal{K}\int_0^{\nu}D_L(u_s,\nu_s)ds\right)\leq 2cm\mathcal{K}\mathfrak{I}F(H_1(u,\nu)). \tag{40}$$

As a consequence, $(2\text{cm}\mathscr{K}\mathfrak{J}) < 1$ is a Ã, sufficient $\mathfrak{J}$. As a result, ψ stands for contraction. The Banach fixed point theorem is now being applied to show that (33) has a single fixed point. $[0, \mathfrak{J}]$ can be used to control (3).

Example 1. We investigate FFDE in credibility space:

$$\begin{cases} {}_0^{\mathscr{C}}D_\nu^\beta u(\nu,\zeta) = \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu,\zeta) + \int_0^\nu f\left((\nu, u(\nu,\zeta)) + \int_0^s k(s, u(\nu,\zeta))\right) d\mathscr{C}_\nu + Bx(\nu)\mathscr{C}x(\nu)d\nu, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.)), \qquad \in \boldsymbol{E}_N, \end{cases}, \tag{41}$$

where states consider values from $U(\subset E_N)$ and space V $(\subset E_N)$ two bounded spaces. The set of all, upper semicontinuously convex, fuzzy numbers on $\mathbf{R}^\mathbf{m}$ is $\mathbf{E_N}$ and $(\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r)$ denotes credibility space.

The state function $u : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r) \longrightarrow U$ is fuzzy coefficient. Fuzzy process $f : [0, \mathfrak{J}] \times U \longrightarrow U$. $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r) \longrightarrow V$ is a regular fuzzy function, $x : [0, \mathfrak{J}] \times (\Theta_1, \mathbf{P}^\mathbf{m}, \mathscr{C}_r) \longrightarrow V$ is a control function, and B is a V to U linear bounded operator. $u_0 \in \mathbf{E_N}$ is an initial value, and $\mathscr{C}_\nu$ is standard Liu process.

Assume $f(\nu, u_\nu) = \tilde{2}\nu u_\nu$, $\mathbf{S}^{-1}(\nu) = e^{-\tilde{2}\nu}$, defining $w_\nu = \mathbf{S}^{-1}(\nu)u_\nu$. Then, the equations of balance become

$$\begin{cases} u_\nu = \nu^{\beta-1}P_\beta(\nu)(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\left[Au(s,\zeta) + \int_0^\nu f\left(s, u(s,\zeta), \int_0^s \mathscr{K}(s, u(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\right]ds, \\ u(0) = u_0 + h(\nu_1, \nu_2, \cdots, \nu_i, u(.) \in \mathbf{E_N}. \end{cases} \tag{42}$$

Therefore, Lemma 17 is satisfied.

$[2]^\beta = [\beta+1, 3-\beta]$ is the β-level, set of fuzzy, number $\tilde{2}$, for all $\beta \in (0,1)$. β-level set of $f(\nu, u_\nu)$ is

$$[f(\nu, u_\nu)]^\beta = \nu\left[(\beta+1)(u_\nu)_m^\beta, (3-\beta)(u_\nu)_m^\beta\right]. \tag{43}$$

Further, we have

$$\begin{aligned} d_L&\left([f(\nu,u_\nu)]^\beta, [f(\nu,\nu_\nu)]^\beta\right) \\ &= d_L\left(\nu\left[(\beta+1)(u_\nu)_m^\beta, (3-\beta)(u_\nu)_n^\beta\right], \nu \right. \\ &\quad \left. \cdot\left[(\beta+1)_m^\beta, (3-\beta)(\nu_\nu)_n^\beta\right]\right) \\ &= \nu \max\left\{(\beta+1)\left|(u_\nu)_m^\beta - (\nu_\nu)_m^\beta\right|,\right. \\ &\quad \left.\cdot(3-\beta)\left|(u_\nu)_n^\beta - (\nu_\nu)_n^\beta\right|\right\} \\ &\le 3\mathfrak{J} \max\left\{\left|(u_\nu)_m^\beta - (\nu_\nu)_m^\beta\right|, \left|(u_\nu)_n^\beta - (\nu_\nu)_n^\beta\right|\right\} \\ &= md_L\left([u_\nu]^\beta, [\nu_\nu]^\beta\right), \end{aligned} \tag{44}$$

where $m = 3\mathfrak{J}$ satisfies an inequality in the (J1) and (J_2) hypotheses. All conditions given in Theorem 18 are fulfilled. Assume that $\tilde{1}$ is the initial value for u_0. The plan set $u^1 = \tilde{2}$. $\tilde{1}$ is $[\tilde{1}] = [\beta-1, 1-\beta]$, $\beta \in (0,1)$ is β-level set of fuzzy numbers $\tilde{1}$. The x_s of (41)'s β-level set is presented.

$$\begin{aligned} [x_s] &= \left[(x_s)_m^\beta, (x_s)_n^\beta\right] \\ &= \left[\left(\tilde{G}_m^\beta\right)^{-1}\left\{(\beta+1) - S_m^\beta(\mathfrak{J}-s)(\beta-1)\right.\right. \\ &\quad \left. - \int_0^{\mathfrak{J}} S_m^\beta(\mathfrak{J}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s\right\}, \left(\tilde{G}_n^\beta\right)^{-1} \\ &\quad \cdot\left\{(3-\beta) - S_n^\beta(\mathfrak{J})(3-\beta)\right. \\ &\quad \left.\left. - \int_0^{\mathfrak{J}} S_n^\beta(\mathfrak{J}-s)s(3-\beta)(u_s)_n^\beta d\mathscr{C}_s\right\}\right]. \end{aligned} \tag{45}$$

This expression is then substituted into (42) to get the β-level of u_ν:

$$
\begin{aligned}
[u_\nu]^\beta = \Bigg[& S_m^\beta(\mathfrak{T})(\beta-1) + \int_0^{\mathfrak{T}} S_m^\beta(\mathfrak{T}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s \\
& + \int_0^{\mathfrak{T}} S_m^\beta(\mathfrak{T}-s)B\left(\tilde{G}_m^\beta\right)^{-1}\Big\{(\beta+1) - S_m^\beta(\mathfrak{T})(\beta-1) \\
& - \int_0^{\mathfrak{T}} S_m^\beta(\mathfrak{T}-s)s(\beta+1)(u_s)_m^\beta d\mathscr{C}_s\Big\} ds, S_n^\beta(\mathfrak{T})(1-\beta) \\
& + \int_0^{\mathfrak{T}} S_n^\beta(\mathfrak{T}-s)s(1-\beta)(u_s)_n^\beta d\mathscr{C}_s \\
& + \int_0^{\mathfrak{T}} S_n^\beta(\mathfrak{T}-s)B\left(\tilde{G}_n^\beta\right)^{-1}\Big\{(3-\beta) - S_r^\beta(\mathfrak{T})(1-\beta) \\
& - \int_0^{\mathfrak{T}} S_n^\beta(\mathfrak{T}-s)s(3-\beta)(u_s)_n^\beta d\mathscr{C}_s\Big\} ds\Bigg] \\
= {} & [(\beta+1), (3,-\beta)] = \left[\tilde{2}\right]^\beta.
\end{aligned} \tag{46}
$$

Following that, conditions in Theorem 20 have been fulfilled. As a result, (41) on $[0, T]$ can be controlled.

4. Definition of Stability in Credibility

We shall provide a concept of credibility stability for FFDEs driven by the Liu process in this part.

Definition 21. The FDE 1 is said to be stability in credibility if for, any two, solutions u_ν and v_ν corresponding to different initial values $u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)$ and $v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)$, we have

$$
\lim_{|u_0 - v_0| \longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} = 1, \text{for } all\ \nu \geq 0, \tag{47}
$$

where ε is any given number and $\varepsilon > 0$.

Example 2. Take the FFDE to better understand the concept of credibility stability.

$$
\begin{aligned}
u_\nu = {} & \nu^{\beta-1}P_\beta(\nu)(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\Big[Au(s,\zeta) \\
& + \int_0^\nu f\left(s, u(s,\zeta), \int_0^s \mathscr{K}(s, u(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Big]ds,
\end{aligned}
$$

$$
\begin{aligned}
v_\nu = {} & \nu^{\beta-1}P_\beta(\nu)(v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)) \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\mathfrak{g}_i(s, x(s))ds \\
& + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)\Big[Av(s,\zeta) \\
& + \int_0^\nu f\left(s, v(s,\zeta), \int_0^s \mathscr{K}(s, v(s,\zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Big]ds,
\end{aligned} \tag{48}
$$

respectively. Then, we have

$$
\begin{aligned}
|u_\nu - v_\nu| = {} & \big|\left(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right) \\
& - \left(v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)\right)\big|.
\end{aligned} \tag{49}
$$

Deduce to, for any given $\varepsilon > 0$, we always have

$$
\begin{aligned}
& \lim_{\left|\left(u_0 + h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\right) - \left(v_0 + h(\nu_1,\nu_2,\cdots,\nu_p,v(.))\right)\right| \longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} \\
& = \lim_{\left|\left(u_0 + h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\right) - \left(v_0 + h(\nu_1,\nu_2,\cdots,\nu_p,v(.))\right)\right| \longrightarrow 0} \mathscr{C}_r\left\{\left|\left(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)\right) - \left(v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)\right)\right| < \varepsilon\right\} = 1, \forall \nu \geq 0.
\end{aligned} \tag{50}
$$

As a result, the credibility of FFDE is stable.

Definition 22. The n-dimensional FDE 1 is called stable in credibility, if for any two solutions u_ν and v_ν corresponding to different initial values $u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)$ and $v_0 + h(\nu_1, \nu_2, \cdots, \nu_p, v(.)$, we have

$$
\lim_{\left\|\left(u_0 + h(\nu_1,\nu_2,\cdots,\nu_p,u(.))\right) - \left(v_0 + h(\nu_1,\nu_2,\cdots,\nu_p,v(.))\right)\right\| \longrightarrow 0} \mathscr{C}_r\{|u_\nu - v_\nu| < \varepsilon\} = 1, \forall \nu \geq 0. \tag{51}
$$

Example 3. Take an m-dimensional FFDE:

$$
\begin{aligned}
{}_0^{\mathscr{C}}D_\nu^\beta u(\nu,\zeta) = {} & \mathfrak{g}_i(\nu, u(\nu)) + Au(\nu,\zeta) \\
& + \int_0^\nu f\left((\nu, u(\nu,\zeta)), \int_0^s k(s, u(\nu,\zeta))\right)d\mathscr{C}_\nu \\
& + Bx(\nu)\mathscr{C}x(\nu)d\nu.
\end{aligned} \tag{52}
$$

The two solutions corresponding to different initial values are

$$\begin{aligned} u_\nu = {} & \nu^{\beta-1}P_\beta(\nu)(u_0 + \mathfrak{g}(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)g_i(s, x(s))ds \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)[Au(s, \zeta) \\ & + \int_0^\nu f\left(s, u(s, \zeta), \int_0^s \mathscr{K}(s, u(s, \zeta))d\mathscr{C}_s\right) + B(s)\mathscr{C}(s)\Big] ds \end{aligned}$$

$$\begin{aligned} \nu_\nu = {} & \nu^{\beta-1}P_\beta(\nu)(\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.)) \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)\mathfrak{g}_i(s, x(s))ds \\ & + \int_0^\nu (\nu - s)^{\beta-1}P_\beta(\nu - s)[A\nu(s, \zeta) \\ & + \int_0^\nu f\left(s, \nu(s, \zeta), \int_0^s \mathscr{K}(s, \nu(s, \zeta))d\mathscr{C}_s\right) \\ & + B(s)\mathscr{C}(s)]ds, \end{aligned} \tag{53}$$

respectively. Then, we have

$$\|u_\nu - \nu_\nu\| = \left\|(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) - (\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))\right\|. \tag{54}$$

As a result, we always have

$$\begin{aligned} & \lim_{(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0} \mathscr{C}_r\{|u_\nu - \nu_\nu| < \varepsilon\} \\ & = \lim_{|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0} \mathscr{C}_r\left\{\left|(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) - (\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))\right| < \varepsilon\right\} = 1, \forall \nu \geq 0. \end{aligned} \tag{55}$$

Thus, m-dimensional FFDE is stability in credibility.

Note that some fuzzy differential equations driven by the Liu process are not stable in credibility. It will be demonstrated in the following example.

5. Theorems of Stability in Credibility

In this part, we will discuss the necessary criteria for a FFDE driven by the Liu process to achieve credibility stability.

Theorem 23. *Assume the FFDE 1 for each initial value has a unique solution. Then, it is stable in credibility space, if coefficients $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ satisfy strongly Lipschitz condition*

$$\begin{aligned} & D(f(\nu, u) - f(\nu, \nu)) + (\mathfrak{g}(\nu, u) + \mathfrak{g}(\nu, \nu)) \\ & \leq L(\nu)D(u - \nu), \forall u, \nu \in \mathbf{R}^\mathbf{m}, \nu \geq 0, \end{aligned} \tag{56}$$

for some integrable function $L(\nu)$ on $[0, +\infty)$.

Proof. Let u_ν and ν_ν be two solutions corresponding to differential initial values $(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.))$ and $(\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))$, respectively. Then, for each $\vartheta \in \Theta_1$,

$$\begin{aligned} & D(u_\nu - \nu_\nu) \\ & = D(f(\nu, u_t)d\nu - f(\nu, \nu_\nu)d\nu + D(\mathfrak{g}(\nu, u_\nu)d\mathscr{C}_\nu - g(\nu, \nu_\nu)d\mathscr{C}_\nu) \\ & = D((f(\nu, u_\nu) - f(\nu, \nu_\nu))d\nu + D((g(\nu, u_\nu) - \mathfrak{g}(\nu, \nu_\nu))d\mathscr{C}_\nu) \\ & \leq D((f(\nu, u_\nu) - f(\nu, \nu_\nu))d\nu) + D((\mathfrak{g}(\nu, u_\nu) - g(\nu, \nu_\nu))d\mathscr{C}_\nu) \\ & \leq L(\nu)D(u_\nu - \nu_\nu)d\nu + DL(t)(u_t - \nu_\nu)d\mathscr{C}_\nu \\ & \leq L(\nu)D(u_t - \nu_\nu)d\nu + DL(\nu)|\mathscr{K}(\vartheta)|(u_\nu - \nu_\nu)d\nu \\ & = L(t)(1 + |K(\vartheta)|)D(u(\nu) - \nu(\nu)), \end{aligned} \tag{57}$$

where $\mathscr{K}(\vartheta)$ is the Lipschitz constant of the Liu process. When we take integral on both sides of equation (57),

$$\begin{aligned} D(u_\nu - \nu_\nu) \leq {} & D\big((u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ & - (\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))\big) \exp \\ & \cdot \left(1 + |\mathscr{K}(\vartheta)| \int_0^\nu L(s)ds\right). \end{aligned} \tag{58}$$

□

For any given $\varepsilon > 0$, we always have

$$\begin{aligned} & \mathscr{C}_r\{|u_\nu - \nu_\nu| < \varepsilon\} \\ & \geq \big\{\big|(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ & \quad - (\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))\big| \exp\left(1 + |\mathscr{K}(\vartheta)| \int_0^\nu L(s)ds\right) < \varepsilon\big\}. \end{aligned} \tag{59}$$

Since

$$\begin{aligned} & \mathscr{C}_r\big\{\big|(u_0 + h(\nu_1, \nu_2, \cdots, \nu_p, u(.)) \\ & - (\nu_0 + h(\nu_1, \nu_2, \cdots, \nu_p, \nu(.))\big| \exp \\ & \cdot \left(1 + |\mathscr{K}(\vartheta)| \int_0^\nu L(s)ds\right) < \varepsilon\big\} \longrightarrow 1, \end{aligned} \tag{60}$$

as $|u_0 - \nu_0| \longrightarrow 0$, we obtain

$$\lim_{|(u_0+h(\nu_1,\nu_2,\cdots,\nu_p,u(.))-(\nu_0+h(\nu_1,\nu_2,\cdots,\nu_p,\nu(.))|\longrightarrow 0} \mathscr{C}_r\{|u_\nu - \nu_\nu| < \varepsilon\} = 1. \tag{61}$$

Hence, the FFDE is stability in credibility. If it is not easy to determine whether or not $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ satisfy strong

Lipschitz condition, the following corollary can be used to determine whether the FFDE is stable in credibility space.

Corollary 24. *Assume $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ be bounded real value functions on $[0, +\infty)$. If $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$ have derivatives with respect to u and satisfy*

$$\left|f'_u(\nu, u)\right| + \left|\mathfrak{g}'_u(\nu, u)\right| \le L(\nu), \forall \ge 0, \tag{62}$$

for some integrable function $L(\nu)$ on $[0, +\infty)$, then FFDE 1 is stability in credibility.

Proof. For the bounded real valued functions $f(\nu, u)$ and $\mathfrak{g}(\nu, u)$,

$$|f(\nu, u)| + |\mathfrak{g}(\nu, u)| < \mathscr{K}(1 + |u|), \tag{63}$$

where $\mathscr{K}$ is constant which satisfy $|f(\nu, u)| + |\mathfrak{g}(\nu, u)| < \mathscr{K}$. We can derive from the mean value theorem that

$$\begin{aligned} &\left|f\left(\nu, u'\right) - f\left(\nu, u''\right)\right| + \left|g\left(\nu, u'\right) - \mathfrak{g}\left(\nu, u''\right)\right| \\ &\quad = f'_u(\nu, \xi)\left|u' - u''\right| + \mathfrak{g}'_u(\nu, \eta)\left|u' - u''\right| \\ &\quad \le L(\nu)\left|u' - u''\right| + L(\nu)\left|u' - u''\right| = 2L(\nu)\left|u' - u''\right|, \end{aligned} \tag{64}$$

where $\xi, \eta \in (u' - u)$ existence-uniqueness theorem demonstrates that FFDE has a unique solution. We can deduce from Theorem 23 that FFDE is stable in credibility. Different from Theorem 23 and Corollary 24, we have below corollary when FFDE is general linear FFDE driven by the Liu process. □

Corollary 25. *Suppose that $u_{1\nu}$, $u_{2\nu}$, $\nu_{1\nu}$, and $\nu_{2\nu}$ are bounded functions, with respect to ν on $[0, +\infty)$. If $u_{1\nu}$ and $\nu_{1\nu}$ are integrable, on $[0, +\infty)$, then linear FDE driven by Liu process*

$$du_\nu = (u_{1\nu}u_\nu + u_{2\nu})d\nu + (\nu_{1\nu}u_\nu + \nu_{2\nu})d\mathscr{C}_\nu, \tag{65}$$

is stability in credibility.

Proof. For the linear FFDE 7, we have $f(\nu, x) = u_{1\nu}x + u_{2\nu}$ and $\mathfrak{g}(\nu, x) = \nu_{1\nu}x + \nu_{2\nu}$, since

$$\begin{aligned} &|u_{1\nu}u_\nu + u_{2\nu}| + |\nu_{1\nu}\nu_\nu + \nu_{2\nu}| \\ &\quad \le |u_{1\nu}||u_\nu| + |u_{2\nu}| + |\nu_{1\nu}||u_\nu| + |\nu_{2\nu}| \\ &\quad < \mathscr{K}|u_\nu| + \mathscr{K} + \mathscr{K}|u_\nu| + \mathscr{K} = 2\mathscr{K}(|u_\nu| + 1), \\ &|(u_{1\nu}u_\nu + u_{2\nu}) - (u_{1\nu}\nu_\nu + u_{2\nu})| + |(\nu_{1\nu}u_\nu + \nu_{2\nu}) - (\nu_{1\nu}\nu_\nu + \nu_{2\nu})| \\ &\quad = |u_{1\nu}(u_\nu - \nu_\nu)| + |\nu_{1\nu}(u_\nu + \nu_\nu)| \\ &\quad \le |u_{1\nu}||u_\nu - \nu_\nu| + |\nu_{1\nu}||u_\nu + \nu_\nu| \\ &\quad = (|u_{1\nu}| + |\nu_{1\nu}|)|(u_\nu - \nu_\nu)| \le 2\mathscr{K}(u_\nu - \nu_\nu), \end{aligned} \tag{66}$$

where $\mathscr{K}$ is a constant which make $u_{1\nu} < \mathscr{K}, u_{2\nu} < \mathscr{K}$, $\nu_{1\nu} < \mathscr{K}, \nu_{2\nu} < \mathscr{K}$ hold. The existence-uniqueness theorem shows that FDE 7 has a unique solution. Since $L(\nu) = |u_{1\nu}| + |\nu_{1\nu}|$ is integrable function on $[0, +\infty)$, from Theorem 23, the credibility of FFDE can be determined. □

According to Definition 22, Theorem 23 can be used to n-dimensional FFDEs driven by the Liu process.

Theorem 26. *Assume that each initial value of the n-dimensional FFDE 1 has a unique solution. If coefficients $f(\nu, u)$ and $g(\nu, u)$ satisfy Lipschitz's strong condition,then it is stable in credibility space:*

$$\begin{aligned} &\|f(\nu, u) - f(\nu, \nu)\| + \|\mathfrak{g}(\nu, u) - \mathfrak{g}(\nu, \nu)\| \\ &\quad \le L(\nu)\|u - \nu\|, for \forall u, \nu \in \mathbf{R^m}, \nu \ge 0, \end{aligned} \tag{67}$$

for some integrable function $L(\nu)$ on $[0, +\infty)$.

6. Conclusion

Accurate controllability for FFDEs can be used as a standard when analyzing controllability for semilinear integro-differential equations in the credibility space and fuzzy delay integro-differential equations. Therefore, the research's theoretical conclusions can be applied to construct stochastic extensions on credibility space. The FFDEs driven by the Liu process have an important role in both theory and practice as a technique for dealing with dynamic systems in a fuzzy environment. There have been some proposed stability approaches for FFDEs driven by the Liu process up until now. This is a rewarding field with numerous research projects that can lead to a variety of applications and theories. We hope to learn more about fuzzy fractional evolution problems in future projects. We can discover uniqueness and existence with uncertainty using the Caputo derivative. Future work could include expanding on the mission concept, including observability, and generalizing other activities. This is an interesting area with a lot of study going on that could lead to a lot of different applications and theories. This is a path in which we intend to invest significant resources.

Data Availability

Data is original and references are given where required.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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