

Research Article

Bernardi Integral Operator and Its Application to the Fourth Hankel Determinant

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In recent years, the theory of operators got the attention of many authors due to its applications in different fields of sciences and engineering. In this paper, making use of the Bernardi integral operator, we define a new class of starlike functions associated with the sine functions. For our new function class, extended Bernardi's theorem is studied, and the upper bounds for the fourth Hankel determinant are determined.

1. Introduction

Let \mathcal{H} be the family of holomorphic (or analytic) functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathfrak{A}_n \subset \mathcal{H}$ such that $f \in \mathfrak{A}_n$ has the series representation:

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j. \quad (1)$$

Let \mathcal{S} be the subfamily of $\mathfrak{A}_1 \equiv \mathfrak{A}$ containing univalent functions in \mathfrak{D} . Despite the fact that function theory was first proposed in 1851, it only became a viable research topic in 1916. Many academics attempted to prove or refute the conjecture $|a_n| \leq n$, which was initially proven by de Branges in 1985, and as a result, they identified multiple subfamilies of the class \mathcal{S} that are connected to various image domains. The starlike, convex, and close-to-convex functions are among those families which are defined by

$$\begin{aligned} \mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathfrak{D} \right\}, \\ \mathcal{C} &= \left\{ f \in \mathcal{S} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in \mathfrak{D} \right\}, \\ \mathcal{R} &= \left\{ f \in \mathcal{S} : \Re \left(\frac{zf'(z)}{g(z)} \right) > 0, g(z) \in \mathcal{S}^*, z \in \mathfrak{D} \right\}. \end{aligned} \quad (2)$$

Let f and g be the two analytic functions in \mathfrak{D} ; then, f is subordinate to g , denoted by

$$f(z) \prec g(z), (z \in \mathfrak{D}), \quad (3)$$

if there exists a Schwarz function $w(z)$ satisfying the conditions:

$$w(0) = 0, |w(z)| < 1, (z \in \mathfrak{D}), \quad (4)$$

such that

$$f(z) = g(w(z)), (z \in \mathfrak{D}). \tag{5}$$

Let $p(z) = 1 + c_1z + c_2z^2 + \dots$ be an analytic and regular function in \mathfrak{D} with $p(0) = 1, \Re p(z) > 0$, satisfying the criteria:

$$p(z) < \left(\frac{1 + Az}{1 + Bz} \right), -1 \leq B < A, -1 < A < 1. \tag{6}$$

Then, this function is referred to as the Janowski function which is represented by $\mathcal{P}(A, B)$. Geometrically, $p(z) \in \mathcal{P}(A, B) \iff p(0) = 1$, and $p(\mathfrak{U}_d)$ is inside the domain specified by

$$\Omega(A, B) = \left\{ \omega : \left| \omega - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}, \tag{7}$$

having diameter end points:

$$\frac{1 - A}{1 - B} = p(-1), \frac{1 + A}{1 + B} = p(1). \tag{8}$$

Let $\mathcal{S}^*(A, B)$ be the class of functions $\kappa(z)$, where $\kappa(0) = 0 = \kappa'(0) - 1$ are holomorphic in \mathfrak{U}_d and meet the following requirements:

$$\kappa(z) \in \mathcal{S}^*(A, B) \iff \frac{z\kappa'(z)}{\kappa(z)} \in \mathcal{P}(A, B). \tag{9}$$

Distinct subclasses of analytic functions associated with various image domains have been introduced by many scholars. For example, Cho et al. [1] and Dziok et al. [2] discussed various properties of starlike functions related to Bell numbers and a shell-like curve connected with Fibonacci numbers, respectively. Similarly, Kumar and Ravichandran [3] and Mendiratta et al. [4] investigated subclasses of starlike functions associated with rational and exponential functions, respectively. Kanas and Raducanu [5] and Sharma et al. [6] explored some subclasses of analytic functions related to conic and cardioid domains, respectively. Raina and Sokól [7] investigated some important properties related to a certain class of starlike functions. Sokól and Stankiewicz [8] discussed radius problems of some subclasses of strongly starlike functions. Recently, Cho et al. [9] explored a family of starlike functions related to the sine function, which is defined as follows:

$$\mathcal{S}_s^* = \left\{ f \in \mathfrak{A} \sim \frac{zf'(z)}{f(z)} < 1 + \sin z \right\}. \tag{10}$$

The q th Hankel determinant for $q \geq 1$ and $n \geq 1$ of the functions f is introduced by Noonan and Thomas [10], which is given by

$$\Delta_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2(q-1)} \end{vmatrix} \quad (a_1 = 1). \tag{11}$$

The following options are provided for some special choices of n and q :

(1) For $q = 2, n = 1$,

$$\Delta_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, a_1 = 1, \tag{12}$$

is the famed Fekete-Szegő functional.

(2) For $q = 2, n = 2$,

$$\Delta_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \tag{13}$$

is the second Hankel determinant.

There are relatively few findings in the literature in connection with the Hankel determinant for functions belonging to the general family \mathcal{S} . Hayman [11] established the well-known sharp inequality:

$$|\Delta_{q,n}(f)| \leq \lambda \sqrt{n}, \tag{14}$$

where λ is the absolute constant. Similarly for the same class \mathcal{S} , it was obtained in [12] that

$$|\Delta_{2,2}(f)| \leq \lambda, 1 \leq \lambda \leq \frac{11}{3}. \tag{15}$$

For different subclasses of the set \mathcal{S} of univalent functions, the growth of $|\Delta_{q,n}(f)|$ has been estimated many times. For example, Janteng [13] investigated the sharp bounds of $\Delta_{2,2}(f)$ for the classes $\mathcal{S}^*, \mathfrak{C}$, and \mathfrak{R} as given below:

$$|\Delta_{2,2}(f)| \leq \begin{cases} 1, & f \in \mathcal{S}^*, \\ \frac{1}{8}, & f \in \mathfrak{C}, \\ \frac{4}{9}, & f \in \mathfrak{R}. \end{cases} \tag{16}$$

The sharp bound of $\Delta_{2,2}(f)$ for the class of close-to-convex functions is unknown. On the other hand, Krishna and Reddy [14] calculated a precision estimate of $\Delta_{2,2}(f)$ for the Bazilevic function class.

(3) For $q = 3, n = 1$,

$$\Delta_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \quad (17)$$

is the third Hankel determinant.

The calculations in (17) represent that estimating $|\Delta_{3,1}(f)|$ is significantly more difficult than estimating the bound of $|\Delta_{2,2}(f)|$. In the first paper of Babalola [15] on $\Delta_{3,1}(f)$, he obtained the upper bound of $|\Delta_{3,1}(f)|$ for the classes \mathcal{S}^* , \mathcal{C} , and \mathcal{R} . Later, some more contributions have been made by different authors to calculate the bounds of $|\Delta_{3,1}(f)|$ for different subclasses of analytic and univalent functions. Zaprawa [16] enhanced the results of Babalola [15] and demonstrated that

$$|\Delta_{3,1}(f)| \leq \begin{cases} 1, & f \in \mathcal{S}^*, \\ \frac{49}{540}, & f \in \mathcal{C}, \\ \frac{41}{60}, & f \in \mathcal{R}. \end{cases} \quad (18)$$

He also observed that the bounds are still not sharp.

(4) For $q = 4, n = 1$,

$$\Delta_{4,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} \quad (19)$$

is the fourth Hankel determinant.

Since $f \in \mathcal{S}$ and $a_1 = 1$, thus $\Delta_{4,1}(f) = a_7\Lambda_1 - a_6\Lambda_2 + a_5\Lambda_3 - a_4\Lambda_4$, where

$$\begin{aligned} a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) &= \Lambda_1, \\ a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2) &= \Lambda_2, \\ a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3) &= \Lambda_3, \\ a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3) &= \Lambda_4. \end{aligned} \quad (20)$$

In the last few years, many articles have been published to investigate the upper bounds for the second-order Hankel determinant $\Delta_{2,2}(f)$, third-order Hankel determinant $\Delta_{3,1}(f)$, and fourth Hankel determinant $\Delta_{4,1}(f)$. For the functions with bounded turning, Arif et al. [17, 18] estimated the bound for the fourth- and fifth-order Hankel determinants. Khan et al. [19] also addressed this issue and derived upper bounds for the third- and fourth-order Hankel determinants for a class of functions with bounded turning that are related to sine functions. For more study about the Hankel determinant, we refer to [19–29].

In geometric function theory (GFT), especially in the category of univalent functions, integral and differential operators are extremely helpful and important. Convolution of certain analytic functions has been used to introduce certain differential and integral operators. This approach is developed to facilitate further exploration of geometric features of analytic and univalent functions. Libera and Bernardi were the ones who investigated the classes of starlike, convex, and close-to-convex functions by introducing the idea of integral operators. Recently, some researchers have shown a keen interest in this field and developed various features of the integral and differential operators. Srivastava et al. [30] investigated a new family of complex-order analytic functions by using the fractional q -calculus operator. Mahmood et al. [31] looked at a group of analytic functions that were defined using q -integral operators. Using the q -analogue of the Ruscheweyh-type operator, Arif et al. [32] constructed a family of multivalent functions. Srivastava [33] presented a review on basic (or q -) calculus operators, fractional q -calculus operators, and their applications in GFT and complex analysis. This review article has been proven very helpful to investigate some new subclasses from different viewpoints and perspectives [34–40].

Inspired from the above recent developments, in this study, we investigate the inclusion of the Bernardi integral operator in the class of starlike function associated with sine function in \mathfrak{D} . The Bernardi integral operator $\mathfrak{F}(z): \mathfrak{A} \rightarrow \mathfrak{A}$ was defined by Bernardi [41], which is given by the following relation:

$$\mathfrak{F}(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} g(t) dt, \quad \gamma > -1. \quad (21)$$

In the first part of the study, we extend Bernardi’s theorem to a certain class \mathcal{S}_s^* of univalent starlike functions in \mathfrak{D} . Particularly, we prove that if $g \in \mathcal{S}_s^*$, then $\mathfrak{F}(z) \in \mathcal{S}_s^*$. In the second part of the study, we investigate the upper bounds for the fourth-order Hankel determinant $\Delta_{4,1}(f)$ with respect to the function class \mathcal{S}_s^* associated with the sine function.

2. Main Results

In order to obtain our desired results, we first need the following lemmas.

Lemma 1. *Let M and N be holomorphic functions in \mathfrak{U}_d such that N maps \mathfrak{U}_d onto many sheeted starlike regions with $M(0) = N(0) = 0$ and $M'(0) = N'(0) = 1$. If $(M'(z)/N'(z)) \in \mathcal{S}_s^*$, then $(M(z)/N(z)) \in \mathcal{S}_s^*$.*

Proof. We know that

$$\frac{M'(z)}{N'(z)} \in \mathcal{S}_s^* \iff \frac{M'(z)}{N'(z)} = 1 + \sin w(z) < 1 + \sin z. \quad (22)$$

Also, $\sigma(z) = 1 + \sin z$ maps $|z| < r$ onto the disc $|\sigma(z) - 1| < \sin(1)$. But $M'(z)/N'(z)$ takes values in the same disc, and therefore,

$$\left| \frac{M'(z)}{N'(z)} - 1 \right| < \sin(1), |z| < r, 0 < r < 1. \quad (23)$$

Choose $\Lambda(z)$ so that

$$N'(z)\Lambda(z) = M'(z) - N'(z). \quad (24)$$

Then, $|\Lambda(z)| < \sin(1)$. Fix z_0 in \mathfrak{U}_d . Let \mathcal{L} be the segment joining 0 and $N(z_0)$, which lies in one sheet of the starlike image of \mathfrak{U}_d by N . Let \mathcal{L}^{-1} be the preimage of \mathcal{L} under N . Then,

$$\begin{aligned} |M(z_0) - N(z_0)| &= \left| \int_0^{z_0} (M'(t) - N'(t)) dt \right| \\ &= \left| \int_{\mathcal{L}^{-1}} N'(t)\Lambda(t) dt \right| < \sin(1) \int_{\mathcal{L}^{-1}} |dN(t)| \\ &= \sin(1) |N(z_0)|. \end{aligned} \quad (25)$$

That is,

$$\left| \frac{M(z)}{N(z)} - 1 \right| < \sin(1). \quad (26)$$

This implies that

$$\frac{M(z)}{N(z)} < 1 + \sin z, \quad (27)$$

and hence,

$$\frac{M(z)}{N(z)} \in \mathcal{S}_s^*. \quad (28)$$

Lemma 2 (see [12]). Let $M(z)$ and $N(z)$ be regular in \mathfrak{D} and $N(z)$ map \mathfrak{D} onto many sheeted starlike regions:

$$\begin{aligned} M(0) &= N(0) \\ &= 0, \frac{M'(0)}{N'(0)} \\ &= 1, \Re \left(\frac{M'(z)}{N'(z)} \right) > 0. \end{aligned} \quad (29)$$

Then,

$$\Re \left(\frac{M(z)}{N(z)} \right) > 0. \quad (30)$$

Lemma 3. Let $g \in \mathcal{S}_s^*$ such that

$$\mathcal{F}(z) = \int_0^z t^{\gamma-1} g(t) dt. \quad (31)$$

Then, \mathcal{F} is $(1 + \gamma)$ -valent starlike for $\gamma = 1, 2, 3, \dots$, in \mathfrak{D} .

Proof. The proof is analogous to the one given in [41] and hence omitted. \square

Lemma 4. Let $g \in \mathcal{S}_s^*$ and $\mathfrak{F}(z) = (1 + \gamma)z^\gamma \mathcal{F}(z)$ for $\gamma = 1, 2, 3, \dots$, where $\mathcal{F}(z)$ is given by (31) in Lemma 3. Then, $\mathfrak{F}(z) \in \mathcal{S}_s^*$.

Proof. Let

$$\mathfrak{F}'(z) = \frac{1 + \gamma}{z} g(z) - \frac{\gamma}{z} \mathfrak{F}(z). \quad (32)$$

Then,

$$\frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} = \frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} \cdot \frac{z^\gamma}{z^\gamma} = \frac{z^\gamma g(z) - \gamma \mathfrak{F}(z)}{\mathfrak{F}(z)} = \frac{M(z)}{N(z)}, \quad (33)$$

where

$$M(z) = z^\gamma g(z) - \gamma \mathfrak{F}(z), N(z) = \mathfrak{F}(z). \quad (34)$$

\square

By Lemma 3, $N \ll \mathcal{F}$ is $(1 + \gamma)$ -valent starlike for $\gamma = 1$,

$$\begin{aligned} 2c_2 &= c_1^2 + (4 - c_1^2), \\ 4c_3 &= c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z. \end{aligned} \quad (38)$$

2, 3, ... in \mathfrak{D} :

$$\frac{M'(0)}{N'(0)} = \frac{zg'(0)}{g(0)} = 1, \tag{35}$$

and since $g \in \mathcal{S}_s^*$,

$$\frac{M'(z)}{N'(z)} = \frac{zg'(z)}{g(z)} \in \mathcal{S}_s^*. \tag{36}$$

Lemma 6 (see [43]). *Let $p(z) \in \mathcal{P}$; then,*

$$\begin{aligned} |c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4| &\leq 2, \\ |c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5| &\leq 2, \\ |c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6| &\leq 2, \\ |c_n| &\leq 2, \text{ for } n = 1, 2, 3, \dots \end{aligned} \tag{39}$$

Lemma 7 (see [44]). *Let $p(z) \in \mathcal{P}$; then,*

$$\begin{aligned} \left| c_2 - \frac{|c_1^2|}{2} \right| &\leq 2 - \frac{|c_1^2|}{2}, \\ |c_{n+k} - \mu c_n c_k| &< 2, 0 \leq \mu \leq 1, \\ |c_{n+2k} - \mu c_n c_k^2| &\leq 2(1 + 2\mu). \end{aligned} \tag{40}$$

Now, we are in position to present our main results.

Theorem 8. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then*

$$|a_2| \leq \frac{1}{\beta_2}, \tag{41}$$

$$|a_3| \leq \frac{1}{2\beta_3}, \tag{42}$$

$$|a_4| \leq \frac{0.344}{\beta_4}, \tag{43}$$

$$|a_5| \leq \frac{3}{8\beta_5}, \tag{44}$$

$$|a_6| \leq \frac{67}{120\beta_6}, \tag{45}$$

$$|a_7| \leq \frac{5587}{10800\beta_7}. \tag{46}$$

Proof. Since $\mathfrak{F}(z) \in \mathcal{S}_s^*$, according to the definition of subordination, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} = 1 + \sin w(z). \tag{47}$$

From Lemma 1, we can get the conclusion:

$$\frac{M(z)}{N(z)} = \frac{zg'(z)}{g(z)} \in \mathcal{S}_s^*. \tag{37}$$

□

Lemma 5 (see [42]). *If $p(z) \in \mathcal{P}$, then there exists some x, z with $|x| \leq 1, |z| \leq 1$, such that*

Now,

$$\begin{aligned} \frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)} &= 1 + \beta_2 a_2 z + (2\beta_3 a_3 - \beta_2^2 a_2^2) z^2 \\ &+ (\beta_2^3 a_2^3 - 3\beta_2 \beta_3 a_2 a_3 + 3\beta_4 a_4) z^3 \\ &+ (4\beta_5 a_5 - \beta_2^4 a_2^4 + 4\beta_2^2 \beta_3 a_2^2 a_3 - 4\beta_2 \beta_4 a_2 a_4 - 2\beta_2^4 a_2^4) z^4 \\ &+ (5\beta_6 a_6 - 5\beta_2 \beta_5 a_2 a_5 + \beta_2^5 a_2^5 - 5\beta_3 \beta_4 a_3 a_4 \\ &- 5\beta_2^3 \beta_3 a_2^3 a_3 + 5a_2^2 a_4 \beta_2^2 \beta_4 + 5\beta_2 \beta_3^2 a_2 a_3^2) z^5 \\ &+ (6\beta_7 a_7 - 6\beta_2 \beta_6 a_2 a_6 + 6\beta_2^2 \beta_5 a_2^2 a_5 - 6\beta_3 \beta_5 a_3 a_5 \\ &+ 12\beta_2 \beta_3 \beta_4 a_2 a_3 a_4 - \beta_2^6 a_2^6 - 6\beta_2^3 \beta_4 a_2^3 a_4 - 3\beta_4^2 a_4^2 \\ &+ 2\beta_3^3 a_3^3 - 9\beta_2^2 \beta_3^2 a_2^2 a_3^2 + 6\beta_2^4 \beta_3 a_2^4 a_3) z^6 + \dots, \end{aligned} \tag{48}$$

where $\beta_n = (n + \gamma)/(1 + \gamma)$. We define a function:

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \dots \tag{49}$$

It is easy to see that $p(z) \in \mathcal{P}$ and

$$w(z) = \frac{p(z) + 1}{p(z) - 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \dots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \dots}. \tag{50}$$

On the other hand,

$$\begin{aligned} 1 + \sin w(z) &= 1 + \left(\frac{c_1}{2}\right)z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{5c_3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)z^3 \\ &+ \left(\frac{5c_1^2c_2}{16} - \frac{c_1^4}{32} - \frac{c_1c_3}{2} - \frac{c_2^2}{4} + \frac{c_4}{2}\right)z^4 \\ &+ \left(\frac{c_5}{2} - \frac{c_1c_4}{2} - \frac{c_2c_3}{2} + \frac{5c_1^2c_3}{16} + \frac{c_1c_2^2}{16} - \frac{c_1^3c_2}{8} + \frac{c_1^5}{3840}\right)z^5 \\ &+ \left(\frac{c_6}{2} - \frac{c_2c_4}{2} - \frac{c_1c_5}{2} + \frac{5c_1c_2c_3}{8} + \frac{5c_3^2}{48} - \frac{c_3^2}{4} \right. \\ &\left. - \frac{5c_1^6}{512} + \frac{c_1^4c_2}{768} - \frac{3c_1^2c_2^2}{16} + \frac{5c_1^2c_4}{16} - \frac{c_1^3c_3}{8}\right)z^6 + \dots \end{aligned} \tag{51}$$

When the coefficients of z, z^2, z^3 are compared between the equations (51) and (48), then we get

$$a_2 = \frac{c_1}{2\beta_2}, a_3 = \frac{c_2}{4\beta_3}, a_4 = \frac{1}{\beta_4} \left(\frac{c_3}{6} - \frac{c_1c_2}{24} - \frac{c_1^3}{144} \right), \tag{52}$$

$$a_5 = \frac{1}{\beta_5} \left(\frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32} \right), \tag{53}$$

$$a_6 = \frac{1}{\beta_6} \left(-\frac{3c_1 c_4}{80} - \frac{7c_3 c_2}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10} \right), \tag{54}$$

$$a_7 = \frac{1}{\beta_7} \left(\frac{c_1^2 c_4}{480} + \frac{c_1 c_2 c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2 c_2^2}{3840} - \frac{109c_1^4 c_2}{11520} - \frac{c_1 c_5}{30} - \frac{5c_2 c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3 c_3}{144} \right). \tag{55}$$

Using Lemma 6, we can simply obtain

$$|a_2| \leq \frac{1}{\beta_2} \tag{56}$$

$$|a_3| \leq \frac{1}{2\beta_3}.$$

$$\mathfrak{F}z = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} g(t) dt = \sum_{n=1}^{\infty} \frac{b_n}{\beta_n} z^n, \tag{57}$$

with $b_1 = 1$ and

$$\beta_n = \frac{n + \gamma}{1 + \gamma}. \tag{58}$$

If $\mathfrak{F}(z) = \sum_{n=1}^{\infty} A_n z^n$, then by comparing like powers of z, z^2, \dots, z^n , we have

$$A_1 = 1, A_2 = \frac{b_2}{\beta_2}, A_3 = \frac{b_3}{\beta_3}, \dots, A_n = \frac{b_n}{\beta_n}. \tag{59}$$

For sharpness, if we take

$$g(z) = 1 + \sin(z) = 1 + z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots, \tag{60}$$

and thus $b_2 = 1, b_3 = 0$, and $b_4 = -1/6$, then $A_2 = 1/\beta_2$. This shows that the obtained second coefficient bound is sharp.

Again, by Lemma 6,

$$|a_4| = \frac{1}{\beta_4} \left| \frac{c_3}{6} - \frac{c_1^3}{144} - \frac{c_1 c_2}{24} \right| = \frac{1}{\beta_4} \left| \frac{1}{6} \left[c_3 - \frac{c_1 c_2}{3} \right] + \frac{c_1}{72} \left[c_2 - \frac{c_1^2}{2} \right] \right|. \tag{61}$$

Let $c_1 = c$, with $c \in [0, 2]$; then, by Lemma 7, we can get

$$|a_4| = \frac{1}{\beta_4} \left| \frac{1}{6} \left[c_3 - \frac{c_1 c_2}{3} \right] + \frac{c_1}{72} \left[c_2 - \frac{c_1^2}{2} \right] \right| \leq \frac{1}{\beta_4} \left\{ \frac{1}{3} + \frac{c}{72} \left[2 - \frac{c^2}{2} \right] \right\}. \tag{62}$$

Now, suppose that

$$F(c) = \frac{1}{\beta_4} \left\{ \frac{1}{3} + \frac{c}{72} \left[2 - \frac{c^2}{2} \right] \right\}. \tag{63}$$

Then obviously,

$$F'(c) = \frac{1}{\beta_4} \left\{ \frac{1}{36} - \frac{c^2}{48} \right\}. \tag{64}$$

Setting $F'(c) = 0$, we can get $c = 2\sqrt{3}/2$, and hence, the maximum value of $F(c)$ is given by

$$|a_4| \leq F\left(\frac{2\sqrt{3}}{2}\right) = \frac{1}{\beta_4} \left\{ \frac{1}{3} + \frac{\sqrt{3}}{162} \right\} = \frac{0.344}{\beta_4}. \tag{65}$$

Also,

$$|a_5| = \frac{1}{\beta_5} \left| \frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32} \right| = \frac{1}{\beta_5} \left| \frac{1}{8} \left[c_4 - \frac{c_1 c_3}{3} \right] - \frac{c_1^2}{576} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{c_2}{32} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{7c_1^2 c_2}{576} \right|. \tag{66}$$

Let $c_1 = c$, with $c \in [0, 2]$; then, again by Lemma 7,

$$|a_5| \leq \frac{1}{\beta_5} \left\{ \frac{1}{4} + \frac{5c^2}{576} \left[2 - \frac{c^2}{2} \right] + \frac{1}{16} \left[2 - \frac{c^2}{2} \right] + \frac{7c^2}{288} \right\}. \tag{67}$$

$$|a_6| = \frac{1}{\beta_6} \left| -\frac{3c_1 c_4}{80} - \frac{7c_3 c_2}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10} \right| = \frac{1}{\beta_6} \left| \frac{1}{24} \left[c_5 - \frac{9c_1 c_4}{10} \right] + \frac{11c_1^3}{2400} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{43c_1 c_2}{960} \left[c_2 - \frac{c_1^2}{2} \right] + \frac{211c_1^3 c_2}{14400} + \frac{7[c_5 - c_3 c_2]}{120} \right|. \tag{71}$$

Assume that

$$F(c) = \frac{1}{\beta_5} \left\{ \frac{1}{4} + \frac{5c^2}{576} \left[2 - \frac{c^2}{2} \right] + \frac{1}{16} \left[2 - \frac{c^2}{2} \right] + \frac{7c^2}{288} \right\}. \quad (68)$$

Obviously, we meet the requirement:

$$F'(c) = \frac{1}{\beta_5} \left\{ -\frac{7c}{144} - \frac{5c^3}{288} \right\} \leq 0. \quad (69)$$

Take $c_1 = c$, with $c \in [0, 2]$; then, according to Lemma 7, we have

$$|a_6| \leq \frac{1}{\beta_6} \left\{ \frac{7}{60} + \frac{1}{12} + \frac{11c^3}{2400} \left[2 - \frac{c^2}{2} \right] + \frac{43}{240} \left[2 - \frac{c^2}{2} \right] + \frac{211c^3}{7200} \right\}. \quad (72)$$

Suppose that

$$F(c) = \frac{1}{\beta_6} \left\{ \frac{7}{60} + \frac{1}{12} + \frac{11c^3}{2400} \left[2 - \frac{c^2}{2} \right] + \frac{43}{240} \left[2 - \frac{c^2}{2} \right] + \frac{211c^3}{7200} \right\}. \quad (73)$$

Then obviously,

$$F'(c) = \frac{1}{\beta_6} \left\{ -\frac{c}{240} + \frac{277c^2}{2400} - \frac{55c^4}{4800} \right\}, \quad (74)$$

We see that $F''(0) < 0$, and we get the maximum value at $c = 0$:

$$|a_6| \leq F(0) = \frac{67}{120\beta_6}. \quad (75)$$

Finally,

$$\begin{aligned} |a_7| &= \frac{1}{\beta_7} \left| \frac{c_1^2 c_4}{480} + \frac{c_1 c_2 c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2 c_2^2}{3840} - \frac{109c_1^4 c_2}{11520} \right. \\ &\quad \left. - \frac{c_1 c_5}{30} - \frac{5c_2 c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_1^3 c_3}{144} + \frac{c_6}{12} \right| \\ &= \frac{1}{\beta_7} \left| \frac{-37c_1^6}{691200} - \frac{25c_1^2 c_2^2}{5760} - \frac{c_1 c_5}{30} + \frac{c_1^2 [c_4 - c_1^2]}{480} \right. \\ &\quad \left. + \frac{c_1 c_2 [c_3 - c_1 c_2]}{480} + \frac{c_1^3 [c_3 - c_1 c_2]}{144} - \frac{29c_1^4 [c_2 - c_1^2/2]}{11520} \right. \\ &\quad \left. + \frac{5c_2^3 [c_2 - c_1^2/2]}{1152} + \frac{[c_6 - 5/8c_2 c_4]}{12} \right|. \end{aligned} \quad (76)$$

Again, taking $c_1 = c$, with $c \in [0, 2]$, and using the result of

So the function $F(c)$ attains its maximum value at $c = 0$, and it is given by

$$|a_5| \leq F(0) = \frac{3}{8\beta_5}. \quad (70)$$

Next,

Lemma 7, we can obtain

$$|a_7| \leq \frac{1}{\beta_7} \left\{ \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{25c^2}{1440} + \frac{29c^4 [2 - c^2/2]}{11520} \right. \\ \left. + \frac{c^3}{72} + \frac{5[2 - c^2/2]}{288} + \frac{37c^6}{691200} \right\}. \quad (77)$$

Let

$$F(c) = \frac{1}{\beta_7} \left\{ \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{25c^2}{1440} + \frac{29c^4 [2 - c^2/2]}{11520} \right. \\ \left. + \frac{c^3}{72} + \frac{5[2 - c^2/2]}{288} + \frac{37c^6}{691200} \right\}. \quad (78)$$

Then obviously, $F'(c) \geq 0$. As a result, the function $F(c)$ attains its maximum value at $c = 2$. Hence,

$$|a_7| \leq F(2) \leq \frac{5587}{10800\beta_7}. \quad (79)$$

□

Theorem 9. If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have

$$|a_3 - a_2^2| \leq \frac{1}{2\beta_3}. \quad (80)$$

Proof. From (52), we can write

$$|a_3 - a_2^2| = \left| \frac{c_2}{4\beta_3} - \frac{c_1^2}{4\beta_2^2} \right|. \quad (81)$$

Using Lemma 5, we get

$$|a_3 - a_2^2| = \left| \frac{(\beta_2^2 - 2\beta_3)c_1^2}{8\beta_3\beta_2^2} - \frac{x(4 - c_1^2)}{8\beta_3} \right|. \quad (82)$$

We suppose that $|x| = t \in [0, 1]$, and $c_1 = c \in [0, 2]$. Also, if we apply the triangle inequality to the above equation, then we get

$$|a_3 - a_2^2| \leq \frac{(\beta_2^2 - 2\beta_3)c^2}{8\beta_3\beta_2^2} + \frac{t(4 - c^2)}{8\beta_3}. \quad (83)$$

Assume that

$$F(c, t) = |a_3 - a_2| \leq \frac{(\beta_2^2 - 2\beta_3)c^2}{8\beta_3\beta_2^2} + \frac{t(4 - c^2)}{8\beta_3}. \quad (84)$$

Obviously, we can write

$$\frac{\partial F}{\partial t} = \frac{(4 - c^2)}{8\beta_3} \geq 0, \quad (85)$$

$F(c, t)$ is increasing on $[0, 1]$. Therefore, at $t = 1$, the function $F(c, t)$ will obtain its maximum value:

$$\max F(c, t) = F(c, 1) = \frac{(\beta_2^2 - 2\beta_3)c^2}{8\beta_3\beta_2^2} + \frac{(4 - c^2)}{8\beta_3}. \quad (86)$$

Let us take

$$\begin{aligned} G(c) &= \frac{(\beta_2^2 - 2\beta_3)c^2}{8\beta_3\beta_2^2} + \frac{(4 - c^2)}{8\beta_3}, \\ G'(c) &= \frac{(\beta_2^2 - 2\beta_3)c}{4\beta_3\beta_2^2} - \frac{c}{4\beta_3} = \frac{-c}{2\beta_2^2}, \\ G''(c) &= \frac{-1}{2\beta_2^2} < 0. \end{aligned} \quad (87)$$

It is clear that $G(c)$ is decreasing on $[0, 2]$. So at $c = 0$, the function $G(c)$ will obtain its maximum value:

$$|a_3 - a_2| \leq G(0) = \frac{1}{2\beta_3}. \quad (88)$$

This complete the proof. \square

Theorem 10. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$|a_2a_3 - a_4| \leq \frac{1}{3\beta_4}. \quad (89)$$

Proof. From (52), we can write

$$|a_2a_3 - a_4| = \left| \frac{c_1^3}{144\beta_4} + \frac{c_1c_2}{24\beta_4} + \frac{c_1c_2}{8\beta_2\beta_3} - \frac{c_3}{6\beta_4} \right|. \quad (90)$$

From Lemma 5, we can deduce that

$$|a_2a_3 - a_4| = \left| \frac{(9\beta_4 - 2\beta_2\beta_3)c_1^3}{144\beta_2\beta_3\beta_4} + \frac{c_1x^2(4 - c_1^2)}{24\beta_4} - \frac{(4 - c_1^2)(1 - |x|^2)z}{12\beta_4} \right|. \quad (91)$$

We suppose that $|x| = t \in [0, 1]$, and $c_1 = c \in [0, 2]$. Once again, if we apply the triangle inequality to the above

equation, then we get

$$|a_2a_3 - a_4| \leq \frac{(9\beta_4 - 2\beta_2\beta_3)c^3}{144\beta_2\beta_3\beta_4} + \frac{ct^2(4 - c^2)}{24\beta_4} + \frac{(4 - c^2)(1 - t^2)z}{12\beta_4}. \quad (92)$$

Suppose that

$$\begin{aligned} F(c, t) &= \frac{(9\beta_4 - 2\beta_2\beta_3)c^3}{144\beta_2\beta_3\beta_4} \\ &+ \frac{ct^2(4 - c^2)}{24\beta_4} + \frac{(4 - c^2)(1 - t^2)z}{12\beta_4}. \end{aligned} \quad (93)$$

Then, we get

$$\frac{\partial F}{\partial t} = \frac{(c - 2)t(4 - c^2)}{12\beta_4} < 0. \quad (94)$$

The above expression shows that $F(c, t)$ is a decreasing function about t on the closed interval $[0, 1]$. This implies that $F(c, t)$ will attain its maximum value at $t = 0$, which is

$$\max F(c, t) = F(c, 0) = \frac{(9\beta_4 - 2\beta_2\beta_3)c^3}{144\beta_2\beta_3\beta_4} + \frac{(4 - c^2)}{12\beta_4}. \quad (95)$$

Now define

$$\begin{aligned} G(c) &= \frac{(9\beta_4 - 2\beta_2\beta_3)c^3}{144\beta_2\beta_3\beta_4} + \frac{(4 - c^2)}{12\beta_4}, \\ G'(c) &= \frac{(9\beta_4 - 2\beta_2\beta_3)c^2}{48\beta_2\beta_3\beta_4} - \frac{c}{6\beta_4}, \\ G''(c) &= -\frac{1}{6\beta_4}. \end{aligned} \quad (96)$$

Since $G''(c) < 0$, the function $G(c)$ has maximum value at $c = 0$. That is,

$$|a_2a_3 - a_4| = G(c) = G(0) \leq \frac{1}{3\beta_4}, \quad (97)$$

and this completes the proof. \square

Theorem 11. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$|a_2a_4 - a_3^2| \leq \frac{1}{3\beta_2\beta_4} + \frac{1}{4\beta_3^2}. \quad (98)$$

Proof. Again from (52), we can write

$$|a_2a_4 - a_3^2| = \left| \frac{-c_1^2c_2}{48\beta_2\beta_4} - \frac{c_1^4}{288\beta_2\beta_4} + \frac{c_1c_3}{12\beta_2\beta_4} - \frac{c_2^2}{16\beta_3^2} \right|. \quad (99)$$

Using the result of Lemma 5, we can obtain

$$|a_2a_4 - a_3^2| = \left| \frac{c_1 [c_3 - c_1^3/24]}{12\beta_2\beta_4} - \frac{c_1^2c_2}{48\beta_2\beta_4} - \frac{c_2^2}{16\beta_3^2} \right|. \quad (100)$$

Also, by Lemma 7, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{c}{6\beta_2\beta_4} + \frac{c^2}{24\beta_2\beta_4} + \frac{1}{4\beta_3^2}, \\ &= \frac{1}{6\beta_2\beta_4} \left(c + \frac{c^2}{4} \right) + \frac{1}{4\beta_3^2} \\ &= \frac{1}{6\beta_2\beta_4} H(c) + \frac{1}{4\beta_3^2}, \end{aligned} \quad (101)$$

where

$$H(c) = c + \frac{c^2}{4}. \quad (102)$$

Clearly $H(c)$ is an increasing function about c on the closed interval $[0, 2]$. This means that $H(c)$ will attain its maximum value at $c = 2$, which is $H(c) \leq 3$. Thus,

$$|a_2a_4 - a_3^2| \leq \frac{1}{2\beta_2\beta_4} + \frac{1}{4\beta_3^2}. \quad (103)$$

□

Theorem 12. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$|a_2a_5 - a_3a_4| \leq \frac{7}{36\beta_3\beta_4} + \frac{0.8156}{2\beta_2\beta_5}. \quad (104)$$

Proof. From (52) and (53), we have

$$\begin{aligned} |a_2a_5 - a_3a_4| &= \left| \frac{c_2}{4\beta_3\beta_4} \left\{ \frac{c_1^3}{144} + \frac{c_1c_2}{24} - \frac{c_3}{6} \right\} \right. \\ &\quad \left. - \frac{c_1}{2\beta_2\beta_5} \left\{ \frac{-5c_1^4}{1152} + \frac{c_1^2c_2}{192} + \frac{c_1c_3}{24} + \frac{c_2^2}{32} - \frac{c_4}{8} \right\} \right| \\ &= \left| \frac{c_2}{4\beta_3\beta_4} \left\{ \frac{c_1^3}{144} - \frac{1}{6} \left(c_3 - \frac{c_1c_2}{4} \right) \right\} \right. \\ &\quad \left. - \frac{c_1}{2\beta_2\beta_5} \left\{ \frac{c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4}{32} \right. \right. \\ &\quad \left. \left. + \frac{19c_1^2}{192} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{c_1 [c_3 - 2/3c_1^3]}{48} - \frac{3c_4}{32} \right\} \right|. \end{aligned} \quad (105)$$

Using the result of Lemma 7, we can write

$$\begin{aligned} |a_2a_5 - a_3a_4| &\leq \frac{1}{2\beta_3\beta_4} \left\{ \frac{c^3}{144} + \frac{2}{6} \right\} + \frac{1}{2\beta_2\beta_5} \left\{ \frac{c}{4} + \frac{19c^3}{96} - \frac{19c^5}{384} + \frac{c^2}{24} \right\} \\ &= \frac{1}{2\beta_3\beta_4} H(c) + \frac{1}{2\beta_2\beta_5} M(c), \end{aligned} \quad (106)$$

where

$$H(c) = \left\{ \frac{1}{3} + \frac{c^3}{144} \right\}, M(c) = \left\{ \frac{c}{4} + \frac{c^2}{24} + \frac{19c^3}{96} - \frac{19c^5}{384} \right\}. \quad (107)$$

Obviously,

$$M'(c) = \frac{1}{4} + \frac{c}{12} + \frac{19c^2}{32} + \frac{95c^4}{384}. \quad (108)$$

For $M'(c) = 0$, we can get $c = 1.71468508801$ and consequently $M''(1.71468508801) = -2.8693$. As $M''(0) < 0$, the maximum value at $c = 0$ is

$$M(0) \leq 0.8156. \quad (109)$$

Also,

$$H'(c) = \frac{c^2}{48}, \quad (110)$$

where $H(c)$ attains its maximum value at $c = 2$, so

$$H(c) = H(2) \leq \frac{7}{18}. \quad (111)$$

Using the results of (109) and (111) in (106), we can get

$$|a_2a_5 - a_3a_4| \leq \frac{7}{36\beta_3\beta_4} + \frac{0.8156}{2\beta_2\beta_5}. \quad (112)$$

□

Theorem 13. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$|a_5 - a_2a_4| \leq \frac{7}{18\beta_2\beta_4} + \frac{0.50}{\beta_5}. \quad (113)$$

Proof. From (52) and (53), we have

$$\begin{aligned}
 |a_5 - a_2 a_4| &= \left| \frac{1}{2\beta_2\beta_4} \left\{ \frac{c_1^4}{144} + \frac{c_1^2 c_2}{24} - \frac{c_1 c_3}{6} \right\} \right. \\
 &\quad \left. - \frac{1}{\beta_5} \left\{ \frac{-5c_1^4}{1152} + \frac{c_1^2 c_2}{192} + \frac{c_1 c_3}{24} + \frac{c_2^2}{32} - \frac{c_4}{8} \right\} \right| \\
 &= \left| \frac{1}{2\beta_2\beta_4} \left\{ \frac{c_1^4}{144} - \frac{c_1}{6} \left(c_3 - \frac{c_1 c_2}{4} \right) \right\} \right. \\
 &\quad \left. - \frac{1}{\beta_5} \left\{ \frac{c_1^4 + c_2^2 + 2c_1 c_3 - 3c_1^2 c_2 - c_4}{32} \right. \right. \\
 &\quad \left. \left. + \frac{19c_1^2}{192} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{c_1 [c_3 - 2/3c_1^3]}{48} - \frac{3c_4}{32} \right\} \right|. \tag{114}
 \end{aligned}$$

Letting $|x| = t \in [0, 1]$ and $c_1 = c \in [0, 2]$ and using the results of Lemmas 6 and 7, we get

$$|a_5 - a_2 a_4| \leq \frac{1}{2\beta_2\beta_4} \left\{ \frac{c^4}{144} + \frac{2c}{6} \right\} + \frac{1}{\beta_5} \left\{ \frac{1}{4} + \frac{c}{24} + \frac{19c^2}{96} - \frac{19c^4}{384} \right\}. \tag{115}$$

Suppose that

$$F(c) = \frac{1}{2\beta_2\beta_4} H(c) + \frac{1}{\beta_5} M(c), \tag{116}$$

where

$$H(c) = \frac{c^4}{144} + \frac{c}{3}, M(c) = \left\{ \frac{1}{4} + \frac{c}{24} + \frac{19c^2}{96} - \frac{19c^4}{384} \right\}. \tag{117}$$

We see that $H'(c) \geq 0$ and the maximum value of $H(c)$ can be attained at $c = 2$, which is $H(2) \leq 7/9$. Also,

$$M'(c) = \frac{1}{24} + \frac{19c}{48} - \frac{19c^3}{96}. \tag{118}$$

If we set $M'(c) = 0$, then we get $c = 1.46416723$. Consequently, $M''(1.46416723) = -0.86$. As $M''(0) < 0$, the maximum value at $c = 0$ is given by $M(0) \leq 0.50$. Hence,

$$|a_5 - a_2 a_4| \leq \frac{7}{18\beta_2\beta_4} + \frac{0.50}{\beta_5}. \tag{119}$$

□

Theorem 14. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$|a_5 a_3 - a_4^2| \leq \frac{85}{324\beta_4^2} + \frac{0.507900}{2\beta_2\beta_5}. \tag{120}$$

Proof. From (52) and (53), we have

$$\begin{aligned}
 |a_5 a_3 - a_4^2| &= \left| -\frac{1}{\beta_4^2} \left\{ \frac{c_1^6}{20736} + \frac{c_1^4 c_2}{1728} - \frac{c_1^3 c_3}{432} + \frac{c_1^2 c_2^2}{576} - \frac{c_1 c_2 c_3}{72} + \frac{c_3^2}{36} \right\} \right. \\
 &\quad \left. - \frac{c_2}{4\beta_3\beta_5} \left\{ \frac{-5c_1^4}{1152} + \frac{c_1^2 c_2}{192} + \frac{c_1 c_3}{24} + \frac{c_2^2}{32} - \frac{c_4}{8} \right\} \right| \\
 &= \left| -\frac{1}{\beta_4^2} \left\{ \frac{c_1^6}{20736} + \frac{c_2^2}{36} - \frac{c_1^3}{432} \left(c_3 - \frac{c_1 c_2}{4} \right) \right. \right. \\
 &\quad \left. \left. - \frac{c_1 c_2}{72} \left(c_3 - \frac{c_1 c_2}{8} \right) \right\} - \frac{c_2}{4\beta_3\beta_5} \left\{ \frac{c_1^4 + c_2^2 + 2c_1 c_3 - 3c_1^2 c_2 - c_4}{32} \right. \right. \\
 &\quad \left. \left. + \frac{19c_1^2}{192} \left[c_2 - \frac{c_1^2}{2} \right] - \frac{3c_4}{32} - \frac{c_1 [c_3 - 2/3c_1^3]}{48} \right\} \right|. \tag{121}
 \end{aligned}$$

Now, using the results of Lemmas 6 and 7, we obtain

$$\begin{aligned}
 |a_5 a_3 - a_4^2| &\leq \frac{1}{\beta_4^2} \left\{ \frac{1}{9} + \frac{c}{18} + \frac{c^3}{216} + \frac{c^6}{20736} \right\} \\
 &\quad + \frac{1}{2\beta_3\beta_5} \left\{ \frac{1}{4} + \frac{c}{24} + \frac{19c^2}{96} - \frac{19c^4}{384} \right\},
 \end{aligned}$$

$$|a_5 a_3 - a_4^2| \leq \frac{1}{\beta_4^2} H(c) + \frac{1}{2\beta_2\beta_5} M(c), \tag{122}$$

where

$$H(c) = \frac{1}{9} + \frac{c}{18} + \frac{c^3}{216} + \frac{c^6}{20736}, M(c) = \frac{1}{4} + \frac{c}{24} + \frac{19c^2}{96} - \frac{19c^4}{384}. \tag{123}$$

It is clear that $H(c)$ is an increasing function, so it attains its maximum value at $c = 2$, which is

$$H(2) \leq \frac{85}{324\beta_4^2}. \tag{124}$$

Also, for all $c \in [0, 2]$, we have

$$\begin{aligned}
 M'(c) &= \frac{1}{24} + \frac{19c}{48} - \frac{19c^3}{96}, \\
 M''(c) &= \frac{19}{48} - \frac{19c^2}{32}.
 \end{aligned} \tag{125}$$

When we set $M'(c) = 0$, then we get $c = 1.464167$. Obviously,

$$M''(c) = M''(1.464167) = -0.87703 < 0, \tag{126}$$

and it attains its maximum value at $c = 1.464167$, which is given by

$$M(1.464167) \leq 0.507900686. \tag{127}$$

Hence,

$$|a_5 a_3 - a_4^2| \leq \frac{85}{324 \beta_4^2} + \frac{0.507900}{2 \beta_2 \beta_5}, \tag{128}$$

which completes the proof of Theorem 14. \square

Theorem 15. *If the function $f(z) \in \mathcal{S}_s^*$ and is of the form (1), then we have*

$$\begin{aligned} |\Delta_{4,1}(f)| \leq & \frac{2.9161 \times 10^{-2}}{\beta_4^4} + \frac{7.0313 \times 10^{-2}}{\beta_5^3} + \frac{4489}{28800 \beta_3 \beta_6^2} \\ & + \frac{0.006204}{\beta_4^2 \beta_6} + \frac{5587}{86400 \beta_3^3 \beta_7} + \frac{5.7491 \times 10^{-2}}{\beta_4^2 \beta_7} \\ & + \frac{4.7616 \times 10^{-2}}{\beta_2 \beta_3 \beta_5^2} + \frac{0.10567}{\beta_2 \beta_4 \beta_5^2} + \frac{2.8228 \times 10^{-2}}{\beta_2 \beta_4^2 \beta_5} \\ & + \frac{7.2391 \times 10^{-2}}{\beta_2 \beta_4^2 \beta_6} + \frac{0.0735}{\beta_3 \beta_4^2 \beta_5} + \frac{469}{8640 \beta_3^2 \beta_4 \beta_6} \\ & + \frac{5587}{57600 \beta_3 \beta_5 \beta_7} + \frac{0.16287}{\beta_4 \beta_5 \beta_6} \\ & + \frac{5587}{64800 \beta_2 \beta_3 \beta_4 \beta_7} + \frac{0.11384}{\beta_2 \beta_3 \beta_5 \beta_6}. \end{aligned} \tag{129}$$

Proof. We know that

$$\begin{aligned} \Delta_{4,1}(f) = & a_7 \{ a_3 (a_2 a_4 - a_3^2) - a_4 (a_4 - a_2 a_3) + a_5 (a_3 - a_2^2) \} \\ & - a_6 \{ a_3 (a_2 a_5 - a_3 a_4) - a_4 (a_5 - a_2 a_4) \\ & + a_6 (a_3 - a_2^2) \} + a_5 \{ a_3 (a_3 a_5 - a_4^2) \\ & - a_5 (a_5 - a_2 a_4) + a_6 (a_4 - a_2 a_3) \} - a_4 \{ a_4 (a_3 a_5 - a_4^2) \\ & - a_5 (a_2 a_5 - a_3 a_4) + a_6 (a_4 - a_2 a_3) \}, \end{aligned} \tag{130}$$

which further implies that

$$|\Delta_{4,1}(f)| = \left| \begin{aligned} & a_7 \{ a_3 (a_2 a_4 - a_3^2) - a_4 (a_4 - a_2 a_3) + a_5 (a_3 - a_2^2) \} \\ & - a_6 \{ a_3 (a_2 a_5 - a_3 a_4) - a_4 (a_5 - a_2 a_4) + a_6 (a_3 - a_2^2) \} \\ & + a_5 \{ a_3 (a_3 a_5 - a_4^2) - a_5 (a_5 - a_2 a_4) + a_6 (a_4 - a_2 a_3) \} \\ & - a_4 \{ a_4 (a_3 a_5 - a_4^2) - a_5 (a_2 a_5 - a_3 a_4) + a_6 (a_4 - a_2 a_3) \} \end{aligned} \right|. \tag{131}$$

Using the triangle inequality, we can write

$$\begin{aligned} |\Delta_{4,1}(f)| \leq & |a_7| |a_3| |a_2 a_4 - a_3^2| + |a_4| |a_7| |a_4 - a_2 a_3| \\ & + |a_5| |a_7| |a_3 - a_2^2| + |a_6| |a_3| |a_2 a_5 - a_3 a_4| \\ & + |a_4| |a_6| |a_5 - a_2 a_4| + |a_6|^2 |a_3 - a_2^2| \\ & + |a_5| |a_3| |a_3 a_5 - a_4^2| + |a_5|^2 |a_5 - a_2 a_4| \\ & + |a_5| |a_6| |a_4 - a_2 a_3| + |a_4|^2 |a_3 a_5 - a_4^2| \\ & + |a_4| |a_5| |a_2 a_5 - a_3 a_4| + |a_4| |a_6| |a_4 - a_2 a_3|. \end{aligned} \tag{132}$$

By substituting the results of (41), (80), (89), (98), (104), (113), (120), and (128) in (132), we can get the desired result in (129). \square

3. Conclusion

In the present investigation, first, we have extended the well-known Bernardi theorem to a specific class \mathcal{S}_s^* of univalent starlike functions in the open unit disk \mathfrak{D} . We have proven that if g is a starlike univalent function in the unit disk \mathfrak{D} and $g \in \mathcal{S}_s^*$, then $\mathfrak{F}(z) \in \mathcal{S}_s^*$, where

$$\mathfrak{F}(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} g(t) dt, \gamma > -1. \tag{133}$$

Additionally, we have estimated the upper bounds of the fourth-order Hankel determinant for the functions class \mathcal{S}_s^* associated with the sine function systematically.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

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