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Research Article

Novel Investigation of Fractional-Order Cauchy-Reaction Diffusion Equation Involving Caputo-Fabrizio Operator

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In this article, the new iterative transform technique and homotopy perturbation transform method are applied to calculate the fractional-order Cauchy-reaction diffusion equation solution. Yang transformation is mixed with the new iteration method and homotopy perturbation method in these methods. The fractional derivative is considered in the sense of Caputo-Fabrizio operator. The convection-diffusion models arise in physical phenomena in which energy, particles, or other physical properties are transferred within a physical process via two processes: diffusion and convection. Four problems are evaluated to demonstrate, show, and verify the present methods' efficiency. The analytically obtained results by the present method suggest that the method is accurate and simple to implement.

1. Introduction

The convection-diffusion equation is a mixture of convection and diffusion equations and identifies physical processes where energy, particles, or other physical properties are transmitted inside a physical process due to two process steps: diffusion and convection. In standard form, the convection-diffusion model is written as follows:

$$\frac{\partial \mathbb{U}}{\partial \mathfrak{T}} = \nabla \cdot (D \cdot \nabla \mathbb{U}) - \nabla \cdot \left(\vec{v}\mathbb{U}\right) + R, \tag{1}$$

where D is the diffusivity, $\mathbb{U}$ is the variable term, such as thermal diffusivity for heat flow or mass diffusion coefficient for particle, and $\vec{v}$ is the average velocity that the volume is travelling. For instance, in convection, u might be the density of river in salt and then the flow velocity of water $\vec{v}$. For example, in a calm lake, $\vec{v}$ would be the average velocity of bubbles rising to the surface due to buoyancy, and $\mathbb{U}$ would be the concentration of small bubbles. R defines "sinks" or "sources" of the quantity $\mathbb{U}$. For a chemical species, $R > 0$ indicates that a chemical reaction is increasing the number of the species, while $R > 0$ indicates that a chemical reaction is decreasing the number of the species. If thermal energy is generated by friction, $R > 0$ may occur in heat transport. ∇ denotes gradient, while $\nabla\cdot$ denotes divergence. Previously, different techniques have been applied to investigate these models such as Adomian's decomposition technique [1], variational iteration technique [2], Bessel collocation technique [3], and homotopy perturbation technique [4].

In recent decades, fractional derivatives have been used to interpret many physical problems mathematically, and these representations have produced excellent results in modelling real-world issues. Many basic definitions of fractional operators were given by Riesz, Riemann-Liouville, Hadamard, Weyl, Grunwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio, and Atangana-Baleanu, among others [5–8]. Over the last few years, many nonlinear equations have been developed and widely used in nonlinear physical sciences like chemistry, biology, mathematics, and different branches of physics like plasma physics, condensed matter physics, fluid mechanics, field theory, and nonlinear optics. The exact outcome of nonlinear equations is crucial in determining the characteristics and behaviour of physical processes. Still, it is impossible to find exact results when dealing with linear equations. Many useful methods have been applied to investigate nonlinear fractional partial differential equations, for example, analytical solutions with the help of natural decomposition method of fractional-order heat and wave equations [9], fractional-order partial differential equations with proportional delay [10], fractional-order hyperbolic telegraph equation [11] and fractional-order diffusion equations [12], the variational iterative transform method [13], the homotopy perturbation transform method [14, 15], the homotopy analysis transform method [16, 17], reduced differential transform method [18, 19], q-homotopy analysis transform method [20–24], the finite element technique [25], the finite difference technique [26], and so on [27–30].

Daftardar-Gejji and Jafari developed a new iterative method of analysis for solving nonlinear equations in 2006 [31, 32]. It is the first application of Laplace transformation in iterative technique by Jafari et al. Iterative Laplace transformation method [33] was introduced as a simple method for estimating approximate effects of the fractional partial differential equation system. Iterative Laplace transformation method (NITM) is used to solve linear and nonlinear partial differential equations such as fractional-order Fornberg Whitham equations [34], time-fractional Zakharov Kuznetsov equation [35], and fractional-order Fokker Planck equations [36].

In 1999, He developed the homotopy perturbation method (HPM) [37], which combines the homotopy technique, and the standard perturbation method has been broadly utilized to both linear and nonlinear models [38–40]. The homotopy perturbation method is important because it eliminates the need for a small parameter in the model, eliminating the disadvantages of traditional perturbation techniques. The main goal of this paper is to use HPM to solve nonlinear fractional-order Cauchy-reaction diffusion equation using a newly introduced integral transformation known as the "Yang transform" [41]. The suggested technique is applied to analyse two well-known nonlinear partial differential equations. In the context of a quickly convergent series, we obtain a power series solution, and only a few iterations are required to obtain very efficient solutions. There is no need for a discretization technique or linearization for the nonlinear equations, and just a few few can yield a result that can be quickly estimated to utilize these methods.

2. Basic Definitions

We provide the fundamental definitions that will be used throughout the article. For the purpose of simplification, we write the exponential decay kernel as, $K(\mathfrak{T}, \varrho) = e^{[-\wp(\mathfrak{T}-\varrho/1-\wp)]}$.

Definition 1. The Caputo-Fabrizio derivative is given as follows [42]:

$$ {}^{\mathrm{CF}}D^{\wp}_{\mathfrak{T}}[\mathbb{P}(\mathfrak{T})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{T}} \mathbb{P}'(\varrho)K(\mathfrak{T},\varrho)d\varrho, n-1<\wp\le n. \tag{2}$$

$N(\wp)$ is the normalization function with $N(0)=N(1)=1$.

$$ {}^{\mathrm{CF}}D^{\wp}_{\mathfrak{T}}[\mathbb{P}(\mathfrak{T})] = \frac{N(\wp)}{1-\wp}\int_0^{\mathfrak{T}} [\mathbb{P}(\mathfrak{T})-\mathbb{P}(\varrho)]K(\mathfrak{T},\varrho)d\varrho. \tag{3}$$

Definition 2. The fractional integral Caputo-Fabrizio is given as [42]

$$ {}^{\mathrm{CF}}I^{\wp}_{\mathfrak{T}}[\mathbb{P}(\mathfrak{T})] = \frac{1-\wp}{N(\wp)}\mathbb{P}(\mathfrak{T}) + \frac{\wp}{N(\wp)}\int_0^{\mathfrak{T}} \mathbb{P}(\varrho)d\varrho, \mathfrak{T}\ge 0, \wp\in(0,1]. \tag{4}$$

Definition 3. For $N(\wp)=1$, the following result shows the Caputo-Fabrizio derivative of Laplace transformation [42]:

$$ L\left[{}^{\mathrm{CF}}D^{\wp}_{\mathfrak{T}}[\mathbb{P}(\mathfrak{T})]\right] = \frac{\nu L[\mathbb{P}(\mathfrak{T})-\mathbb{P}(0)]}{\nu+\wp(1-\nu)}. \tag{5}$$

Definition 4. The Yang transformation of $\mathbb{P}(\mathfrak{T})$ is expressed as [42]

$$ \mathbb{Y}[\mathbb{P}(\mathfrak{T})] = \chi(\nu) = \int_0^{\infty} \mathbb{P}(\mathfrak{T})e^{-\frac{\mathfrak{T}}{\nu}}d\mathfrak{T}.\mathfrak{T}>0, \tag{6}$$

Remark 5. Yang transformation of few useful functions is defined as below.

$$\begin{aligned} \mathbb{Y}[1] &= \nu, \\ \mathbb{Y}[\mathfrak{T}] &= \nu^2, \\ \mathbb{Y}\left[\mathfrak{T}^i\right] &= \Gamma(i+1)\nu^{i+1}. \end{aligned} \tag{7}$$

Lemma 6 (Laplace-Yang duality). *Let the Laplace transformation of* $\mathbb{P}(\mathfrak{T})$ *be* $F(\nu)$, *and then,* $\chi(\nu)=F(1/\nu)$ *[43].*

Proof. From Equation (5), we can achieve another type of the Yang transformation by putting $\mathfrak{T}/\nu=\zeta$ as

$$ L[\mathbb{P}(\mathfrak{T})] = \chi(\nu) = \nu\int_0^{\infty} \mathbb{P}(\nu\zeta)e^{\zeta}d\zeta.\zeta>0. \tag{8}$$

Since $L[\mathbb{P}(\mathfrak{T})]=F(\nu)$, this implies that

$$ F(\nu) = L[\mathbb{P}(\mathfrak{T})] = \int_0^{\infty} \mathbb{P}(\mathfrak{T})e^{-\nu\mathfrak{T}}d\mathfrak{T}. \tag{9}$$

Put $\mathfrak{T}=\zeta/\nu$ in (8), and we have

$$ F(\nu) = \frac{1}{\nu}\int_0^{\infty} \mathbb{P}\left(\frac{\zeta}{\nu}\right)e^{\zeta}d\zeta. \tag{10}$$

Thus, from Equation (7), we achieve

$$F(\nu) = \chi\left(\frac{1}{\nu}\right). \quad (11)$$

Also from Equations (5) and (8), we achieve

$$F\left(\frac{1}{\nu}\right) = \chi(\nu). \quad (12)$$

The connections (10) and (11) represent the duality link between the Laplace and Yang transformation. □

Lemma 7. *Let $\mathbb{P}(\mathfrak{I})$ be a continuous function; then, the Caputo-Fabrizio derivative Yang transformation of $\mathbb{P}(\mathfrak{I})$ is define by [43]*

$$\mathbb{Y}[\mathbb{P}(\mathfrak{I})] = \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \nu\mathbb{P}(0)]}{1 + \wp(\nu - 1)}. \quad (13)$$

Proof. The Caputo-Fabrizio fractional Laplace transformation is given by

$$L[\mathbb{P}(\mathfrak{I})] = \frac{L[\nu\mathbb{P}(\mathfrak{I}) - \mathbb{P}(0)]}{\nu + \wp(1 - \nu)}. \quad (14)$$

Also, we have that the connection among Laplace and Yang property, i.e., $\chi(\nu) = F(1/\nu)$. To achieve the necessary result, we substitute ν by $1/\nu$ in Equation (13), and we get

$$\begin{aligned} \mathbb{Y}[\mathbb{P}(\mathfrak{I})] &= \frac{1/\nu\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \mathbb{P}(0)]}{1/\nu + \wp(1 - 1/\nu)}, \\ \mathbb{Y}[\mathbb{P}(\mathfrak{I})] &= \frac{\mathbb{Y}[\mathbb{P}(\mathfrak{I}) - \nu\mathbb{P}(0)]}{1 + \wp(\nu - 1)}. \end{aligned} \quad (15)$$

The proof is completed. □

3. Algorithm of the HPTM

The procedure of general nonlinear Caputo-Fabrizio fractional partial differential equations is through HPTM. Let us take a general nonlinear Caputo-Fabrizio partial differential equations with nonlinear function $N(\mathbb{U}(\varphi, \mathfrak{I}))$ and linear fractional $L(\mathbb{U}(\varphi, \mathfrak{I}))$ as [43]

$$\begin{cases} \mathrm{CF}D_{\mathfrak{I}}^{\rho}\mathbb{V}(\varphi, \mathfrak{I}) + L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I})) = g(\varphi, \mathfrak{I}), \\ \mathbb{V}(\varphi, 0) = h(\varphi), \end{cases} \quad (16)$$

where the term $g(\varphi, \mathfrak{I})$ shows the source function. Using Yang transformation to Equation (16), one can obtain

$$\begin{aligned} &\frac{\mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{I}) - \nu\mathbb{V}(\varphi, 0)]}{1 + \wp(\nu - 1)} \\ &\quad = -\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})], \\ &\mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{I})] = \nu h(\varphi) - (1 + \wp(\nu - 1)) \\ &\quad \cdot [\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})]. \end{aligned} \quad (17)$$

Implementing inverse Yang transformation, we obtain

$$\begin{aligned} \mathbb{V}(\varphi, \mathfrak{I}) &= \mathbb{V}(\varphi, 0) - \mathbb{Y}^{-1}[1 + \wp(\nu - 1) \\ &\quad \cdot [\mathbb{Y}[L(\mathbb{V}(\varphi, \mathfrak{I})) + N(\mathbb{V}(\varphi, \mathfrak{I}))] + \mathbb{Y}[g(\varphi, \mathfrak{I})]], \end{aligned} \quad (18)$$

where the term $\mathbb{V}(\varphi, \mathfrak{I})$ shows the source function and with the initial condition. Now, we apply homoptopy perturbation method.

$$\mathbb{V}(\varphi, \mathfrak{I}) = \sum_{i=0}^{\infty} \rho^i \mathbb{V}_i(\varphi, \mathfrak{I}). \quad (19)$$

We decompose the nonlinear term $N(\mathbb{V}(\varphi, \mathfrak{I}))$ as

$$N(\mathbb{V}(\varphi, \mathfrak{I})) = \sum_{i=0}^{\infty} \rho^i H_i(\mathbb{V}), \quad (20)$$

where $H_i(\mathbb{V})$ represents the He's polynomial and is calculated through the following formula:

$$H_i(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \cdots, \mathbb{V}_i) = \frac{1}{\Gamma(i+1)} \frac{\partial^i}{\partial \rho^i} \left[N\left(\sum_{i=0}^{\infty} \rho^i \mathbb{V}_i \right) \right]_{\rho=0}, i = 1, 2, 3. \quad (21)$$

Substituting Equations (19) and (20) in Equation (18), we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} \rho^i \mathbb{V}_i(\varphi, \mathfrak{I}) &= \mathbb{V}(\varphi, \mathfrak{I}) - \rho\left(\mathbb{Y}^{-1}\left[(1 + \wp(\nu - 1))\mathbb{Y} \right.\right. \\ &\quad \left.\left. \cdot \left[L \sum_{i=0}^{\infty} \rho_i^{i\mathbb{V}}(\varphi, \mathfrak{I}) + N \sum_{i=0}^{\infty} \rho^i H_i(\mathbb{V}) \right] \right] \right). \end{aligned} \quad (22)$$

We obtain the following terms by coefficients comparing of ρ in (22):

$$\begin{aligned} \rho^0 &: \mathbb{V}_0(\varphi, \mathfrak{I}) = \mathbb{V}(\varphi, \mathfrak{I}), \\ \rho^1 &: \mathbb{V}_1(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1 + \wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_0(\varphi, \mathfrak{I})) + H_0(\mathbb{V})]], \\ \rho^2 &: \mathbb{V}_2(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1 + \wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_1(\varphi, \mathfrak{I})) + H_1(\mathbb{V})]], \\ \rho^3 &: \mathbb{V}_3(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1 + \wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_2(\varphi, \mathfrak{I})) + H_2(\mathbb{V})]], \\ &\vdots \\ \rho^i &: \mathbb{V}_i(\varphi, \mathfrak{I}) = \mathbb{Y}^{-1}[(1 + \wp(\nu - 1))\mathbb{Y}[L(\mathbb{V}_i(\varphi, \mathfrak{I})) + H_i(\mathbb{V})]]. \end{aligned} \quad (23)$$

As a result, the obtained solution of Equation (16) can be written as follows:

$$\mathbb{V}(\varphi, \mathfrak{T}) = \mathbb{V}_0(\varphi, \mathfrak{T}) + \mathbb{V}_1(\varphi, \mathfrak{T}) + \cdots. \tag{24}$$

4. Error Analysis and Convergence

The following theorems are fundamental on the techniques address the original models [16] error analysis and convergence.

Theorem 8. *Let $\mathbb{V}(\varphi, \mathfrak{T})$ be the actual result of (16), and let $\mathbb{V}_i(\varphi, \mathfrak{T}) \in H$ and $\sigma \in (0, 1)$, where H denotes the Hilbert space. Then, the achieved result $\sum_{i=0}^{\infty} \mathbb{V}_i(\varphi, \mathfrak{T})$ will converge $\mathbb{V}(\varphi, \mathfrak{T})$ if $\mathbb{V}_i(\varphi, \mathfrak{T}) \le \mathbb{V}_{i-1}(\varphi, \mathfrak{T}) \forall i > A$, i.e., for any $\omega > 0 \exists A > 0$, such that $\|\mathbb{V}_{i+n}(\varphi, \mathfrak{T})\| \le \beta, \forall i, n \in N$ [43].*

Proof. We make a sequence of $\sum_{i=0}^{\infty} \mathbb{V}_i(\varphi, \mathfrak{T})$.

$$\begin{aligned}
C_0(\varphi, \mathfrak{T}) &= \mathbb{V}_0(\varphi, \mathfrak{T}),\\
C_1(\varphi, \mathfrak{T}) &= \mathbb{V}_0(\varphi, \mathfrak{T}) + \mathbb{V}_1(\varphi, \mathfrak{T}),\\
C_2(\varphi, \mathfrak{T}) &= \mathbb{V}_0(\varphi, \mathfrak{T}) + \mathbb{V}_1(\varphi, \mathfrak{T}) + \mathbb{V}_2(\varphi, \mathfrak{T}),\\
C_3(\varphi, \mathfrak{T}) &= \mathbb{V}_0(\varphi, \mathfrak{T}) + \mathbb{V}_1(\varphi, \mathfrak{T}) + \mathbb{V}_2(\varphi, \mathfrak{T}) + \mathbb{V}_3(\varphi, \mathfrak{T}),\\
&\vdots\\
C_i(\varphi, \mathfrak{T}) &= \mathbb{V}_0(\varphi, \mathfrak{T}) + \mathbb{V}_1(\varphi, \mathfrak{T}) + \mathbb{V}_2(\varphi, \mathfrak{T}) + \cdots + \mathbb{V}_i(\varphi, \mathfrak{T}).
\end{aligned} \tag{25}$$

To provide the correct outcome, we have to demonstrate that $C_i(\varphi, \mathfrak{T})$ forms a "Cauchy sequence." Take, for example,

$$\begin{aligned}
\|C_{i+1}(\varphi, \mathfrak{T}) - C_i(\varphi, \mathfrak{T})\| &= \|\mathbb{V}_{i+1}(\varphi, \mathfrak{T})\| \le \sigma \|\mathbb{V}_i(\varphi, \mathfrak{T})\|\\
&\le \sigma^2 \|\mathbb{V}_{i-1}(\varphi, \mathfrak{T})\| \le \sigma^3 \|\mathbb{V}_{i-2}(\varphi, \mathfrak{T})\| \cdots\\
&\le \sigma_{i+1} \|\mathbb{V}_0(\varphi, \mathfrak{T})\|.
\end{aligned} \tag{26}$$

For $i, n \in N$, we acquire

$$\begin{aligned}
&\|C_i(\varphi, \mathfrak{T}) - C_n(\varphi, \mathfrak{T})\| = \|\mathbb{V}_{i+n}(\varphi, \mathfrak{T})\|\\
&\quad = \|C_i(\varphi, \mathfrak{T}) - C_{i-1}(\varphi, \mathfrak{T}) + (C_{i-1}(\varphi, \mathfrak{T}) - C_{i-2}(\varphi, \mathfrak{T}))\\
&\qquad + (C_{i-2}(\varphi, \mathfrak{T}) - C_{i-3}(\varphi, \mathfrak{T})) + \cdots + (C_{n+1}(\varphi, \mathfrak{T}) - C_n(\varphi, \mathfrak{T}))\|\\
&\quad \le \|C_i(\varphi, \mathfrak{T}) - C_{i-1}(\varphi, \mathfrak{T})\| + \|C_{i-1}(\varphi, \mathfrak{T}) - C_{i-2}(\varphi, \mathfrak{T})\|\\
&\qquad + \|C_{i-2}(\varphi, \mathfrak{T}) - C_{i-3}(\varphi, \mathfrak{T})\| + \cdots + \|C_{n+1}(\varphi, \mathfrak{T}) - C_n(\varphi, \mathfrak{T})\|\\
&\quad \le \sigma^i \|\mathbb{V}_0(\varphi, \mathfrak{T})\| + \sigma^{i-1} \|\mathbb{V}_0(\varphi, \mathfrak{T})\| + \cdots + \sigma^{i+1} \|\mathbb{V}_0(\varphi, \mathfrak{T})\|\\
&\quad = \|\mathbb{V}_0(\varphi, \mathfrak{T})\| \left(\sigma^i + \sigma^{i-1} + \sigma^{i+1}\right)\\
&\quad = \|\mathbb{V}_0(\varphi, \mathfrak{T})\| \frac{1 - \sigma^{i-n}}{1 - \sigma^{i+1}} \sigma^{n+1}.
\end{aligned} \tag{27}$$

Since $0 < \sigma < 1$, and $\mathbb{V}_0(\varphi, \mathfrak{T})$ is bounded, let us take $\beta = 1 - \sigma / (1 - \sigma_{i-n}) \sigma^{n+1} \|\mathbb{V}_0(\varphi, \mathfrak{T})\|$. Thus, $\{\mathbb{V}_i(\varphi, \mathfrak{T})\}_{i=0}^{\infty}$ forms a "Cauchy sequence" in H. It follows that the sequence $\{\mathbb{V}_i(\varphi, \mathfrak{T})\}_{i=0}^{\infty}$ is a convergent sequence with the limit $\lim_{i \longrightarrow \infty} \mathbb{V}_i(\varphi, \mathfrak{T}) = \mathbb{V}(\varphi, \mathfrak{T})$ for $\exists \mathbb{V}(\varphi, \mathfrak{T}) \in \mathscr{H}$. Hence, this ends the proof. □

Theorem 9. *Let $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{T})$ is finite and $\mathbb{V}(\varphi, \mathfrak{T})$ represents the obtained series solution. Let $\sigma > 0$ such that $\|\mathbb{V}_{h+1}(\varphi, \mathfrak{T})\| \le \|\mathbb{V}_h(\varphi, \mathfrak{T})\|$; then, the following relation gives the maximum absolute error [43].*

$$\left\| \mathbb{V}(\varphi, \mathfrak{T}) - \sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{T}) \right\| < \frac{\sigma^{k+1}}{1 - \sigma} \|\mathbb{V}_0(\varphi, \mathfrak{T})\|. \tag{28}$$

Proof. Since $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{T})$ is finite, this implies that $\sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{T}) < \infty$.

Consider

$$\begin{aligned}
\left\| \mathbb{V}(\varphi, \mathfrak{T}) - \sum_{h=0}^{k} \mathbb{V}_h(\varphi, \mathfrak{T}) \right\| &= \left\| \sum_{h=k+1}^{\infty} \mathbb{V}_h(\varphi, \mathfrak{T}) \right\|\\
&\le \sum_{h=k+1}^{\infty} \|\mathbb{V}_h(\varphi, \mathfrak{T})\|\\
&\le \sum_{h=k+1}^{\infty} \sigma^h \|\mathbb{V}_0(\varphi, \mathfrak{T})\|\\
&\le \sigma^{k+1} \left(1 + \sigma + \sigma^2 + \cdots\right) \|\mathbb{V}_0(\varphi, \mathfrak{T})\|\\
&\le \frac{\sigma^{k+1}}{1 - \sigma} \|\mathbb{V}_0(\varphi, \mathfrak{T})\|.
\end{aligned} \tag{29}$$

This ends the theorem's proof. □

5. The General Procedure of NITM

The general solution of fractional-order partial differential equation is as follows:

$$\begin{aligned}
&{}^{CF}D_{\mathfrak{T}}^{\wp} \mathbb{V}(\varphi, \mathfrak{T}) + N\mathbb{V}(\varphi, \mathfrak{T}) + M\mathbb{V}(\varphi, \mathfrak{T})\\
&\quad = h(\varphi, \mathfrak{T}), i \in N, i - 1 < \wp \le i,
\end{aligned} \tag{30}$$

where N is nonlinear and M linear functions.

With the initial condition

$$\mathbb{V}^k(\varphi, 0) = g_k(\varphi), k = 0, 1, 2, \cdots, i - 1, \tag{31}$$

implementing the Yang transformation of Equation (30), we get

$$\mathbb{Y}\left[D_{\mathfrak{T}}^{\wp} \mathbb{V}(\varphi, \mathfrak{T})\right] + \mathbb{Y}[N\mathbb{V}(\varphi, \mathfrak{T}) + M\mathbb{V}(\varphi, \mathfrak{T})] = \mathbb{Y}[h(\varphi, \mathfrak{T})]. \tag{32}$$

Applying the Yang differentiation is given to

$$\begin{aligned}
\mathbb{Y}[\mathbb{V}(\varphi, \mathfrak{T})] &= \nu \mathbb{V}(\varphi, 0) + (1 + \wp(\nu - 1)) \mathbb{Y}[h(\varphi, \mathfrak{T})]\\
&\quad - (1 + \wp(\nu - 1)) \mathbb{Y}[N\mathbb{V}(\varphi, \mathfrak{T}) + M\mathbb{V}(\varphi, \mathfrak{T})].
\end{aligned} \tag{33}$$

Using inverse Yang transformation Equation (32), we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = \mathbb{Y}^{-1}[\{\nu\mathbb{V}(\varphi,0) + (1+\wp(\nu-1))\mathbb{Y}[h(\varphi,\mathfrak{J})]\}] - \mathbb{Y}^{-1}[(1+\wp(\nu-1))\mathbb{Y}[N\mathbb{V}(\varphi,\mathfrak{J}) + M\mathbb{V}(\varphi,\mathfrak{J})]]. \tag{34}$$

By iterative method, we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = \sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J}), \tag{35}$$

$$N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) = \sum_{i=0}^{\infty}N[\mathbb{V}_i(\varphi,\mathfrak{J})]. \tag{36}$$

The nonlinear term N is identified as

$$N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) = \mathbb{V}_0(\varphi,\mathfrak{J}) + N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right). \tag{37}$$

Putting Equations (35)–(37) in Equation (34), we have obtain the following solution:

$$\begin{aligned}
\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[\left(1+\wp(\nu-1)\right)\left(\sum_{i=0}^{\infty}s^{2-\varphi+i}u^i(\varphi,0) + \mathbb{Y}[h(\varphi,\mathfrak{J})]\right)\right] - \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y} \cdot \left[N\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{\infty}\mathbb{V}_i(\varphi,\mathfrak{J})\right)\right]\right],\\
\mathbb{V}_0(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}[\nu\mathbb{V}(\varphi,0) + (1+\wp(\nu-1))\mathbb{Y}(g(\varphi,\mathfrak{J}))],\\
\mathbb{V}_1(\varphi,\mathfrak{J}) &= -\mathbb{Y}^{-1}[(1+\wp(\nu-1))\mathbb{Y}[N[\mathbb{V}_0(\varphi,\mathfrak{J})] + M[\mathbb{V}_0(\varphi,\mathfrak{J})]]],\\
\mathbb{V}_{m+1}(\varphi,\mathfrak{J}) &= -\mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left[-N\left(\sum_{i=0}^{i}\mathbb{V}_i(\varphi,\mathfrak{J})\right) - M\left(\sum_{i=0}^{i}\mathbb{V}_i(\varphi,\mathfrak{J})\right)\right]\right], m \ge 1.
\end{aligned} \tag{38}$$

Lastly, Equations (30) and (31) provide the i-term solution in series form which is expressed as

$$\mathbb{V}(\varphi,\mathfrak{J}) \cong \mathbb{V}_0(\varphi,\mathfrak{J}) + \mathbb{V}_1(\varphi,\mathfrak{J}) + \mathbb{V}_2(\varphi,\mathfrak{J}) + \cdots.., + \mathbb{V}_i(\varphi,\mathfrak{J}), i = 1, 2, \cdots. \tag{39}$$

Example 10. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$${}^{CF}D_{\mathfrak{J}}^{\wp}\mathbb{V}(\varphi,\mathfrak{J}) = D_{\mathfrak{J}}^{2}\mathbb{V}(\varphi,\mathfrak{J}) - \mathbb{V}(\varphi,\mathfrak{J}), 0 < \wp \le 1, (\varphi,\mathfrak{J}) \in \Omega \subset R^2, \tag{40}$$

with initial and boundary conditions

$$\begin{aligned}
&\mathbb{V}(\varphi,0) = e^{-\varphi} + \varphi = g(\varphi), \mathbb{V}(0,\mathfrak{J}) = 1 = f_0(\mathfrak{J}),\\
&\frac{\partial\mathbb{V}(0,\mathfrak{J})}{\partial\mathfrak{J}} = e^{-\mathfrak{J}} - 1 = f_1(\mathfrak{J}), \varphi, \mathfrak{J} \in R.
\end{aligned} \tag{41}$$

The methodology consists of applying Yang transformation first on both side in (40) and utilizing the differentiation property of Yang transformation, and we have

$$\mathbb{Y}[\mathbb{V}(\varphi,\mathfrak{J})] = \nu(e^{-\varphi} + \varphi) + (1+\wp(\nu-1))\mathbb{Y}\left[D_{\mathfrak{J}}^{2}\mathbb{V} - \mathbb{V}\right]. \tag{42}$$

Using Yang inverse transform, we get

$$\mathbb{V}(\varphi,\mathfrak{J}) = (e^{-\varphi} + \varphi) + \mathbb{Y}^{-1}\left((1+\wp(\nu-1))\mathbb{Y}\left[D_{\mathfrak{J}}^{2}\mathbb{V} - \mathbb{V}\right]\right). \tag{43}$$

Now, we apply the new iterative transform method

$$\begin{aligned}
\mathbb{V}_0(\varphi,\mathfrak{J}) &= e^{-\varphi} + \varphi,\\
\mathbb{V}_1(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_0 - \mathbb{V}_0\right\}\right] = -\varphi\{1+\wp\mathfrak{J}-\wp\},\\
\mathbb{V}_2(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_1 - \mathbb{V}_1\right\}\right]\\
&= \varphi\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\},\\
\mathbb{V}_3(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_2 - \mathbb{V}_2\right\}\right]\\
&= -\varphi\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\\
&\vdots
\end{aligned} \tag{44}$$

The series type solution is given as

$$\mathbb{V}(\varphi,\mathfrak{J}) = \mathbb{V}_0(\varphi,\mathfrak{J}) + \mathbb{V}_1(\varphi,\mathfrak{J}) + \mathbb{V}_2(\varphi,\mathfrak{J}) + \mathbb{V}_3(\varphi,\mathfrak{J}) + \cdots\mathbb{V}_i(\varphi,\mathfrak{J}). \tag{45}$$

The approximate solution is achieved as

$$\begin{aligned}\mathbb{V}(\varphi,\mathfrak{J}) = e^{-\varphi} + \varphi\bigg\{ & 1-\{1+\wp\mathfrak{J}-\wp\} \\ & + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} \\ & - \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2}\right. \\ & \left. + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots\bigg\}. \end{aligned} \tag{46}$$

Now applying the HPTM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{J}) = (e^{-\varphi}+\varphi) + p \cdot \left\{ \mathbb{Y}^{-1}\left((1+\wp(\nu-1))\mathbb{Y}\left[\sum_{i=0}^{\infty} p^i H_i(\mathbb{V})\right]\right)\right\}, \tag{47}$$

where the polynomials represent the nonlinear functions are $H_i(\mathbb{V})$. For instance, the terms of He's polynomials are achieved through the recursive relationship $H_i(\mathbb{V}) = D^2_{\mathfrak{J}}\mathbb{V}_i - \mathbb{V}_i$, $\forall n \in N$. Now, as the correspond power coefficients of p is comparison on both sides, the following solution is obtained as follows:

$$\begin{aligned} p^0 &: \mathbb{V}_0(\varphi,\mathfrak{J}) = e^{-\varphi} + \varphi, \\ p^1 &: \mathbb{V}_1(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(\mathbb{V}))\}\right] = -\varphi\{1+\wp\mathfrak{J}-\wp\}, \\ p^2 &: \mathbb{V}_2(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(\mathbb{V}))\}\right] \\ &\qquad = \varphi\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}, \\ p^3 &: \mathbb{V}_3(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_2(\mathbb{V}))\}\right] \\ &\qquad = -\varphi\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}, \\ &\vdots \end{aligned} \tag{48}$$

Then, the homotopy perturbation method series form solution is defined as

$$\mathbb{V}(\varphi,\mathfrak{J}) = \sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{J}). \tag{49}$$

The analytical result of the above equation is defined as

$$\begin{aligned}\mathbb{V}(\varphi,\mathfrak{J}) = e^{-\varphi} + \varphi\bigg\{ & 1-\{1+\wp\mathfrak{J}-\wp\} \\ & + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} \\ & - \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3\right. \\ & \left. + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots\bigg\} = \varphi\sum_{i=0}^{\infty}\frac{(\mathfrak{J}^{\wp})^i}{\Gamma(i\wp+1)}, \\ \mathbb{V}(\varphi,\mathfrak{J}) = e^{-\varphi} + \varphi E_{\wp}(\mathfrak{J}^{\wp}). & \end{aligned} \tag{50}$$

The exact result of the above equation is

$$\mathbb{V}(\varphi,\mathfrak{J}) = e^{-\varphi} + \varphi e^{-\mathfrak{J}}. \tag{51}$$

Figure 1 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 2 shows separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 3, the graph shows the different fractional-order $\wp$ of Example 10.

Example 11. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$$ {}^{CF}D^{\wp}_{\mathfrak{J}}\mathbb{V}(\varphi,\mathfrak{J}) = D^2_{\mathfrak{J}}\mathbb{V}(\varphi,\mathfrak{J}) - \left(1+4\varphi^2\right)\mathbb{V}(\varphi,\mathfrak{J}), 0<\wp\le 1, (\varphi,t) \in \Omega \subset R^2, \tag{52}$$

with initial condition

$$\mathbb{V}(\varphi,0) = e^{\varphi^2}. \tag{53}$$

and the exact result is given as

$$\mathbb{V}(\varphi,\mathfrak{J}) = e^{\varphi^2+1}. \tag{54}$$

Now, we apply the new iterative transform method

$$\begin{aligned} \mathbb{V}_0(\varphi,\mathfrak{J}) &= e^{\varphi^2}, \\ \mathbb{V}_1(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D^2_{\mathfrak{J}}\mathbb{V}_0(\varphi,\mathfrak{J}) - \left(1+4\varphi^2\right)\mathbb{V}_0(\varphi,\mathfrak{J})\right\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\}, \\ \mathbb{V}_2(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D^2_{\mathfrak{J}}\mathbb{V}_1(\varphi,\mathfrak{J}) - \left(1+4\varphi^2\right)\mathbb{V}_1(\varphi,\mathfrak{J})\right\}\right] \\ &= e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}, \\ \mathbb{V}_3(\varphi,\mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D^2_{\mathfrak{J}}\mathbb{V}_2(\varphi,\mathfrak{J}) - \left(1+4\varphi^2\right)\mathbb{V}_2(\varphi,\mathfrak{J})\right\}\right] \\ &= e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}, \\ &\vdots \end{aligned} \tag{55}$$

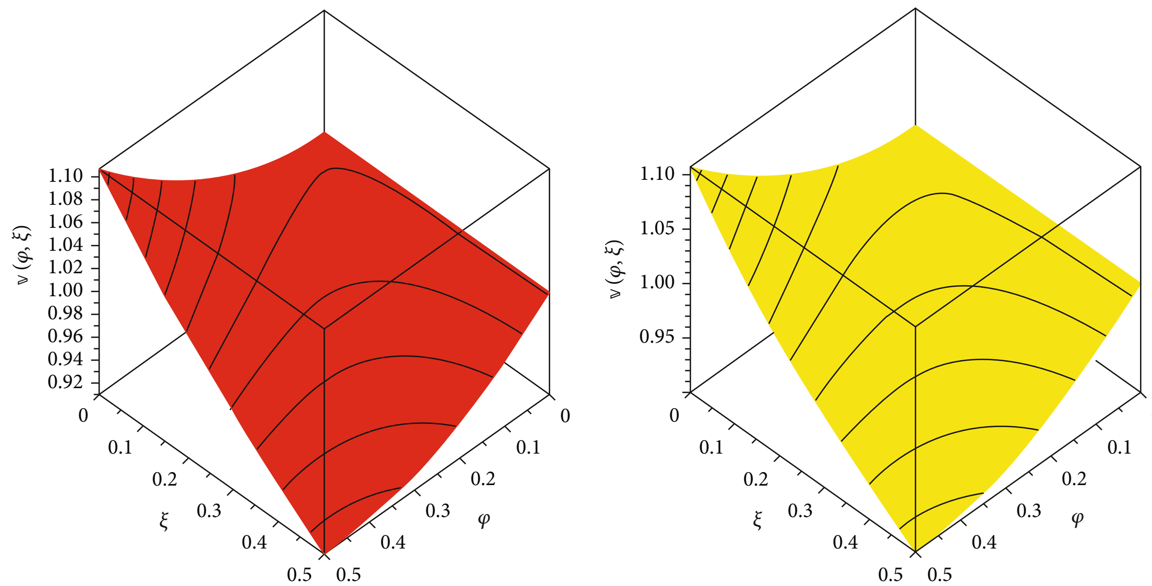

Figure 1: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

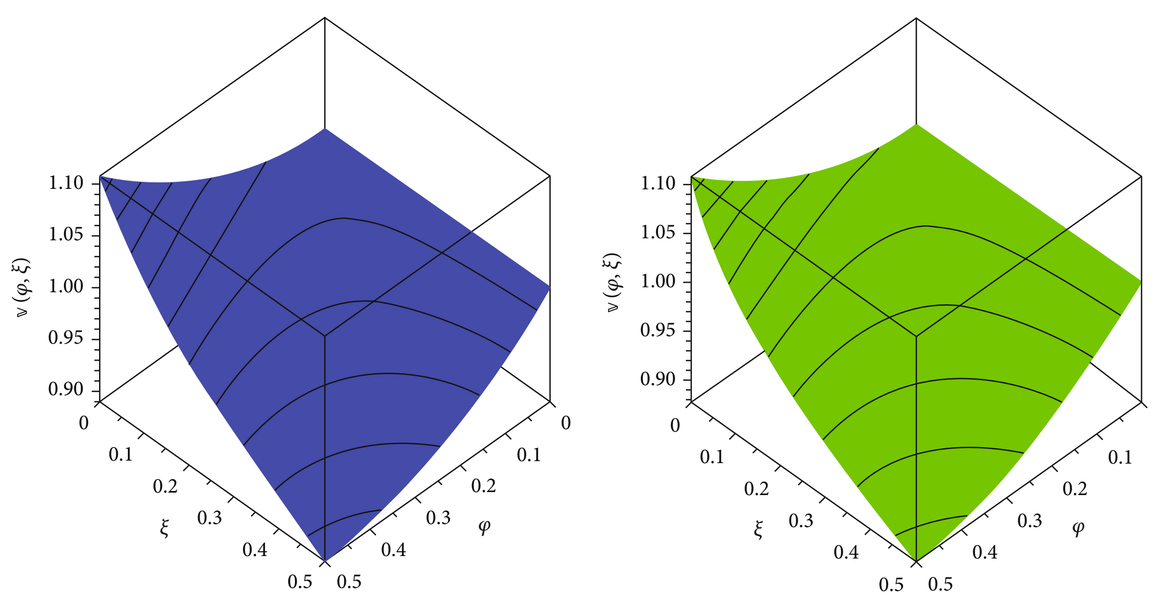

Figure 2: Different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

The series type solution is given as

$$\mathbb{V}(\varphi, \mathfrak{I}) = \mathbb{V}_0(\varphi, \mathfrak{I}) + \mathbb{V}_1(\varphi, \mathfrak{I}) + \mathbb{V}_2(\varphi, \mathfrak{I}) + \mathbb{V}_3(\varphi, \mathfrak{I}) + \cdots \mathbb{V}_i(\varphi, \mathfrak{I}). \tag{56}$$

The approximate solution of the above equation is defined as

$$\begin{aligned}\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi^2}\Bigg\{ & 1 + \{1 + \wp\mathfrak{I} - \wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\} \\ & + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\} + \cdots \Bigg\},\end{aligned}$$

$$\mathbb{V}(\varphi, \mathfrak{I}) = e^{\varphi^2} E_{\wp}\left(\mathfrak{I}^{\wp}\right). \tag{57}$$

Now by applying homotopy perturbation transform method, we get

$$\begin{aligned}&\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi, \mathfrak{I}) \\ &\quad = e^{\varphi^2} + p\left\{\mathbb{Y}^{-1}\left(\left(1 + \wp(\nu - 1)\right)\mathbb{Y}\left[\sum_{i=0}^{\infty} p^i H_i(w)\right]\right)\right\}.\end{aligned} \tag{58}$$

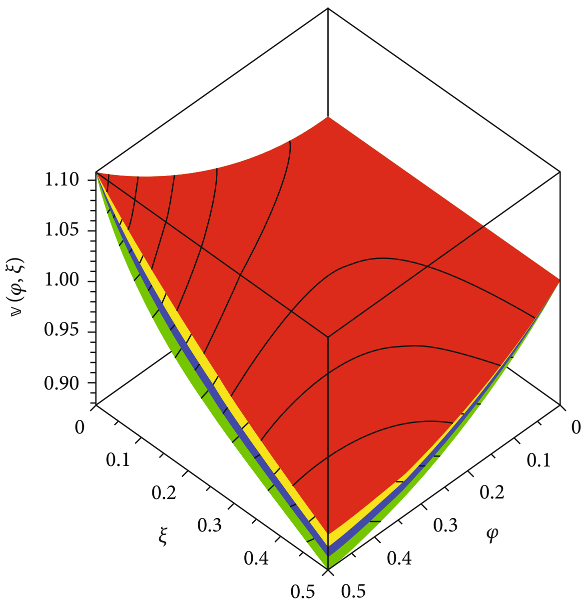

Figure 3: The different fractional-order $\wp$ of Example 10.

Comparing the coefficients of power p, we get

$$\begin{aligned}
p^0 &: \mathbb{V}_0(\varphi,\mathfrak{I}) = e^{\varphi^2},\\
p^1 &: \mathbb{V}_1(\varphi,\mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_0(w)])\right\} = e^{\varphi^2}\{1+\wp\mathfrak{I}-\wp\},\\
p^2 &: \mathbb{V}_2(\varphi,\mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_1(w)])\right\}\\
&\quad = e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\},\\
p^3 &: \mathbb{V}_3(\varphi,\mathfrak{I}) = \left\{\mathbb{Y}^{-1}((1+\wp(\nu-1))\mathbb{Y}[H_2(w)])\right\}\\
&\quad = e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\},\\
&\vdots
\end{aligned} \tag{59}$$

The HPTM series solution is given as

$$\begin{aligned}
\mathbb{V}(\varphi,\mathfrak{I}) &= \sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{I}),\\
\mathbb{V}(\varphi,\mathfrak{I}) &= e^{\varphi^2}\Bigg\{1 + \{1+\wp\mathfrak{I}-\wp\}\\
&\quad + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\\
&\quad + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2}\right.\\
&\quad \left. + \frac{\wp^3\mathfrak{I}^3}{3!}\right\}+\cdots\Bigg\},\\
\mathbb{V}(\varphi,\mathfrak{I}) &= e^{\varphi^2} E_{\wp}(\mathfrak{I}^{\wp}).
\end{aligned} \tag{60}$$

Now $\wp = 1$; then, the actual result of Equation (52) is $\mathbb{V}(\varphi,\mathfrak{I}) = e^{\varphi^2+\mathfrak{I}}$.

Figure 4 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 5 shows the separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 6, the graph shows the different fractional-order $\wp$ of Example 11.

Example 12. Consider fractional-order Cauchy-reaction diffusion equation [44]

$$^{\mathrm{CF}}D^{\wp}_{\mathfrak{I}}\mathbb{V}(\varphi,\mathfrak{I}) = D^2_{\mathfrak{I}}\mathbb{V}(\varphi,\mathfrak{I}) + 2\mathfrak{I}\mathbb{V}(\varphi,\mathfrak{I}),\ 0<\wp\le 1,\ (\varphi,\mathfrak{I})\in\Omega\subset R^2, \tag{61}$$

with initial condition

$$\mathbb{V}(\varphi,0) = e^{\varphi}. \tag{62}$$

The exact result is

$$\mathbb{V}(\varphi,\mathfrak{I}) = e^{\varphi+\mathfrak{I}+\mathfrak{I}^2}. \tag{63}$$

By using the Yang transformation, we get

$$\mathbb{V}(\varphi,\mathfrak{I}) = (e^{\varphi}) + \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left(D^2_{\mathfrak{I}}\mathbb{V}(\varphi,\mathfrak{I}) + 2\mathfrak{I}\mathbb{V}(\varphi,\mathfrak{I})\right)\right]. \tag{64}$$

Now, we apply the new iterative transform method

$$\begin{aligned}
\mathbb{V}_0(\varphi,\mathfrak{I}) &= e^{\varphi},\\
\mathbb{V}_1(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D^2_{\mathfrak{I}}\mathbb{V}_0(\varphi,\mathfrak{I}) + 2\mathfrak{I}\mathbb{V}_0(\varphi,\mathfrak{I})\right\}\right]\\
&= e^{\varphi}\left(\{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right),\\
\mathbb{V}_2(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D^2_{\mathfrak{I}}\mathbb{V}_1(\varphi,\mathfrak{I}) + 2\mathfrak{I}\mathbb{V}_1(\varphi,\mathfrak{I})\right\}\right]\\
&= e^{\varphi}\Bigg(\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\\
&\quad + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\}\Bigg),\\
&\vdots
\end{aligned} \tag{65}$$

The series type solution is given as

$$\mathbb{V}(\varphi,\mathfrak{I}) = \mathbb{V}_0(\varphi,\mathfrak{I}) + \mathbb{V}_1(\varphi,\mathfrak{I}) + \mathbb{V}_2(\varphi,\mathfrak{I}) + \mathbb{V}_3(\varphi,\mathfrak{I}) + \cdots \mathbb{V}_i(\varphi,\mathfrak{I}). \tag{66}$$

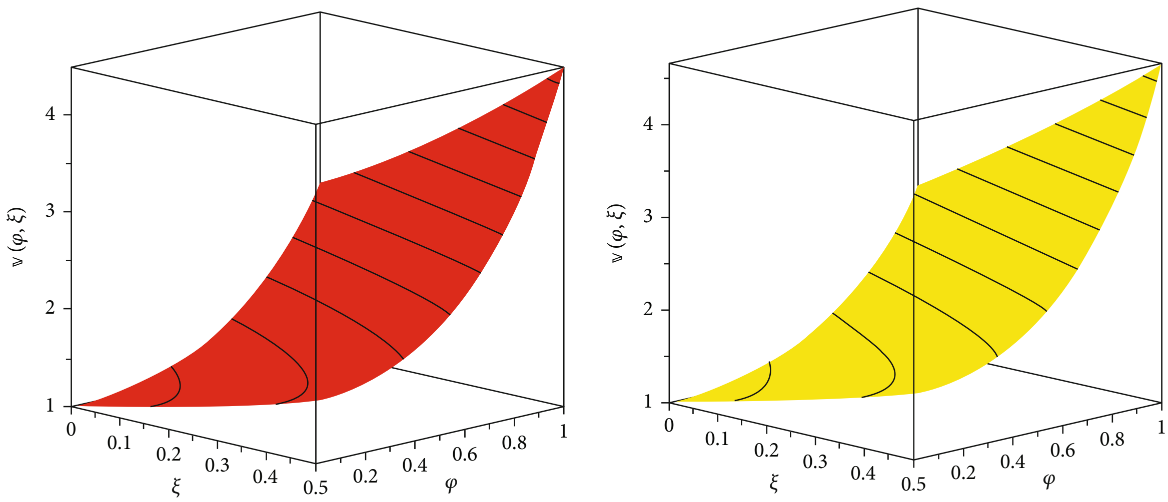

FIGURE 4: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

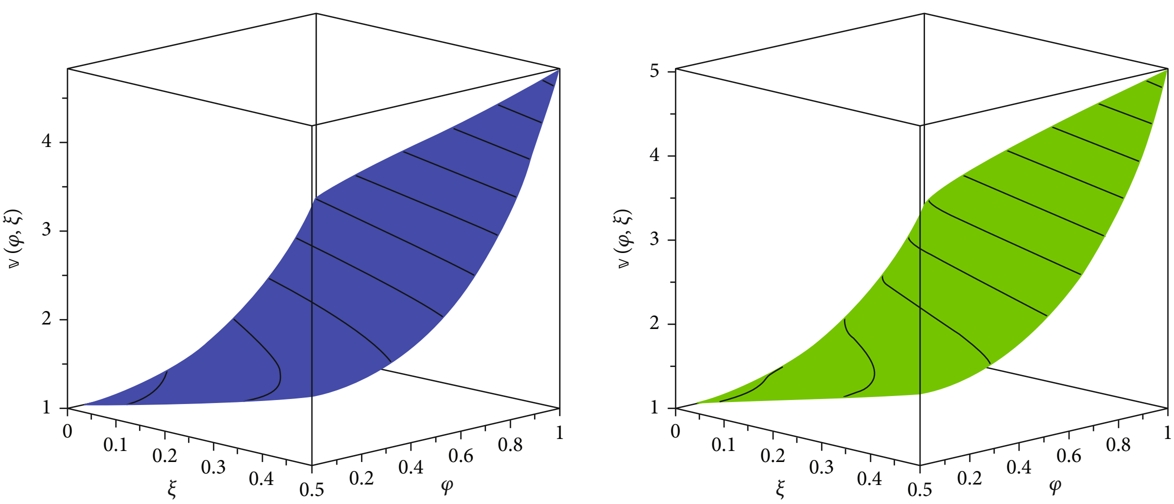

FIGURE 5: The different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

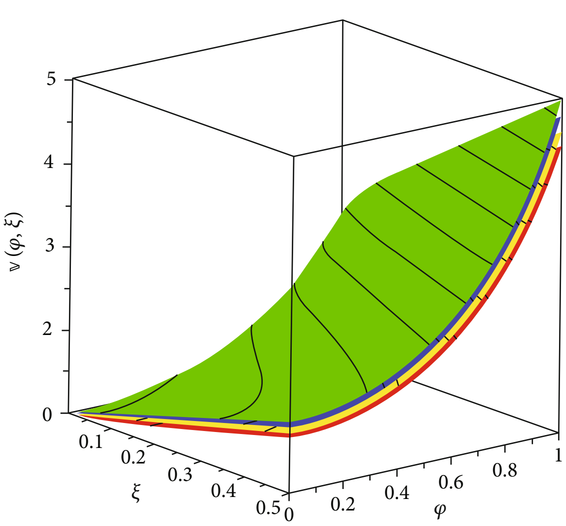

FIGURE 6: The different fractional-order $\wp$ of Example 10.

The approximate solution of the above equation is defined as

$$\mathbb{V}(\varphi,\mathfrak{I}) = e^{\varphi} + e^{\varphi}\left(\{1+\wp\mathfrak{I}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right. \\ + e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{I} + (1-\wp)^2 + \frac{\wp^2\mathfrak{I}^2}{2}\right\}\right. \\ \left.\left. + \left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\}\right) + \cdot \tag{67}$$

Now, using HPM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{I}) = e^{\varphi} + p\left[\mathbb{Y}^{-1}\left\{(1+\wp(\nu-1))\mathbb{Y}\left(\sum_{i=0}^{\infty} p^i H_i(w)\right)\right\}\right]. \tag{68}$$

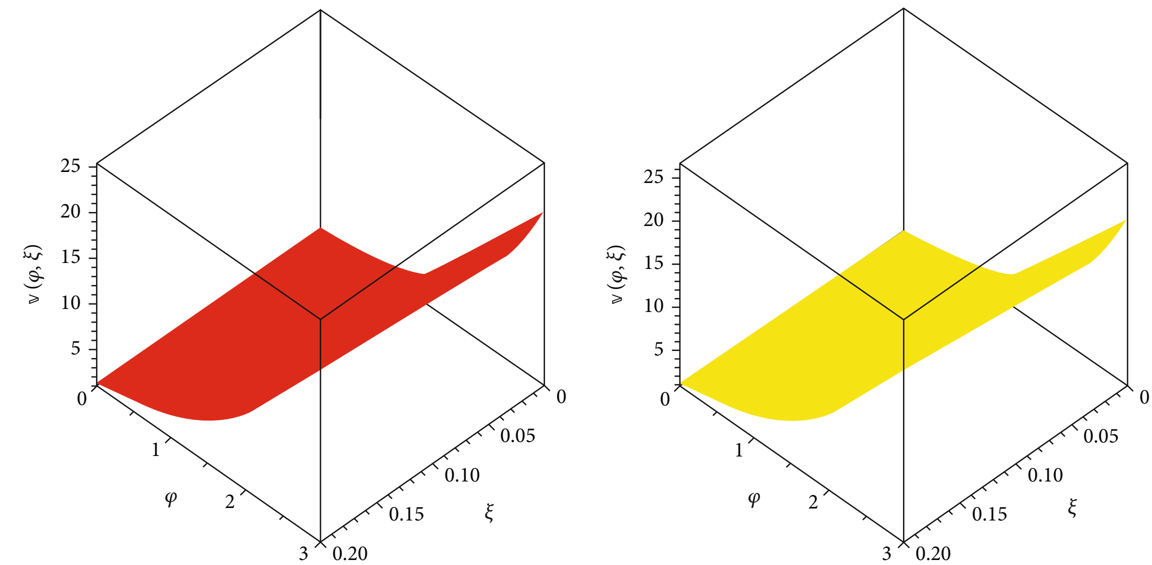

FIGURE 7: (a) $\wp = 1$ and (b) the fractional-order $\wp = 0.8$ of Example 10.

Comparing the coefficients of power p, we get

$$
\begin{aligned}
p^0 &: \mathbb{V}_0(\varphi, \mathfrak{J}) = e^{\varphi},\\
p^1 &: \mathbb{V}_1(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(w))\}\right]\\
&= e^{\varphi}\left(\{1+\wp\mathfrak{J}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right),\\
p^2 &: \mathbb{V}_2(\varphi, \mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(w))\}\right]\\
&= e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right.\\
&\quad \left. + \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\right).
\end{aligned} \tag{69}
$$

Proceeding in this path, the rest of the $\mathbb{V}n(\varphi, \mathfrak{J})$ for $n \geq 3$ component can be completely recovered and the series solution can therefore be absolutely determined. Eventually, we approximate the numerical solution $\mathbb{V}(\varphi, \mathfrak{J})$ to the truncated series.

$$
\begin{aligned}
\mathbb{V}(\varphi, \mathfrak{J}) &= \lim_{N\longrightarrow\infty} \sum_{n=1}^{N} \mathbb{V}_i(\varphi, \mathfrak{J}),\\
\mathbb{V}(\varphi, \mathfrak{J}) &= e^{\varphi} + e^{\varphi}\left(\{1+\wp\mathfrak{J}-\wp\} + \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right)\\
&\quad + e^{\varphi}\left(\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}\right.\\
&\quad \left. + \left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}\right) + \cdots.
\end{aligned} \tag{70}
$$

Now for $\wp = 1$, the closed form of the above series is

$$
\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi+\mathfrak{J}+\mathfrak{J}^2}. \tag{71}
$$

Figure 7 shows the analytical solution of two methods at different fractional-order $\wp = 1$ and 0.8, and Figure 8 shows the separate fractional-order at $\wp = 0.6$ and 0.4 with close contact with each other. In Figure 9, the graph shows the different fractional-order $\wp$ of Example 12.

Example 13. Consider fractional-order Cauchy-reaction diffusion equation as [44]

$$
\begin{aligned}
{}^{\mathrm{CF}}D_{\mathfrak{J}}^{\wp}\mathbb{V}(\varphi, \mathfrak{J}) &= D_{\mathfrak{J}}^{2}\mathbb{V}(\varphi, \mathfrak{J}) - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}\\
&\quad \cdot (\varphi, \mathfrak{J}), 0 < \wp \leq 1, (\varphi, \mathfrak{J}) \in \Omega \subset R^2,
\end{aligned} \tag{72}
$$

with initial condition

$$
\mathbb{V}(\varphi, 0) = e^{\varphi^2}. \tag{73}
$$

The exact result is

$$
\mathbb{V}(\varphi, \mathfrak{J}) = e^{\varphi^2+\mathfrak{J}^2}. \tag{74}
$$

Now, we apply the new iterative transform method

$$
\begin{aligned}
\mathbb{V}_0(\varphi, \mathfrak{J}) &= e^{\varphi^2},\\
\mathbb{V}_1(\varphi, \mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_0(\varphi, \mathfrak{J})\right.\right.\\
&\quad \left.\left. - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}_0(\varphi, \mathfrak{J})\right\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\},\\
\mathbb{V}_2(\varphi, \mathfrak{J}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{J}}^{2}\mathbb{V}_1(\varphi, \mathfrak{J})\right.\right.\\
&\quad \left.\left. - \left(4\varphi^2 - 2\mathfrak{J} + 2\right)\mathbb{V}_1(\varphi, \mathfrak{J})\right\}\right]\\
&= e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\},
\end{aligned}
$$

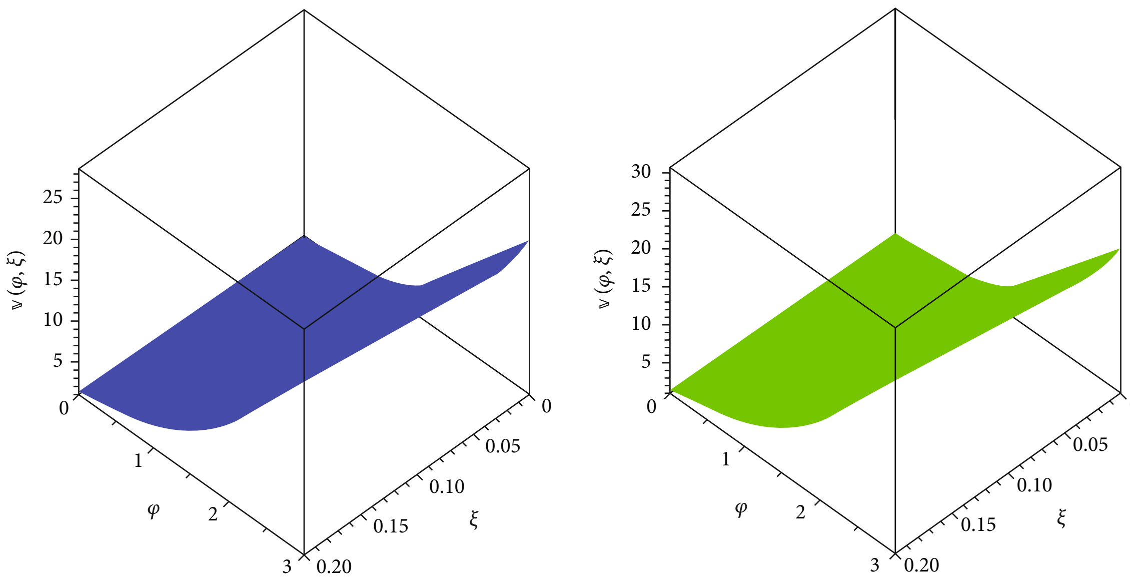

Figure 8: The different fractional-order of $\wp = 0.6$ and 0.4 of Example 10.

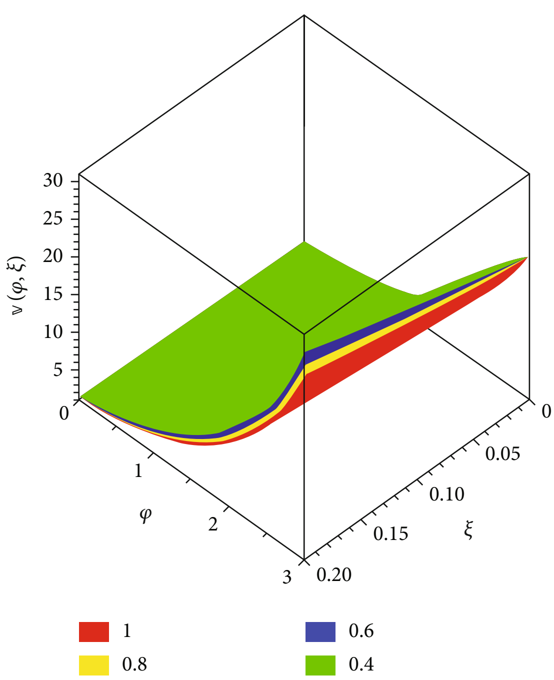

Figure 9: The different fractional-order $\wp$ of Example 10.

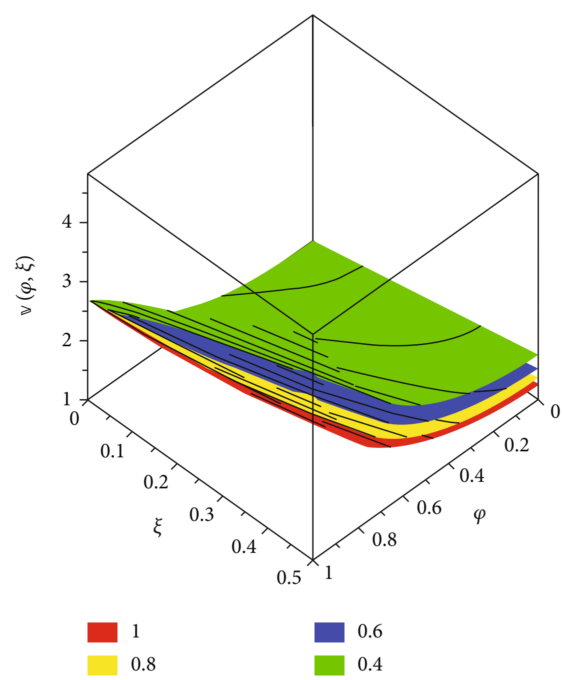

Figure 10: The different fractional-order $\wp$ of Example 13.

$$
\begin{aligned}
\mathbb{V}_3(\varphi,\mathfrak{I}) &= \mathbb{Y}^{-1}\left[(1+\wp(\nu-1))\mathbb{Y}\left\{D_{\mathfrak{I}}^{2}\mathbb{V}_2(\varphi,\mathfrak{I})\right.\right.\\
&\qquad\left.\left. -\left(4\varphi^2-2\mathfrak{I}+2\right)\mathbb{V}_2(\varphi,\mathfrak{I})\right\}\right]\\
&= e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{I} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{I}^2}{2} + \frac{\wp^3\mathfrak{I}^3}{3!}\right\},\\
&\;\;\vdots
\end{aligned}
\tag{75}
$$

The series type solution is given as

$$
\begin{aligned}
\mathbb{V}(\varphi,\mathfrak{I}) &= \mathbb{V}_0(\varphi,\mathfrak{I}) + \mathbb{V}_1(\varphi,\mathfrak{I}) + \mathbb{V}_2(\varphi,\mathfrak{I})\\
&\quad + \mathbb{V}_3(\varphi,\mathfrak{I}) + \cdots \mathbb{V}_i(\varphi,\mathfrak{I}).
\end{aligned}
\tag{76}
$$

The approximate solution of the above example is

$$\begin{aligned}\mathbb{V}(\varphi,\mathfrak{J}) &= e^{\varphi^2} + e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\} + e^{\varphi^2} \\ &\cdot \left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} \\ &+ e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots.\end{aligned} \tag{77}$$

Now, using the HPM, we get

$$\sum_{i=0}^{\infty} p^i \mathbb{V}_i(\varphi,\mathfrak{J}) = e^{\varphi^2} + p\left[\mathbb{Y}^{-1}\left\{(1+\wp(\nu-1))\mathbb{Y}\left(\sum_{i=0}^{\infty} p^i H_i(w)\right)\right\}\right]. \tag{78}$$

Comparing the coefficients of power p, we have

$$\begin{aligned}p^0 &: \mathbb{V}_0(\varphi,\mathfrak{J}) = e^{\varphi^2}, \\ p^1 &: \mathbb{V}_1(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_0(w))\}\right] = e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\}, \\ p^2 &: \mathbb{V}_2(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_1(w))\}\right] \\ &= e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\}, \\ p^3 &: \mathbb{V}_3(\varphi,\mathfrak{J}) = \left[\mathbb{Y}^{-1}\{(1+\wp(\nu-1))\mathbb{Y}(H_2(w))\}\right] \\ &= e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\}.\end{aligned} \tag{79}$$

Similarly, the remainder of the $\mathbb{V}_i(\varphi,\mathfrak{J})$ components for $n \geq 4$ can be completely achieved, thereby fully evaluating the series solutions. Finally, we estimate the approximate result $\mathbb{V}(\varphi,\mathfrak{J})$ by truncated sequence

$$\begin{aligned}\mathbb{V}(\varphi,\mathfrak{J}) &= \lim_{N\longrightarrow\infty} \sum_{n=1}^{N} \mathbb{V}_i(\varphi,\mathfrak{J}), \\ \mathbb{V}(\varphi,\mathfrak{J}) &= e^{\varphi^2} + e^{\varphi^2}\{1+\wp\mathfrak{J}-\wp\} \\ &+ e^{\varphi^2}\left\{(1-\wp)2\wp\mathfrak{J} + (1-\wp)^2 + \frac{\wp^2\mathfrak{J}^2}{2}\right\} \\ &+ e^{\varphi^2}\left\{(1-\wp)^2 3\wp\mathfrak{J} + (1-\wp)^3 \right. \\ &\left. + \frac{3\wp^2(1-\wp)\mathfrak{J}^2}{2} + \frac{\wp^3\mathfrak{J}^3}{3!}\right\} + \cdots.\end{aligned} \tag{80}$$

Figure 10 shows the analytical solution of two methods at different fractional-order $\wp = 1$, 0.8, 0.6, and 0.4 of Example 13. The special case for $\wp = 1$, and the above problem close form is given as

$$\mathbb{V}(\varphi,\mathfrak{J}) = e^{\varphi^2+\mathfrak{J}^2}. \tag{81}$$

6. Conclusion

The homotopy perturbation transform technique and the Iterative transform method are used in this article to obtain numerical solutions for the fractional-order Cauchy-reaction diffusion equation, which is broadly used in applied sciences as a problem for spatial effects. In physical models, the techniques produce a series of form results that converge quickly. The obtained results in this article are expected to be useful for further analysis of complicated nonlinear physical problems. The calculations for these techniques are very simple and straightforward. As a result, we can conclude that these techniques can be used to solve a variety of nonlinear fractional-order partial differential equation schemes.

Data Availability

The numerical data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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