Research Article

Necessary Conditions for the Existence of Global Solutions to Nonlinear Fractional Differential Inequalities and Systems

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Received 25 November 2021; Revised 24 December 2021; Accepted 27 December 2021; Published 22 January 2022

Academic Editor: Salah Mahmoud Boulaaras

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In this paper, fractional differential inequalities and systems of fractional differential inequalities involving fractional derivatives in the sense of Caputo are investigated. Namely, necessary conditions for the existence of global solutions are obtained. Our approach is based on the test function method and some integral inequalities.

1. Introduction and Main Results

In several studies, the usefulness of fractional derivatives in the mathematical modeling of various phenomena from physics and engineering has been demonstrated (see, e.g., [1–7], and the references therein). Due to this fact, the study of fractional differential equations has received a great deal of attention from many researchers. The existence of solutions is one of the most important topics of fractional differential equations. The study of sufficient conditions for the existence of solutions has been investigated by many authors using different approaches from functional analysis (see, e.g., [8–18], and the references therein). The study of necessary conditions for the existence of global solutions in the context of fractional differential equations has been initiated by Kirane and his collaborators (see, e.g., [19–24], and the references therein).

In [19], Furati and Kirane investigated the system of fractional differential equations:

$$\begin{cases} u_1'(t) + {}^C D_0^{\alpha_1} u_1(t) \geq |u_2(t)|^{p_2}, & t > 0, \\ u_2'(t) + {}^C D_0^{\alpha_2} u_2(t) \geq |u_1(t)|^{p_1}, & t > 0, \\ (u_1(0), u_2(0)) = (u_1^{(0)}, u_2^{(0)}), \end{cases} \quad (1)$$

where $0 < \alpha_1, \alpha_2 < 1$, $u_1^{(0)}, u_2^{(0)} > 0$, and $p_1, p_2 > 1$. Here, ${}^C D_0^{\alpha_i}$, $i \in \{1, 2\}$, denotes the Caputo fractional derivative of order α_i . Namely, it was shown that if

$$1 - \frac{1}{p_1 p_2} \leq \max \left\{ \alpha_1 + \frac{\alpha_2}{p_1}, \alpha_2 + \frac{\alpha_1}{p_2} \right\}, \quad (2)$$

then (1) admits no global solution.

Motivated by Furati and Kirane [19], in this paper, we first consider the fractional differential inequality

$$\begin{cases} u'(t) + {}^C D_0^{\alpha} u(t) \geq \rho(t) |u(a(t))|^p + \mu(t), & t > 0, \\ u(0) = u_0, \end{cases} \quad (3)$$

where $0 < \alpha < 1$, $p > 1$, $u_0 \in \mathbb{R}$, $\rho \in C([0, \infty))$, $\rho > 0$, $\mu \in L_{loc}^1([0, \infty))$, and $\mu \geq 0$. It is supposed that $a : [0, \infty) \rightarrow [0, \infty)$ is a C^1 function and satisfies

(A1) $a'(t) > 0$ for all $t \geq 0$

(A2) $a(0) = 0$

(A3) $a(t) \geq t$ for all $t \geq 0$

Our aim is to study the influence of $a(t)$ on the large time behavior of solutions. By a global solution to (3), we mean a function $u \in AC([0, \infty))$ (an absolutely continuous function) satisfying the fractional differential inequality in

(3) for almost everywhere $t > 0$, and the initial condition $u(0) = u_0$. Our aim is to derive sufficient conditions for which (3) admits no global solution. Namely, the following result is obtained.

Theorem 1. Suppose that

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T [\rho(a^{-1}(t))(a^{-1})'(t)]^{-1/p-1} dt} = +\infty. \quad (4)$$

Then (3) admits no global solution.

We provide below some examples where (4) is satisfied.

Example 2. Consider problem (3) with $a(t) = t$, $\rho \equiv 1$, $\mu \equiv 0$, and $u_0 > 0$. Then,

$$\frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T [\rho(a^{-1}(t))(a^{-1})'(t)]^{-1/p-1} dt} = u_0 T^{\alpha/p-1} \rightarrow +\infty, \text{ as } T \rightarrow +\infty. \quad (5)$$

Hence, by Theorem 1, we deduce that for all $p > 1$, (3) admits no global solution.

Example 3. Consider problem (3) with $a(t) = t$, $\rho \equiv 1$, $\mu \equiv 1$, and $u_0 \in \mathbb{R}$. Then

$$\begin{aligned} \frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T [\rho(a^{-1}(t))(a^{-1})'(t)]^{-1/p-1} dt} &= u_0 T^{\alpha(p/p-1-1)} \\ &+ \frac{1}{2} T^{\alpha p/p-1} \rightarrow +\infty, \text{ as } T \rightarrow +\infty. \end{aligned} \quad (6)$$

Hence, by Theorem 1, we deduce that for all $p > 1$, (3) admits no global solution.

Example 4. Consider problem (3) with $u_0 > 0$ and

$$a(t) = (t+1)^\gamma - 1, \rho(t) = (t+1)^\beta, \mu(t) = t^\delta, \quad (7)$$

where

$$\gamma \geq 1, -1 < \delta < -\alpha, \gamma(1-\alpha) < 1 + \beta, p > \max \left\{ \frac{1+\beta}{\gamma}, 1 \right\}. \quad (8)$$

In this case, after elementary calculations, we obtain

$$\frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T [\rho(a^{-1}(t))(a^{-1})'(t)]^{-1/p-1} dt} \sim C u_0 T^{(\beta+1-\gamma(1-\alpha))/\gamma(p-1)} \rightarrow +\infty, \text{ as } T \rightarrow +\infty, \quad (9)$$

where $C > 0$ is a constant independent of T . Hence, by Theorem 1, we deduce that for all $p > \max \{1 + \beta/\gamma, 1\}$, (3) admits no global solution.

Example 5. Consider problem (3) with $u_0 \in \mathbb{R}$ and

$$a(t) = (t+1)^\gamma - 1, \rho(t) = (t+1)^\beta, \mu(t) = t^\delta, \quad (10)$$

where

$$\gamma \geq 1, \delta > -\alpha, \gamma(1-\alpha) < 1 + \beta, p > \max \left\{ \frac{1+\beta}{\gamma}, \frac{\delta+1-1+\beta/\gamma}{\alpha+\delta}, 1 \right\}. \quad (11)$$

In this case, after elementary calculations, we obtain

$$\frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T [\rho(a^{-1}(t))(a^{-1})'(t)]^{-1/p-1} dt} \sim C T^{p(\alpha+\delta) - (\delta+1-1+\beta/\gamma)/p-1} \rightarrow +\infty, \text{ as } T \rightarrow +\infty, \quad (12)$$

where $C > 0$ is a constant independent of T . Hence, by Theorem 1, we deduce that for all $p > \max \{1 + \beta/\gamma, \delta + 1 - 1 + \beta/\gamma/\alpha + \delta, 1\}$, (3) admits no global solution.

In the second part of this paper, we extend the previous study to the system of fractional differential inequalities:

$$\begin{cases} u_1'(t) + {}^C D_0^{\alpha_1} u_1(t) \geq \rho_2(t) |u_2(a_2(t))|^{p_2} + \mu_2(t), & t > 0, \\ u_2'(t) + {}^C D_0^{\alpha_2} u_2(t) \geq \rho_1(t) |u_1(a_1(t))|^{p_1} + \mu_1(t), & t > 0, \\ (u_1(0), u_2(0)) = (u_1^{(0)}, u_2^{(0)}), \end{cases} \quad (13)$$

where for $i \in \{1, 2\}$, $0 < \alpha_i < 1$, $p_i > 1$, $u_i^{(0)} \in \mathbb{R}$, $\rho_i \in C([0, \infty))$, $\rho_i > 0$, $\mu_i \in L_{\text{loc}}^1([0, \infty))$, and $\mu_i \geq 0$. Moreover, it is supposed that $a_i : [0, \infty) \rightarrow [0, \infty)$ is a C^1 function and satisfies (A1)–(A3). Notice that in the special case $\rho_i \equiv 1$, $a_i(t) = t$, and $\mu_i \equiv 0$, (13) reduces to (1).

By a global solution to (13), we mean a pair of functions $(u_1, u_2) \in AC([0, \infty)) \times AC([0, \infty))$ satisfying the fractional differential inequalities in (13) for almost everywhere $t > 0$, and the initial condition $(u_1(0), u_2(0)) = (u_1^{(0)}, u_2^{(0)})$.

We have the following result.

Theorem 6.

(i) Let $u_1^{(0)} \geq 0$ and $u_2^{(0)} \in \mathbb{R}$. If

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha_1 p_1 + \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} \right)}{\left(\int_0^T [\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t)]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2-1} \left(\int_0^T [\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t)]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2-1}} = +\infty, \quad (14)$$

then (13) admits no global solution

(ii) Let $u_1^{(0)} \in \mathbb{R}$ and $u_2^{(0)} \geq 0$. If

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha_2 p_2 + \alpha_1 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_2(t) dt + T^{1-\alpha_1} u_1^{(0)} \right)}{\left(\int_0^T [\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t)]^{-1/p_1-1} dt \right)^{p_2(p_1-1)/p_1 p_2-1} \left(\int_0^T [\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t)]^{-1/p_2-1} dt \right)^{p_2-1/p_1 p_2-1}} = +\infty, \quad (15)$$

then (13) admits no global solution

We provide below some examples for which (14) or (15) are satisfied.

Example 7. Consider (1), that is, System (13) with $a_i(t) = t$, $\rho_i \equiv 1$, $\mu_i \equiv 0$, and $u_i^{(0)} > 0$, for all $i \in \{1, 2\}$. Then

$$\frac{T^{\alpha_1 p_1 + \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} \right)}{\left(\int_0^T [\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t)]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2-1} \left(\int_0^T [\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t)]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2-1}} = T^{\alpha_1 p_1 + \alpha_2 / p_1 p_2 - 1} u_2^{(0)} \longrightarrow +\infty, \text{ as } T \longrightarrow +\infty. \quad (16)$$

Hence, by Theorem 6, we deduce that for all $p_1, p_2 > 1$, (1) admits no global solution. This improves [19] (Theorem 1), where the nonexistence of a global solution was obtained only when $1 - 1/p_1 p_2 \leq \max \{ \alpha_1 + \alpha_2 / p_1, \alpha_2 + \alpha_1 / p_2 \}$.

Example 8. Consider System (13) with

$$\begin{aligned} a_1(t) &= t(t+2), a_2(t) = t, \rho_1(t) = \rho_2(t) = (t+1)^3, \mu_1(t) = \mu_2(t) = t^\delta, \\ \alpha_1 &= \alpha_2 = \alpha, -1 < \delta < -\alpha, u_1^{(0)}, u_2^{(0)} > 0, p_1 > 2, p_2 > 4. \end{aligned} \quad (17)$$

In this case, after elementary calculations, we obtain

$$\begin{aligned} & \frac{T^{\alpha_1 p_1 + \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} \right)}{\left(\int_0^T [\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t)]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2-1} \left(\int_0^T [\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t)]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2-1}} \sim C T^{(\alpha+3)p_1 + \alpha+1/p_1 p_2-1} \longrightarrow +\infty, \text{ as } T \longrightarrow +\infty, \quad (18) \\ & \frac{T^{\alpha_2 p_2 + \alpha_1 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_2(t) dt + T^{1-\alpha_1} u_1^{(0)} \right)}{\left(\int_0^T [\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t)]^{-1/p_1-1} dt \right)^{p_2(p_1-1)/p_1 p_2-1} \left(\int_0^T [\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t)]^{-1/p_2-1} dt \right)^{p_2-1/p_1 p_2-1}} \sim C T^{\alpha(p_1+1)+p_2/p_1 p_2-1} \longrightarrow +\infty, \text{ as } T \longrightarrow +\infty. \quad (19) \end{aligned}$$

Hence, by Theorem 6, we deduce that for all $p_1 > 2$ and $p_2 > 4$, (13) admits no global solution.

The rest of this paper is organized as follows. In Section 2, we recall briefly some notions related to fractional calculus and provide some lemmas that will be used in the proofs of our main results. In Section 3, we prove Theorems 1 and 6.

2. Preliminaries

We first recall some basic notions and properties related to fractional calculus (see, e.g., [25, 26]).

Let $T > 0$ and $I = [0, T]$. The left-sided Riemann-Liouville fractional integral of order $\sigma > 0$ of a function $f \in L^1(I)$ is defined by

$$(I_0^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} f(s) ds, \quad (20)$$

for almost everywhere $t \in I$, where Γ is the Gamma function. The right-sided Riemann-Liouville fractional integral of order $\sigma > 0$ of a function $f \in L^1(I)$ is defined by

$$(I_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} f(s) ds, \quad (21)$$

for almost everywhere $t \in I$.

The Caputo fractional derivative of order $\sigma \in (0, 1)$ of a function $f \in AC(I)$ is defined by

$${}^C D_0^\sigma f(t) = \left(I_0^{1-\sigma} f' \right)(t) = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} f'(s) ds, \quad (22)$$

for almost everywhere $t \in I$.

Lemma 9 (see [25], Lemma 2.7). *Let $\sigma > 0$, $r, s \geq 1$, and $1/r + 1/s \leq 1 + \sigma$ ($r \neq 1$, $s \neq 1$, in the case $1/r + 1/s = 1 + \sigma$). If $(f, g) \in L^r(I) \times L^s(I)$, then*

$$\int_0^T (I_0^\sigma f)(t) g(t) dt = \int_0^T f(t) (I_T^\sigma g)(t) dt. \quad (23)$$

For $\theta \gg 1$ (θ is sufficiently large), let

$$\phi(t) = T^{-\theta} (T-t)^\theta, \quad t \in I. \quad (24)$$

The following results can be found in [19].

Lemma 10. *Let $\sigma > 0$. Then*

$$(I_T^\sigma \phi)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\sigma+\theta+1)} T^{-\theta} (T-t)^{\theta+\sigma}, \quad (25)$$

$$(I_T^\sigma \phi)'(t) = -\frac{\Gamma(\theta+1)}{\Gamma(\sigma+\theta)} T^{-\theta} (T-t)^{\theta+\sigma-1}$$

for all $t \in I$.

3. Proofs of the Main Results

The proofs of our main results are based on the test function method developed by Mitidieri and Pohozaev [27].

Proof of Theorem 1. We use the contradiction argument. Namely, suppose that $u \in AC([0, \infty))$ is a global solution to (3). Multiplying the fractional differential inequality in (3) by the function ϕ defined by (24) with $\theta \gg 1$, and integrating over $(0, T)$, $T \gg 1$, we obtain

$$\begin{aligned} & \int_0^T \rho(t) |u(a(t))|^p \phi(t) dt + \int_0^T \mu(t) \phi(t) dt \\ & \leq \int_0^T u'(t) \phi(t) dt + \int_0^T {}^C D_0^\alpha u(t) \phi(t) dt. \end{aligned} \quad (26)$$

By the initial condition $u(0) = u_0$, we have

$$\int_0^T u'(t) \phi(t) dt = u(T) \phi(T) - u_0 \phi(0) - \int_0^T u(t) \phi'(t) dt. \quad (27)$$

□

Notice that by the definition of ϕ , we have $\phi(T) = 0$ and $\phi(0) = 1$. Therefore, it holds that

$$\int_0^T u'(t) \phi(t) dt = -u_0 - \int_0^T u(t) \phi'(t) dt. \quad (28)$$

Next, using Lemma 9, we obtain

$$\int_0^T {}^C D_0^\alpha u(t) \phi(t) dt = \int_0^T \left(I_0^{1-\alpha} u' \right)(t) \phi(t) dt = \int_0^T u'(t) (I_T^{1-\alpha} \phi)(t) dt. \quad (29)$$

Integrating by parts and using the initial condition $u(0) = u_0$, we obtain

$$\begin{aligned} & \int_0^T u'(t) (I_T^{1-\alpha} \phi)(t) dt = u(T) (I_T^{1-\alpha} \phi)(T) - u_0 (I_T^{1-\alpha} \phi)(0) \\ & \quad - \int_0^T u(t) (I_T^{1-\alpha} \phi)'(t) dt. \end{aligned} \quad (30)$$

On the other hand, by Lemma 10, we have $(I_T^{1-\alpha} \phi)(T) = 0$ and $(I_T^{1-\alpha} \phi)(0) = \Gamma(\theta+1)/\Gamma(2-\alpha+\theta) T^{1-\alpha}$. Therefore,

$$\int_0^T u'(t) (I_T^{1-\alpha} \phi)(t) dt = -\frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} u_0 T^{1-\alpha} - \int_0^T u(t) (I_T^{1-\alpha} \phi)'(t) dt. \quad (31)$$

Combining (29) with (31), we deduce that

$$\int_0^T {}^C D_0^\alpha u(t) \phi(t) dt = -\frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} u_0 T^{1-\alpha} - \int_0^T u(t) (I_T^{1-\alpha} \phi)'(t) dt. \quad (32)$$

Hence, it follows from (28) and (29) that

$$\begin{aligned} & \int_0^T u'(t) \phi(t) dt + \int_0^T {}^C D_0^\alpha u(t) \phi(t) dt \\ &= -\left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} T^{1-\alpha}\right) u_0 - \int_0^T u(t) \phi'(t) dt \\ & \quad - \int_0^T u(t) (I_T^{1-\alpha} \phi)'(t) dt. \end{aligned} \quad (33)$$

Consider now the terms from the left side of (26). By a change of variable, using the properties (A1)–(A3), and the decay property of ϕ , we obtain

$$\begin{aligned} \int_0^T \rho(t) |u(a(t))|^p \phi(t) dt &= \int_0^{a(T)} \rho(a^{-1}(t)) |u(t)|^p \phi(a^{-1}(t)) (a^{-1})'(t) dt \\ &\geq \int_0^T \rho(a^{-1}(t)) |u(t)|^p \phi(t) (a^{-1})'(t) dt. \end{aligned} \quad (34)$$

Next, by (24), and using that $\mu \geq 0$ (we have also $\mu \in L_{\text{loc}}^1([0, \infty))$), we obtain

$$\begin{aligned} \int_0^T \mu(t) \phi(t) dt &= T^{-\theta} \int_0^T \mu(t) (T-t)^\theta dt \\ &\geq T^{-\theta} \int_0^{T/2} \mu(t) (T-t)^\theta dt \\ &\geq 2^{-\theta} \int_0^{T/2} \mu(t) dt. \end{aligned} \quad (35)$$

Combining (34) with (35), we deduce that

$$\begin{aligned} & \int_0^T \rho(t) |u(a(t))|^p \phi(t) dt + \int_0^T \mu(t) \phi(t) dt \\ &\geq \int_0^T \rho(a^{-1}(t)) |u(t)|^p \phi(t) (a^{-1})'(t) dt + 2^{-\theta} \int_0^{T/2} \mu(t) dt. \end{aligned} \quad (36)$$

Hence, it follows from (26), (33), and (36) that

$$\begin{aligned} & \int_0^T |u(t)|^p \psi(t) dt + 2^{-\theta} \int_0^{T/2} \mu(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} T^{1-\alpha}\right) u_0 \\ &\leq \int_0^T |u(t)| |\phi'(t)| dt + \int_0^T |u(t)| |(I_T^{1-\alpha} \phi)'(t)| dt, \end{aligned} \quad (37)$$

where

$$\psi(t) = \rho(a^{-1}(t)) (a^{-1})'(t) \phi(t), \quad t \in (0, T). \quad (38)$$

Next, using Young's inequality, we obtain

$$\int_0^T |u(t)| |\phi'(t)| dt \leq \frac{1}{2} \int_0^T |u(t)|^p \psi(t) dt + C \int_0^T \psi(t)^{-1/p-1} |\phi'(t)|^{p/p-1} dt, \quad (39)$$

$$\begin{aligned} \int_0^T |u(t)| |(I_T^{1-\alpha} \phi)'(t)| dt &\leq \frac{1}{2} \int_0^T |u(t)|^p \psi(t) dt \\ &\quad + C \int_0^T \psi(t)^{-1/p-1} |(I_T^{1-\alpha} \phi)'(t)|^{p/p-1} dt, \end{aligned} \quad (40)$$

where C denotes a positive constant (independent of T) whose value may change from line to line. Then, it follows from (37), (39), and (40) that

$$2^{-\theta} \int_0^{T/2} \mu(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} T^{1-\alpha}\right) u_0 \leq C(I_1(T) + I_2(T)), \quad (41)$$

where

$$\begin{aligned} I_1(T) &= \int_0^T \psi(t)^{-1/p-1} |\phi'(t)|^{p/p-1} dt, \\ I_2(T) &= \int_0^T \psi(t)^{-1/p-1} |(I_T^{1-\alpha} \phi)'(t)|^{p/p-1} dt. \end{aligned} \quad (42)$$

Let us estimate the terms $I_i(T)$, $i = 1, 2$. For all $t \in (0, T)$, by (24) and (38), we have

$$\begin{aligned} \psi(t)^{-1/p-1} |\phi'(t)|^{p/p-1} &= [\rho(a^{-1}(t)) (a^{-1})'(t) \phi(t)]^{-1/p-1} |\phi'(t)|^{p/p-1} \\ &= C [\rho(a^{-1}(t)) (a^{-1})'(t)]^{-1/p-1} T^{\theta/p-1} \\ &\quad \cdot (T-t)^{-\theta/p-1} T^{-\theta p/p-1} (T-t)^{(\theta-1)p/p-1} \\ &\leq C [\rho(a^{-1}(t)) (a^{-1})'(t)]^{-1/p-1} T^{-\theta} T^{\theta-p/p-1} \\ &= C [\rho(a^{-1}(t)) (a^{-1})'(t)]^{-1/p-1} T^{-p/p-1}. \end{aligned} \quad (43)$$

Integrating over $(0, T)$, we obtain

$$I_1(T) \leq T^{-p/p-1} \int_0^T [\rho(a^{-1}(t)) (a^{-1})'(t)]^{-1/p-1} dt. \quad (44)$$

Next, by (38) and Lemma 10, for all $t \in (0, T)$, we have

$$\begin{aligned} \psi(t)^{-1/p-1} |(I_T^{1-\alpha} \phi)'(t)|^{p/p-1} &= C \left[\rho(a^{-1}(t)) (a^{-1})'(t) T^{-\theta} (T-t)^{\theta} \right]^{-1/p-1} \\ &\quad \cdot \left[T^{-\theta} (T-t)^{\theta-\alpha} \right]^{p/p-1} \\ &= C \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} T^{-\theta} (T-t)^{\theta-\alpha p/p-1} \\ &\leq C \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} T^{-\alpha p/p-1}. \end{aligned} \quad (45)$$

Integrating over $(0, T)$, we obtain

$$I_2(T) \leq T^{-\alpha p/p-1} \int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt. \quad (46)$$

Therefore, it follows from (41), (44), and (46) that

$$\begin{aligned} 2^{-\theta} \int_0^{T/2} \mu(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha+\theta)} T^{1-\alpha} \right) u_0 \\ \leq C \left(T^{-p/p-1} \int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt \right. \\ \left. + T^{-\alpha p/p-1} \int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt \right) \\ \leq CT^{-\alpha p/p-1} \int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt, \end{aligned} \quad (47)$$

which yields

$$\int_0^{T/2} \mu(t) dt + T^{1-\alpha} u_0 \leq CT^{-\alpha p/p-1} \int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt. \quad (48)$$

Hence, we deduce that

$$\frac{T^{\alpha p/p-1} \left(\int_0^{T/2} \mu(t) dt + T^{1-\alpha} u_0 \right)}{\int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt} \leq C, \quad (49)$$

which contradicts (4). The proof is completed.

Proof of Theorem 6. Suppose that $(u_1, u_2) \in AC([0, \infty)) \times A C([0, \infty))$ is a global solution to (13). Multiplying the first fractional differential inequality in (13) by the function ϕ defined by (24) with $\theta \gg 1$, and integrating over $(0, T)$, $T \gg 1$, we obtain

$$\begin{aligned} \int_0^T \rho_2(t) |u_2(a_2(t))|^{p_2} \phi(t) dt + \int_0^T \mu_2(t) \phi(t) dt \\ \leq \int_0^T u_1'(t) \phi(t) dt + \int_0^T {}^C D_0^{\alpha_1} u_1(t) \phi(t) dt. \end{aligned} \quad (50)$$

□

Following the same steps of the Proof of Theorem 1, we obtain

$$\begin{aligned} \int_0^T u_1'(t) \phi(t) dt + \int_0^T {}^C D_0^{\alpha_1} u_1(t) \phi(t) dt \\ = - \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_1+\theta)} T^{1-\alpha_1} \right) u_1^{(0)} \\ - \int_0^T u_1(t) \phi'(t) dt - \int_0^T u_1(t) \left(I_T^{1-\alpha_1} \phi \right)'(t) dt, \end{aligned} \quad (51)$$

$$\begin{aligned} \int_0^T \rho_2(t) |u_2(a_2(t))|^{p_2} \phi(t) dt + \int_0^T \mu_2(t) \phi(t) dt \\ \geq \int_0^T \rho_2(a_2^{-1}(t)) |u_2(t)|^{p_2} \phi(t) (a_2^{-1})'(t) dt \\ + 2^{-\theta} \int_0^{T/2} \mu_2(t) dt. \end{aligned} \quad (52)$$

Hence, it follows from (50), (51), and (52) that

$$\begin{aligned} \int_0^T |u_2(t)|^{p_2} \psi_2(t) dt + 2^{-\theta} \int_0^{T/2} \mu_2(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_1+\theta)} T^{1-\alpha_1} \right) u_1^{(0)} \\ \leq \int_0^T |u_1(t)| |\phi'(t)| dt + \int_0^T |u_1(t)| \left| \left(I_T^{1-\alpha_1} \phi \right)'(t) \right| dt, \end{aligned} \quad (53)$$

where

$$\psi_2(t) = \rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \phi(t), \quad t \in (0, T). \quad (54)$$

Arguing similarly with the second inequality in (13), we obtain

$$\begin{aligned} \int_0^T |u_1(t)|^{p_1} \psi_1(t) dt + 2^{-\theta} \int_0^{T/2} \mu_1(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_2+\theta)} T^{1-\alpha_2} \right) u_2^{(0)} \\ \leq \int_0^T |u_2(t)| |\phi'(t)| dt + \int_0^T |u_2(t)| \left| \left(I_T^{1-\alpha_2} \phi \right)'(t) \right| dt, \end{aligned} \quad (55)$$

where

$$\psi_1(t) = \rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \phi(t), \quad t \in (0, T). \quad (56)$$

Next, by Hölder's inequality, we have

$$\begin{aligned} \int_0^T |u_1(t)| |\phi'(t)| dt \leq \left(\int_0^T |u_1(t)|^{p_1} \psi_1(t) dt \right)^{1/p_1} \\ \cdot \left(\int_0^T \psi_1(t)^{-1/p_1-1} |\phi'(t)|^{p_1/p_1-1} dt \right)^{p_1-1/p_1}, \end{aligned} \quad (57)$$

$$\begin{aligned} \int_0^T |u_1(t)| \left| \left(I_T^{1-\alpha_1} \phi \right)'(t) \right| dt &\leq \left(\int_0^T |u_1(t)|^{p_1} \psi_1(t) dt \right)^{1/p_1} \\ &\cdot \left(\int_0^T \psi_1(t)^{-1/p_1-1} \left| \left(I_T^{1-\alpha_1} \phi \right)'(t) \right|^{p_1/p_1-1} dt \right)^{p_1-1/p_1}. \end{aligned} \quad (58)$$

Similarly, we have

$$\begin{aligned} \int_0^T |u_2(t)| \left| \phi'(t) \right| dt &\leq \left(\int_0^T |u_2(t)|^{p_2} \psi_2(t) dt \right)^{1/p_2} \\ &\cdot \left(\int_0^T \psi_2(t)^{-1/p_2-1} \left| \phi'(t) \right|^{p_2/p_2-1} dt \right)^{p_2-1/p_2}, \\ \int_0^T |u_2(t)| \left| \left(I_T^{1-\alpha_2} \phi \right)'(t) \right| dt &\leq \left(\int_0^T |u_2(t)|^{p_2} \psi_2(t) dt \right)^{1/p_2} \\ &\cdot \left(\int_0^T \psi_2(t)^{-1/p_2-1} \left| \left(I_T^{1-\alpha_2} \phi \right)'(t) \right|^{p_2/p_2-1} dt \right)^{p_2-1/p_2}. \end{aligned} \quad (59)$$

$$\quad (60)$$

For $i \in \{1, 2\}$, let

$$\begin{aligned} J_i &= \int_0^T |u_i(t)|^{p_i} \psi_i(t) dt, K_{\psi_i}(T) \\ &= \left(\int_0^T \psi_i(t)^{-1/p_i-1} \left| \phi'(t) \right|^{p_i/p_i-1} dt \right)^{p_i-1/p_i}, \\ L_{\psi_i}(T) &= \left(\int_0^T \psi_i(t)^{-1/p_i-1} \left| \left(I_T^{1-\alpha_i} \phi \right)'(t) \right|^{p_i/p_i-1} dt \right)^{p_i-1/p_i}. \end{aligned} \quad (61)$$

It follows from (53), (57), and (58) that

$$\begin{aligned} J_2 + 2^{-\theta} \int_0^{T/2} \mu_2(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_1+\theta)} T^{1-\alpha_1} \right) u_1^{(0)} \\ \leq J_1^{1/p_1} \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right). \end{aligned} \quad (62)$$

Similarly, by (55), (59), and (60), we obtain

$$\begin{aligned} J_1 + 2^{-\theta} \int_0^{T/2} \mu_1(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_2+\theta)} T^{1-\alpha_2} \right) u_2^{(0)} \\ \leq J_2^{1/p_2} \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right). \end{aligned} \quad (63)$$

Now, we consider the case $u_1^{(0)} \geq 0$ and $u_2^{(0)} \in \mathbb{R}$. In this

case, using (62) and (63), we obtain

$$\begin{aligned} J_1 + 2^{-\theta} \int_0^{T/2} \mu_1(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_2+\theta)} T^{1-\alpha_2} \right) u_2^{(0)} \\ \leq J_1^{1/p_1} \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{1/p_2} \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right). \end{aligned} \quad (64)$$

On the other hand, using Young's inequality, we obtain

$$\begin{aligned} J_1^{1/p_1} \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{1/p_2} \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right) \\ \leq \frac{1}{p_1 p_2} J_1 + \frac{p_1 p_2 - 1}{p_1 p_2} \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{p_1/p_1 p_2 - 1} \\ \times \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right)^{p_1 p_2 / p_1 p_2 - 1}. \end{aligned} \quad (65)$$

Therefore, combining (64) with (65), it holds that

$$\begin{aligned} 2^{-\theta} \int_0^{T/2} \mu_1(t) dt + \left(1 + \frac{\Gamma(\theta+1)}{\Gamma(2-\alpha_2+\theta)} T^{1-\alpha_2} \right) u_2^{(0)} \\ \leq C \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{p_1/p_1 p_2 - 1} \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right)^{p_1 p_2 / p_1 p_2 - 1}, \end{aligned} \quad (66)$$

which yields

$$\begin{aligned} \int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} &\leq C \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{p_1/p_1 p_2 - 1} \\ &\cdot \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right)^{p_1 p_2 / p_1 p_2 - 1}. \end{aligned} \quad (67)$$

Next, using the estimates (44) and (46), we obtain (after elementary calculations)

$$\begin{aligned} \left(K_{\psi_1}(T) + L_{\psi_1}(T) \right)^{p_1/p_1 p_2 - 1} \left(K_{\psi_2}(T) + L_{\psi_2}(T) \right)^{p_1 p_2 / p_1 p_2 - 1} \\ \leq C T^{-\alpha_1 p_1 - \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^T \left[\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \right]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2 - 1} \\ \times \left(\int_0^T \left[\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \right]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2 - 1}. \end{aligned} \quad (68)$$

Combining (67) with (68), we obtain

$$\begin{aligned} T^{\alpha_1 p_1 + \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} \right) \\ \left(\int_0^T \left[\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \right]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2 - 1} \left(\int_0^T \left[\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \right]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2 - 1} \leq C, \end{aligned} \quad (69)$$

which contradicts (14).

In the second case, when $u_1^{(0)} \in \mathbb{R}$ and $u_2^{(0)} \geq 0$, following the same steps as used in the previous case, we obtain

$$\frac{T^{\alpha_2 p_2 + \alpha_1 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_2(t) dt + T^{1-\alpha_1} u_1^{(0)} \right)}{\left(\int_0^T \left[\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \right]^{-1/p_1-1} dt \right)^{p_2(p_1-1)/p_1 p_2 - 1} \left(\int_0^T \left[\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \right]^{-1/p_2-1} dt \right)^{p_2-1/p_1 p_2 - 1}} \leq C, \quad (70)$$

which contradicts (15). The proof is completed.

4. Conclusion

In this paper, problems (3) and (13) are investigated. Namely, using the test function method and some integral inequalities, sufficient conditions for the nonexistence of global solutions (or equivalently, necessary conditions for the existence of global solutions) to the considered problems are obtained. For problem (3), we proved that (see Theorem 1) under assumptions (A1), (A2), and (A3), if

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha p/p-1} \left(u_0 T^{1-\alpha} + \int_0^{T/2} \mu(t) dt \right)}{\int_0^T \left[\rho(a^{-1}(t)) (a^{-1})'(t) \right]^{-1/p-1} dt} = +\infty, \quad (71)$$

then (3) admits no global solution. For the system of fractional differential inequalities (13), always under assumptions (A1), (A2), and (A3), we proved that (see Theorem 6), if $u_1^{(0)} \geq 0$ and

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha_1 p_1 + \alpha_2 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_1(t) dt + T^{1-\alpha_2} u_2^{(0)} \right)}{\left(\int_0^T \left[\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \right]^{-1/p_1-1} dt \right)^{p_1-1/p_1 p_2 - 1} \left(\int_0^T \left[\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \right]^{-1/p_2-1} dt \right)^{p_1(p_2-1)/p_1 p_2 - 1}} = +\infty, \quad (72)$$

or $u_2^{(0)} \geq 0$ and

$$\limsup_{T \rightarrow +\infty} \frac{T^{\alpha_2 p_2 + \alpha_1 p_1 p_2 / p_1 p_2 - 1} \left(\int_0^{T/2} \mu_2(t) dt + T^{1-\alpha_1} u_1^{(0)} \right)}{\left(\int_0^T \left[\rho_1(a_1^{-1}(t)) (a_1^{-1})'(t) \right]^{-1/p_1-1} dt \right)^{p_2(p_1-1)/p_1 p_2 - 1} \left(\int_0^T \left[\rho_2(a_2^{-1}(t)) (a_2^{-1})'(t) \right]^{-1/p_2-1} dt \right)^{p_2-1/p_1 p_2 - 1}} = +\infty, \quad (73)$$

then (13) admits no global solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors made equal contributions and read and supported the last original copy.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Al Imam Mohammad Ibn Saud

Islamic University for funding this work through Research Group no. RG-21-09-01.

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