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Research Article

The Study of Solutions of Several Systems of Nonlinear Partial Differential Difference Equations

Hong Yan Xu,[1,2] **Meiying Yu**,[3] **and Keyu Zhang**[3]

[1]*College of Arts and Sciences, Suqian University, Suqian, Jiangsu 223800, China*
[2]*School of Mathematics and Computer Science, Shangrao Normal University, Shangrao Jiangxi 334001, China*
[3]*School of Mathematics, Qilu Normal University, Shandong, Jinan 250200, China*

Correspondence should be addressed to Meiying Yu; zkyymy@126.com

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Our main aim is to describe the entire solutions of several systems of $\begin{cases} [\alpha_1 f_1(z)]^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ [\beta_1 f_2(z)]^2 + [\beta_2 f_1(z+c)]^2 = 1, \end{cases}$ $\begin{cases} (\alpha_1 \partial f_1/\partial z_1)^{n_1} + [\alpha_2 f_2(z+c)]^{m_1} = 1, \\ (\beta_1 \partial f_2/\partial z_1)^{n_2} + [\beta_2 f_1(z+c)]^{m_2} = 1, \end{cases}$ and $\begin{cases} (\alpha_1 \partial f_1/\partial z_1)^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ (\beta_1 \partial f_2/\partial z_1)^2 + [\beta_2 f_1(z+c)]^2 = 1, \end{cases}$ $\begin{cases} (\alpha_1 \partial f_1/\partial z_1)^2 + [\alpha_2 f_2(z+c) + \alpha_3 f_1(z)]^2 = 1, \\ (\beta_1 \partial f_2/\partial z_1)^2 + [\beta_2 f_1(z+c) + \beta_3 f_2(z)]^2 = 1, \end{cases}$ where $\alpha_j, \beta_j (j = 1, 2, 3)$ are nonzero constants in $\mathbb{C}$ and $m_j, n_j (j = 1, 2)$ are positive integers. We obtain several theorems on the existence and the forms of solutions for these systems, which are some improvements and supplements of the previous theorems given by Xu and Cao, Gao, and Liu and Yang. Moreover, we give some examples to explain the existence of solutions for such systems.

1. Introduction

As everyone knows, the study of the existence of solutions for Fermat type equations has always been an important and interesting problem. The famous Fermat's Last Theorem has attracted the attention of many mathematical scholars [1, 2]. About 60 years ago or even earlier, Montel [3] and Gross [4] had considered the equation $f^m + g^m = 1$ and obtained that the entire solutions of $f^2 + g^2 = 1$ are $f = \cos \zeta(z)$, $g = \sin \zeta(z)$ for the case $m = 2$, where $\zeta(z)$ is an entire function, and this equation does not admit any nonconstant entire solution for any positive integer $m > 2$.

With the establishment and rapid development of Nevanlinna value distribution theory for meromorphic functions and theirs difference [5–7], Liu [8] in 2009, Liu et al. [9] in 2012, and Liu and Yang [10] in 2013 studied some complex Fermat type difference and Fermat type differential difference equations and obtained some results.

Theorem 1 (see [9], Theorem 1.1). *The transcendental entire solutions with finite order of*

$$f(z)^2 + f(z+c)^2 = 1 \tag{1}$$

must satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = (4k+1)\pi/2c$, with k an integer.

Theorem 2 (see [9], Theorem 1.3). *The transcendental entire solutions with finite order of*

$$f'(z)^2 + f(z+c)^2 = 1 \tag{2}$$

must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = (2k+1)\pi$, with k an integer.

After that, Gao [11] in 2016 extended Theorem 2 from complex differential difference equation to the system of complex differential difference equations.

Theorem 3 (see [11], Theorem 1.1). *Suppose that (f_1, f_2) is a pair of finite-order transcendental entire solutions for the system of differential difference equations*

$$\begin{cases} [f_{1'}(z)]^2 + f_2(z+c)^2 = 1, \\ [f_{2'}(z)]^2 + f_1(z+c)^2 = 1. \end{cases} \tag{3}$$

Then, (f_1, f_2) satisfies

$$(f_1, f_2) = (\sin(z - bi), \sin(z - b_1 i)), \tag{4}$$

or

$$(f_1, f_2) = (\sin(z + bi), \sin(z + b_1 i)), \tag{5}$$

where b, b_1 are constants and $c = k\pi$, where k is an integer.

In recent, Xu and Cao [12, 13] further discussed the solutions for some Fermat type PDDEs and obtained the following:

Theorem 4 (see [13], Theorem 1.4). *Let $c \in \mathbb{C}^n - \{0\}$. Then, any nonconstant entire solution with finite order of the equation*

$$f(z)^2 + f(z+c)^2 = 1 \tag{6}$$

has the form of $f(z) = \cos(L(z) + B)$, where L is a linear function of the form $L(z) = a_1 z_1 + \cdots + a_n z_n$ on $\mathbb{C}^n$ such that $L(c) = -\pi/2 - 2k\pi, k \in \mathbb{Z}$, and B is a constant on $\mathbb{C}$.

Theorem 5 (see [13], Theorem 1.1). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then,*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1 \tag{7}$$

does not have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Theorem 6 (see [13], Theorem 1.2). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any transcendental entire solution with finite order of the PDDE*

$$\left(\frac{\partial f}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1 \tag{8}$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1} = 1$ and B is a constant on $\mathbb{C}$; in the special case whenever $c_1 = 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.

By analyzing Theorems 3–6, a natural question is as follows: *What will happen about the transcendental entire solutions for the system of the PDDEs of Fermat type*? Although many scholars have paid considerable attention to the complex difference equation with a single variable and the complex Fermat type difference equation in recent years, a series of important and meaningful results (including [7, 14–22]) were obtained, however, to our knowledge, there were not much results about the complex difference equation in several complex variables. Of course, the references involving the results of systems of complex PDDEs are even less.

This manuscript is aimed at studying the solutions of several Fermat type systems involving both difference operator and partial differential. We establish four theorems on the forms of solutions for several systems of Fermat type PDDEs, which are improvement of the previous theorems given by Liu et al., Gao, and Xu and Cao [8, 9, 11, 13]. We mainly employ the Nevanlinna value distribution theory and difference Nevanlinna theory of several complex variables in this article, and the readers can refer to [23, 24]. Now, we start to state our main results below.

Theorem 7. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$, and $m_j, n_j (j = 1, 2) \in \mathbb{N}_+$. If the Fermat type PDDE system*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1}\right)^{n_1} + [\alpha_2 f_2(z+c)]^{m_1} = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1}\right)^{n_2} + [\beta_2 f_1(z+c)]^{m_2} = 1 \end{cases} \tag{9}$$

satisfies one of the conditions

(i) $m_1 m_2 > n_1 n_2$

(ii) $n_j > m_j / m_j - 1$ and $m_j \geq 2$, $j = 1, 2$

then system (9) does not exist any pair of finite-order transcendental entire solution.

Remark 8. Here, we say that (f, g) is a pair of finite-order transcendental entire solution for

$$\begin{cases} f^{n_1} + g^{m_1} = 1, \\ f^{n_2} + g^{m_2} = 1, \end{cases} \tag{10}$$

if f, g are transcendental entire functions satisfying the above system and $\rho = \max\{\rho(f), \rho(g)\} < \infty$.

Remark 9. We list an example to demonstrate that the condition $m_j \geq 2$ in Theorem 7 cannot be removed. Let

$$\begin{aligned} f_1 = f_2 = 1 + \frac{1}{4}c_1^2 - \frac{1}{4}z_1^2 + \left(\frac{c_1}{2c_2}z_2 + b + e^{(2\pi i/c_2)z_2}\right)(z_1 - c_1) \\ - \left[\frac{c_1}{2c_2}(z_2 - c_2) + b + e^{(2\pi i/c_2)z_2}\right]^2, \end{aligned} \tag{11}$$

where $c_1, b \in \mathbb{C}$ and $c_2 \neq 0$. Thus, (f_1, f_2) satisfies the system (9) with $n_1 = n_2 = 2$, $m_1 = m_2 = 1$, and $\alpha_j = \beta_j = 1$, $j = 1, 2$.

Theorem 10. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$. If the system of Fermat type difference equations*

$$\begin{cases} [\alpha_1 f_1(z)]^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ [\beta_1 f_2(z)]^2 + [\beta_2 f_1(z+c)]^2 = 1 \end{cases} \tag{12}$$

admits a pair of finite-order transcendental entire solution (f_1, f_2), then $\alpha_1^2/\beta_2^2 = \beta_1^2/\alpha_2^2 = 1$, and (f_1, f_2) have the *following forms*

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{2\alpha_1}, \frac{\beta_2}{\beta_1}\frac{A_{21}e^{L(z)+B_0} + A_{22}e^{-(L(z)+B_0)}}{2\alpha_1}\right), \tag{13}$$

where $L(z) = a_1z_1 + a_2z_2$, $a_1, a_2, B_0 \in \mathbb{C}$, *and* A_{21}, A_{22}, c *satisfy one of the following cases.*

(i) $L(c) = k\pi i$, $A_{21} = -i$ *and* $A_{22} = i$ *or* $A_{21} = i$ *and* $A_{22} = -i$, k *is a integer*

(ii) $L(c) = (2k \pm 1/2)\pi i$, $A_{21} = -1$ *and* $A_{22} = -1$, *or* $A_{21} = 1$ *and* $A_{22} = 1$.

Remark 11. From Theorem 10, we can conclude that f_1, f_2 have the following relationships

(i) $f_2 = \eta f_1$

(ii) $f_2 = i\eta e^{L(z)+B_1} - e^{-(L(z)+B_1)}/2\alpha_1$, where $\eta = \pm\beta_2/\beta_1$ and $f_1(z) = e^{L(z)+B_1} + e^{-(L(z)+B_1)}/2\alpha_1$.

Now, two examples can verify the existence of solutions for (12).

Example 1. Let c_1, c_2 and $L(z) = a_1z_1 + a_2z_2$ satisfy $L(c) = a_1 c_1 + a_2c_2 = (2k \pm 1/2)\pi i$, and $B_0 \in \mathbb{C}$. Then, the function

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{4}, -\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{2}\right) \tag{14}$$

satisfies the system (12) with $\alpha_1 = 2$, $\alpha_2 = 1$, and $\beta_1 = \beta_2 = 1$.

Example 2. Let c_1, c_2 and $L(z) = a_1z_1 + a_2z_2$ satisfy $L(c) = a_1 c_1 + a_2c_2 = k\pi i$, and $B_0 \in \mathbb{C}$. Then, the function

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-L(z)-B_0}}{2}, \frac{1}{3}\frac{e^{L(z)+B_0} - e^{-L(z)-B_0}}{2i}\right) \tag{15}$$

satisfies the system (12) with $\alpha_1 = 1 = \beta_2$ and $\alpha_2 = 3 = \beta_1$.

Theorem 12. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $\alpha_j, \beta_j (j = 1, 2) \in \mathbb{C} - \{0\}$. *If the system of Fermat type PDDEs*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1}\right)^2 + [\alpha_2 f_2(z+c)]^2 = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1}\right)^2 + [\beta_2 f_1(z+c)]^2 = 1 \end{cases} \tag{16}$$

admits a pair of finite-order transcendental entire solution (f_1, f_2), *then* $(\alpha_1\alpha_2)^2 = (\beta_1\beta_2)^2$ *and* (f_1, f_2) *is the form of*

$$(f_1, f_2) = \left(\frac{A_{11}e^{L(z)+B_0} + A_{12}e^{-(L(z)+B_0)}}{2\beta_2}, \eta\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2\beta_2}\right), \tag{17}$$

where $L(z) = a_1z_1 + a_2z_2$, B_0 *is a constant in* $\mathbb{C}$, *and* $a_1, c, A_{11}, A_{12}, \eta$ *satisfy* $a_1^2 = -\beta_2^2/\alpha_1^2 = -\alpha_2^2/\beta_1^2$ *and one of the following cases*

(i) $L(c) = 2k\pi i$, *and* $\eta = -1$, $A_{11} = -i$, $A_{12} = i$, *or* $\eta = 1$, $A_{11} = i$, $A_{12} = -i$

(ii) $L(c) = (2k + 1)\pi i$, *and* $\eta = -1$, $A_{11} = i$, $A_{12} = -i$, *or* $\eta = 1$, $A_{11} = -i$, $A_{12} = i$

(iii) $L(c) = (2k + 1/2)\pi i$, *and* $\eta = 1$, $A_{11} = -1$, $A_{12} = -1$, *or* $\eta = -1$, $A_{11} = 1$, $A_{12} = 1$

(iv) $L(c) = (2k - 1/2)\pi i$, *and* $\eta = 1$, $A_{11} = 1$, $A_{12} = 1$, *or* $\eta = -1$, $A_{11} = -1$, $A_{12} = -1$.

Here, two examples can verify the existence of solutions for (16).

Example 3. Let $(a_1, a_2) = (i, \pi)$, $A_{11} = -i$, $A_{12} = i$, $\eta = -1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = iz_1 + \pi z_2$ and

$$(f_1(z), f_2(z)) = \left(-i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{4}, -i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2}\right). \tag{18}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, i)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = 1$, and $\beta_2 = 2$.

Example 4. Let $(a_1, a_2) = (1, -\pi i)$, $A_{11} = -1$, $A_{12} = -1$, $\eta = 1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = z_1 - \pi i z_2$ and

$$(f_1(z), f_2(z)) = \left(-\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{4i}, -\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{2i}\right). \tag{19}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi i, 1/2)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = i$, and $\beta_2 = 2i$.

Example 5. Let $(a_1, a_2) = (2i, i)$, $A_{11} = i$, $A_{12} = -i$, $\eta = -1$, and $B_0 \in \mathbb{C}$. That is, $L(z) = 2iz_1 + iz_2$ and

$$(f_1(z), f_2(z)) = \left(i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{4}, -i\frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{8}\right). \tag{20}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, -\pi)$, $\alpha_1 = 1$, $\beta_1 = 2$, $\alpha_2 = 4$, and $\beta_2 = 2$.

Example 6. Let $(a_1, a_2) = (3, 1)$, $A_{11} = 1$, $A_{12} = 1$, $\eta = i$, and $B_0 \in \mathbb{C}$. That is, $L(z) = 3z_1 + z_2$ and

$$(f_1(z), f_2(z)) = \left(\frac{e^{L(z)+B_0} + e^{-(L(z)+B_0)}}{6}, i \frac{e^{L(z)+B_0} - e^{-(L(z)+B_0)}}{18} \right). \tag{21}$$

Thus, (f_1, f_2) satisfies the system (16) with $(c_1, c_2) = (\pi, -\pi)$, $\alpha_1 = i$, $\alpha_2 = 9$, $\beta_1 = 3i$, and $\beta_2 = 3$.

Theorem 13. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $\alpha_j, \beta_j (j = 1, 2, 3) \in \mathbb{C} - \{0\}$. Let (f_1, f_2) be a pair of transcendental entire solutions of finite order for the system*

$$\begin{cases} \left(\alpha_1 \dfrac{\partial f_1}{\partial z_1} \right)^2 + [\alpha_2 f_2(z+c) + \alpha_3 f_1(z)]^2 = 1, \\ \left(\beta_1 \dfrac{\partial f_2}{\partial z_1} \right)^2 + [\beta_2 f_1(z+c) + \beta_3 f_2(z)]^2 = 1. \end{cases} \tag{22}$$

Then, (f_1, f_2) is one of the forms

$$(f_1, f_2) = \left(\frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\alpha_1} + e^{\eta z_2} G_1(z_2), \pm \frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\beta_1} + e^{\eta z_2} G_2(z_2) \right), \tag{23}$$

or

$$(f_1, f_2) = \left(\frac{e^{iL(z)+B_0} - e^{-iL(z)-B_0}}{2ia_1\alpha_1} + e^{\eta z_2} G_1(z_2), \pm \frac{e^{iL(z)+B_0} + e^{-iL(z)-B_0}}{2a_1\beta_1} + e^{\eta z_2} G_2(z_2) \right), \tag{24}$$

where $L(z) = a_1 z_1 + a_2 z_2$, $a_1 (\neq 0), a_2, B_0 \in \mathbb{C}$, and $G_1(z_2), G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and $a_1, a_2, \alpha_j, \beta_j, \eta, c_1, c_2$ satisfy $e^{4iL(c)} = 1$ and the following conditions

(C_1) $\eta = 0$ if $\alpha_2\beta_2 = \alpha_3\beta_3$, and $\eta = \log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2$ if $\alpha_2\beta_2 \neq \alpha_3\beta_3$

(C_2) $[\beta_1/\alpha_2(a_1 - \alpha_3/\alpha_1)]^2 = [\alpha_1/\beta_2(a_1 - \beta_3/\beta_1)]^2 = 1$, or $[\beta_1/\alpha_2(a_1 - \alpha_3/\alpha_1)]^2 = [\alpha_1/\beta_2(a_1 + \beta_3/\beta_1)]^2 = 1$

$$(f_1(z), f_2(z)) = \left(e^{\log(-1)/2c_2 z_2} G_1(z_2) + D_1, e^{\log(-1)/2c_2 z_2} G_2(z_2) + D_2 \right), \tag{25}$$

where $D_1 = \alpha_2\xi_2 - \beta_3\xi_1/2\alpha_2\beta_2$, $D_2 = \beta_2\xi_1 - \alpha_3\xi_2/2\alpha_2\beta_2$, $\xi_1 = \pm 1$, and $\xi_2 = \pm 1$;

$$(f_1(z), f_2(z)) = \left(b_1 z_1 + \gamma_1 z_2 + G_1(z_2), -\frac{\alpha_3}{\alpha_2} b_1 z_1 + \gamma_2 z_2 + G_2(z_2) \right), \tag{26}$$

where $G_1(z_2), G_2(z_2)$ are stated as in ((23)) and ((24)), and $b_1 (\neq 0), \gamma_1, \gamma_2$ satisfy

$$\gamma_1 = \frac{\alpha_2 b_3 - \beta_3 b_2 - (\alpha_2\beta_2 + \alpha_3\beta_3) b_1 c_1}{2\alpha_2\beta_2 c_2}, \tag{27}$$

$$\gamma_2 = \frac{\beta_2 b_2 - \alpha_3 b_3 + (\alpha_3\beta_2 + \beta_3\alpha_2) b_1 c_1}{2\alpha_2\beta_2 c_2}, \tag{28}$$

$$b_2^2 + (\alpha_1 b_1)^2 = 1, b_3^2 + \left(\beta_1 \frac{\alpha_3}{\alpha_2} b_1 \right)^2 = 1. \tag{29}$$

Here, five examples can verify the existence of solutions for (22).

Example 7. Let $B_0 \in \mathbb{C}$, $a_1 = 1$, $a_2 = 1$, and

$$(f_1(z), f_2(z)) = \left(e^{i(z_1+z_2)+B_0} - e^{-i(z_1+z_2)-B_0}/4i + e^{iz_2}, e^{i(z_1+z_2)+B_0} - e^{-i(z_1+z_2)-B_0}/2i - 4e^{iz_2} \right). \tag{30}$$

Thus, (f_1, f_2) satisfies system (22) with $(c_1, c_2) = (\pi, \pi)$, $\alpha_1 = 2$, $\beta_1 = 1$, $\alpha_2 = -1$, $\beta_2 = 4$, $\alpha_3 = 4$, and $\beta_3 = 1$.

Example 8. Let $B_0 \in \mathbb{C}$, $a_1 = i$, $a_2 = 1$, and

$$f_1(z_1, z_2) = e^{i(iz_1+z_2)+B_0} - e^{-i(iz_1+z_2)-B_0}/-2 + e^{\log[-(1+2i)]/2\pi z_2} e^{iz_2},$$
$$f_2(z_1, z_2) = \frac{e^{i(iz_1+z_2)+B_0} + e^{-i(iz_1+z_2)-B_0}}{-2} + \frac{i}{2} e^{\log[-(1+2i)]/2\pi z_2} e^{iz_2}. \tag{31}$$

Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (1/2\pi, \pi)$, $\alpha_1 = 1$, $\beta_1 = i$, $\alpha_2 = 2$, $\beta_2 = i - 2$, $\alpha_3 = -i$, and $\beta_3 = 2$.

Example 9. Let $\alpha_1 \in \mathbb{C}$ and $(f_1(z), f_2(z)) = (e^{3\pi i/2 z_2} + 1, e^{3\pi i/2 z_2})$. Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (c_1, 1)$, $c_1 \in \mathbb{C}$, $\alpha_2 = 2$, $\alpha_2 = 1$, $\beta_2 = 1$, and $\beta_3 = -2$.

Example 10. Let (f_1, f_2) be of the forms

$$(f_1(z), f_2(z)) = \left(e^{iz_2} + \frac{i-1}{4\pi} z_2, 2e^{iz_2} + \frac{1-i}{2\pi} z_2 + \frac{1+i}{2} \right). \tag{32}$$

Thus, (f_1, f_2) satisfies the system (22) with $(c_1, c_2) = (c_1, \pi)$, $c_1 \in \mathbb{C}$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\beta_2 = 2i$, $\beta_3 = i$, and $\alpha_1, \beta_1 \in \mathbb{C}$.

Example 11. Let

$$f_1(z_1, z_2) = \frac{1}{2} z_1 + \frac{\sqrt{15} - 4\sqrt{3} - 4}{16} z_2 + e^{2\pi i z_2},$$
$$f_2(z_1, z_2) = -\frac{1}{4} z_1 + \frac{8 - \sqrt{15} + 4\sqrt{3}}{32} z_2 - \frac{1}{2} e^{2\pi i z_2} + \frac{4\sqrt{3} + \sqrt{15}}{32}. \tag{33}$$

Thus, (f_1, f_2) satisfies system (22) with $c = (c_1, c_2) = (1, 1)$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 2$, $\beta_2 = 2$, $\alpha_3 = 1$, and $\beta_3 = 4$.

2. Proof of Theorem 7

Proof. Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (9). Here, let us consider two cases below. □

Case 1. $m_1 m_2 > n_1 n_2$. Owing to Refs. [23, 24], we have the following facts that

$$m\left(r, \frac{f_j(z)}{f_j(z+c)}\right) = S\left(r, f_j\right), j = 1, 2 \tag{34}$$

hold for all $r > 0$ outside of a possible exceptional set $E_j \subset [1, +\infty)$ of finite logarithmic measure $\int_{E_j} dt/t < \infty$. Due to the above fact, we have

$$\begin{aligned} T\left(r, f_j\right) = m\left(r, f_j\right) &\le m\left(r, \frac{f_j(z)}{f(z+c)}\right) + m\left(r, f_j(z+c)\right) + \log 2 \\ &= m\left(r, f_j(z+c)\right) + S\left(r, f_j\right) \\ &= T\left(r, f_j(z+c)\right) + S\left(r, f_j\right), j = 1, 2, \end{aligned} \tag{35}$$

for all $r \in E =: E_1 \cup E_2$. By the Mokhon'ko theorem ([25], Theorem 3.4) and the Logarithmic Derivative Lemma [26], it yields from (35) that

$$\begin{aligned} m_1 T(r, f_2) &\le m_1 T(r, f_2(z+c)) + S(r, f_2) \\ &= T(r, [\alpha_2 f_2(z+c)]^{m_1}) + S(r, f_2) \\ &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1}\right)^{n_1} - 1\right) + S(r, f_2) \\ &= n_1 T\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2) \\ &= n_1 m\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2) \\ &\le n_1 \left(m\left(r, \frac{\partial f_1/\partial z_1}{f_1}\right) + m(r, f_1)\right) + S(r, f_1) + S(r, f_2) \\ &= n_1 T(r, f_1) + S(r, f_1) + S(r, f_2), \end{aligned} \tag{36}$$

for all $r \in E$. Similarly, we also get

$$m_2 T(r, f_1) \le n_2 T(r, f_2) + S(r, f_1) + S(r, f_2), r \in E. \tag{37}$$

Thus, we conclude from (36) and (37) that

$$(m_1 m_2 - n_1 n_2) T\left(r, f_j\right) \le S(r, f_1) + S(r, f_2), r \in E. \tag{38}$$

By combining with the condition that $m_1 m_2 > n_1 n_2$ and f_1, f_2 being transcendental functions, we obtain a contradiction.

Case 2. $n_j > m_j/m_j - 1$ and $m_j \ge 2$, $j = 1, 2$. Thus, it is easy to get that $m_j > n_j/n_j - 1$. In view of the Nevanlinna second fundamental theorem, the difference logarithmic derivative lemma in several complex variables [23, 24], we thus obtain from (9) that

$$\begin{aligned} (n_1 - 1) T\left(r, \frac{\partial f_1}{\partial z_1}\right) &\le \bar{N}\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &+ \sum_{q=1}^{n_1} \boxtimes \bar{N}\left(r, \frac{1}{\partial f_1/\partial z_1 - w_q/\alpha_1}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &\le \bar{N}\left(r, \frac{1}{(\alpha_1 \partial f_1/\partial z_1)^{n_1} - 1}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\ &\le \bar{N}\left(r, \frac{1}{f_2(z+c)}\right) + S(r, f_1) \le T(r, f_2(z+c)) \\ &+ S(r, f_1) + S(r, f_2), \end{aligned} \tag{39}$$

where w_q is a roots of $w^{n_1} - 1 = 0$. Similarly, we also have

$$(n_2 - 1) T\left(r, \frac{\partial f_2}{\partial z_1}\right) \le T(r, f_1(z+c)) + S(r, f_1) + S(r, f_2). \tag{40}$$

In addition, by applying the Mokhon'ko theorem in several complex variables ([25], Theorem 3.4) for (9), we can conclude

$$\begin{aligned} m_1 T(r, f_2(z+c)) &= T(r, [\alpha_2 f_2(z+c)]^{m_1}) + S(r, f_2) \\ &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1}\right)^{n_1} - 1\right) + S(r, f_2) \\ &= n_1 T\left(r, \frac{\partial f_1}{\partial z_1}\right) + S(r, f_1) + S(r, f_2). \end{aligned} \tag{41}$$

Similarly, we also get

$$m_2 T(r, f_1(z+c)) = n_2 T\left(r, \frac{\partial f_2}{\partial z_1}\right) + S(r, f_1) + S(r, f_2). \tag{42}$$

Due to $m_j > n_j/n_j - 1$, it follows from (39)–(42) that

$$\begin{aligned} &\left(m_1 - \frac{n_1}{n_1 - 1}\right) T(r, f_2(z+c)) \le S(r, f_1) + S(r, f_2), \\ &\left(m_2 - \frac{n_2}{n_2 - 1}\right) T(r, f_1(z+c)) \le S(r, f_1) + S(r, f_2), \end{aligned} \tag{43}$$

and this is a contradiction with f_1, f_2 being transcendental functions.

Therefore, Theorem 7 is proved.

3. The Proof of Theorem 10

Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (12). We firstly rewrite the system (12) as

$$\begin{cases} [\alpha_1 f_1 + i\alpha_2 f_2(z+c)][\alpha_1 f_1 - i\alpha_2 f_2(z+c)] = 1, \\ [\beta_1 f_2 + i\beta_2 f_1(z+c)][\beta_1 f_2 - i\beta_2 f_1(z+c)] = 1. \end{cases} \quad (44)$$

By applying the Hadamard factorization theorem (can be found in [27, 28]), then there exist two polynomials p_1, p_2 such that

$$\begin{cases} \alpha_1 f_1 + i\alpha_2 f_2(z+c) = e^{p_1}, \\ \alpha_1 f_1 - i\alpha_2 f_2(z+c) = e^{-p_1}, \\ \beta_1 f_2 + i\beta_2 f_1(z+c) = e^{p_2}, \\ \beta_1 f_2 - i\beta_2 f_1(z+c) = e^{-p_2}. \end{cases} \quad (45)$$

Thus, we have from (45) that

$$\begin{cases} \alpha_1 f_1 = \dfrac{e^{p_1} + e^{-p_1}}{2}, \\ \alpha_2 f_2(z+c) = \dfrac{e^{p_1} - e^{-p_1}}{2i}, \\ \beta_1 f_2 = \dfrac{e^{p_2} + e^{-p_2}}{2}, \\ \beta_2 f_1(z+c) = \dfrac{e^{p_2} - e^{-p_2}}{2i}, \end{cases} \quad (46)$$

which implies

$$\frac{\alpha_1}{\beta_2 i} e^{p_1(z+c)+p_2} + \frac{\alpha_1}{\beta_2} i e^{p_1(z+c)-p_2} - e^{2p_1(z+c)} \equiv 1, \quad (47)$$

$$\frac{\beta_1}{\alpha_2 i} e^{p_2(z+c)+p_1} + \frac{\beta_1}{\alpha_2} i e^{p_2(z+c)-p_1} - e^{2p_2(z+c)} \equiv 1. \quad (48)$$

By applying [29], Lemma 3.1 (can be found in [30]), for (47) and (48), we have that

$$\begin{aligned} &\alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \text{or} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ &\beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2, \text{or} \beta_1 e^{p_2(z+c)+p_1} \equiv \alpha_2 i. \end{aligned} \quad (49)$$

Here, four cases will be discussed below.

Case 1.

$$\begin{cases} \alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \\ \beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2. \end{cases} \quad (50)$$

Thus, we can conclude from (50) that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_2(z+c) - p_1(z) \equiv C_2$; here and below, C_1, C_2 are constants. So, this leads to $p_1(z) = L(z) + B_1, p_2(z) = L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$ and a_1, a_2, B_1, B_2 are constants. Thus, by virtue of (47)–(50)), it yields that

$$\begin{cases} \alpha_1 i e^{L(c)+B_1-B_2} \equiv \beta_2, \\ \beta_1 i e^{L(c)-B_1+B_2} \equiv \alpha_2, \\ \alpha_1 e^{-L(c)-B_1+B_2} \equiv \beta_2 i, \\ \beta_1 e^{-L(c)+B_1-B_2} \equiv \alpha_2 i, \end{cases} \quad (51)$$

which implies

$$\frac{\alpha_1^2}{\beta_2^2} = \frac{\beta_1^2}{\alpha_2^2} = 1, e^{4L(c)} = 1, e^{B_1-B_2} = \frac{\beta_2}{\alpha_1 i} e^{-L(c)}. \quad (52)$$

In view of (46), let

$$f_1(z) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1}, f_2(z) = \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_1}. \quad (53)$$

If $e^{L(c)} = 1$, i.e., $L(c) = 2k\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = \alpha_1/\beta_2 i$. Thus,

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_1} = \frac{e^{L(z)+B_1} e^{B_2-B_1} + e^{-L(z)-B_1} e^{B_1-B_2}}{2\beta_1} \\ &= \frac{\beta_2}{\beta_1} \frac{i e^{L(z)+B_1} - i e^{-L(z)-B_1}}{2\alpha_1}. \end{aligned} \quad (54)$$

If $e^{L(c)} = -1$, i.e., $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = -\alpha_1/\beta_2 i$. Thus,

$$f_2(z) = \frac{e^{L(z)+B_1} e^{B_2-B_1} + e^{-L(z)-B_1} e^{B_1-B_2}}{2\beta_1} = -\frac{\beta_2}{\beta_1} \frac{i e^{L(z)+B_1} - i e^{-L(z)-B_1}}{2\alpha_1}. \quad (55)$$

If $e^{L(c)} = i$, i.e., $L(c) = (2k + 1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = -\alpha_1/\beta_2$. Thus,

$$f_2(z) = -\frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = -\frac{\beta_2}{\beta_1} f_1(z). \quad (56)$$

If $e^{L(c)} = -i$, i.e., $L(c) = (2k - 1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_2-B_1} = \alpha_1/\beta_2$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = \frac{\beta_2}{\beta_1} f_1(z). \quad (57)$$

Case 2.

$$\begin{cases} \alpha_1 i e^{p_1(z+c)-p_2} \equiv \beta_2, \\ \beta_1 e^{p_1-p_2(z+c)} \equiv \alpha_2 i. \end{cases} \quad (58)$$

Thus, it yields from (58) that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_1(z) + p_2(z+c) \equiv C_2$. Hence, we obtain that $p_1(z+2c) + p_1(z) \equiv C_1 + C_2$, and this is a contradiction with p_1 is not a constant.

Case 3.

$$\begin{cases} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ \beta_1 i e^{p_2(z+c)-p_1} \equiv \alpha_2. \end{cases} \quad (59)$$

Thus, it yields from (59) that $p_1(z+c)+p_2(z) \equiv C_1$ and $p_2(z+c)-p_1(z) \equiv C_2$. Hence, we obtain that $p_2(z+2c)+p_2(z) \equiv C_1 + C_2$, and this is a contradiction with p_2 is not a constant.

Case 4.

$$\begin{cases} \alpha_1 e^{p_1(z+c)+p_2} \equiv \beta_2 i, \\ \beta_1 i e^{p_2(z+c)+p_1} \equiv \alpha_2. \end{cases} \quad (60)$$

Thus, it yields from (60) that $p_2(z)-p_1(z+c) \equiv C_1$ and $p_1(z)-p_2(z+c) \equiv C_2$. Hence, we obtian that $p_1(z) = L(z) + B_1, p_2(z) = -L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$, a_1, a_2, B_1, B_2 are constants. By virtue of (47),(48), (60), it yields that

$$\begin{cases} \alpha_1 e^{L(c)+B_1+B_2} \equiv \beta_2 i, \\ \beta_1 e^{-L(c)+B_1+B_2} \equiv \alpha_2 i, \\ \alpha_1 e^{-L(c)-B_1-B_2} \equiv \beta_2 i, \\ \beta_1 e^{L(c)-B_1-B_2} \equiv \alpha_2 i, \end{cases} \quad (61)$$

which implies

$$\frac{\alpha_1^2}{\beta_2^2} = \frac{\beta_1^2}{\alpha_2^2} = 1, e^{4L(c)} = 1, e^{B_1+B_2} = \frac{\beta_2}{\alpha_1 i} e^{-L(c)} = \frac{\alpha_2 i}{\beta_1} e^{L(c)}. \quad (62)$$

In view of (46), let

$$f_1(z) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1}, f_2(z) = \frac{e^{-L(z)+B_2} + e^{L(z)-B_2}}{2\beta_1}. \quad (63)$$

If $e^{L(c)} = 1$, i.e., $L(c) = 2k\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = \beta_2/\alpha_1 i = \alpha_2/\beta_1 i$. Thus,

$$f_2(z) = \frac{e^{L(z)+B_1} e^{-B_2-B_1} + e^{-L(z)-B_1} e^{B_1+B_2}}{2\beta_1} = \frac{\beta_2}{\beta_1} \frac{-ie^{L(z)+B_1} + ie^{-L(z)-B_1}}{2\alpha_1}. \quad (64)$$

If $e^{L(c)} = -1$, i.e., $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = -\beta_2/\alpha_1 i = -\alpha_2/\beta_1 i$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{ie^{L(z)+B_1} - ie^{-L(z)-B_1}}{2\alpha_1}. \quad (65)$$

If $e^{L(c)} = i$, i.e., $L(c) = (2k+1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1+B_2} = \beta_2/\alpha_1 = -\alpha_2/\beta_1$. Thus,

$$f_2(z) = \frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = \frac{\beta_2}{\beta_1} f_1(z). \quad (66)$$

If $e^{L(c)} = -i$, i.e., $L(c) = (2k-1/2)\pi i$, $k \in \mathbb{Z}$, then $e^{B_1-B_2} = -\beta_2/\alpha_1 = \alpha_2/\beta_1$. Thus,

$$f_2(z) = -\frac{\beta_2}{\beta_1} \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_1} = -\frac{\beta_2}{\beta_1} f_1(z). \quad (67)$$

Therefore, this completes the proof of Theorem 10.

4. The Proof of Theorem 12

Proof. Let (f_1, f_2) be a pair of finite-order transcendental entire functions satisfying (16). Firstly, (16) may be represented as the following form:

$$\begin{cases} \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + i\alpha_2 f_2(z+c)\right]\left[\alpha_1 \frac{\partial f_1}{\partial z_1} - i\alpha_2 f_2(z+c)\right] = 1, \\ \left[\beta_1 \frac{\partial f_2}{\partial z_1} + i\beta_2 f_1(z+c)\right]\left[\beta_1 \frac{\partial f_2}{\partial z_1} - i\beta_2 f_1(z+c)\right] = 1. \end{cases} \quad (68)$$

By the Hadamard factorization theorem (can be found in [27, 28]), there are two nonconstant polynomials p_1, p_2 satisfying

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1} + i\alpha_2 f_2(z+c) = e^{p_1}, \\ \alpha_1 \frac{\partial f_1}{\partial z_1} - i\alpha_2 f_2(z+c) = e^{-p_1}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} + i\beta_2 f_1(z+c) = e^{p_2}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} - i\beta_2 f_1(z+c) = e^{-p_2}. \end{cases} \quad (69)$$

In view of (69), it yields that

$$\begin{cases} \alpha_1 \frac{\partial f_1}{\partial z_1} = \frac{e^{p_1} + e^{-p_1}}{2}, \\ \alpha_2 f_2(z+c) = \frac{e^{p_1} - e^{-p_1}}{2i}, \\ \beta_1 \frac{\partial f_2}{\partial z_1} = \frac{e^{p_2} + e^{-p_2}}{2}, \\ \beta_2 f_1(z+c) = \frac{e^{p_2} - e^{-p_2}}{2i}, \end{cases} \quad (70)$$

which implies

$$\frac{\alpha_1}{\beta_2 i} \frac{\partial p_2}{\partial z_1} e^{p_1(z+c)+p_2} + \frac{\alpha_1}{\beta_2 i} \frac{\partial p_2}{\partial z_1} e^{p_1(z+c)-p_2} - e^{2p_1(z+c)} \equiv 1, \quad (71)$$

$$\frac{\beta_1}{\alpha_2 i} \frac{\partial p_1}{\partial z_1} e^{p_2(z+c)+p_1} + \frac{\beta_1}{\alpha_2 i} \frac{\partial p_1}{\partial z_1} e^{p_2(z+c)-p_1} - e^{2p_2(z+c)} \equiv 1. \quad (72)$$

Obviously, $\partial p_1/\partial z_1 \equiv 0$. Otherwise, $e^{2p_2(z+c)} \equiv 1$. This leads to a contradiction with p_1 is not a constant. Similarly, $\partial p_2/\partial z_1 \equiv 0$. Thus, due to [29], Lemma 3.1 (can be found in [30]), (71), and (72), we obtain that

$$\begin{aligned} &\frac{\alpha_1}{\beta_2 i}\frac{\partial p_2}{\partial z_1} e^{p_1(z+c)+p_2} \equiv 1 \text{or} \frac{\alpha_1}{\beta_2 i}\frac{\partial p_2}{\partial z_1} e^{p_1(z+c)-p_2(z)} \equiv 1, \\ &\frac{\beta_1}{\alpha_2 i}\frac{\partial p_1}{\partial z_1} e^{p_2(z+c)+p_1} \equiv 1 \text{or} \frac{\beta_1}{\alpha_2 i}\frac{\partial p_1}{\partial z_1} e^{p_2(z+c)-p_1} \equiv 1. \end{aligned} \tag{73}$$

□

Hence, four cases will be discussed below.

Case 1.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1} e^{p_1(z+c)+p_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1} e^{p_2(z+c)+p_1} \equiv 1. \end{cases} \tag{74}$$

Thus, it follows from (74) that $p_1(z+c)+p_2 \equiv C_1$ and $p_2(z+c)+p_1 \equiv C_2$. These lead to $p_1(z+2c)-p_1 \equiv C_1 - C_2$ and $p_2(z+c)-p_2 \equiv C_2 - C_1$. Hence, we obtain that $p_1(z) = L(z) + B_1$, $p_2(z) = -L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$, $a_1 (\neq 0)$, a_2, B_1, B_2 are constants. By combining with (71)–(74), we have

$$\begin{cases} \dfrac{\alpha_1 a_1}{\beta_2} i e^{L(c)+B_1+B_2} \equiv 1, \\ \dfrac{\beta_1 a_1}{\alpha_2 i} e^{-L(c)+B_1+B_2} \equiv 1, \\ \dfrac{\alpha_1 a_1}{\beta_2} i e^{-L(c)-B_1-B_2} \equiv 1, \\ \dfrac{\beta_1 a_1}{\alpha_2 i} e^{L(c)-B_1-B_2} \equiv 1, \end{cases} \tag{75}$$

and this leads to

$$a_1^2 = -\frac{\beta_2^2}{\alpha_1^2} = -\frac{\alpha_2^2}{\beta_1^2}, e^{4L(c)} = 1, e^{B_1+B_2} = \frac{\beta_2}{\alpha_1 a_1 i} e^{-L(c)} = \frac{\alpha_2 i}{\beta_1 a_1} e^{L(c)}. \tag{76}$$

Subcase 1. If $e^{L(c)} = 1$, then $L(c) = 2k\pi i$ and $e^{B_1+B_2} = \beta_2/\alpha_1 a_1 i = \alpha_2 i/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = i\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} - e^{-L(z)-B_1}}{2\alpha_2 i} = \frac{e^{L(z)-B_2}e^{B_1+B_2} - e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2 i} = \frac{-\beta_2 i/\alpha_1 a_1 e^{L(z)-B_2} - \beta_2 i/-\alpha_1 a_1 e^{-L(z)+B_2}}{2\alpha_2 i} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{77}$$

Subcase 2. If $e^{L(c)} = -1$, then $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$ and $e^{B_1+B_2} = -\beta_2/\alpha_1 a_1 i = -\alpha_2 i/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = -i\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2 i} = \frac{-e^{L(z)-B_2}e^{B_1+B_2} + e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2 i} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{78}$$

Subcase 3. If $e^{L(c)} = i$, then $L(c) = (2k+1/2)\pi i$, $k \in \mathbb{Z}$, and $e^{B_1+B_2} = -\beta_2/\alpha_1 a_1 = -\alpha_2/\beta_1 a_1$. Due to (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = \frac{e^{L(z)-B_2} + e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} - e^{-L(z)-B_1}}{2\alpha_2} = \frac{-e^{L(z)-B_2}e^{B_1+B_2} - e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2} = -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \end{aligned} \tag{79}$$

Subcase 4. If $e^{L(c)} = -i$, then $L(c) = (2k - 1/2)\pi i$, $k \in \mathbb{Z}$, and $e^{B_1+B_2}\beta_2/\alpha_1 a_1 = \alpha_2/\beta_1 a_1$. Due to (70), we have that

$$f_1 = \frac{e^{-L(z)+B_2+L(c)} - e^{L(z)-B_2-L(c)}}{2\beta_2 i} = -\frac{e^{L(z)-B_2} + e^{-L(z)+B_2}}{2\beta_2},$$
$$f_2 = \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2} = \frac{e^{L(z)-B_2}e^{B_1+B_2} + e^{-L(z)+B_2}e^{-B_1-B_2}}{2\alpha_2} = -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)-B_2} - e^{-L(z)+B_2}}{2\beta_2}. \quad (80)$$

Case 2.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)+p_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)-p_1} \equiv 1. \end{cases} \quad (81)$$

Thus, it yields from (81) that $p_1(z+c) + p_2 \equiv C_1$ and $p_2(z+c) - p_1 \equiv C_2$. We have that $p_2(z+2c) + p_2 \equiv C_1 + C_2$, and this leads to a contradiction with p_2 being not constant.

Case 3.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)-p_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)+p_1} \equiv 1. \end{cases} \quad (82)$$

Since $p_1(z), p_2(z)$ are polynomials, then from (82), it follows that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_2(z+c) + p_1(z) \equiv C_2$. This means $p_1(z+2c) + p_1(z) \equiv C_1 + C_2$, and this is a contradiction because $p_1(z)$ is not a constant.

Case 4.

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}\dfrac{\partial p_2}{\partial z_1}e^{p_1(z+c)-p_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}\dfrac{\partial p_1}{\partial z_1}e^{p_2(z+c)-p_1} \equiv 1. \end{cases} \quad (83)$$

Then, from (83), it yields that $p_1(z+c) - p_2 \equiv C_1$ and $p_2(z+c) - p_1 \equiv C_2$, and this leads to $p_1(z+2c) - p_1 \equiv C_1 + C_2$ and $p_2(z+2c) - p_2 \equiv C_2 + C_1$. Thus, it follows that $p_1(z) = L(z) + B_1, p_2(z) = L(z) + B_2$, where $L(z) = a_1 z_1 + a_2 z_2$, $a_1(\neq 0), a_2, B_1, B_2$ are constants in $\mathbb{C}$. In view of (71), (72), and (83), we have

$$\begin{cases} \dfrac{\alpha_1}{\beta_2 i}a_1 e^{L(c)+B_1-B_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}a_1 e^{L(c)-B_1+B_2} \equiv 1, \\ \dfrac{\alpha_1}{\beta_2 i}a_1 e^{-L(c)-B_1+B_2} \equiv 1, \\ \dfrac{\beta_1}{\alpha_2 i}a_1 e^{-L(c)+B_1-B_2} \equiv 1, \end{cases} \quad (84)$$

which implies

$$a_1^2 = -\frac{\beta_2^2}{\alpha_1^2} = -\frac{\alpha_2^2}{\beta_1^2}, e^{4L(c)} = 1, e^{B_1-B_2} = \frac{\beta_2 i}{\alpha_1 a_1}e^{-L(c)} = \frac{\beta_1 a_1}{\alpha_2 i}e^{L(c)}. \quad (85)$$

Subcase 4.1. If $e^{L(c)} = 1$, then $L(c) = 2k\pi i$, $k \in \mathbb{Z}$, and $e^{B_1-B_2} = \beta_2 i/\alpha_1 a_1 = \beta_1 a_1/\alpha_2 i$. By virtue of (70), it follows that

$$f_1 = \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = -i\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2},$$
$$\begin{aligned} f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} - e^{-L(z)-B_1}}{2\alpha_2 i} \\ &= \frac{e^{L(z)+B_2}e^{B_1-B_2} - e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2 i} \\ &= \frac{\beta_2 i/\alpha_1 a_1 e^{L(z)+B_2} - \beta_2 i/\alpha_1 a_1 e^{-L(z)-B_2}}{2\alpha_2 i} \\ &= -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \end{aligned} \quad (86)$$

Subcase 4.2. If $e^{L(c)} = -1$, then $L(c) = (2k+1)\pi i$, $k \in \mathbb{Z}$, and $e^{B_1-B_2} = -\beta_2 i/\alpha_1 a_1 = -\beta_1 a_1/\alpha_2 i$. By virtue of (70), it follows that

$$f_1 = \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = i\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2},$$

$$\begin{aligned} f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{-e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2 i} \\ &= \frac{-e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2 i} \\ &= -\frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \end{aligned} \tag{87}$$

Subcase 4.3. If $e^{L(c)} = i$, then $L(c) = (2k + 1/2)\pi i$ and $e^{B_1-B_2} = \beta_2/\alpha_1 a_1 = \alpha_2/\beta_1 a_1$. By virtue of (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = -\frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = -\frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2} \\ &= -\frac{e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \end{aligned} \tag{88}$$

Subcase 4.4. If $e^{L(c)} = -i$, then $L(c) = (2k - 1/2)\pi i$ and $e^{B_1-B_2} = -\beta_2/\alpha_1 a_1 = -\alpha_2/\beta_1 a_1$. By virtue of (70), we have that

$$\begin{aligned} f_1 &= \frac{e^{L(z)+B_2-L(c)} - e^{-L(z)-B_2+L(c)}}{2\beta_2 i} = \frac{e^{L(z)-B_2} + e^{-L(z)+B_2}}{2\beta_2}, \\ f_2 &= \frac{e^{L(z)+B_1-L(c)} - e^{-L(z)-B_1+L(c)}}{2\alpha_2 i} = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2\alpha_2} \\ &= \frac{e^{L(z)+B_2}e^{B_1-B_2} + e^{-L(z)-B_2}e^{-B_1+B_2}}{2\alpha_2} = \frac{\alpha_1 a_1}{\alpha_2}\frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2\beta_2}. \end{aligned} \tag{89}$$

Hence, the proof of Theorem 12 is completed.

5. The Proof of Theorem 13

Proof. Assume that (f_1, f_2) is a pair of finite-order transcendental entire functions satisfying (22). Thus, let us discuss two following cases.

(i) Suppose that $\partial f_1/\partial z_1$ is transcendental, then $\alpha_2 f_2(z + c) + \alpha_3 f_1$ is transcendental. Noting that α_j, β_j are nonzero constants, we next prove that $\beta_2 f_1(z + c) + \beta_3 f_2$ and $\beta_1 \partial f_2/\partial z_1$ are transcendental

Suppose that $\alpha_2 \partial f_2(z + c)/\partial z_1 + \alpha_3 \partial f_1/\partial z_1$ is not transcendental. Since $\partial f_1/\partial z_1$ is transcendental, then $\partial f_2(z + c)/\partial z_1$ and $\partial f_2(z)/\partial z_1$ are transcendental. By observing the second equation of (22), we can conclude that $\beta_2 f_1(z + c) + \beta_3 f_2$ is transcendental.

Suppose that $\alpha_2 \partial f_2(z + c)/\partial z_1 + \alpha_3 \partial f_1/\partial z_1$ is transcendental. If $\partial f_2(z + c)/\partial z_1$ is transcendental, similar to the above argument, $\beta_2 f_1(z + c) + \beta_3 f_2$ and $\partial f_2/\partial z_1$ are transcendental. If $\partial f_2(z + c)/\partial z_1$ is not transcendental, it thus leads to that $\partial f_2/\partial z_1$ is not transcendental. From (22), we thus get that $\beta_2 f_1(z + c) + \beta_3 f_2$ is not transcendental. Thus, it yields that $\beta_2 \partial f_1(z + c)/\partial z_1 + \beta_3 \partial f_2/\partial z_1$ is not transcendental. This is a contradiction with $\partial f_1(z + c)/\partial z_1$ is transcendental and $\partial f_2/\partial z_1$ is not transcendental.

Hence, if $\partial f_1/\partial z_1$ is transcendental, then $\alpha_2 f_2(z + c) + \alpha_3 f_1$, $\beta_2 f_1(z + c) + \beta_3 f_2$, and $\partial f_2/\partial z_1$ are transcendental. Hence, system (22) can be represented as

$$\begin{cases} \left[\alpha_1 \dfrac{\partial f_1}{\partial z_1} + i[\alpha_2 f_2(z + c) + \alpha_3 f_1]\right]\left[\alpha_1 \dfrac{\partial f_1}{\partial z_1} - i[\alpha_2 f_2(z + c) + \alpha_3 f_1]\right] = 1, \\ \left[\beta_1 \dfrac{\partial f_2}{\partial z_1} + i[\beta_2 f_1(z + c) + \beta_3 f_2]\right]\left[\beta_1 \dfrac{\partial f_2}{\partial z_1} - i[\beta_2 f_1(z + c) + \beta_3 f_2]\right] = 1. \end{cases} \tag{90}$$

Thus, by the Hadamard factorization theorem (can be found in [27, 28]), there are two nonconstant polynomials p, q such that

$$\begin{cases} \alpha_1 \dfrac{\partial f_1}{\partial z_1} + i[\alpha_2 f_2(z + c) + \alpha_3 f_1] = e^{ip}, \\ \alpha_1 \dfrac{\partial f_1}{\partial z_1} - i[\alpha_2 f_2(z + c) + \alpha_3 f_1] = e^{-ip}, \\ \beta_1 \dfrac{\partial f_2}{\partial z_1} + i[\beta_2 f_1(z + c) + \beta_3 f_2] = e^{iq}, \\ \beta_1 \dfrac{\partial f_2}{\partial z_1} - i[\beta_2 f_1(z + c) + \beta_3 f_2] = e^{-iq}. \end{cases} \tag{91}$$

In view of (91), it yields that

$$\begin{cases} \alpha_1 \dfrac{\partial f_1}{\partial z_1} = \dfrac{e^{ip} + e^{-ip}}{2}, \\ \alpha_2 f_2(z + c) + \alpha_3 f_1 = \dfrac{e^{ip} - e^{-ip}}{2i}, \\ \beta_1 \dfrac{\partial f_2}{\partial z_1} = \dfrac{e^{iq} + e^{-iq}}{2}, \\ \beta_2 f_1(z + c) + \beta_3 f_2 = \dfrac{e^{iq} - e^{-iq}}{2i}, \end{cases} \tag{92}$$

which implies

$$\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1} - \frac{\alpha_3}{\alpha_1}\right)e^{i(p+q(z+c))} + \frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1} - \frac{\alpha_3}{\alpha_1}\right)e^{i(q(z+c)-p)} - e^{2iq(z+c)} \equiv 1, \tag{93}$$

$$\frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1} - \frac{\beta_3}{\beta_1}\right)e^{i(q+p(z+c))} + \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1} - \frac{\beta_3}{\beta_1}\right)e^{i(p(z+c)-q)} - e^{2ip(z+c)} \equiv 1. \tag{94}$$

Obviously, $\partial p/\partial z_1 \neq \alpha_3/\alpha_1$. Otherwise, we have that $-e^{2iq(z+c)} \equiv 1$, and this leads to a contradiction since q is not a constant. Similarly, $\partial q/\partial z_1 \neq \beta_3/\beta_1$. Thus, due to [29], Lemma 3.1 (can be found in [30]), and in view of

(93) and (94), we can deduce that

$$\frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i[q(z+c)-p]}\equiv 1, \text{or} \frac{\beta_1}{\alpha_2}\left(\frac{\partial p}{\partial z_1}-\frac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1,$$
$$\frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1, \text{or} \frac{\alpha_1}{\beta_2}\left(\frac{\partial q}{\partial z_1}-\frac{\beta_3}{\beta_1}\right)e^{i[q+p(z+c)]}\equiv 1. \tag{95}$$

□

Now, let us consider the following four cases.

Case 1.

$$\begin{cases}\dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i[q(z+c)-p]}\equiv 1,\\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1.\end{cases} \tag{96}$$

Then, (96) can lead to that $q(z+c)-p\equiv C_1$ and $p(z+c)-q\equiv C_2$. Thus, we obtain that $p(z+2c)-p\equiv C_2+C_1$ and $q(z+2c)-q\equiv C_1+C_2$. Hence, we can conclude that $p(z)=L(z)+B_1, q(z)=L(z)+B_2$, where $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants. By combining with (93)–(96), we have

$$\begin{cases}\dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{i(L(c)+B_2-B_1)}\equiv 1,\\ \dfrac{\alpha_1}{\beta_2}\left(a_1-\dfrac{\beta_3}{\beta_1}\right)e^{i(L(c)+B_1-B_2)}\equiv 1,\\ \dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{-i(L(c)+B_2-B_1)}\equiv 1,\\ \dfrac{\alpha_1}{\beta_2}\left(a_1-\dfrac{\beta_3}{\beta_1}\right)e^{-i(L(c)+B_1-B_2)}\equiv 1.\end{cases} \tag{97}$$

This means that

$$\left[\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)\right]^2=\left[\frac{\alpha_1}{\beta_2}\left(a_1-\frac{\beta_3}{\beta_1}\right)\right]^2=1, e^{4iL(c)}=1, e^{i(B_1-B_2)}=\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{iL(c)}. \tag{98}$$

By combining with (92), f_1, f_2 have the following forms:

$$f_1(z)=\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_1(z_2), f_2(z)=\frac{e^{i(L(z)+B_2)}-e^{-i(L(z)+B_2)}}{2ia_1\beta_1}+\varphi_2(z_2), \tag{99}$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are entire functions of finite order in z_2. Substituting the above expressions into (92), we can deduce that

$$\begin{cases}\alpha_2\varphi_2(z_2+c_2)+\alpha_3\varphi_1(z_2)=0,\\ \beta_2\varphi_1(z_2+c_2)+\beta_3\varphi_2(z_2)=0.\end{cases} \tag{100}$$

This leads to

$$\varphi_1(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\varphi_1(z_2), \varphi_2(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\varphi_2(z_2). \tag{101}$$

Due to (101), we have

$$\varphi_1(z_2)=e^{\eta z_2}G_1(z_2), \varphi_2(z_2)=e^{\eta z_2}G_2(z_2), \tag{102}$$

where $G_1(z_2), G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and in (102), $\eta=0$, if $\alpha_2\beta_2=\alpha_3\beta_3$, and $\eta=\log(\alpha_2\beta_2)-\log(\alpha_3\beta_3)/2c_2$, if $\alpha_2\beta_2\neq\alpha_3\beta_3$. Further, in view of (100) and (102), we have $G_2(z_2)=-\alpha_3/\alpha_2G_1(z_2)$; if $\alpha_2\beta_2\neq\alpha_3\beta_3$, we have $G_2(z_2)=-\alpha_3/\alpha_2G_1(z_2)$.

If $e^{iL(c)}=1$, it follows from (97) that $e^{i(B_1-B_2)}=\pm 1$. Thus, it yields that

$$\begin{aligned} f_2(z)&=\frac{e^{i(L(z)+B_2)}-e^{-i(L(z)+B_2)}}{2ia_1\beta_1}+\varphi_2(z_2)\\ &=\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}e^{i(B_2-B_1)}-e^{-i(L(z)+B_1)}e^{i(B_1-B_2)}}{2ia_1\alpha_1}+\varphi_2(z_2)\\ &=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2).\end{aligned} \tag{103}$$

If $e^{iL(c)}=-1$, it follows from (97) that $e^{2i(B_1-B_2)}=1$. Thus, similar to the above argument, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \tag{104}$$

If $e^{iL(c)}=i$, it follows from (97) that $e^{2i(B_1-B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{105}$$

If $e^{iL(c)}=-i$, it follows from (97) that $e^{2i(B_1-B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{106}$$

Case 2.

$$\begin{cases}\dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i(q(z+c)-p(z))}\equiv 1,\\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i(q(z)+p(z+c))}\equiv 1.\end{cases} \tag{107}$$

We thus get from (107) that $p(z+c)+q(z)\equiv C_2$ and $q(z+c)-p\equiv C_1$. This means $q(z+2c)+q(z)\equiv C_1+C_2$, and this yields a contradiction with q being not a constant.

Case 3.

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i[p(z+c)-q]}\equiv 1. \end{cases} \tag{108}$$

We thus get from (108) that $q(z+c)+p(z)\equiv C_1$ and $p(z+c)-q(z)\equiv C_2$. So, we conclude that $p(z+2c)+p(z)\equiv C_1+C_2$, and this leads to a contradiction with p being not a constant.

Case 4.

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(\dfrac{\partial p}{\partial z_1}-\dfrac{\alpha_3}{\alpha_1}\right)e^{i[p+q(z+c)]}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(\dfrac{\partial q}{\partial z_1}-\dfrac{\beta_3}{\beta_1}\right)e^{i[q+p(z+c)]}\equiv 1. \end{cases} \tag{109}$$

Then, it follows from (109) that $p+q(z+c)\equiv C_1$ and $q+p(z+c)\equiv C_2$. These yield that $p(z+2c)-p\equiv C_1+C_2$ and $q(z+2c)-q\equiv C_2+C_1$, which leads to $p=L(z)+B_1, q=-L(z)+B_2$, where $L(z)=a_1z_1+a_2z_2$, a_1, a_2, B_1, B_2 are constants. In view of (93), (94), and (109), we have

$$\begin{cases} \dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{-i(L(c)-B_1-B_2)}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(-a_1-\dfrac{\beta_3}{\beta_1}\right)e^{i(L(c)+B_1+B_2)}\equiv 1, \\ \dfrac{\beta_1}{\alpha_2}\left(a_1-\dfrac{\alpha_3}{\alpha_1}\right)e^{i(L(c)-B_1-B_2)}\equiv 1, \\ \dfrac{\alpha_1}{\beta_2}\left(-a_1-\dfrac{\beta_3}{\beta_1}\right)e^{-i(L(c)+B_1+B_2)}\equiv 1. \end{cases} \tag{110}$$

In view of (110), it follows that

$$\left[\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)\right]^2=\left[\frac{\alpha_1}{\beta_2}\left(a_1+\frac{\beta_3}{\beta_1}\right)\right]^2=1, e^{4iL(c)}=1, e^{i(B_1+B_2)}=\frac{\beta_1}{\alpha_2}\left(a_1-\frac{\alpha_3}{\alpha_1}\right)e^{iL(c)}. \tag{111}$$

By combining with (92), f_1, f_2 are of the following forms

$$f_1(z)=\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_1(z_2), f_2(z)=\frac{e^{i(-L(z)+B_2)}-e^{i(L(z)-B_2)}}{2ia_1\beta_1}+\varphi_2(z_2), \tag{112}$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are finite-order entire functions in z_2. By using the same argument as in Case 1, we have (102).

If $e^{iL(c)}=1$, it follows from (110) that $e^{2i(B_1+B_2)}=1$. Thus, we can deduce that

$$\begin{aligned} f_2(z) &= \frac{e^{i(-L(z)+B_2)}-e^{i(L(z)-B_2)}}{2ia_1\beta_1}+\varphi_2(z_2) \\ &= \frac{\alpha_1}{\beta_2}\frac{-e^{i(L(z)+B_1)}e^{-i(B_1+B_2)}+e^{-i(L(z)+B_1)}e^{i(B_1+B_2)}}{2ia_1\alpha_1}+\varphi_2(z_2) \\ &= \pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \end{aligned} \tag{113}$$

If $e^{iL(c)}=-1$, it follows from (110) that $e^{2i(B_1+B_2)}=1$. We have that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}-e^{-i(L(z)-B_1)}}{2ia_1\alpha_1}+\varphi_2(z_2). \tag{114}$$

If $e^{iL(c)}=i$, it follows from (110) that $e^{2i(B_1+B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{115}$$

If $e^{iL(c)}=-i$, it follows from (110) that $e^{2i(B_1+B_2)}=-1$. Thus, we obtain that

$$f_2(z)=\pm\frac{\alpha_1}{\beta_2}\frac{e^{i(L(z)+B_1)}+e^{-i(L(z)-B_1)}}{2a_1\alpha_1}+\varphi_2(z_2). \tag{116}$$

Therefore, from (102)–(106) and (113)–(116), we can prove the conclusions (23) and (24) of Theorem 13.

(i)Assume that $\partial f_1/\partial z_1=0$. Thus, from (22), it follows that

$$f_1(z)=\phi_1(z_2), \alpha_2 f_2(z+c)+\alpha_3 f_1(z)\equiv\xi_1, \xi_1=\pm 1. \tag{117}$$

This leads to $\partial f_2/\partial z_1=0$. We thus get from (22) that

$$f_2(z)=\phi_2(z_2), \beta_2 f_1(z+c)+\beta_3 f_2(z)\equiv\xi_2, \xi_2=\pm 1. \tag{118}$$

By combining with (117) and (118), it yields

$$\begin{cases} \alpha_2\phi_2(z_2+c_2)+\alpha_3\phi_1(z_2)\equiv\xi_1, \\ \beta_2\phi_1(z_2+c_2)+\beta_3\phi_2(z_2)\equiv\xi_2, \end{cases} \tag{119}$$

which implies that

$$\phi_1(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\phi_1(z_2)+\frac{\alpha_2\xi_2-\beta_3\xi_1}{\alpha_2\beta_2}, \phi_2(z_2+2c_2)=\frac{\alpha_3\beta_3}{\alpha_2\beta_2}\phi_2(z_2)+\frac{\beta_2\xi_1-\alpha_3\xi_2}{\alpha_2\beta_2}. \tag{120}$$

If $\alpha_2\beta_2 = \alpha_3\beta_3$, then from (120), it follows that

$$\phi_1(z_2 + 2c_2) = \phi_1(z_2) + \frac{\alpha_2\xi_2 - \beta_3\xi_1}{\alpha_2\beta_2}, \phi_2(z_2 + 2c_2) = \phi_2(z_2) + \frac{\beta_2\xi_1 - \alpha_3\xi_2}{\alpha_2\beta_2}, \tag{121}$$

which implies that

$$\phi_1(z_2) = G_1(z_2) + \gamma_1 z_2, \phi_2(z_2) = G_2(z_2) + \gamma_2 z_2, \tag{122}$$

where $G_1(z_2)$, $G_2(z_2)$ are entire period functions of finite order with period $2c_2$, and

$$\gamma_1 = \frac{\alpha_2\xi_2 - \beta_3\xi_1}{2c_2\alpha_2\beta_2}, \gamma_2 = \frac{\beta_2\xi_1 - \alpha_3\xi_2}{2c_2\alpha_2\beta_2}. \tag{123}$$

If $\alpha_2\beta_2 \neq \alpha_3\beta_3$, then from (120), it follows that

$$\begin{aligned}\phi_1(z_2) &= e^{\log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2 z_2} G_1(z_2) + D_1, \phi_2(z_2)\\ &= e^{\log(\alpha_2\beta_2) - \log(\alpha_3\beta_3)/2c_2 z_2} G_2(z_2) + D_2,\end{aligned} \tag{124}$$

where $D_1 = \alpha_2\xi_2 - \beta_3\xi_1/\alpha_2\beta_2 - \alpha_3\beta_3$ and $D_2 = \beta_2\xi_1 - \alpha_3 \xi_2/\alpha_2\beta_2 - \alpha_3\beta_3$. Substituting (124) into (119), it follows that $\alpha_2\beta_2 = -\alpha_3\beta_3$, $G_1(z_2) = \beta_3/\beta_2 i G_2(z_2 - c_2)$ and $G_2(z_2) = \alpha_3/\alpha_2 i G_1(z_2 - c_2)$. Thus, we have

$$\begin{aligned}\phi_1(z_2) &= e^{\log(-1)/2c_2 z_2} G_1(z_2) + \frac{\alpha_2\xi_2 - \beta_3\xi_1}{2\alpha_2\beta_2}, \phi_2(z_2)\\ &= e^{\log(-1)/2c_2 z_2} G_2(z_2) + \frac{\beta_2\xi_1 - \alpha_3\xi_2}{2\alpha_2\beta_2}.\end{aligned} \tag{125}$$

(ii) Suppose that $\partial f_1(z_1, z_2)/\partial z_1 = b_1(\neq 0)$. Then, it yields in view of (22) that

$$f_1(z) = b_1 z_1 + \psi_1(z_2), \alpha_2 f_2(z + c) + \alpha_3 f_1(z) = b_2, b_2^2 + (\alpha_1 b_1)^2 = 1, \tag{126}$$

where $\psi_1(z_2)$ is a transcendental entire function of finite order in z_2. Equation (126) leads to $\partial f_2/\partial z_1 = -\alpha_3/\alpha_2 b_1$. Thus, due to the second equation in (22), we have

$$f_2(z) = -\frac{\alpha_3}{\alpha_2} b_1 z_1 + \psi_2(z_2), \beta_2 f_1(z + c) + \beta_3 f_2(z) = b_3, b_3^2 + \left(\frac{\beta_1\alpha_3}{\alpha_2} b_1\right)^2 = 1, \tag{127}$$

where $\psi_2(z_2)$ is a transcendental entire function of finite order in z_2. Combining with (126) and (127), we can deduce that $\alpha_2\beta_2 = \alpha_3\beta_3$ and

$$\begin{cases}\alpha_2\psi_2(z_2 + c_2) + \alpha_3\psi_1(z_2) = b_2 + \alpha_3 b_1 c_1,\\ \beta_2\psi_1(z_2 + c_2) + \beta_3\psi_2(z_2) = b_3 - \beta_2 b_1 c_1,\end{cases} \tag{128}$$

This means that

$$\psi_1(z_2) = G_1(z_2) + \gamma_1 z_2, \psi_2(z_2) = G_2(z_2) + \gamma_2 z_2, \tag{129}$$

where $G_1(z_2)$, $G_2(z_2)$ are entire period functions of finite order with period $2c_2$ satisfying

$$\begin{aligned}&G_2(z_2 + c_2) + \frac{\alpha_3}{\alpha_2} G_1(z_2) = \frac{\beta_2 b_2 + \alpha_3 b_3}{2\alpha_2\beta_2}, G_1(z_2 + c_2) + \frac{\beta_3}{\beta_2} G_2(z_2) = \frac{\beta_3 b_2 + \alpha_2 b_3}{2\alpha_2\beta_2},\\ &\gamma_1 = \frac{\alpha_2 b_3 - \beta_3 b_2 - 2(\alpha_2\beta_2) b_1 c_1}{2\alpha_2\beta_2 c_2}, \gamma_2 = \frac{\beta_2 b_2 - \alpha_3 b_3 + 2\alpha_3\beta_2 b_1 c_1}{2\alpha_2\beta_2 c_2}.\end{aligned} \tag{130}$$

Hence, from (126)–(129), it is easy to get the cases ((26)) and ((27)) of Theorem 13.

Therefore, the proof of Theorem 13 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that none of the authors have any competing interests in the manuscript.

Authors' Contributions

Conceptualization was contributed by H. Y. Xu; writing-original draft preparation was contributed by H.Y. Xu and K.Y. Zhang; writing-review and editing was contributed by H. Y. Xu and M.Y. Yu; funding acquisition was contributed by H. Y. Xu and K.Y. Zhang.

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