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Research Article

Dual of Modulation Spaces with Variable Smoothness and Integrability

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In this article, we first give a proof for the denseness of the Schwartz class in the modulation spaces with variable smoothness and integrability. Then, we study the dual spaces of such modulation spaces.

1. Introduction

The modulation spaces $M^s_{p,q}$ were introduced by Feichtinger [1] on a locally compact Abelian group in 1983 through short-time Fourier transform. His original motivation for modulation spaces was to introduce a new theory of function spaces and to offer an alternative to the class of Besov spaces. In recent years, it is gradually recognized that the modulation spaces are very useful for studying time-frequency behavior of functions. Therefore, the modulation spaces, α-modulation spaces, and their applications have received a lot of attention and research, such as [2–10] and the references therein. Particularly, in [11–14], Wang and other authors showed that from PDE point of view, the combination of frequency-uniform decomposition operators and Banach function spaces $\ell^q(X(\mathbb{R}^n))$ is important in making nonlinear estimates, where X is a Banach function space defined on $\mathbb{R}^n$.

On the other hand, function spaces with variable exponents have received extensive attention recently. Even though the study on variable Lebesgue spaces can be traced back to [15, 16] by Orlicz, the modern development started from the article [17] by Kováčik and Rákosník in 1991. In [18], Fan and Zhao obtained the results in [17] again through the method of Musielak-Orlicz spaces. Thereafter, variable Lebesgue and Sobolev spaces have been widely studied (see, for example, [19–23]). In addition, function spaces with variable exponents have a wealth of applications in many fields, such as in fluid dynamics [24], image processing [25], and partial differential equations [26].

The function spaces with variable smoothness and variable integrability were firstly introduced by Diening et al. in [27], where they studied Triebel-Lizorkin spaces with variable exponents $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then, Almeida and Hästö introduced the Besov space with variable smoothness and integrability $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [28]. Since then, many articles about these function spaces appeared, such as [29, 30]. In the past few years, many function spaces with variable exponents have appeared, such as Besov-type spaces with variable exponent, Bessel potential spaces with variable exponent, and Hardy spaces with variable exponent (see [31–35]). Recently, we studied the modulation spaces with variable smoothness and integrability $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and gave some properties about these spaces in [36]. Since the modulation spaces and the function spaces with variable exponents have rich applications, we believe that the modulation spaces with variable exponents will also have many application areas, and we will continue to explore these application areas, especially in partial differential equations and time frequency analysis.

The dual is an important content when we study function spaces; for example, Triebel [37] has obtained duality of the usual Besov spaces and applied it to real interpolation and Sobolev embedding, Izuki [38] has given the duality of

Herz spaces with variable exponent and applied it to characterize the above spaces by wavelet expansions. In [8, 12], the dual of modulation spaces was studied, respectively. In [39], Izuki and Noi were concerned with the dual of Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ with variable exponents. In this paper, we will study the dual of modulation spaces $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$ with variable smoothness and integrability.

The paper is organized as follows. In Section 2, we review some notions and notations about semimodular spaces and function spaces with variable exponents. In the theories of function spaces, the research on denseness of the Schwartz class has always been an important topic, by which we can obtain many conclusions such as duality of function spaces and boundedness of some operators. Therefore, in Section 3, we study the denseness of the Schwartz class in the modulation spaces with variable smoothness and integrability. In Section 4, we give the dual of modulation spaces with variable exponents.

2. Preliminaries

In this section, we review some notions and conventions and state some basic results. Throughout this article, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a positive constant C such that $1/C \le A/B \le C$. The symbol $A \lesssim B$ means that $A \le CB$. The symbol $[s]$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s. We also set $\mathbb{N} \equiv \{1, 2, \cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. We write $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\langle x \rangle_o = 1 + |x_1| + |x_2| + \cdots + |x_n|$ for $x \in \mathbb{R}^n$. It is easy to see that $\langle x \rangle \sim \langle x \rangle_o$. For any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we denote $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, and for $k = (k_1, k_2, \cdots, k_n)$, we denote $|k|_\infty = \max_{i=1,\cdots,n} |k_i|$. We also denote the sequence Lebesgue space by ℓ^p and Lebesgue space by $L^p := L^p(\mathbb{R}^n)$ for which the norm is written by $\|\cdot\|_p$.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz function space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be its strongly topological dual space which is also known as the space of all tempered distributions. For $f \in \mathcal{S}$, we define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$, respectively, by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx,$$
$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi. \tag{1}$$

2.1. Modular Spaces. In what follows, let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. The function spaces studied in this paper fit into the framework semimodular spaces, and we refer to monograph [23] for a detailed exposition of these concepts.

Definition 1. A function $\varrho : X \longrightarrow [0,\infty]$ is called a semimodular on X if it satisfies

$$\varrho(0) = 0. \tag{2}$$

(i) $\varrho(\lambda f) = \varrho(f)$ for all $f \in X$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$ with $|\lambda| = 1$

(ii) $\varrho(\lambda f) = 0$ for all $\lambda > 0$ implies $f = 0$

(iii) $\lambda \mapsto \varrho(\lambda f)$ is left-continuous on $[0, \infty)$ for every $f \in X$

A semimodular ϱ is called a modular if $\varrho(f) = 0$ implies $f = 0$, and it is called continuous if the mapping $\lambda \mapsto \varrho(\lambda f)$ is continuous on $[0, \infty)$ for every $f \in X$. A semimodular ρ can also be qualified by the term (quasi)convex; that is, for all $f, g \in X$ and $\theta \in [0, 1]$, there exists A such that

$$\varrho(\theta f + (1 - \theta) g) \le A[\theta \varrho(f) + (1 - \theta) \varrho(g)], \tag{3}$$

where $A = 1$ in the convex case and $A \in [1,\infty)$ in the quasiconvex case. By semimodular, we can obtain a normed space as follows:

Definition 2. If ρ is a (semi)modular on X, then $X_\varrho := \{f \in X : \exists \lambda > 0, \text{s.t.} \varrho(\lambda f) < \infty\}$ is called a (semi)modular space.

In [11], the authors have proven that the X_ϱ is a (quasi) normed space with the Luxemburg (quasi)norm $\|f\|_\varrho := \inf \{\lambda > 0 : \varrho(f/\lambda) \le 1\}$, where the infimum of the empty set is infinity by definition. The following conclusion can be found in [23], and we omit the proof here.

Theorem 3 (norm-modular unit ball property). *Let ϱ be a semimodular on X and $f \in X$. Then, $\|f\|_\varrho \le 1$ if and only if $\varrho(f) \le 1$. If ϱ is continuous, then $\|f\|_\varrho < 1$ and $\varrho(f) < 1$ are equivalent, so are $\|f\|_\varrho = 1$ and $\varrho(f) = 1$.*

2.2. Function Spaces with Variable Exponents. A measurable function $p(\cdot)$: $\mathbb{R}^n \longrightarrow (0,\infty]$ is called a variable exponent function if it is bounded away from zero; namely, the range of the $p(x)$ is $(c, \infty]$ for some $c > 0$. For a measurable function $p(\cdot)$ and a measurable set $\Omega \subset \mathbb{R}^n$, let $p_\Omega^- := \operatorname{ess\,inf}_\Omega p(x)$ and $p_\Omega^+ := \operatorname{ess\,sup}_\Omega p(x)$. For simplicity, we abbreviate $p^- = p^-_{\mathbb{R}^n}$ and $p^+ = p^+_{\mathbb{R}^n}$.

We denote by $\mathcal{P}_0$ for the set of all measurable functions $p(\cdot)$: $\mathbb{R}^n \longrightarrow (0,\infty)$ such that $0 < p^- \le p^+ < \infty$ and denote by $\mathcal{P}$ for the set of all measurable functions $p(\cdot)$: $\mathbb{R}^n \longrightarrow (0,\infty)$ such that $1 < p^- \le p^+ < \infty$.

In order to make the Hardy-Littlewood maximal function bounded in the variable exponent Lebesgue spaces, one need to add some conditions to the variable exponent function, that is, so-called log-Hölder continuity, which was first introduced in [40].

Definition 4. Let $p(\cdot)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$.

(i) If there exists $c_{\log} > 0$ such that

$$|p(x) - p(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)}, \qquad (4)$$

for all $x, y \in \mathbb{R}^n$, then $p(\cdot)$ is called locally log-Hölder continuous, abbreviated as $p \in C^{\log}_{\text{loc}}$.

(ii) If $p(\cdot)$ is locally log-Hölder continuous and there exists $p_\infty \in \mathbb{R}$ such that

$$|p(x) - p_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}, \qquad (5)$$

for all $x \in \mathbb{R}^n$, then $p(\cdot)$ is called globally log-Hölder continuous, abbreviated as $p \in C^{\log}$.

If a variable exponent $p \in \mathscr{P}$ satisfies $1/p \in C^{\log}$, we say that it belongs to the class $\mathscr{P}^{\log}$. The class $\mathscr{P}^{\log}_0$ is defined similarly.

Remark 5.

(i) One can notice that all functions $p \in C^{\log}_{\text{loc}}$ always belong to L^∞

(ii) Let $p \in \mathscr{P}_0$, then $p \in C^{\log}$ if and only if $1/p \in C^{\log}$. If p satisfies (5), then $p_\infty = \lim_{|x| \longrightarrow \infty} p(x)$

(iii) We define the conjugate exponent function $p'(\cdot)$ by the formula $(1/p(\cdot)) + (1/p'(\cdot)) = 1$. If $p(\cdot)$ is in $C^{\log}$, then $p'(\cdot)$ is also in $C^{\log}$

We define

$$\varphi_p(t) = \begin{cases} t^p, & \text{if } p \in (0, \infty), \\ 0, & \text{if } p = \infty \text{and } t \leq 1, \\ \infty, & \text{if } p = \infty \text{and } t > 1, \end{cases} \qquad (6)$$

and we adopt the convention $1^\infty = 0$ in order that φ_p is left-continuous. The variable exponent modular of a measurable function f on $\mathbb{R}^n$ is defined by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|)dx. \qquad (7)$$

According to Definition 2, one can define the corresponding semimodular space, namely, the variable exponent Lebesgue space which is denoted by $L^{p(\cdot)}(\mathbb{R}^n)$, and the Luxemburg (quasi)norm of the $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}. \qquad (8)$$

Now let us recall the mixed Lebesgue sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ which was introduced by Almeida and Hästö in [28].

Definition 6. Let $p, q \in \mathscr{P}_0$ and Ω be a measurable subset of $\mathbb{R}^n$. The mixed Lebesgue sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(\Omega))$ is the collection of all sequences $\{f_j\}_{j \in \mathbb{N}}$ of $L^{p(\cdot)}(\Omega)$-functions such that

$$\left\| \left\{ f_j \right\}_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\Omega))} := \inf \left\{ \lambda > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{ \frac{f_j \chi_\Omega}{\lambda} \right\}_j \right) \leq 1 \right\} < \infty, \qquad (9)$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{ f_j \right\}_j \right) := \sum_{j \in \mathbb{N}} \inf \left\{ \mu_j > 0 : \varrho_{p(\cdot)}\left(\frac{f_j}{\mu_j^{1/q(\cdot)}} \right) \leq 1 \right\}, \qquad (10)$$

with the convention $\lambda^{1/\infty} = 1$ for all $\lambda > 0$.

Remark 7. Let $p, q \in \mathscr{P}_0$.

(i) If $q^+ < \infty$, then $\inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda^{1/q(\cdot)}) \leq 1\} = \| |f|^{q(\cdot)} \|_{p(\cdot)/q(\cdot)}$, and we use the notation

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(f_j \right)_j \right) = \sum_j \left\| \left| f_j \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)}. \qquad (11)$$

(ii) By Proposition 3.3 of [28], if $q \in (0, \infty]$ is constant, then we have $\|(f_j)_j\|_{\ell^q(L^{p(\cdot)})} = \| \|f_j\|_{p(\cdot)} \|_{\ell^q}$

(iii) In [28], Almeida and Hästö proved that $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasinorm for all $p(\cdot), q(\cdot) \in \mathscr{P}$, and $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm when $(1/p(\cdot)) + (1/q(\cdot)) \leq 1$ pointwise or q is a constant. In [30], Kempka and Vybíral proved that $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm if $p(\cdot), q(\cdot) \in \mathscr{P}$ satisfy either $1 \leq q(x) \leq p(x) \leq \infty$ for almost every $x \in \mathbb{R}^n$ or $p(x) \geq 1$, and $q \in [1, \infty)$ is a constant almost everywhere

To define modulation space with variable exponents, we need some general definitions from the constant exponent case. For $k \in \mathbb{Z}^n$, let Q_k be the unit cube with the center at k; then, $\{Q_k\}_{k \in \mathbb{Z}^n}$ constitutes a decomposition of $\mathbb{R}^n$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $\phi : \mathbb{R}^n \longrightarrow [0, 1]$ be a smooth function satisfying $\phi(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\phi(\xi) = 0$ for $|\xi|_\infty \geq 1$. Let ϕ_k be a translation of ϕ: $\phi_k(\xi) = \phi(\xi - k), k \in \mathbb{Z}^n$. Then, we see that $\phi_k(\xi) = 1$ in Q_k and $\sum_{k \in \mathbb{Z}^n} \phi_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. If we denote $\varphi_k(\xi) = \phi_k(\xi)(\sum_{k \in \mathbb{Z}^n} \phi_k(\xi))^{-1}$, for $k \in \mathbb{Z}^n$, then we have

$$\begin{cases} |\varphi_k(\xi)| \geq c, & \forall \xi \in Q_k, \\ \operatorname{supp} \varphi_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, & , \\ \sum_{k\in\mathbb{Z}^n} \varphi_k(\xi) \equiv 1, & \forall \xi \in \mathbb{R}^n, \\ |D^\alpha \varphi_k(\xi)| \leq C_{|\alpha|}, & \forall \xi \in \mathbb{R}^n, \alpha \in (\mathbb{N} \cup \{0\})^n. \end{cases} \tag{12}$$

We denote $Y = \{\{\varphi_k\}_{k\in\mathbb{Z}^n} : \{\varphi_k\}_{k\in\mathbb{Z}^n} \text{ satisfies}(3)\}.Y$ is nonempty, and for every sequence $\{\varphi_k\}_{k\in\mathbb{Z}^n} \in Y$, one can construct an operator sequence as follows:

$$\square_k := \mathcal{F}^{-1}\varphi_k\mathcal{F}, \quad k \in \mathbb{Z}^n. \tag{13}$$

$\{\square_k\}_{k\in\mathbb{Z}^n}$ are said to be frequency-uniform decomposition operators. Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$; the modulation space can be defined as

$$M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\square_k f\|^q_p \right)^{1/q} < \infty \right\}. \tag{14}$$

Further details about the frequency-uniform decomposition techniques and their applications to PDE can be found in the book [13] and articles [11, 12, 14].

Definition 8. Let $\{\varphi_k\}_{k\in\mathbb{Z}^n} \in Y$ and $\{\square_k\}_{k\in\mathbb{Z}^n}$ be the corresponding frequency-uniform decomposition operators. For $p, q \in \mathcal{P}^{\log}_0$ and $s \in C^{\log}_{\mathrm{loc}}$, the modulation space with variable smoothness and integrability $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is defined to be the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|^{\varphi}_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left(\langle k\rangle^{s(\cdot)} \square_k f \right)_k \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} < \infty. \tag{15}$$

For above modulation space, we can define the following modular:

$$\varrho^{\varphi}_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}}(f) := \varrho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\left(\left(\langle k\rangle^{s(\cdot)} \square_k f \right)_k \right), \tag{16}$$

which can be used to define the norm. In [36], we have shown that the space given by Definition 8 is independent of the choice of $\{\varphi_k\}_{k\in\mathbb{Z}^n} \in Y$ and the corresponding frequency-uniform decomposition operators. Thus, we can choose $\{\varphi_k\}_{k\in\mathbb{Z}^n} \in Y$ according to our requirements, and we will omit φ in the notation of the norm and modular.

3. Density

In [28], the authors showed that the maximal function is not a good tool in the variable exponent space $\ell^{q(\cdot)}(L^{p(\cdot)})$; hence, they used so-called η-functions which were also used in [27]. Similarly, in our article, we define the so-called θ-functions on $\mathbb{R}^n$ by

$$\theta_{k,m}(x) := \frac{r^n_k}{(1 + r_k|x|)^m}, \tag{17}$$

with $k \in \mathbb{Z}^n$, $m > 0$, and $r_k := \sqrt{n}(1 + |k|_\infty)$. Note that $\theta_{k,m} \in L^1$ when $m > n$ and that $\|\theta_{k,m}\|_1 = c_m$ is independent of k. These functions are different from the η-functions since we use the uniform decomposition of $\mathbb{R}^n$ rather than the dyadic decomposition.

Now let us review some useful results about θ-functions which have been proven in [36].

Lemma 9 (see [36]). *Let $s(\cdot) \in C^{\log}_{loc}$ and $k \in \mathbb{Z}^n$; then, there exists a positive constant C such that*

$$\langle k\rangle^{s(x)} \theta_{k,m+R}(x - y) \leq C \langle k\rangle^{s(y)} \theta_{k,m}(x - y), \tag{18}$$

for all $x, y \in \mathbb{R}^n$ and $R \geq c_{\log}(s)$, where $c_{\log}(s)$ is the constant from (4) for $s(\cdot)$.

Remark 10. By Lemma 9, we can move the term inside the convolution as follows:

$$\langle k\rangle^{s(x)} \theta_{k,m+R} * f(x) \leq C\theta_{k,m} * \left(\langle k\rangle^{s(\cdot)} f \right)(x), \tag{19}$$

which helps us to treat the variable smoothness in many cases.

Lemma 11 (see [36]). *Let $p, q \in \mathcal{P}^{\log}$, for $m > n$ and every sequence $\{f_k\}_{k\in\mathbb{Z}^n}$ of L^1_{loc}-functions, there exists a constant $C > 0$ such that*

$$\left\| (\theta_{k,2m} * f_k)_k \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} \leq C \left\| (f_k)_k \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}. \tag{20}$$

Remark 12. In some cases, although we need to require that $p^-, q^- \geq 1$, we can weaken this condition by the following identity:

$$\left\| (f_k)_k \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} = \left\| (|f_k|^r)_k \right\|^{1/r}_{\ell^{q(\cdot)/r}\left(L^{p(\cdot)/r}\right)}. \tag{21}$$

In [36], we have proven that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow M^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and in this section, we will prove that $\mathcal{S}(\mathbb{R}^n)$ is also dense in $M^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. For this purpose, we need the following conclusions as in [41]. The first one is the generalization of Lemma A.6 in [27] and Lemma 3.4 in [36].

Lemma 13. *Let $r > 0$, $k \in \mathbb{Z}^n$, and $m > n$. Then, for all $x, z \in \mathbb{R}^n$ and $g \in \mathcal{S}'$ with $\operatorname{supp} \widehat{g} \subset \{\xi : |\xi - k|_\infty \leq 1\}$, there exists a constant $C = C(r, m, n) > 0$ such that*

$$\frac{|g(x - z)|}{(1 + r_k|z|)^{m/r}} \leq C(\theta_{k,m} * |g|^r(x))^{1/r}. \tag{22}$$

Proof. As in the proof of Lemma 3.4 of [36], for $r_k = \sqrt{n}(1 + |k|_\infty)$, there exists $\nu \in \mathbb{N}$ such that $2^\nu \le r_k < 2^{\nu+1}$, which implies $\{\xi : |\xi - k|_\infty \le 1\} \subset \{\xi : |\xi| \le 2^{\nu+1}\}$. Then, for $u \in \mathbb{Z}^n$ and a fixed dyadic cube $Q = Q_{\nu,u} \coloneqq \{x \in \mathbb{R}^n : 2^{-\nu}u_i \le x_i < 2^{-\nu}(u_i + 1), i = 1, 2, \cdots, n\}$, when $x - z \in Q$, we have

$$|g(x-z)|^r \le \sup_{w\in Q}|g(w)|^r \le C2^{\nu n}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-m}\int_{Q_{\nu,u+l}}|g(y)|^r dy. \tag{23}$$

In addition, for $x - z \in Q_{\nu,u}$ and $y \in Q_{\nu,u+l}$, we have $|x - z - y| \sim 2^{-\nu}|l|$ when l is large enough, which implies $1 + 2^\nu|x - z - y| \sim 1 + |l|$. Since

$$(1 + 2^\nu|x-y|)^m \le (1 + 2^\nu|x-z-y|)^m(1 + 2^\nu|z|)^m, \tag{24}$$

we get

$$\begin{aligned}\sup_{w\in Q}|g(w)|^r &\le C2^{\nu n}\sum_{l\in\mathbb{Z}^n}\int_{Q_{\nu,u+l}}(1+2^\nu|x-z-y|)^{-m}|g(y)|^r dy\\ &\le C2^{\nu n}\sum_{l\in\mathbb{Z}^n}\int_{Q_{\nu,u+l}}(1+2^\nu|x-y|)^{-m}(1+2^\nu|z|)^m|g(y)|^r dy\\ &\le C(1+r_k|z|)^m\sum_{l\in\mathbb{Z}^n}\int_{Q_{\nu,u+l}}r_k^n(1+r_k|x-y|)^{-m}|g(y)|^r dy\\ &\le C(1+r_k|z|)^m\int_{\mathbb{R}^n}r_k^n(1+r_k|x-y|)^{-m}|g(y)|^r dy\\ &= C(1+r_k|z|)^m(\theta_{k,m} * |g|^r)(x).\end{aligned} \tag{25}$$

Hence, for $x - z \in Q_{\nu,u}$, we have

$$\frac{|g(x-z)|}{(1+r_k|z|)^{m/r}} \le C(\theta_{k,m} * |g|^r(x))^{1/r}, \tag{26}$$

where $C = C(r, m, n)$ depends only on r, m, and n. For any $x, z \in \mathbb{R}^n$, there exists a $u' \in \mathbb{Z}^n$ such that $x - z \in Q_{\nu,u'}$. Then, we get the desired conclusion. □

Definition 14.

(i) Let Ω be a compact subset of $\mathbb{R}^n$; then, we denote the space of all elements $f \in \mathcal{S}(\mathbb{R}^n)$ with supp $\mathcal{F}f \subset \Omega$ by $\mathcal{S}_\Omega(\mathbb{R}^n)$

(ii) Let $p(\cdot) \in \mathcal{P}_0^{\log}$ and $\Omega = \{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$; then, we denote by $L_\Omega^{p(\cdot)}(\mathbb{R}^n)$ the space of all sequences $\{f_k\}_{k=0}^\infty$ in $\mathcal{S}'(\mathbb{R}^n)$ such that supp $\mathcal{F}f_k \subset \Omega_k$ and $\|f_k\|_{p(\cdot)} < \infty$ for $k = 0, 1, 2, \cdots$

Lemma 15. *Let $p, q \in \mathcal{P}_0^{\log}$, $s \in C_{loc}^{\log}$, and $\Omega = \{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $\Omega_k \subset \{\xi \in \mathbb{R}^n : |\xi - k|_\infty \le 1\}$. If $0 < r < \min\{p^-, q^-\}$ and $m > 2n + 2c_{\log}(s)\min\{p^-, q^-\}$, then for all $\{f_k\}_{k\in\mathbb{Z}^n} \subset L_\Omega^{p(\cdot)}(\mathbb{R}^n)$, there exists a constant C such that*

$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\langle k\rangle^{s(\cdot-z)}|f_k(\cdot-z)|}{1+|r_kz|^{m/r}}\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le C\left\|\left\{\langle k\rangle^{s(\cdot)}f_k\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{27}$$

Proof. Let $\{f_k\}_{k\in\mathbb{Z}^n} \in L_\Omega^{p(\cdot)}(\mathbb{R}^n)$ and $R \ge c_{\log}(s)$; then, for any $k \in \mathbb{Z}^n$ by Lemmas 9 and 13, we have

$$\begin{aligned}\frac{\langle k\rangle^{s(x-z)}|f_k(x-z)|}{1+|r_kz|^{m/r}} &\le \max\{2^{m/r},1\}\frac{\langle k\rangle^{s(x-z)}|f_k(x-z)|}{(1+|r_kz|)^{m/r}}\\ &\lesssim \max\{2^{m/r},1\}\frac{\langle k\rangle^{s(x)}|f_k(x-z)|}{(1+|r_kz|)^{m-Rr/r}}\\ &\lesssim \max\{2^{m/r},1\}\langle k\rangle^{s(x)}(\theta_{k,m-Rr} * |f_k|^r(x))^{1/r}\\ &\lesssim \max\{2^{m/r},1\}\left(\left(\theta_{k,m-2Rr}(y) * \langle k\rangle^{rs(y)}|f_k(y)|^r\right)(x)\right)^{1/r}.\end{aligned} \tag{28}$$

Therefore, for $0 < r < \min\{p^-, q^-\}$ and $m > 2n + 2Rr$, by Lemma 11, we obtain

$$\begin{aligned}&\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\langle k\rangle^{s(\cdot-z)}|f_k(\cdot-z)|}{1+|r_kz|^{m/r}}\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\\ &\le C\left\|\left\{\left(\theta_{k,m-2Rr}(y) * \langle k\rangle^{rs(y)}|f_k(y)|^r\right)(\cdot)\right\}_{k\in\mathbb{R}^n}\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}\\ &\le C\left\|\left\{\langle k\rangle^{rs(\cdot)}|f_k(\cdot)|^r\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}^{1/r}\\ &= C\left\|\left\{\langle k\rangle^{s(\cdot)}f_k\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},\end{aligned} \tag{29}$$

which completes the proof. □

For a real number s, we denote

$$H_2^s(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n): \|f\|_{H_2^s} = \left\|\left(1+|\cdot|^2\right)^{s/2}(\mathcal{F}f)(\cdot)\right\|_{L^2} < \infty\right\}. \tag{30}$$

Proposition 16. *Let $p, q \in \mathcal{P}_0^{\log}$, $s \in C_{loc}^{\log}$, and $\Omega = \{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $\Omega_k \subset \{\xi \in \mathbb{R}^n : |\xi - k|_\infty \le 1\}$. If $t > (n/2) + ((2n + 3c_{\log}(s)\min\{p^-, q^-\})/\min\{p^-, q^-\})$, then for all $\{f_k\}_{k\in\mathbb{Z}^n} \subset L_\Omega^{p(\cdot)}(\mathbb{R}^n)$ and $\{M_k(\cdot)\}_{k\in\mathbb{Z}^n} \subset H_2^t(\mathbb{R}^n)$, there exists a constant C such that*

$$\begin{aligned}&\left\|\left\{\langle k\rangle^{s(\cdot)}\mathcal{F}^{-1}M_k\mathcal{F}f_k\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\\ &\le C\sup_k\|M_k(r_k\,\cdot)\|_{H_2^t}\left\|\left\{\langle k\rangle^{s(\cdot)}f_k\right\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.\end{aligned} \tag{31}$$

Proof. According to Lemma 15, by the similar argument in the proof of Theorem 4.15 of [41], we have

$$\begin{aligned}&\left|\langle k\rangle^{s(x)}\mathcal{F}^{-1}M_k\mathcal{F}f_k(x-z)\right|\\&\quad\lesssim\int_{\mathbb{R}^n}\langle k\rangle^{s(x)}\frac{\left|\left(\mathcal{F}^{-1}M_k\right)(x-z-y)\right|}{(1+r_k|x-y|)^{(m+Rr)/r}}|f_k(y)|\left(1+|r_k(x-y)|^{(m+Rr)/r}\right)dy\\&\quad\lesssim\int_{\mathbb{R}^n}\frac{\left|\left(\mathcal{F}^{-1}M_k\right)(x-z-y)\right|}{(1+r_k|x-y|)^{m/r}}\langle k\rangle^{s(y)}|f_k(y)|\left(1+|r_k(x-y)|^{(m+Rr)/r}\right)dy\\&\quad\lesssim\sup_{u\in\mathbb{R}^n}\frac{\langle k\rangle^{s(u)}|f_k(u)|}{(1+r_k|x-u|)^{m/r}}\int_{\mathbb{R}^n}\left|\left(\mathcal{F}^{-1}M_k\right)(x-z-y)\right|\left(1+|r_k(x-y)|^{(m+Rr)/r}\right)dy.\end{aligned}\tag{32}$$

Since

$$1+|r_k(x-y)|^{(m+Rr)/r}\lesssim\left(1+|r_k(x-y-z)|^{(m+Rr)/r}\right)\left(1+|r_kz|^{(m+Rr)/r}\right),\tag{33}$$

then for $0<r<\min\{p^-,q^-\}$ and $t>(n/2)+((m+Rr)/r)$, by the same argument in 1.6.3 of [42], we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\left(\mathcal{F}^{-1}M_k\right)(x-z-y)\right|\left(1+|r_k(x-y)|^{(m+Rr)/r}\right)dy\\&\lesssim\int_{\mathbb{R}^n}\left|\left(\mathcal{F}^{-1}M_k\right)(x-z-y)\right|\left(1+|r_k(x-y-z)|^{(m+Rr)/r}\right)\\&\quad\cdot\left(1+|r_kz|^{(m+Rr)/r}\right)dy\lesssim\left(1+|r_kz|^{(m+Rr)/r}\right)\|M_k(r_k\,\cdot)\|_{H_2^t}.\end{aligned}\tag{34}$$

Thus,

$$\sup_{z\in\mathbb{R}^n}\frac{\left|\langle k\rangle^{s(x)}\mathcal{F}^{-1}M_k\mathcal{F}f_k(x-z)\right|}{\left(1+|r_kz|^{(m+Rr)/r}\right)}\lesssim\sup_{z\in\mathbb{R}^n}\frac{\langle k\rangle^{s(x-z)}|f_k(x-z)|}{(1+r_k|z|)^{m/r}}\|M_k(r_k\,\cdot)\|_{H_2^t}.\tag{35}$$

In addition, since

$$\langle k\rangle^{s(x)}\left|\mathcal{F}^{-1}M_k\mathcal{F}f_k(x)\right|\le\sup_{z\in\mathbb{R}^n}\frac{\langle k\rangle^{s(x)}\left|\mathcal{F}^{-1}M_k\mathcal{F}f_k(x-z)\right|}{1+|r_kz|^{(m+Rr)/r}},\tag{36}$$

then together with above inequality, (33) and Lemma 15, we can get the conclusion. □

Remark 17. In fact, in the conclusion of the above lemma, the "r_k" in $\|M_k(r_k\,\cdot)\|_{H_2^t}$ can be replaced by $|\Omega_k|:=\sup_{x,y\in\Omega_k}|x-y|$.

Theorem 18 (density). *Let $p,q\in\mathcal{P}_0^{\log}$ and $s\in C_{loc}^{\log}$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $f\in M_{p(\cdot),q(\cdot)}^{s(\cdot)}$, for $\{\varphi_k\}_{k\in\mathbb{Z}^n}\in Y$ and $N\in\mathbb{Z}_+$, we put

$$f_N=\sum_{|k|\le N}\square_kf=\sum_{|k|\le N}\mathcal{F}^{-1}\varphi_k\mathcal{F}f,\tag{37}$$

where $|k|:=|k_1|+|k_2|+\cdots+|k_n|$ for $k\in\mathbb{Z}^n$. Then, we have $f_N\in M_{p(\cdot),q(\cdot)}^{s(\cdot)}$. In fact, by Proposition 16, we obtain

$$\begin{aligned}\|f_N\|_{M_{p(\cdot),q(\cdot)}^{s(\cdot)}}&=\left\|\left(\langle l\rangle^{s(\cdot)}\square_lf_N\right)_{l\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\&\lesssim\left\|\left(\langle l\rangle^{s(\cdot)}\sum_{|r|_\infty\le1}|\square_l\square_{l+r}f|\right)_{|l|\le N+n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\&\lesssim\left\|\left(\langle l\rangle^{s(\cdot)}\square_lf\right)_{l\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}<\infty.\end{aligned}\tag{38}$$

Consequently,

$$\begin{aligned}\|f-f_N\|_{M_{p(\cdot),q(\cdot)}^{s(\cdot)}}&\lesssim\left\|\left(\langle l\rangle^{s(\cdot)}\sum_{|r|_\infty\le1}|\square_l\square_{l+r}f|\right)_{|l|>N-n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\&\lesssim\left\|\left(\langle l\rangle^{s(\cdot)}\square_lf\right)_{|l|>N-n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\longrightarrow0,\end{aligned}\tag{39}$$

when $N\longrightarrow\infty$, in which the last limit can be deduced by Lemma 2.2 of [43]. Hence, $f_N\longrightarrow f$ in $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$ when $N\longrightarrow\infty$.

Next, we should approximate f_N by some functions in $\mathcal{S}(\mathbb{R}^n)$ for $N\in\mathbb{Z}_+$. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\psi(0)=1$ and supp $\mathcal{F}\psi\subset\{\xi\in\mathbb{R}^n:|\xi|\le1\}$. Then, for any $N\in\mathbb{Z}_+$ and $n\in\mathbb{N}$, we have $\psi(\cdot/n)f_N\in\mathcal{S}(\mathbb{R}^n)$ and

$$\operatorname{supp}\mathcal{F}(f_N-\psi(\cdot/n)f_N)\subset\left\{\xi\in\mathbb{R}^n:|\xi|_\infty\le N+3\right\}.\tag{40}$$

Since $\|1-\psi(\cdot/n)\|_\infty=\|1-\psi\|_\infty<\infty$, we have

$$\lim_{n\longrightarrow\infty}\|f_N-\psi(\cdot/n)f_N\|_{p(\cdot)}=0,\tag{41}$$

by Lemma 3.2.8 of [23]. Now, we prove that $\{\psi(\cdot/n)f_N\}_{n\in\mathbb{N}}\subset\mathcal{S}(\mathbb{R}^n)$ is an approximation of f_N in $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$. By (38), we have

$$\begin{aligned}\|f_N-\psi(\cdot/n)f_N\|_{M_{p(\cdot),q(\cdot)}^{s(\cdot)}}&=\left\|\left(\langle l\rangle^{s(\cdot)}\square_l(f_N-\psi(\cdot/n)f_N)\right)_{l\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\&=\left\|\left(\langle l\rangle^{s(\cdot)}\square_l(f_N-\psi(\cdot/n)f_N)\right)_{|l|_\infty\le N+3}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}.\end{aligned}\tag{42}$$

Let $l_0=(N+3,0,0,\cdots,0)$, then $N+3\le r_{l_0}=\sqrt{n}(N+4)$ and $\{\xi\in\mathbb{R}^n:|\xi|_\infty\le N+3\}\subset\{\xi\in\mathbb{R}^n:|\xi|\le r_{l_0}\}$. For each $l\in\mathbb{Z}^n$ with $|l|_\infty\le N+3$, let $g_l=f_N-\psi(\cdot/n)f_N$; then, $g_l\in L_{\Omega_{l_0}}^{p(\cdot)}$, where $\Omega_{l_0}=\{\xi\in\mathbb{R}^n:|\xi|\le r_{l_0}\}$. By Remark 17 and the embedding properties of $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$, we get

$$\begin{aligned}
&\|f_N-\psi(\cdot/n)f_N\|_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}}\\
&=\left\|\left(\langle l\rangle^{s(\cdot)}\square_l(f_N-\psi(\cdot/n)f_N)\right)_{|l|_\infty\le N+3}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\
&\lesssim\left(\sup_{|l|_\infty\le N+3}\|\varphi_l(|\Omega_{l_0}|\,\cdot)\|_{H_2^t}\right)\left\|\left(\langle l\rangle^{s(\cdot)}(f_N-\psi(\cdot/n)f_N)\right)_{|l|_\infty\le N+3}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\
&\lesssim\left(\sup_{|l|_\infty\le N+3}\|\varphi_l(|\Omega_{l_0}|\,\cdot)\|_{H_2^t}\right)\left\|\left(\langle l\rangle^{s^+}(f_N-\psi(\cdot/n)f_N)\right)_{|l|_\infty\le N+3}\right\|_{\ell^{q^-}\left(L^{p(\cdot)}\right)}\\
&\lesssim\left(\sup_{|l|_\infty\le N+3}\|\varphi_l(|\Omega_{l_0}|\,\cdot)\|_{H_2^t}\right)\left(\sum_{|l|_\infty\le N+3}\langle l\rangle^{s^+q^-}\right)^{1/q^-}\|f_N-\psi(\cdot/n)f_N\|_{p(\cdot)}.
\end{aligned}\tag{43}$$

Then, combining (39) and $\sup_{|l|_\infty\le N+3}\|\varphi_l(|\Omega_{l_0}|\,\cdot)\|_{H_2^t}<\infty$, we obtain

$$\lim_{n\longrightarrow\infty}\|f_N-\psi(\cdot/n)f_N\|_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}}=0.\tag{44}$$

Therefore, $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. □

4. Dual Spaces of $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$

For a quasi-Banach space X, we denote the dual space of X by X^*. In this section, we show that $(M^{s(\cdot)}_{p(\cdot),q(\cdot)})^*=M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)}$ for $p,q\in\mathcal{P}^{\log}$ and $s\in C^{\log}_{\text{loc}}$.

Lemma 19. *Let $p,q\in\mathcal{P}$, $\{f_k\}_{k\in\mathbb{Z}^n}$, and $\{g_k\}_{k\in\mathbb{Z}^n}$ be sequences of locally Lebesgue integrable functions satisfying $\|\{f_k\}_{k\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty$ and $\|\{g_k\}_{k\in\mathbb{Z}^n}\|_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}<\infty$. Then, we have*

$$\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|f_k(x)g_k(x)|dx\le 2\left(1+\frac{1}{p^-}-\frac{1}{p^+}\right)\|\{f_k\}_{k\in\mathbb{Z}^n}\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\|\{g_k\}_{k\in\mathbb{Z}^n}\|_{\ell^{q'(\cdot)}\left(L^{p'(\cdot)}\right)}.\tag{45}$$

The above lemma has been proven in [39]; hence, we omit the proof here.

Proposition 20. *Let $p,q\in\mathcal{P}$ and $s\in C^{\log}_{loc}$. We denote by $\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that there exists $\{f_k\}_{k\in\mathbb{Z}^n}\subset L^{p(\cdot)}(\mathbb{R}^n)$ such that $f=\sum_{k\in\mathbb{Z}^n}\square_k f_k$ and*

$$\left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}<\infty.\tag{46}$$

If we define

$$\|f\|_{\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}}=\inf\left\{\left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}:f\in\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}\right\},\tag{47}$$

then $\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}=M^{s(\cdot)}_{p(\cdot),q(\cdot)}$, and $\|\cdot\|_{\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}}$ is an equivalent norm on $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. Let $f\in M^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and write $f_k=\sum_{|l|_\infty\le1}\square_{k+l}f$; then, $f=\sum_{k\in\mathbb{Z}^n}\square_k f_k$ and

$$\left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\lesssim\|f\|_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}},\tag{48}$$

which implies $M^{s(\cdot)}_{p(\cdot),q(\cdot)}\subset\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}$. On the other hand, for any $f\in\tilde{M}^{s(\cdot)}_{p(\cdot),q(\cdot)}$, by Proposition 16, we have

$$\begin{aligned}
\left\|\left(\langle k\rangle^{s(\cdot)}\square_k f\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}&\lesssim\sum_{|l|_\infty\le1}\left\|\left(\langle k\rangle^{s(\cdot)}\square_{k+l}f_{k+l}\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\
&\lesssim\sum_{|l|_\infty\le1}\left\|\left(\langle k\rangle^{s(\cdot)}f_{k+l}\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\\
&\lesssim\left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)},
\end{aligned}\tag{49}$$

by which we can deduce the conclusion of the proposition. □

Let us define

$$\ell^{q(\cdot)}_{s(\cdot)}\left(\mathbb{Z}^n,L^{p(\cdot)}\right)=\left\{f=\{f_k(x)\}_{k\in\mathbb{Z}^n}:\|f\|_{\ell^{q(\cdot)}_{s(\cdot)}\left(L^{p(\cdot)}\right)}<\infty\right\},\tag{50}$$

where

$$\|f\|_{\ell^{q(\cdot)}_{s(\cdot)}\left(L^{p(\cdot)}\right)}=\left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}.\tag{51}$$

Then, we have the following proposition about dual spaces.

Proposition 21. *Let $p,q\in\mathcal{P}$ and $s\in C^{\log}_{loc}$. Then,*

$$\left(\ell^{q(\cdot)}_{s(\cdot)}\left(\mathbb{Z}^n,L^{p(\cdot)}\right)\right)^*=\ell^{q'(\cdot)}_{-s(\cdot)}\left(\mathbb{Z}^n,L^{p'(\cdot)}\right).\tag{52}$$

Moreover, $g\in(\ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n,L^{p(\cdot)}))^*$ is equivalent to

$$\langle g,f\rangle=\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}g_k(x)f_k(x)dx,\tag{53}$$

for all $f=\{f_k\}_{k\in\mathbb{Z}^n}\in\ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n,L^{p(\cdot)})$, in which

$$g=\{g_k\}_{k\in\mathbb{Z}^n}\in\ell^{q'(\cdot)}_{-s(\cdot)}\left(\mathbb{Z}^n,L^{p'(\cdot)}\right),\quad\|g\|_{\left(\ell^{q(\cdot)}_{s(\cdot)}\left(L^{p(\cdot)}\right)\right)^*}=\|\{g_k\}\|_{\ell^{q'(\cdot)}_{-s(\cdot)}\left(L^{p'(\cdot)}\right)}.\tag{54}$$

Proof. Firstly, by Lemma 19, we have $\left(\ell_{s(\cdot)}^{q(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})\right)^* \supset \ell_{-s(\cdot)}^{q'(\cdot)}(\mathbb{Z}^n, L^{p'(\cdot)})$ and

$$\langle g, f\rangle = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} g_k(x)f_k(x)dx, \tag{55}$$

for all $f=\{f_k\}_{k\in\mathbb{Z}^n}\in\ell_{s(\cdot)}^{q(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$, $g=\{g_k\}_{k\in\mathbb{Z}^n}\in\ell_{-s(\cdot)}^{q'(\cdot)}(\mathbb{Z}^n, L^{p'(\cdot)})$.

On the other hand, for any $g\in\left(\ell_{s(\cdot)}^{q(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})\right)^*$, let us define g_k by

$$\langle g_k, f_k\rangle = \langle g, (0,\cdots,0, f_k, 0, 0,\cdots)\rangle, \tag{56}$$

where $f=\{f_k\}_{k\in\mathbb{Z}^n}\in\ell_{s(\cdot)}^{q(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$. It follows that $g_k\in(L^{p(\cdot)})^*=L^{p'(\cdot)}$, whence

$$\langle g_k, f_k\rangle = \int_{\mathbb{R}^n} g_k(x)f_k(x)dx. \tag{57}$$

Therefore,

$$\langle g, f\rangle = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} g_k(x)f_k(x)dx. \tag{58}$$

We assume that $g_k\neq 0$ for all $k\in\mathbb{Z}^n$. Then, for any $N\in\mathbb{N}$, let us define $f_N=\{f_k\}_{k\in\mathbb{Z}^n}$ as follows: when $|k|>N$, we put $f_k=0$; when $|k|\le N$, we set $g'_k(x)=g_k(x)/\lambda$, where $\lambda=\|(\langle k\rangle^{-s(\cdot)}g_k)_{|k|\le N}\|_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}$, and set

$$\begin{aligned} f_k &= \operatorname{sgn} g'_k(x)\left|\langle k\rangle^{-s(x)q'(x)}g'_k(x)^{q'(x)}\right|^{p'(x)-1/q'(x)} \\ &\quad\times\langle k\rangle^{-s(x)}\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}^{1-p'(x)/q'(x)}. \end{aligned} \tag{59}$$

Then, by Proposition 2.21 of [22], we have

$$\begin{aligned} &\int_{\mathbb{R}^n}\left(\frac{\langle k\rangle^{s(x)q(x)}f_k(x)^{q(x)}}{\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}}\right)^{p(x)/q(x)}dx \\ &= \int_{\mathbb{R}^n}\frac{\left|\langle k\rangle^{-s(x)q'(x)}g'_k(x)^{q'(x)}\right|^{p'(x)/q'(x)}\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}^{p(x)/q(x)-p'(x)/q'(x)}}{\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}^{p(x)/q(x)}}dx \\ &= \int_{\mathbb{R}^n}\left(\frac{\left|\langle k\rangle^{-s(x)q'(x)}g'_k(x)^{q'(x)}\right|}{\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}}\right)^{p'(x)/q'(x)}dx = 1, \end{aligned} \tag{60}$$

which implies

$$\left\|\langle k\rangle^{s(\cdot)q(\cdot)}f_k(\cdot)^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le \left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}. \tag{61}$$

Thus, by the definition of g'_k, we have

$$\begin{aligned} \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(\langle k\rangle^{s(\cdot)}f_k(\cdot)\right)_{|k|\le N}\right) &= \sum_{|k|\le N}\left\|\langle k\rangle^{s(\cdot)q(\cdot)}f_k(\cdot)^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \\ &\le \rho_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}\left(\left(\langle k\rangle^{-s(\cdot)}g'_k(\cdot)\right)_{|k|\le N}\right)\le 1, \end{aligned} \tag{62}$$

from which we can get

$$\|f_N\|_{\ell_{s(\cdot)}^{q(\cdot)}(L^{p(\cdot)})} = \left\|\left(\langle k\rangle^{s(\cdot)}f_k\right)_{|k|\le N}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\le 1. \tag{63}$$

By Proposition 3.5 of [28], we know that $\rho_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}(\cdot)$ is continuous. Therefore, by $\|(\langle k\rangle^{-s(\cdot)}g'_k)_{|k|\le N}\|_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}=1$, we have

$$\sum_{|k|\le N}\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)} = \rho_{\ell^{q'(\cdot)}(L^{p'(\cdot)})}\left(\left(\langle k\rangle^{-s(\cdot)}g'_k\right)_{|k|\le N}\right)=1. \tag{64}$$

Then, it is easy to see that

$$\begin{aligned} \langle g, f_N\rangle &= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} g_k(x)f_k(x)dx = \lambda\sum_{|k|\le N}\int_{\mathbb{R}^n} g'_k(x)f_k(x)dx \\ &= \lambda\left(\sum_{|k|\le N}\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}\right) \\ &\quad\cdot\int_{\mathbb{R}^n}\left(\frac{\left|\langle k\rangle^{-s(x)q'(x)}g'_k(x)^{q'(x)}\right|}{\left\|\langle k\rangle^{-s(\cdot)q'(\cdot)}g'_k(\cdot)^{q'(\cdot)}\right\|_{p'(\cdot)/q'(\cdot)}}\right)^{p'(x)/q'(x)}dx = \lambda. \end{aligned} \tag{65}$$

Hence, for any $N\in\mathbb{N}$, by (61), we have

$$\begin{aligned} &\left\|\left(\langle k\rangle^{-s(\cdot)}g_k\right)_{|k|\le N}\right\|_{\ell^{q'(\cdot)}(L^{p'(\cdot)})} = \lambda = \langle g, f_N\rangle \\ &\le \|g\|_{\left(\ell_{s(\cdot)}^{q(\cdot)}(L^{p(\cdot)})\right)^*}\|f_N\|_{\ell_{s(\cdot)}^{q(\cdot)}(L^{p(\cdot)})}\le\|g\|_{\left(\ell_{s(\cdot)}^{q(\cdot)}(L^{p(\cdot)})\right)^*}, \end{aligned} \tag{66}$$

which implies $g\in\ell_{-s(\cdot)}^{q'(\cdot)}(\mathbb{Z}^n, L^{p'(\cdot)})$ and $\left(\ell_{s(\cdot)}^{q(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})\right)^*\subset\ell_{-s(\cdot)}^{q'(\cdot)}(\mathbb{Z}^n, L^{p'(\cdot)})$. □

Theorem 22 (dual space). *Let $p, q\in\mathscr{P}^{\log}$ and $s\in C_{loc}^{\log}$; then, we have*

$$\left(M_{p(\cdot),q(\cdot)}^{s(\cdot)}\right)^* = M_{p'(\cdot),q'(\cdot)}^{-s(\cdot)}. \tag{67}$$

Proof. Firstly, we prove that $M_{p'(\cdot),q'(\cdot)}^{-s(\cdot)}\subset\left(M_{p(\cdot),q(\cdot)}^{s(\cdot)}\right)^*$. For any $g\in M_{p'(\cdot),q'(\cdot)}^{-s(\cdot)}\subset\mathcal{S}'(\mathbb{R}^n)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$, by Lemma 19, we have

$$|\langle g,\varphi\rangle| = \left|\sum_{k\in\mathbb{Z}^n}\sum_{|l|_\infty\le 1}\langle \square_{k+l}^* g, \square_k\varphi\rangle\right| \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\square_k^* g(x)\square_k\varphi(x)| \cdot dx \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\left|\langle k\rangle^{-s(x)}\square_k^* g(x)\right|\cdot\left|\langle k\rangle^{s(x)}\square_k\varphi(x)\right| \cdot dx \left\|\langle k\rangle^{-s(x)}\square_k^* g(x)\right\|_{\ell^{q'(\cdot)}\left(L^{p'(\cdot)}\right)}\left\|\langle k\rangle^{s(x)}\square_k\varphi(x)\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} \cdot \|g\|_{M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)}}\|f\|_{M^{s(\cdot)}_{p(\cdot),q(\cdot)}}, \tag{68}$$

where $\square_k^* := \mathscr{F}\sigma_k\mathscr{F}^{-1}$. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $M^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain $M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)} \subset (M^{s(\cdot)}_{p(\cdot),q(\cdot)})^*$.

Now, let us prove that $(M^{s(\cdot)}_{p(\cdot),q(\cdot)})^* \subset M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)}$. It is easy to see that, for $f \in M^{s(\cdot)}_{p(\cdot),q(\cdot)}$,

$$f \mapsto \{\square_k f\}_{k\in\mathbb{Z}^n} \in \ell^{q(\cdot)}_{s(\cdot)}\left(\mathbb{Z}^n, L^{p(\cdot)}\right) \tag{69}$$

is an isometric mapping from $M^{s(\cdot)}_{p(\cdot),q(\cdot)}$ into a subspace X of $\ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$. Hence, for any $g \in (M^{s(\cdot)}_{p(\cdot),q(\cdot)})^*$, we can regard it as a continuous functional on X, which can be extended onto $\ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$ with the same norm. Then, by Proposition 21, for any $f = \{f_k\}_{k\in\mathbb{Z}^n} \in \ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$, we have

$$\langle g, f\rangle = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} g_k(x) f_k(x)dx, \tag{70}$$

where $\{g_k\}_{k\in\mathbb{Z}^n} \in \ell^{q'(\cdot)}_{-s(\cdot)}(\mathbb{Z}^n, L^{p'(\cdot)})$ and

$$\|g\|_{\left(M^{s(\cdot)}_{p(\cdot),q(\cdot)}\right)^*} = \left\|\{g_k\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q'(\cdot)}_{-s(\cdot)}\left(L^{p'(\cdot)}\right)}. \tag{71}$$

Since $\{\square_k^*\varphi\}_{k\in\mathbb{Z}^n} \in \ell^{q(\cdot)}_{s(\cdot)}(\mathbb{Z}^n, L^{p(\cdot)})$ for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle g,\varphi\rangle = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} g_k(x)\square_k^*\varphi(x)dx = \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}\square_k g_k(x)\varphi(x)dx, \tag{72}$$

which implies $g = \sum_{k\in\mathbb{Z}^n}\square_k g_k(x)$. Thus, by Proposition 20, we obtain

$$\|g\|_{M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)}} \lesssim \left\|\{g_k\}_{k\in\mathbb{Z}^n}\right\|_{\ell^{q'(\cdot)}_{-s(\cdot)}\left(L^{p'(\cdot)}\right)} = \|g\|_{\left(M^{s(\cdot)}_{p(\cdot),q(\cdot)}\right)^*}, \tag{73}$$

by which we can get $(M^{s(\cdot)}_{p(\cdot),q(\cdot)})^* \subset M^{-s(\cdot)}_{p'(\cdot),q'(\cdot)}$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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