Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4483348, 17 pages
https://doi.org/10.1155/2022/4483348

Research Article

An Existence Study on the Fractional Coupled Nonlinear q-Difference Systems via Quantum Operators along with Ulam–Hyers and Ulam–Hyers–Rassias Stability

Shahram Rezapour,[1,2,3] **Chatthai Thaiprayoon**,[4] **Sina Etemad**,[5] **Weerawat Sudsutad**,[6] **Chernet Tuge Deressa**,[7] **and Akbar Zada**[8]

[1]*Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam*
[2]*Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam*
[3]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[4]*Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, Thailand*
[5]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[6]*Theoretical and Applied Data Integration Innovations Group, Department of Statistics, Faculty of Science, Ramkhamhaeng University, Bangkok 10240, Thailand*
[7]*College of Natural Sciences, Department of Mathematics, Jimma University, Jimma, Ethiopia*
[8]*Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan*

Correspondence should be addressed to Chernet Tuge Deressa; chernet.deressa@ju.edu.et

Received 1 June 2022; Accepted 29 August 2022; Published 24 September 2022

Academic Editor: Anita Tomar

Copyright © 2022 Shahram Rezapour et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we study the existence of solutions and their uniqueness and different kinds of Ulam–Hyers stability for a new class of nonlinear Caputo quantum boundary value problems. Also, we investigate such properties for the relevant generalized coupled q-system involving fractional quantum operators. By using the Banach contraction principle and Leray-Schauder's fixed–point theorem, we prove the existence and uniqueness of solutions for the suggested fractional quantum problems. The Ulam–Hyers stability of solutions in different forms are studied. Finally, some examples are provided for both q-problem and coupled q-system to show the validity of the main results.

1. Introduction

Fractional calculus is one of the most important fields in applied mathematics. In recent years, many researchers have studied different branches of this theory and conducted numerous analyses analytically and numerically. Particularly, in recent decades, we can see some papers on the applications of fixed-point theorems to prove the existence of solutions of fractional boundary value problems [1–4]. Because of the quick developments in fractional calculus, many mathematicians discussed on the theory of q-calculus that is an equivalent of traditional calculus without defining the concept of limit, and also the parameter q refers to quantum. This theory was originally developed by Jackson [5, 6], and it includes many practical aspects in the fields of hypergeometric series, theory of relativity, particle physics, discrete mathematics, quantum mechanics, combinatorics, and complex analysis. For a fundamental introduction of the basic notions of q-calculus, one can refer to [7–9]. In the early years, for finding positive solutions of given q-difference equations in the nonlinear settings, we lead you to study a work published by both El-Shahed and Al-Askar [10] and also a manuscript by Graef and Kong [11].

So later, various mathematical q-difference fractional models of IVPs and BVPs have been presented in which different methods like the lower-upper solutions technique, fixed-point results, and iterative methods have been implemented. For instance, we see q-intego-equation on time scales in [12], q-delay equations in [13], q-integro-equations under the q-integral conditions in [14], singular q-equations in [15], q-sequential symmetric BVPs in [16], q-difference equations having p-Laplacian in [17], four-point q-BVP with different orders in [18], oscillation on q-difference inclusions in [19], etc.

Here, we apply similar techniques to discuss the existence property of solutions for given q-integro-sum-difference FBVPs depending on the quantum operators. This shows an application of fixed-point theory in relation to q-difference theory. This specifies the main contribution of the present reseach.

In 2014, Ahmad et al. [20] studied a q-sequential equation in the nonlinear case via four-point q-integral conditions given by

$$\begin{cases} {}^{C}_{q}\mathbb{D}^{k_1}_{0^+}\left({}^{C}_{q}\mathbb{D}^{k_2}_{0^+}+\sigma\right)u(r)=G(r,u(r)), & (r\in[0,1], q\in(0,1)),\\ u(0)=e_{1q}\mathbb{I}^{s-1}_{0^+}u(b_1), & u(1)=e_{2q}\mathbb{I}^{s-1}_{0^+}u(b_2), \end{cases} \tag{1}$$

so that $k_1, k_2 \in (0,1)$, $b_1, b_2 \in (0,1)$, $s>2$, and $\sigma, e_1, e_2 \in \mathbb{R}$. As well as, $G:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous, and ${}_{q}\mathbb{I}^{s-1}_{0^+}$ indicates the q-RL-integral. These mathematicians extracted different qualitative aspects of solutions for the above q-FBVP by means of the classical methods which are available in fixed-point theory.

In 2015, Etemad et al. [21] focused on the new four-point three-term q-difference FBVP

$$\begin{aligned} &\left({}^{C}_{q}\mathbb{D}^{\rho}_{0^+}u\right)(r)=G\left(r,u(r),{}^{C}_{q}\mathbb{D}^{1}_{0^+}u(r)\right), 0<q<1,\\ &c_1u(0)+d_{1q}{}^{C}\mathbb{D}^{1}_{0^+}u(0)=b_{1q}\mathbb{I}^{\alpha}_{0^+}u(k_1)=b_1\int_0^{k_1}\frac{(k_1-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}u(z)d_qz,\\ &c_2u(1)+d_{2q}{}^{C}\mathbb{D}^{1}_{0^+}u(1)=b_{2q}\mathbb{I}^{\alpha}_{0^+}u(k_2)=b_2\int_0^{k_2}\frac{(k_2-qz)^{(\alpha-1)}}{\Gamma_q(\alpha)}u(z)d_qz, \end{aligned} \tag{2}$$

where $0\le r\le 1$, $1<\rho\le 2$, $\alpha\in(0,2]$, $c_1, c_2, d_1, d_2, b_1, b_2\in\mathbb{R}$, and $k_1, k_2\in(0,1)$ with $k_1<k_2$.

In 2019, two mathematicians named Ntouyas and Samei [22] devoted their attention to investigate the existence property about the multiterm q-integro-difference FBVP

$$\begin{aligned} &{}^{C}_{q}\mathbb{D}^{\rho}_{0^+}u(r)=G\left(r,u(r),(\hbar_1u)(r),(\hbar_2u)(r),{}^{C}_{q}\mathbb{D}^{\rho_1}_{0^+}u(r),{}^{C}_{q}\mathbb{D}^{\rho_2}_{0^+}u(r),\cdots,{}^{C}_{q}\mathbb{D}^{\rho_m}_{0^+}u(r)\right),\\ &u(0)+b_1u(1)=0, u'(0)+b_2u'(1)=0 \end{aligned} \tag{3}$$

where $r\in[0,1]$, $q\in(0,1)$, $\rho\in(1,2)$, $\rho_i\in(0,1)$ with $i=1,2,\cdots,m$, $b_1, b_2\neq -1$, $\hbar_j$ are formulated as

$$\left(\hbar_ju\right)(r)=\int_0^r \nu_j(r,z)u(z)\,\mathrm{d}_qz, \tag{4}$$

for $j=1,2$ and $G:[0,1]\times\mathbb{R}^{m+3}\longrightarrow\mathbb{R}$ is continuous with respect to all variables [22].

In 2020, Phuong et al. [23] formulated a novel extended configuration of the Caputo q-multi-integro-difference equation with two nonlinearity under q-multi-order-integrals conditions

$$\begin{aligned} &\left(m{}^{C}_{q}\mathbb{D}^{\rho}_{0^+}-(m+1){}_{q}\mathbb{I}^{k_1}_{0^+}-(m+2){}_{q}\mathbb{I}^{k_2}_{0^+}\right)u(r)\\ &\quad=b_{1q}\mathbb{I}^{k_3}_{0^+}G_1(r,u(r))+b_{2q}\mathbb{I}^{k_4}_{0^+}G_2(r,u(r)),\\ &u(0)=0, n_q\mathbb{I}^{p_1}_{0^+}u(1)+(n+1)_q\mathbb{I}^{p_2}_{0^+}u(1)+(n+2)_q\mathbb{I}^{p_3}_{0^+}u(1)=0, \end{aligned} \tag{5}$$

where $r\in[0,1]$, $\rho\in(1,2)$, $k_1,k_2,k_3,k_4\in(0,1)$, $p_1,p_2,p_3,m,n>0$, and $b_1,b_2\in\mathbb{R}^{\ge 0}$.

In this paper, inspired by above q-problems, we analyze a structure of the nonlinear Caputo quantum difference fractional boundary problem (or q-CFBVP) in the form

$$\begin{aligned} &{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\coloneqq\mathscr{G}_{\mu}(r), \quad (r\in\mathcal{O}\coloneqq[0,1], q\in(0,1)),\\ &\mu(0)+\mu(\zeta)=\sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\ &{}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\ &{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\gamma_{jq}{}^{R}\mathfrak{J}^{\varrho}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right], \end{aligned} \tag{6}$$

where $\varsigma\in(2,3)$, $\varrho\in(1,2)$, $\zeta\in(0,1)$, $\alpha_j, \beta_j, \gamma_j\in\mathbb{R}^{>0}$, $\omega, \sigma_j>0$ for $j=1,2,\cdots,k$, and $G:\mathcal{O}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous. As the same way, the operators ${}^{C}_{q}\mathfrak{D}^{(\cdot)}_{0^+}$ and ${}_{q}\mathfrak{J}^{(\cdot)}_{0^+}$ denote the q-Caputo derivative and the q-RL integral, respectively. In the direction of the above problem, we consider a coupled system of nonlinear q-CFBVPs with the same q-boundary conditions. In other words, the mentioned fractional q-system is organized as

$$\begin{aligned} &{}^{C}_{q}\mathfrak{D}^{\varsigma_1}_{0^+}\mu(r)=G_1\left(r,\vartheta(r),{}^{R}_{q}\mathfrak{J}^{\omega_1}_{0^+}\vartheta(r)\right): \mathscr{U}_{\vartheta}(r), \quad (r\in\mathcal{O}, q\in(0,1)),\\ &{}^{C}_{q}\mathfrak{D}^{\varsigma_2}_{0^+}\vartheta(r)=G_2\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega_2}_{0^+}\mu(r)\right)\coloneqq\mathscr{V}_{\mu}(r),\\ &\mu(0)+\mu(\zeta)=\sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\ &\vartheta(0)+\vartheta(\zeta)=\sum_{j=1}^{k}\phi_{jq}{}^{R}\mathfrak{J}^{\delta_j}_{0^+}\vartheta(1),\\ &{}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{\varrho}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1), \end{aligned}$$

$$ {}_q^C\mathfrak{D}_{0^+}^{\rho}\vartheta(0) + {}_q^C\mathfrak{D}_{0^+}^{\rho}\vartheta(\zeta) = \sum_{j=1}^{k}\varphi_{j\,q}^{\,R}\mathfrak{J}_{0^+}^{\delta_j}\vartheta(1), $$

$$ {}_q^C\mathfrak{D}_{0^+}^{2}\mu(0) + {}_q^C\mathfrak{D}_{0^+}^{2}\mu(\zeta) = \sum_{j=1}^{k}\gamma_{j\,q}^{\,R}\mathfrak{J}_{0^+}^{\sigma_j}\left[{}_q^C\mathfrak{D}_{0^+}^{2}\mu(1)\right], $$

$$ {}_q^C\mathfrak{D}_{0^+}^{2}\vartheta(0) + {}_q^C\mathfrak{D}_{0^+}^{2}\vartheta(\zeta) = \sum_{j=1}^{k}\eta_{j\,q}^{\,R}\mathfrak{J}_{0^+}^{\delta_j}\left[{}_q^C\mathfrak{D}_{0^+}^{2}\vartheta(1)\right], \quad (7) $$

where $\varsigma_1, \varsigma_2 \in (2,3)$, $\varrho, \rho \in (1,2)$, $\zeta \in (0,1)$, $\alpha_j, \beta_j, \gamma_j, \phi_j, \varphi_j, \eta_j \in \mathbb{R}^{>0}$, $\omega_1, \omega_2, \sigma_j, \delta_j > 0$ for $j = 1, 2, \cdots, k$, and $G_1, G_2 : \mathcal{O} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous.

In other words, we extend our q-CFBVP to a coupled q-difference system and derive the existence and stability results on such a generalized coupled q-CFBVP system. In fact, a large number of researchers have devoted their concentration to the discussion on various categories of Ulam-Hyers stabilities for standard systems of FDEs (or refer to [24, 25]), while a few articles can be found in the literature in which the researchers developed the relevant existence and stability theory in relation to nonlinear fractional q-difference systems.

The present work is assembled as follows: In Section 2, we state some basic materials required to prove our theoretical results. In both Section 3 and Section 4, several criteria and conditions are presented for the desired uniqueness-existence results, along with different classes of stabilities in relation to the proposed q-CFBVPs (6) and (7), respectively, with the help of some known fixed–point theorems. A simulative example, to represent the consistency of our results, is given with each suggested q-CFBVP in the relevant section. We give Section 6 to the presentation of the conclusion of this research work.

2. Preliminaries

The basic notions of q-calculus are collected in this section by assuming $q \in (0,1)$. The q-analogue of $(a_1 - a_2)^k$ is given by

$$ (a_1 - a_2)^{(0)} = 1, (a_1 - a_2)^{(k)} = \prod_{j=0}^{k-1}\left(a_1 - a_2 q^j\right), (a_1, a_2 \in \mathbb{R}, k \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}) \quad (8) $$

Rajkovic et al. [26]. Now, if $k = \varsigma \in \mathbb{R}$, then

$$ (a_1 - a_2)^{(\varsigma)} = a_1^{\varsigma}\prod_{k=0}^{\infty}\frac{1 - (a_2/a_1)q^k}{1 - (a_2/a_1)q^{\varsigma+k}}, (a_1 \neq 0). \quad (9) $$

On the other side, by taking $a_2 = 0$, we have $a_1^{(\varsigma)} = a_1^{\varsigma}$ [26]. A q-number $[a_1]_q$ for $a_1 \in \mathbb{R}$ is defined by

$$ [a_1]_q = \frac{1 - q^{a_1}}{1 - q} = q^{a_1 - 1} + \cdots + q + 1. \quad (10) $$

Accordingly, the Gamma function in the quantum settings is defined by

$$ \Gamma_q(r) = \frac{(1-q)^{(r-1)}}{(1-q)^{r-1}}, (r \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\})), \quad (11) $$

and $\Gamma_q(r+1) = [r]_q\Gamma_q(r)$ [5, 26].

Definition 1 (see [27]). The q-difference-derivative of the given function μ is defined by

$$ \left({}_q\mathfrak{D}_{0^+}\mu\right)(r) = \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_q \mu(r) = \frac{\mu(r) - \mu(qr)}{(1-q)r}, \quad (12) $$

where $({}_q\mathfrak{D}_{0^+}\mu)(0) = \lim_{r \longrightarrow 0}({}_q\mathfrak{D}_{0^+}\mu)(r)$.

Clearly, we have $({}_q\mathfrak{D}_{0^+}^{k}\mu)(r) = {}_q\mathfrak{D}_{0^+}({}_q\mathfrak{D}_{0^+}^{k-1}\mu)(r)$ for all $k \in \mathbb{N}$ and $({}_q\mathfrak{D}_{0^+}^{0}\mu)(r) = \mu(r)$ [27].

Definition 2 (see [27]). The q-integral of the supposed function $\mu \in C([0, m_2], \mathbb{R})$ is defined as

$$ \left({}_q\mathfrak{J}_{0^+}\mu\right)(r) = \int_0^r \mu(v)\,\mathrm{d}_q v = r(1-q)\sum_{j=0}^{\infty}\mu\left(rq^j\right)q^j, \quad (13) $$

if the series is absolutely convergent.

Similarly, $({}_q\mathfrak{J}_{0^+}^{k}\mu)(r) = {}_q\mathfrak{J}_{0^+}({}_q\mathfrak{J}_{0^+}^{k-1}\mu)(r)$ for all $k \geq 1$ and $({}_q\mathfrak{J}_{0^+}^{0}\mu)(r) = \mu(r)$ [27].

Definition 3 (see [27]). By letting $a_1 \in [0, a_2]$, the definite q-integral of the given function $\mu \in C([0, a_2], \mathbb{R})$ is defined by

$$ \begin{aligned} \int_{a_1}^{a_2}\mu(v)\mathrm{d}_q v &= {}_q\mathfrak{J}_{0^+}\mu(a_2) - {}_q\mathfrak{J}_{0^+}\mu(a_1) \\ &= \int_0^{a_2}\mu(v)\mathrm{d}_q v - \int_0^{a_1}\mu(v)\mathrm{d}_q v \\ &= (1-q)\sum_{j=0}^{\infty}\left[a_2\mu\left(a_2 q^j\right) - a_1\mu\left(a_1 q^j\right)\right]q^j, \end{aligned} \quad (14) $$

if the series exists.

By considering μ as a continuous function at $r = 0$, then $({}_q\mathfrak{J}_{0^+}{}_q\mathfrak{D}_{0^+}\mu)(r) = \mu(r) - \mu(0)$ [27]. Furthermore, $({}_q\mathfrak{D}_{0^+}{}_q\mathfrak{J}_{0^+}\mu)(r) = \mu(r)$ for all r.

Definition 4 (see [11, 28]). The ς^{th}-RL-q-integral of $\mu \in \mathscr{C}_{\mathbb{R}}([0,+\infty))$ is defined by

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\mu(r) = \begin{cases} \dfrac{1}{\Gamma_q(\varsigma)}\displaystyle\int_0^r (r-qv)^{(\varsigma-1)}\mu(v)\,\mathrm{d}_q v, & \varsigma > 0, \\ \mu(r), & \varsigma = 0, \end{cases} \tag{15}$$

if integral exists.

One can simply see that the q-semi-group property satisfies as ${}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma_1}({}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma_2}\mu)(r) = {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma_1+\varsigma_2}\mu(r)$ for $\varsigma_1, \varsigma_2 \geq 0$ [28]. Also, for $\zeta > -1$, we have

$$\begin{aligned} {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma} r^{\zeta} &= \frac{\Gamma_q(\zeta+1)}{\Gamma_q(\zeta+\varsigma+1)} r^{\zeta+\varsigma}, \\ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma} 1(r) &= \frac{1}{\Gamma_q(\varsigma+1)} r^{\varsigma}, (r > 0). \end{aligned} \tag{16}$$

Definition 5 (see [11, 28]). Let $\ell - 1 < \varsigma < \ell$, i.e., $\ell = [\varsigma] + 1$. The ς^{th}-Caputo q-derivative of $\mu \in \mathscr{C}_{\mathbb{R}}^{(\ell)}([0,+\infty))$ is defined as

$$ {}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = \frac{1}{\Gamma_q(\ell-\varsigma)}\int_0^r (r-qv)^{(\ell-\varsigma-1)}\,{}_{q}\mathfrak{D}_{0^+}^{\ell}\mu(v)\,\mathrm{d}_q v, \tag{17}$$

if the integral exists.

Note that for $\zeta > -1$, we have

$$\begin{aligned} {}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma} r^{\iota} &= \frac{\Gamma_q(\iota+1)}{\Gamma_q(\iota-\varsigma+1)} r^{\iota-\varsigma}, \\ {}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma} 1(r) &= 0, (r > 0). \end{aligned} \tag{18}$$

Lemma 6 (see [10]). *Let $\ell - 1 < \varsigma < \ell$. Then,*

$$\left({}_{q}^{C}\mathfrak{J}_{0^+}^{\varsigma}\,{}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma}\mu\right)(r) = \mu(r) - \sum_{j=0}^{\ell-1}\frac{r^j}{\Gamma_q(j+1)}\left({}_{q}\mathfrak{D}_{0^+}^{j}\mu\right)(0). \tag{19}$$

By Lemma 6, the general series solution of q-difference FDE ${}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = 0$ is given as $\mu(r) = \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1}r^{\ell-1}$ with $\tilde{c}_0, \cdots, \tilde{c}_{\ell-1} \in \mathbb{R}$ and $\ell = [\varsigma] + 1$ [10]. In this case, we get

$$\left({}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\,{}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma}\mu\right)(r) = \mu(r) + \tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \cdots + \tilde{c}_{\ell-1}r^{\ell-1}. \tag{20}$$

3. Analysis of the Cap-q-Difference FBVP (6)

Let $\mathfrak{A} = \mathscr{C}_{\mathbb{R}}(\mathscr{O})$ be the space of all real-valued continuous functions on $\mathscr{O} = [0, 1]$. Clearly, $\mathfrak{A}$ is a Banach space under the norm $\|\mu\|_{\mathfrak{A}} = \mathrm{Sup}_{r\in\mathscr{O}}|\mu(r)|$ for all members $\mu \in \mathfrak{A}$. In the first step, we provide the following fundamental lemma which presents a characterization of the structure of solutions for the proposed Cap-q-difference FBVP (6)

Remark 7. For convenience, we consider the following nonzero constants:

$$\begin{aligned} W_1 &= 2 - \sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\sigma_j+1)}, \\ W_2 &= \zeta - \sum_{j=1}^{k}\frac{\alpha_j}{\Gamma_q(\sigma_j+2)}, \\ W_3 &= \zeta^2 - \sum_{j=1}^{k}\frac{\alpha_j(1+q)}{\Gamma_q(\sigma_j+3)}, \end{aligned} \tag{21}$$

$$\begin{aligned} W_4 &= -\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\sigma_j+1)}, \\ W_5 &= -\sum_{j=1}^{k}\frac{\beta_j}{\Gamma_q(\sigma_j+2)}, \\ W_6 &= \frac{2\zeta^{2-\rho}}{\Gamma_q(3-\rho)} - \sum_{j=1}^{k}\frac{\beta_j(1+q)}{\Gamma_q(\sigma_j+3)}, \end{aligned} \tag{22}$$

$$\begin{aligned} W_7 &= 2(1+q) - \sum_{j=1}^{k}\frac{\gamma_j(1+q)}{\Gamma_q(\sigma_j+1)}, \\ W_8 &= W_2W_4 - W_1W_5, \\ W_9 &= W_3W_4 - W_1W_6, \end{aligned} \tag{23}$$

$$\begin{aligned} W_{10} &= W_8 - W_2W_4, \\ W_{11} &= W_3W_8 - W_2W_9. \end{aligned} \tag{24}$$

Lemma 8. *Let $\phi_* \in \mathfrak{A}$, $\varsigma \in (2,3)$, $\rho \in (1,2)$, $\zeta \in (0,1)$, $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}^{>0}$, and $\sigma_j > 0$ for $j = 1, 2, \cdots, k$. The solution of the linear Cap-q-difference FBVP*

$$\begin{aligned} &{}_{q}^{C}\mathfrak{D}_{0^+}^{\varsigma}\mu(r) = \phi_*(r), \quad (r \in \mathscr{O}, q \in (0,1)), \\ &\mu(0) + \mu(\zeta) = \sum_{j=1}^{k}\alpha_{jq}{}^{R}\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ &{}_{q}^{C}\mathfrak{D}_{0^+}^{\varrho}\mu(0) + {}_{q}^{C}\mathfrak{D}_{0^+}^{\varrho}\mu(\zeta) = \sum_{j=1}^{k}\beta_{jq}{}^{R}\mathfrak{J}_{0^+}^{\sigma_j}\mu(1), \\ &{}_{q}^{C}\mathfrak{D}_{0^+}^{2}\mu(0) + {}_{q}^{C}\mathfrak{D}_{0^+}^{2}\mu(\zeta) = \sum_{j=1}^{k}\gamma_{jq}{}^{R}\mathfrak{J}_{0^+}^{\sigma_j}\left[{}_{q}^{C}\mathfrak{D}_{0^+}^{2}\mu(1)\right] \end{aligned} \tag{25}$$

is given by

$$\mu(r)=\int_0^r \frac{(r-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,d_qv-\frac{\Theta_1(r)}{W_1W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,d_qv+\frac{\Theta_2(r)}{W_8}\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,d_qv-\frac{\Theta_3(r)}{W_1W_7W_8}\cdot\int_0^\zeta \frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,d_qv+\frac{\Theta_1(r)}{W_1W_8}\cdot\sum_{j=1}^k\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,d_qv-\frac{\Theta_2(r)}{W_8}\cdot\sum_{j=1}^k\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,d_qv+\frac{\Theta_3(r)}{W_1W_7W_8}\cdot\sum_{j=1}^k\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,d_qv, \tag{26}$$

where

$$\begin{aligned}\Theta_1(r)&=rW_1W_4+W_{10},\\ \Theta_2(r)&=rW_1-W_2,\\ \Theta_3(r)&=r^2W_1W_8-rW_1W_9-W_{11},\end{aligned} \tag{27}$$

and W_i are defined in (24).

Proof. Let μ satisfies the linear Cap-q-difference FBVP (25). Then ${}^C_q\mathfrak{D}^\varsigma_{0^+}\mu(r)=\phi_*(r)$. By virtue of $\varsigma\in(2,3)$ and taking ς^{th}-RL-q-integral, we have

$$\mu(r)=\frac{1}{\Gamma_q(\varsigma)}\int_0^r(r-qv)^{(\varsigma-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde c_0+\tilde c_1r+\tilde c_2r^2, \tag{28}$$

where $\tilde c_0,\tilde c_1,\tilde c_2\in\mathbb{R}$ are unknown coefficients that we have to explore them. It is immediately computed that

$${}^C_q\mathfrak{D}^2_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-2)}\int_0^r(r-qv)^{(\varsigma-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde c_2(1+q), \tag{29}$$

$${}^C_q\mathfrak{D}^\varrho_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma-\varrho)}\int_0^r(r-qv)^{(\varsigma-\varrho-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde c_2\frac{2}{\Gamma_q(3-\varrho)}r^{2-\varrho}, \tag{30}$$

$${}^R_q\mathfrak{I}^{\sigma_j}_{0^+}\mu(r)=\frac{1}{\Gamma_q(\varsigma+\sigma_j)}\int_0^r(r-qv)^{(\varsigma+\sigma_j-1)}\phi_*(v)\,\mathrm{d}_qv+\tilde c_0\frac{1}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}+\tilde c_1\frac{1}{\Gamma_q(\sigma_j+2)}r^{\sigma_j+1}+\tilde c_2\frac{1+q}{\Gamma_q(\sigma_j+3)}r^{\sigma_j+2}, \tag{31}$$

$${}^R_q\mathfrak{I}^{\sigma_j}_{0^+}\left[{}^C_q\mathfrak{D}^2_{0^+}\mu(r)\right]=\frac{1}{\Gamma_q(\varsigma+\sigma_j-2)}\int_0^r(r-qv)^{(\varsigma+\sigma_j-3)}\phi_*(v)\,\mathrm{d}_qv+\tilde c_2\frac{1+q}{\Gamma_q(\sigma_j+1)}r^{\sigma_j}. \tag{32}$$

By considering the constants $W_1,\cdots,W_{11}$ given by (24) and by virtue the given boundary conditions implemented on (29)–(32) and by some straightforward computations, we obtain the following coefficients

$$\tilde c_0=\frac{W_2}{W_8}\sum_{j=1}^k\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_2}{W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,\mathrm{d}_qv+\frac{W_{10}}{W_1W_8}\sum_{j=1}^k\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_{10}}{W_1W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv+\frac{W_{11}}{W_1W_7W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_{11}}{W_1W_7W_8}\sum_{j=1}^k\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv, \tag{33}$$

$$\tilde c_1=\frac{W_4}{W_8}\sum_{j=1}^k\alpha_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_4}{W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-1)}}{\Gamma_q(\varsigma)}\phi_*(v)\,\mathrm{d}_qv+\frac{W_1}{W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-\rho-1)}}{\Gamma_q(\varsigma-\rho)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_1}{W_8}\sum_{j=1}^k\beta_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-1)}}{\Gamma_q(\varsigma+\sigma_j)}\phi_*(v)\,\mathrm{d}_qv+\frac{W_9}{W_7W_8}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{W_9}{W_7W_8}\sum_{j=1}^k\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv, \tag{34}$$

$$\tilde c_2=\frac{1}{W_7}\sum_{j=1}^k\gamma_j\int_0^1\frac{(1-qv)^{(\varsigma+\sigma_j-3)}}{\Gamma_q(\varsigma+\sigma_j-2)}\phi_*(v)\,\mathrm{d}_qv-\frac{1}{W_7}\int_0^\zeta\frac{(\zeta-qv)^{(\varsigma-3)}}{\Gamma_q(\varsigma-2)}\phi_*(v)\,\mathrm{d}_qv. \tag{35}$$

By inserting (33), (34), and (35) into (28), we derive equation (26) which is the same desired q-integral solution of the linear Cap-q-difference FBVP (25). The proof is completed. □

Now, consider the following estimates:

$$\begin{aligned}
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_1(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}(|rW_1W_4|+|W_{10}|)\\
&\le |W_1W_4|+|W_{10}| \coloneqq \Theta_1^* > 0,\\
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_2(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}(|rW_1|+|W_2|)\\
&\le |W_1|+|W_2| \coloneqq \Theta_2^* > 0,\\
\mathrm{Sup}_{r\in\mathcal{O}}|\Theta_3(r)| &\le \mathrm{Sup}_{r\in\mathcal{O}}\left(|r^2W_1W_8|+|rW_1W_9|+|W_{11}|\right)\\
&\le |W_1W_8|+|W_1W_9|+|W_{11}| \coloneqq \Theta_3^* > 0.
\end{aligned} \tag{36}$$

In this paper, for convenience in computation, we set

$$ {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(r) = \frac{1}{\Gamma_q(\varsigma)}\int_0^r (r-q\nu)^{(\varsigma-1)}\mathcal{G}_\mu(\nu)\,\mathrm{d}_q\nu. \tag{37}$$

According to Lemma 8, we define the operator $\mathcal{F}:\mathfrak{A}\longrightarrow\mathfrak{A}$ as

$$\begin{aligned}
(\mathcal{F}\mu)(r) ={}& {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(r) + \frac{\Theta_1(r)}{W_1W_8}\\
&\cdot\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^{k}\alpha_{j}{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathcal{G}_\mu(\nu)(1)\right]\\
&+\frac{\Theta_2(r)}{W_8}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}\mathcal{G}_\mu(\nu)(\zeta) - \sum_{j=1}^{k}\beta_{j}{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathcal{G}_\mu(\nu)(1)\right]\\
&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}\mathcal{G}_\mu(\nu)(\zeta) + \sum_{j=1}^{k}\gamma_{j}{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\mathcal{G}_\mu(\nu)(1)\right].
\end{aligned} \tag{38}$$

Notice that the Cap-q-difference FBVP (6) has solutions if and only if $\mathcal{F}$ has fixed points.

To simplify the computations, we set the following notation and the constants

$$\begin{aligned}
\Lambda ={}& \frac{1}{\Gamma_q(\varsigma+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^\varsigma}{\Gamma_q(\varsigma+1)} + \sum_{j=1}^{k}\frac{|\alpha_j|}{\Gamma_q(\varsigma+\sigma_j+1)}\right)\\
&+\frac{\Theta_2^*}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\rho+1)} + \sum_{j=1}^{k}\frac{|\beta_j|}{\Gamma_q(\varsigma+\sigma_j+1)}\right)\\
&+\frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)} + \sum_{j=1}^{k}\frac{|\gamma_j|}{\Gamma_q(\varsigma+\sigma_j-1)}\right).
\end{aligned} \tag{39}$$

3.1. Uniqueness Result. The uniqueness result for the Cap-q-difference FBVP (6) is proved by using the Banach's fixed-point theorem.

Theorem 9. *Assume that $G\in\mathcal{C}(\mathcal{O}\times\mathbb{R}^2,\mathbb{R})$ satisfies the following assumptions.*

($\mathcal{H}_1$) There are $\mathbb{L}_1, \mathbb{L}_2 > 0$ such that

$$|G(r,u_1,v_1)-G(r,u_2,v_2)| \le \mathbb{L}_1|u_1-u_2| + \mathbb{L}_2|v_1-v_2|, \tag{40}$$

for every u_i, $v_i\in\mathbb{R}$, $i=1,2$, and $r\in\mathcal{O}$.

If

$$\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda < 1, \tag{41}$$

where Λ is given in (39), and then the Cap-q-difference FBVP (6) has a unique solution μ in $\mathfrak{A}$.

Proof. We convert the Cap-q-difference FBVP (6) into $\mu=\mathcal{F}\mu$, where $\mathcal{F}$ is defined by (38). By the Banach's contraction principle, we shall guarantee that $\mathcal{F}$ has exactly one fixed point.

At first, we define a bounded, closed convex subset $\mathbb{B}_{Y_1} \coloneqq \{\mu\in\mathfrak{A} : \|\mu\|_{\mathfrak{A}} \le Y_1\} \ne \emptyset$ with

$$Y_1 \ge \frac{\Lambda\mathbb{G}}{1-\left(\mathbb{L}_1+\left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}, \tag{42}$$

where Λ is defined by (39).

Let $\sup_{r\in\mathcal{O}}|\mathcal{G}(r,0,0)| \coloneqq \mathbb{G} < \infty$. The proof will be completed in two steps:

Step 1. $\mathcal{F}\mathbb{B}_{Y_1}\subset\mathbb{B}_{Y_1}$.

Let $\mu\in\mathbb{B}_{Y_1}$ and $r\in\mathcal{O}$. Estimate

$$\begin{aligned}
|(\mathcal{F}\mu)(r)| \le{}& {}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathcal{G}_\mu(\nu)|(r) + \frac{\Theta_1(r)}{|W_1W_8|}\\
&\cdot\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathcal{G}_\mu(\nu)|(1)\right]\\
&+\frac{\Theta_2(r)}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathcal{G}_\mu(\nu)|(1)\right]\\
&+\frac{\Theta_3(r)}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathcal{G}_\mu(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathcal{G}_\mu(\nu)|(1)\right].
\end{aligned} \tag{43}$$

By using the property of integral (16), we get

$${}_q^R\mathfrak{J}_{0^+}^{\omega}|\mu(\nu)|(r) = \frac{1}{\Gamma_q(\omega)}\int_0^r (r-q\nu)^{(\omega-1)}|\mu(\nu)|\mathrm{d}_q\nu \le \frac{r^\omega\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}. \tag{44}$$

From the assumptions ($\mathcal{H}_1$) and (44), we can estimate

$$\begin{aligned}
|\mathcal{G}_\mu(r)| &\le \left|g\left(r,\mu(r),{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right) - g(r,0,0)\right| + |g(r,0,0,0)|\\
&\le \mathbb{L}_1|\mu(r)| + \mathbb{L}_2\left|{}_q^r\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right| + \mathbb{G} \le \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{G}.
\end{aligned} \tag{45}$$

From (45) and by the property of integral (16), we obtain

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathcal{G}_{\mu}(\nu)\right|(r)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)}, \tag{46}$$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathcal{G}_{\mu}(\nu)\right|(\zeta)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)}, \tag{47}$$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-\varrho}\left|\mathcal{G}_{\mu}(\nu)\right|(\zeta)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)}, \tag{48}$$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-2}\left|\mathcal{G}_{\mu}(\nu)\right|(\zeta)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)}, \tag{49}$$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathcal{G}_{\mu}(\nu)\right|(1)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{1}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}, \tag{50}$$

$$ {}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\left|\mathcal{G}_{\mu}(\nu)\right|(1)\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\frac{1}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}. \tag{51}$$

Substituting (46)–(51) into (43), we obtain

$$\begin{aligned}|(\mathcal{F}\mu)(r)|\le&\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\\&\cdot\left[\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)}+\frac{\Theta_1(r)}{|W_1W_8|}\left(\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)}+\sum_{j=1}^{k}\frac{|\alpha_j|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\right.\\&+\frac{\Theta_2(r)}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)}+\sum_{j=1}^{k}\frac{\left|\beta_j\right|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\\&\left.+\frac{\Theta_3(r)}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)}+\sum_{j=1}^{k}\frac{\left|\gamma_j\right|}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}\right)\right].\end{aligned} \tag{52}$$

Then,

$$|(\mathcal{F}\mu)(r)|\le\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu\|_{\mathfrak{A}}+\mathbb{G}\right]\Lambda, \tag{53}$$

which implies that $\|\mathcal{F}\mu\|_{\mathfrak{A}}\le Y_1$. Thus, $\mathcal{F}\mathbb{B}_{Y_1}\subset\mathbb{B}_{Y_1}$.

Step 2. $\mathcal{F}:\mathfrak{A}\longrightarrow\mathfrak{A}$ is a contraction.

Let μ, $\vartheta\in\mathfrak{A}$. For each $r\in\mathcal{O}$, we have

$$\begin{aligned}|(\mathcal{F}\mu)(r)-(\mathcal{F}\vartheta)(r)|\le&\frac{\Theta_1(r)}{|W_1W_8|}\\&\cdot\left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(\zeta)+\sum_{j=1}^{k}|\alpha_j|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(1)\right]\\&+\frac{\Theta_2(r)}{|W_8|}\left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-\varrho}\left|\mathcal{G}_{\vartheta}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(\zeta)+\sum_{j=1}^{k}\left|\beta_j\right|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(1)\right]\\&+\frac{\Theta_3(r)}{|W_1W_7W_8|}\left[{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma-2}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(\zeta)+\sum_{j=1}^{k}\left|\gamma_j\right|{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(1)\right]\\&+{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|(r).\end{aligned} \tag{54}$$

By $(\mathcal{H}_1)$, it follows that

$$\begin{aligned}\left|\mathcal{G}_{\mu}(\nu)-\mathcal{G}_{\vartheta}(\nu)\right|&\le\left|g\left(r,\mu(r),{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right)-g\left(r,\vartheta(r),{}_{q}^{R}\mathfrak{J}_{0^+}^{\varsigma}\vartheta(r)\right)\right|\\&\le\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu-\vartheta\|_{\mathfrak{A}}.\end{aligned} \tag{55}$$

Hence, by inserting (55) into (54) and using the property of integral (16), we get

$$\begin{aligned}|(\mathcal{F}\mu)(r)-(\mathcal{F}\vartheta)(r)|\le&\left[\left(\mathbb{L}_1+\frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\|\mu-\vartheta\|_{\mathfrak{A}}\right]\\&\cdot\left[\frac{r^{\varsigma}}{\Gamma_q(\varsigma+1)}+\frac{\Theta_1(r)}{|W_1W_8|}\left(\frac{\zeta^{\varsigma}}{\Gamma_q(\varsigma+1)}+\sum_{j=1}^{k}\frac{|\alpha_j|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\right.\\&+\frac{\Theta_2(r)}{|W_8|}\left(\frac{\zeta^{\varsigma-\varrho}}{\Gamma_q(\varsigma-\varrho+1)}+\sum_{j=1}^{k}\frac{\left|\beta_j\right|}{\Gamma_q\left(\varsigma+\sigma_j+1\right)}\right)\\&\left.+\frac{\Theta_3(r)}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma-2}}{\Gamma_q(\varsigma-1)}+\sum_{j=1}^{k}\frac{\left|\gamma_j\right|}{\Gamma_q\left(\varsigma+\sigma_j-1\right)}\right)\right],\end{aligned} \tag{56}$$

which implies that $\|\mathcal{F}\mu-\mathcal{F}\vartheta\|_{\mathfrak{A}}\le(\mathbb{L}_1+(\mathbb{L}_2/\Gamma_q(\omega+1)))\Lambda$ $\|\mu-\vartheta\|_{\mathfrak{A}}$.

In view of (41), $(\mathbb{L}_1+(\mathbb{L}_2/\Gamma_q(\omega+1)))\Lambda<1$, and we conclude that $\mathcal{F}$ is a contraction. Hence, in accordance with the Banach's contraction principle, the Cap-q-difference FBVP (6) has a unique solution $\mu\in\mathfrak{A}$. □

3.2. Existence Result. The second result is based on the Leray-Schauder's nonlinear alternative theorem.

Lemma 10 (Leray-Schauder's nonlinear alternative theorem [29]). *Let M be a Banach space, C be its closed convex subset, and X be an open set in C such that $0\in X$. Let $G:\bar{X}\longrightarrow C$ be a continuous and compact function. Then either (i) there is $\mu\in\bar{X}$ such that $\mu=G(\mu)$ or (ii) there are $\mu\in\partial X$ and $\varrho\in(0,1)$ such that $\mu=\varrho G(\mu)$.*

Theorem 11. *Let $G \in \mathscr{C}(\mathcal{O} \times \mathbb{R}^2, \mathbb{R})$ satisfies the following assumptions:*

($\mathscr{H}_2$) There is continuous nondecreasing functions $\mathbb{Y}: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, p_1, $p_2 \in \mathscr{C}(\mathscr{J}, \mathbb{R}^+)$ such that

$$|G(r,u,v)| \le p_1(r)\mathbb{Y}(|u|) + p_2(r)|v|, \forall (r,u,v) \in \mathcal{O} \times \mathbb{R}^2, \tag{57}$$

where $\bar{p}_i = \sup_{r \in \mathscr{J}}\{p_i(r)\}$, $i = 1, 2$.

($\mathscr{H}_3$) There is $\mathbb{M}^* > 0$ such that

$$\frac{\left(1 - \left(\Lambda \bar{p}_2 / \Gamma_q(\omega+1)\right)\right)\mathbb{M}^*}{\Lambda_1 \bar{p}_1 \mathbb{Y}(\mathbb{M}^*)} > 1. \tag{58}$$

Then the Cap-q-difference FBVP (6) has at least one solution μ in $\mathfrak{A}$.

Proof. Consider $\mathscr{F}$ as (38). In the first step, we will prove that $\mathscr{F}$ corresponds bounded sets (balls) to bounded ones in $\mathfrak{A}$. For each positive real constant Y_2, $\mathbb{B}_{Y_2} := \{\mu \in \mathfrak{A} : \|\mu\| \le Y_2\}$ is a bounded set (ball) in $\mathfrak{A}$. Let $\mu \in \mathbb{B}_{Y_2}$. We have

$$\begin{aligned}
|(\mathscr{F}\mu)(r)| \le {}& {}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathscr{G}_\mu(v)|(r) + \frac{\Theta_1(r)}{|W_1 W_8|} \\
&\cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{\Theta_2(r)}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{\Theta_3(r)}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathscr{G}_\mu(v)|(1)\right].
\end{aligned} \tag{59}$$

From ($\mathscr{H}_2$) and (44) in Theorem 9, we obtain

$$\begin{aligned}
\left|G\left(r, \mu(r), {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right)\right| &\le p_1(r)\mathbb{Y}(|\mu|) + p_2(r)\left|{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mu(r)\right| \\
&\le \bar{p}_1\mathbb{Y}(Y_2) + \frac{\bar{p}_2 Y_2}{\Gamma_q(\omega+1)} := \bar{g}.
\end{aligned} \tag{60}$$

By the same process in Theorem 9, we can estimate

$$\|(\mathscr{F}\mu)(r)\|_{\mathfrak{A}} \le \Lambda \bar{g}. \tag{61}$$

Further, it will be investigated that $\mathscr{F}$ corresponds bounded sets to equicontinuous sets of $\mathfrak{A}$.

Let r_1, $r_2 \in \mathcal{O}$ with $r_1 < r_2$ and $\mu \in \mathbb{B}_{Y_2}$, where $\mathbb{B}_{Y_2}$ is a bounded set in $\mathfrak{A}$. Also, we obtain

$$\begin{aligned}
|(\mathscr{F}\mu)(r_2) - (\mathscr{F}\mu)(r_1)| \le {}& \left|{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathscr{G}_\mu(v)(r_2) - {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathscr{G}_\mu(v)\right|(r_1)| \\
&+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathscr{G}_\mu(v)|(1)\right] \\
\le {}& \left|\frac{1}{\Gamma_q(\varsigma)}\int_{r_1}^{r_2}(r_2 - qv)^{(\varsigma-1)}G_\mu(v)d_qv\right| \\
&+ \left|\frac{1}{\Gamma_q(\varsigma)}\int_0^{r_1}\left[(r_2 - qv)^{(\varsigma-1)} - (r_1 - qv)^{(\varsigma-1)}\right]G_\mu(v)d_qv\right| \\
&+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathscr{G}_\mu(v)|(1)\right] \\
\le {}& \frac{\bar{g}}{\Gamma_q(\varsigma)}\left[\left|\int_{r_1}^{r_2}(r_2 - qv)^{(\varsigma-1)}d_qv\right| + \left|\int_0^{r_1}\left[(r_2 - qv)^{(\varsigma-1)} - (r_1 - qv)^{(\varsigma-1)}\right]d_qv\right|\right] \\
&+ \frac{|\Theta_1(r_2) - \Theta_1(r_1)|}{|W_1 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_2(r_2) - \Theta_2(r_1)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\rho}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\mathscr{G}_\mu(v)|(1)\right] \\
&+ \frac{|\Theta_3(r_2) - \Theta_3(r_1)|}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\mathscr{G}_\mu(v)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\mathscr{G}_\mu(v)|(1)\right].
\end{aligned} \tag{62}$$

Obviously, the above inequality goes to zero as $r_2 - r_1 \longrightarrow 0$, independent of $\mu \in \mathbb{B}_{Y_2}$. Hence, by helping the Arzelá-Ascoli theorem, $\mathscr{F} : \mathfrak{A} \longrightarrow \mathfrak{A}$ is completely continuous.

Now, we prove that there is an open set $\mathscr{D} \subset \mathfrak{A}$ such that $\mu \ne \kappa\mathscr{F}(\mu)$ for $\kappa \in (0,1)$ and $x\mu \in \partial\mathscr{D}$.

Let $\mu \in \mathfrak{A}$ satisfies $\mu = \kappa\mathscr{F}\mu$ for each $\kappa \in (0,1)$. So, for $r \in \mathcal{O}$, by following the calculations applied in proving the boundedness of $\mathscr{F}$, we have

$$|\mu(r)| = |\kappa(\mathscr{F}\mu)(r)| \le \Lambda\left[\bar{p}_1\left(\|\mu\|_{\mathfrak{A}}\right) + \frac{\bar{p}_2\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}\right]. \tag{63}$$

It yields that

$$\|\mu\|_{\mathfrak{A}} \le \bar{p}_1\Lambda\mathbb{Y}\left(\|\mu\|_{\mathfrak{A}}\right) + \frac{\bar{p}_2\Lambda\|\mu\|_{\mathfrak{A}}}{\Gamma_q(\omega+1)}. \tag{64}$$

Consequently, we obtain

$$\frac{\left[\Gamma_q(\omega+1)-\bar{p}_2\Lambda\right]\|\mu\|_{\mathfrak{A}}}{\bar{p}_1\Lambda\Gamma_q(\omega+1)\mathbb{Y}\left(\|\mu\|_{\mathfrak{A}}\right)}\le 1. \tag{65}$$

From $(\mathscr{H}_3)$, there is $\mathbb{M}^*>0$ such that $\|\mu\|_{\mathfrak{A}}\neq\mathbb{M}^*$. Let

$$\mathscr{D}:=\{\mu\in\mathfrak{A}:\|\mu\|\le\mathbb{M}^*+1\},\quad \mathscr{U}=\mathscr{D}\cup\mathbb{B}_{\Upsilon_2}. \tag{66}$$

Notice that $\mathscr{F}:\bar{\mathscr{U}}\longrightarrow\mathfrak{A}$ is completely continuous. For the sake of the choice of $\mathscr{D}$, $\nexists x\in\partial\mathscr{D}$ such that $\mu=\kappa\mathscr{F}\mu$ for some $\kappa\in(0,1)$. Therefore, by Lemma 10, we find out that $\mathscr{F}$ has the fixed point $\mu\in\bar{\mathscr{U}}$ which implies that the Cap-q-difference FBVP (6) has at least one solution on $\mathscr{O}$. □

3.3. On the Stability Property for (6). Stability analysis is one of the most important parts of each research in the field of existence of solution of fractional boundary value problems. For instances, we can mention to such a stability analysis in some newly published works including [24, 25, 30–32]. In this subsection, we introduce some concepts of stabilities for the Cap-q-difference FBVP (6). These definitions were extracted from [33].

Let $\epsilon>0$, $G:\mathscr{O}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous and $\theta:\mathscr{O}\longrightarrow\mathbb{R}^+$ be a nondecreasing mapping. Assume that

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\le\varepsilon, \tag{67}$$

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\le\theta(r), \tag{68}$$

$$\left|{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)-G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)\right|\le\varepsilon\theta(r). \tag{69}$$

Definition 12. The Cap-q-difference FBVP (6) is called Ulam-Hyers stable if $\exists C_G\in\mathbb{R}^+$ s.t. $\forall\varepsilon>0$ and every solution $\mu\in\mathfrak{U}$ of (67), a solution $\kappa\in\mathfrak{U}$ of (6) exists s.t.

$$|\mu(r)-\kappa(r)|\le C_G\epsilon,\ r\in\mathscr{O}. \tag{70}$$

Definition 13. The Cap-q-difference FBVP (6) is called generalized Ulam-Hyers stable if $\exists P\in\mathscr{C}(\mathbb{R}^+,\mathbb{R}^+)$, $P(0)=0$ s.t. $\forall\mu\in\mathfrak{U}$ fulfilling (67), a solution $\kappa\in\mathfrak{U}$ of (6) exists s.t.

$$|\mu(r)-\kappa(r)|\le P(\varepsilon),\ r\in\mathscr{O}. \tag{71}$$

Definition 14. The Cap-q-difference FBVP (6) is Ulam-Hyers-Rassias stable w.r.t. θ if $\exists C_\theta\in\mathbb{R}^+$ s.t. $\forall\varepsilon>0$ and every solution $\mu\in\mathfrak{U}$ of (69), $\exists$ a solution $\kappa\in\mathfrak{U}$ of (6) s.t.

$$|\mu(r)-\kappa(r)|\le C_\theta\theta(r)\varepsilon,\ r\in\mathscr{O}. \tag{72}$$

Definition 15. The Cap-q-difference FBVP (6) is termed generalized Ulam-Hyers-Rassias stable w.r.t. θ if $\exists C_\theta\in\mathbb{R}^+$ s.t. for every solution $\mu\in\mathfrak{U}$ of (68), $\exists$ a solution $\kappa\in\mathfrak{U}$ of (6) s.t.

$$|\mu(r)-\kappa(r)|\le C_\theta\theta(r),\ r\in\mathscr{O}. \tag{73}$$

Remark 16. $\mu\in\mathfrak{U}$ is a solution of (67) if $\exists\omega_\varsigma\in\mathfrak{U}$ (dependent on μ) s.t.

$$\begin{aligned}&(b_1)\,{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)+\omega_\varsigma(r),\ r\in\mathscr{O},\\&(b_2)\,|\omega_\varsigma(r)|\le\varepsilon.\end{aligned} \tag{74}$$

Lemma 17. *If $\mu\in\mathfrak{U}$ satisfies (67), then*

$$|\mu(r)-\lambda(r)|\le\Lambda\varepsilon, \tag{75}$$

where Λ is given as in (39) and $\lambda(r)$ is introduced in the proof.

Proof. Let μ satisfie (67). By (b_1) of Remark 16, there is $\omega_\varsigma\in\mathfrak{U}$ (dependent on μ) such that

$$\begin{aligned}&{}^{C}_{q}\mathfrak{D}^{\varsigma}_{0^+}\mu(r)=G\left(r,\mu(r),{}^{R}_{q}\mathfrak{J}^{\omega}_{0^+}\mu(r)\right)+\omega_\varsigma(r),\quad(r\in\mathscr{O},q\in(0,1)),\\&\mu(0)+\mu(\zeta)=\sum_{j=1}^{k}\alpha_{jq}\,{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\&{}^{C}_{q}\mathfrak{D}^{\mathrm{Q}}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{\mathrm{Q}}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\beta_{jq}\,{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\mu(1),\\&{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(0)+{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(\zeta)=\sum_{j=1}^{k}\gamma_{jq}\,{}^{R}\mathfrak{J}^{\sigma_j}_{0^+}\left[{}^{C}_{q}\mathfrak{D}^{2}_{0^+}\mu(1)\right].\end{aligned} \tag{76}$$

Then, the solution of (76) is given as

$$\begin{aligned}\mu(r)={}&{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)(r)+\frac{\Theta_1(r)}{W_1W_8}\cdot\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\mathscr{G}_\mu(v)(\zeta)+\sum_{j=1}^{k}\alpha_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+\frac{\Theta_2(r)}{W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\mathrm{Q}}_{0^+}\mathscr{G}_\mu(v)(\zeta)-\sum_{j=1}^{k}\beta_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}\mathscr{G}_\mu(v)(\zeta)+\sum_{j=1}^{k}\gamma_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}\mathscr{G}_\mu(v)(1)\right]\\&+{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\omega_\varsigma(r)+\frac{\Theta_1(r)}{W_1W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma}_{0^+}\omega_\varsigma(\zeta)+\sum_{j=1}^{k}\alpha_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\omega_\varsigma(1)\right]\\&+\frac{\Theta_2(r)}{W_8}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma-\mathrm{Q}}_{0^+}\omega_\varsigma(\zeta)-\sum_{j=1}^{k}\beta_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j}_{0^+}\omega_\varsigma(1)\right]\\&+\frac{\Theta_3(r)}{W_1W_7W_8}\left[-{}^{R}_{q}\mathfrak{J}^{\varsigma-2}_{0^+}\omega_\varsigma(\zeta)+\sum_{j=1}^{k}\gamma_{jq}\,{}^{R}\mathfrak{J}^{\varsigma+\sigma_j-2}_{0^+}\omega_\varsigma(1)\right].\end{aligned} \tag{77}$$

For convenience, consider $\lambda(r)$ for the terms that are independent of $\omega_\varsigma(r)$. That is,

$$\lambda(r) = {}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathscr{G}_\mu(v)(r) + \frac{\Theta_1(r)}{W_1 W_8}\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma}\mathscr{G}_\mu(v)(\zeta) + \sum_{j=1}^{k}\alpha_j {}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathscr{G}_\mu(v)(1)\right] + \frac{\Theta_2(r)}{W_8}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}\mathscr{G}_\mu(v)(\zeta) - \sum_{j=1}^{k}\beta_j {}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}\mathscr{G}_\mu(v)(1)\right] + \frac{\Theta_3(r)}{W_1 W_7 W_8}\left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}\mathscr{G}_\mu(v)(\zeta) + \sum_{j=1}^{k}\gamma_j {}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}\mathscr{G}_\mu(v)(1)\right]. \tag{78}$$

Therefore, (77) can be rewritten and by using (b_2) of Remark 16, we have

$$|\mu(r) - \lambda(r)| \le {}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\omega_\varsigma(r)| + \frac{\Theta_1(r)}{|W_1 W_8|} \cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\omega_\varsigma(1)|\right] + \frac{\Theta_2(r)}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-\varrho}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\beta_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j}|\omega_\varsigma(1)|\right] + \frac{\Theta_3(r)}{|W_1 W_7 W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma-2}|\omega_\varsigma(\zeta)| + \sum_{j=1}^{k}|\gamma_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma+\sigma_j-2}|\omega_\varsigma(1)|\right] \le \Lambda\varepsilon. \tag{79}$$

This inequality completes the proof. □

Theorem 18. *Let $(\mathscr{H}_1)$ and*

$$\left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda < 1, \tag{80}$$

to be held. Then, the Cap-q-difference FBVP (6) is both Ulam-Hyers and generalized Ulam-Hyers stable.

Proof. Let $\mu \in \mathfrak{U}$ satisfies (67) and κ fulfills the Cap-q-difference FBVP (6) given as

$$\begin{aligned}
&{}_q^C\mathfrak{D}_{0^+}^{\varsigma}\kappa(r) = G\left(r, \kappa(r), {}_q^R\mathfrak{J}_{0^+}^{\omega}\kappa(r)\right), \quad (r \in \mathcal{O}, q \in (0,1)),\\
&\kappa(0) + \kappa(\zeta) = \sum_{j=1}^{k}\alpha_j {}_q^R\mathfrak{J}_{0^+}^{\sigma_j}\kappa(1),\\
&{}_q^C\mathfrak{D}_{0^+}^{\varrho}\kappa(0) + {}_q^C\mathfrak{D}_{0^+}^{\varrho}\kappa(\zeta) = \sum_{j=1}^{k}\beta_j {}_q^R\mathfrak{J}_{0^+}^{\sigma_j}\kappa(1),\\
&{}_q^C\mathfrak{D}_{0^+}^{2}\kappa(0) + {}_q^C\mathfrak{D}_{0^+}^{2}\kappa(\zeta) = \sum_{j=1}^{k}\gamma_j {}_q^R\mathfrak{J}_{0^+}^{\sigma_j}\left[{}_q^C\mathfrak{D}_{0^+}^{2}\kappa(1)\right].
\end{aligned} \tag{81}$$

By the previous lemma, let

$$|\mu(r) - \kappa(r)| \le |\mu(r) - \lambda(r)| + |\lambda(r) - \kappa(r)|. \tag{82}$$

By using Lemma 17 in (82), we have

$$|\mu(r) - \kappa(r)| \le \Lambda\epsilon + \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda|\mu(r) - \kappa(r)|. \tag{83}$$

For $r \in \mathcal{O}$, we have

$$\|\mu - \kappa\|_{\mathfrak{U}} \le \Lambda\epsilon + \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda\|\mu - \kappa\|_{\mathfrak{U}}. \tag{84}$$

After simplification, we get

$$\|\mu - \kappa\|_{\mathfrak{U}} \le \left(\frac{\Lambda}{1 - \left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}\right)\epsilon. \tag{85}$$

Thus

$$|\mu(r) - \kappa(r)| \le C_G\epsilon, \tag{86}$$

where

$$C_G = \frac{\Lambda}{1 - \left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda}. \tag{87}$$

Thus, the Cap-q-difference FBVP (6) is Ulam-Hyers stable.

In the sequel, the function $P(\epsilon) = C_G\epsilon$ implies that the Cap-q-difference FBVP (6) is generalized Ulam-Hyers stable and $P(0) = 0$.

Now, we add another condition.

$(\mathscr{A}_1)$ Consider an increasing map $\pi_\varsigma \in \mathscr{C}(\mathcal{O}, \mathbb{R}^+)$. Then, there is $\xi_{\pi_\varsigma} > 0$ such that

$${}_q^R\mathfrak{J}_{0^+}^{\varsigma}\pi_\varsigma(r) \le \xi_{\pi_\varsigma}\pi_\varsigma(r). \tag{88}$$

□

Remark 19. Under the hypotheses $(\mathscr{H}_1)$ and $(\mathscr{A}_1)$ and (80), the Cap-q-difference FBVP (6) is the Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stable.

4. Analysis of the Cap-q-Difference System (7)

Here, we continue to discuss the existence and uniqueness results for the proposed system (7). In view of the assumptions of Section 3 for the Banach space $\mathfrak{A}$, the norm considered on the product space $\mathfrak{A} \times \mathfrak{A}$ is $\|(\mu, \vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} = \|\mu\|_{\mathfrak{A}} + \|\vartheta\|_{\mathfrak{A}}$ which implies that $(\mathfrak{A} \times \mathfrak{A}, \|(\mu, \vartheta)\|_{\mathfrak{A}\times\mathfrak{A}})$ is a Banach space.

Remark 20. For convenience, and based on the given parameters in (7), we have nonzero constants:

$$
\begin{aligned}
\bar{W}_1 &= 2 - \sum_{j=1}^{k} \frac{\phi_j}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_2 &= \zeta - \sum_{j=1}^{k} \frac{\phi_j}{\Gamma_q(\delta_j + 2)}, \\
\bar{W}_3 &= \zeta^2 - \sum_{j=1}^{k} \frac{\phi_j(1+q)}{\Gamma_q(\delta_j + 3)}, \\
\bar{W}_4 &= - \sum_{j=1}^{k} \frac{\varphi_j}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_5 &= - \sum_{j=1}^{k} \frac{\varphi_j}{\Gamma_q(\delta_j + 2)}, \\
\bar{W}_6 &= \frac{2\zeta^{2-\rho}}{\Gamma_q(3-\rho)} - \sum_{j=1}^{k} \frac{\varphi_j(1+q)}{\Gamma_q(\delta_j + 3)}, \\
\bar{W}_7 &= 2(1+q) - \sum_{j=1}^{k} \frac{\eta_j(1+q)}{\Gamma_q(\delta_j + 1)}, \\
\bar{W}_8 &= \bar{W}_2\bar{W}_4 - \bar{W}_1\bar{W}_5, \\
\bar{W}_9 &= \bar{W}_3\bar{W}_4 - \bar{W}_1\bar{W}_6, \\
\bar{W}_{10} &= \bar{W}_8 - \bar{W}_2\bar{W}_4, \\
\bar{W}_{11} &= \bar{W}_3\bar{W}_8 - \bar{W}_2\bar{W}_9, \\
\bar{\Theta}_1(r) &= r\bar{W}_1\bar{W}_4 + \bar{W}_{10}, \\
\bar{\Theta}_2(r) &= r\bar{W}_1 - \bar{W}_2, \\
\bar{\Theta}_3(r) &= r^2\bar{W}_1\bar{W}_8 - r\bar{W}_1\bar{W}_9 - \bar{W}_{11}.
\end{aligned}
\tag{89}
$$

Keeping in mind Lemma 8, consider the operator $\mathcal{S} : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A} \times \mathfrak{A}$ as

$$
\mathcal{S}(x, y)(r) := (\mathcal{S}_1(\mu, \vartheta)(r), \mathcal{S}_2(\mu, \vartheta)(r)), \tag{90}
$$

where

$$
\begin{aligned}
\mathcal{S}_1(\mu, \vartheta)(r) &= {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}\mathcal{U}_\vartheta(\nu)(r) + \frac{\Theta_1(r)}{W_1W_8} \\
&\cdot \left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}\mathcal{U}_\vartheta(\nu)(\zeta) + \sum_{j=1}^{k} \alpha_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}\mathcal{U}_\vartheta(\nu)(1) \right] \\
&+ \frac{\Theta_2(r)}{W_8} \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}\mathcal{U}_\vartheta(\nu)(\zeta) - \sum_{j=1}^{k} \beta_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}\mathcal{U}_\vartheta(\nu)(1) \right] \\
&+ \frac{\Theta_3(r)}{W_1W_7W_8} \left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}\mathcal{U}_\vartheta(\nu)(\zeta) + \sum_{j=1}^{k} \gamma_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}\mathcal{U}_\vartheta(\nu)(1) \right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_2(\mu, \vartheta)(r) &= {}_q^R\mathfrak{J}_{0^+}^{\varsigma_2}\mathcal{V}_\mu(\nu)(r) + \frac{\bar{\Theta}_1(r)}{\bar{W}_1\bar{W}_8} \\
&\cdot \left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma_2}\mathcal{V}_\mu(\nu)(\zeta) + \sum_{j=1}^{k} \phi_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_2+\delta_j}\mathcal{V}_\mu(\nu)(1) \right] \\
&+ \frac{\bar{\Theta}_2(r)}{\bar{W}_8} \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_2-\rho}\mathcal{V}_\mu(\nu)(\zeta) - \sum_{j=1}^{k} \varphi_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_2+\delta_j}\mathcal{V}_\mu(\nu)(1) \right] \\
&+ \frac{\bar{\Theta}_3(r)}{\bar{W}_1\bar{W}_7\bar{W}_8} \left[-{}_q^R\mathfrak{J}_{0^+}^{\varsigma_2-2}\mathcal{V}_\mu(\nu)(\zeta) + \sum_{j=1}^{k} \eta_{jq}{}^R\mathfrak{J}_{0^+}^{\varsigma_2+\delta_j-2}\mathcal{V}_\mu(\nu)(1) \right].
\end{aligned}
\tag{91}
$$

Before proceeding, consider the following estimates

$$
\begin{aligned}
\operatorname{Sup}_{r\in\mathcal{O}}\left|\bar{\Theta}_1(r)\right| &:= \bar{\Theta}_1^*, \\
\operatorname{Sup}_{r\in\mathcal{O}}\left|\Theta_2(r)\right| &:= \bar{\Theta}_2^*, \\
\operatorname{Sup}_{r\in\mathcal{O}}\left|\Theta_3(r)\right| &:= \bar{\Theta}_3^*.
\end{aligned}
\tag{92}
$$

To simplify, we also set the following notation and the constants

$$
\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^{k} \frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)} \right) \\
&+ \frac{\Theta_2^*}{|W_8|}\left(\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^{k} \frac{|\beta_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)} \right) \\
&+ \frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^{k} \frac{|\gamma_j|}{\Gamma_q(\varsigma_1+\sigma_j-1)} \right), \\
\Lambda_2 &= \frac{1}{\Gamma_q(\varsigma_2+1)} + \frac{\bar{\Theta}_1^*}{|\bar{W}_1\bar{W}_8|}\left(\frac{\zeta^{\varsigma_2}}{\Gamma_q(\varsigma_2+1)} + \sum_{j=1}^{k} \frac{|\phi_j|}{\Gamma_q(\varsigma_2+\delta_j+1)} \right) \\
&+ \frac{\bar{\Theta}_2^*}{|\bar{W}_8|}\left(\frac{\zeta^{\varsigma_2-\rho}}{\Gamma_q(\varsigma_2-\rho+1)} + \sum_{j=1}^{k} \frac{|\varphi_j|}{\Gamma_q(\varsigma_2+\delta_j+1)} \right) \\
&+ \frac{\bar{\Theta}_3^*}{|\bar{W}_1\bar{W}_7\bar{W}_8|}\left(\frac{\zeta^{\varsigma_2-2}}{\Gamma_q(\varsigma_2-1)} + \sum_{j=1}^{k} \frac{|\eta_j|}{\Gamma_q(\varsigma_2+\delta_j-1)} \right).
\end{aligned}
\tag{93}
$$

4.1. Uniqueness Result. In this step, we shall establish the existence of a unique solution to the coupled system of nonlinear q-CFBVPs (7), by the Banach's contraction principle.

Theorem 21. *Let $G_1, G_2 : \mathcal{O} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous. Assume that*

($\mathcal{H}_4$) There exist positive constants $\mathcal{L}_i, \mathcal{K}_i, i = 1, 2$ such that for each $r \in [0, 1]$ and $u_i, v_i, \bar{u}_i, \bar{v}_i \in \mathbb{R}$, and for $i = 1, 2$

$$
\begin{aligned}
|G_1(r, u_1, v_1) - G_1(r, u_2, v_2)| &\le \mathcal{L}_1|u_1 - u_2| + \mathcal{L}_2|v_1 - v_2|, \\
|G_2(r, \bar{u}_1, \bar{v}_1) - G_2(r, \bar{u}_2, \bar{v}_2)| &\le \mathcal{K}_1|\bar{u}_1 - \bar{u}_2| + \mathcal{K}_2|\bar{v}_1 - \bar{v}_2|.
\end{aligned}
\tag{94}
$$

Then the coupled system of nonlinear q-CFBVPs (7) has a solution on $\mathcal{O}$ provided that

$$\Omega := \max\left\{\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1, \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\right\} < 1. \quad (95)$$

Proof. We transform the coupled system of nonlinear q-CFBVPs (7) into a fixed-point problem $(\mu, \vartheta)(r) = \mathcal{S}(\mu, \vartheta)(r)$, where $\mathcal{S}$ is an operator as (90).

Let $\sup_{r\in\mathcal{O}}|G_1(r,0,0)| := \mathbb{M}_{\mathcal{U}} < \infty$ and $\sup_{r\in\mathcal{O}}|G_2(r,0,0)| := \mathbb{M}_{\mathcal{V}} < \infty$. Next, we set $\mathbb{B}_{Y_3} := \{(\mu,\vartheta) \in \mathfrak{A}\times\mathfrak{A} : \|\mu,\vartheta\|_{\mathfrak{A}\times\mathfrak{A}} \le Y_3\}$ with

$$Y_3 \ge \frac{\mathbb{M}_{\mathcal{U}}\Lambda_1 + \mathbb{M}_{\mathcal{V}}\Lambda_2}{1-\Omega}. \quad (96)$$

Note that $\mathbb{B}_{Y_3}$ is a bounded convex closed set in $\mathfrak{A}$.

Step 1. $\mathcal{S}\mathbb{B}_{Y_3} \subset \mathbb{B}_{Y_3}$.

For each $(\mu,\vartheta) \in \mathbb{B}_{Y_3}$ and $r \in \mathcal{O}$, and by using the condition $(\mathcal{H}_4)$ and (44), we have

$$\begin{aligned}
|\mathcal{U}_\vartheta(r)| &\le \left|G_1\left(r, \vartheta(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right) - G_1(r,0,0)\right| + |G_1(r,0,0)| \\
&\le \mathcal{L}_1|\vartheta(r)| + \mathcal{L}_2\left|{}_q^R\mathfrak{J}_{0^+}^{\omega_1}\vartheta(r)\right| \\
&\quad + \mathbb{M}_{\mathcal{U}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}, \\
|\mathcal{V}_\mu(r)| &\le \left|G_2\left(r, \mu(r), {}_q^R\mathfrak{J}_{0^+}^{\omega_2}\mu(r)\right) - G_2(r,0,0)\right| + |G_2(r,0,0)| \\
&\le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}.
\end{aligned} \quad (97)$$

Then, we get

$$\begin{aligned}
|\mathcal{S}_1(\mu,\vartheta)(r)| &\le {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(r) + \frac{|\Theta_1(r)|}{|W_1W_8|} \\
&\quad \cdot \left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\
&\quad + \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right] \\
&\quad + \frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_\vartheta(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_\vartheta(v)|(1)\right], \\
&\le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\
&\quad + \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] + \frac{\Theta_3^*}{|W_1W_7W_8|} \\
&\quad \left.\cdot\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right] \\
&\quad \times\left[\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\right].
\end{aligned} \quad (98)$$

Hence

$$\|\mathcal{S}_1(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\Lambda_1. \quad (99)$$

Similarly, we find that

$$\|\mathcal{S}_2(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2. \quad (100)$$

Consequently, we have

$$\begin{aligned}
\|\mathcal{S}(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} &\le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{U}}\Lambda_1 \\
&\quad + \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2 \\
&\le \Omega Y_3 + \mathbb{M}_{\mathcal{U}}\Lambda_1 + \mathbb{M}_{\mathcal{V}}\Lambda_2 \le Y_3,
\end{aligned} \quad (101)$$

which implies that $\mathcal{S}\mathbb{B}_{Y_3} \subset \mathbb{B}_{Y_3}$.

Step 2. We show that $\mathcal{S} : \mathfrak{A}\times\mathfrak{A} \longrightarrow \mathfrak{A}\times\mathfrak{A}$ is a contraction.

Using condition $(\mathcal{H}_4)$, for any $(\mu_1,\vartheta_1), (\mu_2,\vartheta_2) \in \mathfrak{A}\times\mathfrak{A}$ and for each $r \in \mathcal{O}$, we have

$$\begin{aligned}
&|\mathcal{S}_1(\mu_1,\vartheta_1)(r) - \mathcal{S}_1(\mu_2,\vartheta_2)(r)| \\
&\le \frac{|\Theta_1(r)|}{|W_1W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + \frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + \frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1-2}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(\zeta) + \sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{J}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(1)\right] \\
&\quad + {}_q^R\mathfrak{J}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_1}(v) - \mathcal{U}_{\vartheta_2}(v)|(r) \\
&\le \left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)} + \sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right. \\
&\quad + \frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)} + \sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right] \\
&\quad \left.+ \frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)} + \sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right] \\
&\quad \times\left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\|\vartheta_1 - \vartheta_2\|_{\mathfrak{A}},
\end{aligned} \quad (102)$$

and therefore

$$\|\mathcal{S}_1(\mu_1,\vartheta_1) - \mathcal{S}_1(\mu_2,\vartheta_2)\|_{\mathfrak{A}} \le \left(\mathcal{L}_1 + \frac{\mathcal{L}_2}{\Gamma_q(\omega_1+1)}\right)\Lambda_1\|\vartheta_1 - \vartheta_2\|_{\mathfrak{A}}. \quad (103)$$

Similarly, we get

$$\|\mathcal{S}_2(\mu_1,\vartheta_1)-\mathcal{S}_2(\mu_2,\vartheta_2)\|_{\mathfrak{A}}\le\left(\mathcal{K}_1+\frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu_1-\mu_2\|_{\mathfrak{A}}. \tag{104}$$

From (103) and (104), it yields

$$\|\mathcal{S}(\mu_1,\vartheta_1)-\mathcal{S}(\mu_2,\vartheta_2)\|_{\mathfrak{A}\times\mathfrak{A}}\le\Omega\left(\|\vartheta_1-\vartheta_2\|_{\mathfrak{A}}+\|\mu_1-\mu_2\|_{\mathfrak{A}}\right). \tag{105}$$

As $\Omega<1$, by (95), the operator $\mathcal{S}$ is a contraction. The Banach's contraction principle implies the existence of unique solution for the coupled system of nonlinear q-CFBVPs (7) on $[0.1]$. □

4.2. Existence Result. We get help from Lemma 10 to complete the main result of this subsection.

Theorem 22. *Let G_1, $G_2:\mathcal{O}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous. Assume that*

($\mathcal{H}_4$) There exist nonnegative continuous maps $x_i(r), y_i(r)\in C(\mathcal{O},\mathbb{R}^+\cup\{0\})$, for $i=1,2,3$ such that

$$\begin{aligned}&|G_1(r,u,v)|\le x_1(r)+x_2(r)|u|+x_3(r)|v|,(r,u,v)\in\left(\mathcal{O},\mathbb{R}^2\right),\\&|G_2(r,\bar{u},\bar{v})|\le y_1(r)+y_2(r)|\bar{u}|+y_3(r)|\bar{v}|,(r,\bar{u},\bar{v})\in\left(\mathcal{O},\mathbb{R}^2\right),\end{aligned} \tag{106}$$

with $x_i^*=\sup_{r\in\mathcal{O}}\{x_i(t)\}$ and $y_i^*=\sup_{r\in\mathcal{O}}\{y_i(t)\}$.

Then the coupled system of nonlinear q-CFBVPs (7) has at least one solution on $\mathcal{O}$.

Proof. Here, the process of the proof will be continued during four steps as follows.

Step 1. $\mathcal{S}$ is continuous.

Let μ_n and ϑ_n be two sequences such that $\mu_n\longrightarrow\mu$ and $\vartheta_n\longrightarrow\vartheta$ in $\mathfrak{A}$. Then for each $r\in\mathcal{O}$, we get

$$\begin{aligned}|\mathcal{S}_1(\mu_n,\vartheta_n)(r)-\mathcal{S}_1(\mu,\vartheta)(r)|&\le\frac{|\Theta_1(r)|}{|W_1W_8|}\\&\cdot\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(1)\right]\\&+\frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1-\varrho}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(1)\right]\\&+\frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1-2}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(1)\right],\\&+{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1}|\mathcal{U}_{\vartheta_n}(v)-\mathcal{U}_\vartheta(v)|(r)\le\left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)}+\frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)}+\sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right.\\&+\frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)}+\sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right]\\&\left.+\frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)}+\sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right]\|\mathcal{U}_{\vartheta_n}-\mathcal{U}_\vartheta\|_{\mathfrak{A}},\end{aligned} \tag{107}$$

and therefore

$$\|\mathcal{S}_1(\mu_n,\vartheta_n)-\mathcal{S}_1(\mu,\vartheta)\|_{\mathfrak{A}}\le\Lambda_1\left\|\mathcal{U}_{\vartheta_n}-\mathcal{U}_\vartheta\right\|_{\mathfrak{A}}. \tag{108}$$

Similarly, we get

$$\|\mathcal{S}_2(\mu_n,\vartheta_n)-\mathcal{S}_2(\mu,\vartheta)\|_{\mathfrak{A}}\le\Lambda_2\left\|\mathcal{V}_{\mu_n}-\mathcal{V}_\mu\right\|_{\mathfrak{A}}. \tag{109}$$

From (108) and (109), it yields

$$\|\mathcal{S}(\mu_n,\vartheta_n)-\mathcal{S}(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}}\le\Lambda_1\left\|\mathcal{U}_{\vartheta_n}-\mathcal{U}_\vartheta\right\|_{\mathfrak{A}}+\Lambda_2\left\|\mathcal{V}_{\mu_n}-\mathcal{V}_\mu\right\|_{\mathfrak{A}}. \tag{110}$$

Since the continuity of G_1 and G_2 imply that of $\mathcal{U}_\vartheta$, $\mathcal{V}_\mu$, so we have $\|\mathcal{U}_{\vartheta_n}-\mathcal{U}_\vartheta\|_{\mathfrak{A}}\longrightarrow0$ and $\|\mathcal{V}_{\mu_n}-\mathcal{V}_\mu\|_{\mathfrak{A}}\longrightarrow0$ as $n\longrightarrow\infty$; and $\mathcal{S}$ is continuous.

Step 2. $\mathcal{S}$ is uniformly bounded.

We prove that for $Y_4>0$, there exists $\mathcal{N}_{\mathcal{S}}>0$ such that for every $(\mu,\vartheta)\in\mathbb{B}_{Y_4}$, where

$$\mathbb{B}_{Y_4}=\left\{(\mu,\vartheta)\in\mathfrak{A}\times\mathfrak{A}:\|(x,y)\|_{\mathfrak{A}\times\mathfrak{A}}<Y_4\right\}, \tag{111}$$

we get $\|S(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}}\le\mathcal{N}_{\mathcal{S}}$.

Using the condition ($\mathcal{H}_5$) and (16), we have

$$\begin{aligned}|\mathcal{U}_\vartheta(r)|&=\left|G_1\left(r,\vartheta(r),{}_q^R\mathfrak{I}_{0^+}^{\omega_1}\vartheta(r)\right)\right|\le x_1(t)+x_2(t)|\vartheta(r)|\\&\quad+x_3(t)\left|{}_q^R\mathfrak{I}_{0^+}^{\omega_1}\vartheta(r)\right|\le x_1^*+\left(x_2^*+\frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}},\\|\mathcal{V}_\mu(r)|&=\left|G_2\left(r,\mu(r),{}_q^R\mathfrak{I}_{0^+}^{\omega_2}\mu(r)\right)\right|\le y_1^*+\left(y_2^*+\frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}.\end{aligned} \tag{112}$$

Then, we get

$$\begin{aligned}|\mathcal{S}_1(\mu,\vartheta)(r)|&\le{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(r)+\frac{|\Theta_1(r)|}{|W_1W_8|}\\&\cdot\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1}|\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k|\alpha_j|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right]\\&+\frac{|\Theta_2(r)|}{|W_8|}\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1-\rho}|\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k\left|\beta_j\right|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j}|\mathcal{U}_\vartheta(v)|(1)\right]\\&+\frac{|\Theta_3(r)|}{|W_1W_7W_8|}\left[{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1-2}|\mathcal{U}_\vartheta(v)|(\zeta)+\sum_{j=1}^k\left|\gamma_j\right|{}_q^R\mathfrak{I}_{0^+}^{\varsigma_1+\sigma_j-2}|\mathcal{U}_\vartheta(v)|(1)\right]\\&\le\left[\frac{r^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)}+\frac{\Theta_1^*}{|W_1W_8|}\left(\frac{\zeta^{\varsigma_1}}{\Gamma_q(\varsigma_1+1)}+\sum_{j=1}^k\frac{|\alpha_j|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right)\right.\\&+\frac{\Theta_2^*}{|W_8|}\left[\frac{\zeta^{\varsigma_1-\varrho}}{\Gamma_q(\varsigma_1-\varrho+1)}+\sum_{j=1}^k\frac{\left|\beta_j\right|}{\Gamma_q(\varsigma_1+\sigma_j+1)}\right]\\&\left.+\frac{\Theta_3^*}{|W_1W_7W_8|}\left(\frac{\zeta^{\varsigma_1-2}}{\Gamma_q(\varsigma_1-1)}+\sum_{j=1}^k\frac{\left|\gamma_j\right|}{\Gamma_q(\varsigma_1+\sigma_j-1)}\right)\right]\\&\times\left[x_1^*+\left(x_2^*+\frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right].\end{aligned} \tag{113}$$

Hence

$$\|\mathcal{S}_1(\mu,\vartheta)\|_{\mathfrak{A}} \le \Lambda_1\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right]. \quad (114)$$

Similarly, we find that

$$\|\mathcal{S}_2(\mu,\vartheta)\|_{\mathfrak{A}} \le \left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{\Gamma_q(\omega_2+1)}\right)\Lambda_2\|\mu\|_{\mathfrak{A}} + \mathbb{M}_{\mathcal{V}}\Lambda_2. \quad (115)$$

Consequently, we have

$$\begin{aligned}\|\mathcal{S}(\mu,\vartheta)\|_{\mathfrak{A}\times\mathfrak{A}} &\le \Lambda_1\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right] \\ &\quad + \Lambda_2\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right] \coloneqq \mathcal{N}_{\mathcal{S}}.\end{aligned} \quad (116)$$

Then, $\mathcal{S}$ is uniformly bounded.

Step 3. $\mathcal{S}$ maps bounded sets into equi-continuous sets of $\mathfrak{A}$. Let $r_1, r_2 \in \mathcal{O}$ such that $r_1 < r_2$ and $(\mu,\vartheta) \in \mathbb{B}_{Y_4}$ where $\mathbb{B}_{Y_4}$ is defined as in Step 2. Then we have

$$\begin{aligned}
&|\mathcal{S}_1(\mu,\vartheta)(r_2) - \mathcal{S}_1(\mu,\vartheta)(r_1)| \\
&\le \left|{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}\mathcal{U}_\vartheta(\nu)(r_2) - {}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}\mathcal{U}_\vartheta(\nu)(r_1)\right| + \frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|} \\
&\quad\cdot\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\le \frac{1}{\Gamma_q(\varsigma_1)}\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right] \\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-q\nu)^{(\varsigma_1-1)}d_q\nu\right| + \left|\int_0^{r_1}\left[(r_2-q\nu)^{(\varsigma_1-1)} - (r_1-q\nu)^{(\varsigma_1-1)}\right]d_q\nu\right|\right] \\
&\quad+ \frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right], \\
&|\mathcal{S}_2(\mu,\vartheta)(r_2) - \mathcal{S}_2(\mu,\vartheta)(r_1)| \\
&\le \frac{1}{\Gamma_q(\varsigma_2)}\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right] \\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-q\nu)^{(\varsigma_2-1)}d_q\nu\right| + \left|\int_0^{r_1}\left[(r_2-q\nu)^{(\varsigma_2-1)} - (r_1-q\nu)^{(\varsigma_2-1)}\right]d_q\nu\right|\right] \\
&\quad+ \frac{\left|\bar{\Theta}_1(r_2)-\bar{\Theta}_1(r_1)\right|}{\left|\bar{W}_1\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\phi_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right] \\
&\quad+ \frac{\left|\bar{\Theta}_2(r_2)-\bar{\Theta}_2(r_1)\right|}{\left|\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2-\rho}_{0^+}\left|\mathcal{V}_\mu\right|(\nu)(\zeta) + \sum_{j=1}^{k}\left|\varphi_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right] \\
&\quad+ \frac{\left|\bar{\Theta}_3(r_2)-\bar{\Theta}_3(r_1)\right|}{\left|\bar{W}_1\bar{W}_7\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2-2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\eta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j-2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right],
\end{aligned} \quad (117)$$

which implies that

$$\begin{aligned}
&|\mathcal{S}(\mu,\vartheta)(r_2) - \mathcal{S}(\mu,\vartheta)(r_1)| \le \frac{1}{\Gamma_q(\varsigma_1)} \\
&\quad\cdot\left[x_1^* + \left(x_2^* + \frac{x_3^*}{\Gamma_q(\omega_1+1)}\right)\|\vartheta\|_{\mathfrak{A}}\right] \\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-q\nu)^{(\varsigma_1-1)}d_q\nu\right| + \left|\int_0^{r_1}\left[(r_2-q\nu)^{(\varsigma_1-1)} - (r_1-q\nu)^{(\varsigma_1-1)}\right]d_q\nu\right|\right] \\
&\quad+ \frac{1}{\Gamma_q(\varsigma_2)}\left[y_1^* + \left(y_2^* + \frac{y_3^*}{\Gamma_q(\omega_2+1)}\right)\|\mu\|_{\mathfrak{A}}\right] \\
&\quad\cdot\left[\left|\int_{r_1}^{r_2}(r_2-q\nu)^{(\varsigma_2-1)}d_q\nu\right| + \left|\int_0^{r_1}\left[(r_2-q\nu)^{(\varsigma_2-1)} - (r_1-q\nu)^{(\varsigma_2-1)}\right]d_q\nu\right|\right] \\
&\quad+ \frac{|\Theta_1(r_2)-\Theta_1(r_2)|}{|W_1W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}|\alpha_j|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{\left|\bar{\Theta}_1(r_2)-\bar{\Theta}_1(r_1)\right|}{\left|\bar{W}_1\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\phi_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right] \\
&\quad+ \frac{|\Theta_2(r_2)-\Theta_2(r_1)|}{|W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-\varrho}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\beta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{\left|\bar{\Theta}_2(r_2)-\bar{\Theta}_2(r_1)\right|}{\left|\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2-\rho}_{0^+}\left|\mathcal{V}_\mu\right|(\nu)(\zeta) + \sum_{j=1}^{k}\left|\varphi_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right] \\
&\quad+ \frac{|\Theta_3(r_2)-\Theta_3(r_1)|}{|W_1W_7W_8|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_1-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(\zeta) + \sum_{j=1}^{k}\left|\gamma_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_1+\sigma_j-2}_{0^+}|\mathcal{U}_\vartheta(\nu)|(1)\right] \\
&\quad+ \frac{\left|\bar{\Theta}_3(r_2)-\bar{\Theta}_3(r_1)\right|}{\left|\bar{W}_1\bar{W}_7\bar{W}_8\right|}\left[{}^{R}_{q}\mathfrak{J}^{\varsigma_2-2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(\zeta) + \sum_{j=1}^{k}\left|\eta_j\right|{}^{R}_{q}\mathfrak{J}^{\varsigma_2+\delta_j-2}_{0^+}\left|\mathcal{V}_\mu(\nu)\right|(1)\right].
\end{aligned} \quad (118)$$

The right-hand side tends to 0 as $r_2 \longrightarrow r_1$, which is independent of $(\mu,\vartheta) \in \mathbb{B}_{Y_4}$. By helping the Arzelá-Ascoli theorem, $S : \mathfrak{A} \longrightarrow \mathfrak{A}$ is completely continuous.

Step 4. The set $\mathfrak{B} = \{(\mu,\vartheta) \in \mathfrak{A}\times\mathfrak{A} : (\mu,\vartheta) = \kappa S(\mu,\vartheta), \kappa \in (0,1]\}$ is bounded.

Let $(\mu,\vartheta) \in \mathfrak{B}$. Then $(\mu,\vartheta) = \kappa S(\mu,\vartheta)$ for some $\kappa \in (0,1]$. Thus, for any $r \in \mathcal{O}$, by using the computations of Step 2, we have

$$\|\mathcal{S}(\mu,\vartheta)(r)\|_{\mathfrak{A}\times\mathfrak{A}} \le \mathcal{N}_{\mathcal{S}}. \quad (119)$$

This means that $\mathfrak{B}$ is bounded. Consequently, by Lemma 10, $\mathcal{S}$ has a fixed point and so a solution to the coupled system of nonlinear q-CFBVPs (7). □

5. Numerical Examples

In this section, we provide some illustrative examples of the exactness and applicability of our main results.

Example 1. (i) Consider the Cap-q-difference FBVP of the form

$${}^{C}_{0.8}\mathfrak{D}^{2.5}_{0^+}\mu(r) = G\left(r, \mu(r), {}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right), \quad (r \in \mathcal{O}, q \in (0,1)),$$

$$\mu(0) + \mu(0.4) = \sum_{j=1}^{2}\left(\frac{12-4j}{10}\right){}^{R}_{0.8}\mathfrak{J}^{3j/10}_{0^+}\mu(1),$$

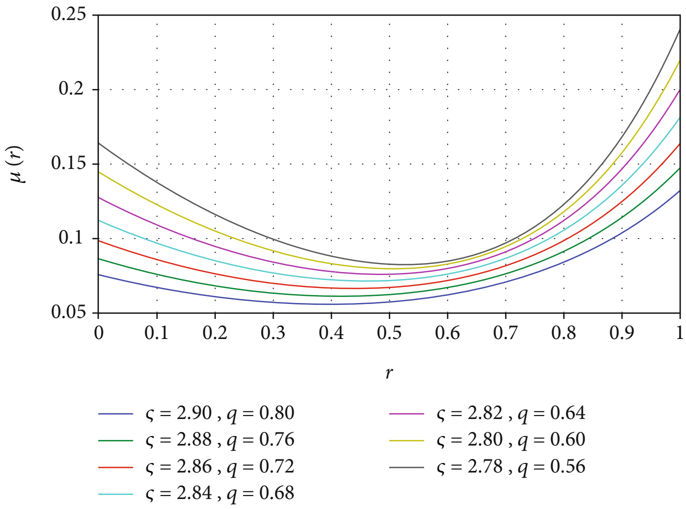

FIGURE 1: The exact solution $\mu(r)$ of (120) for $r \in [0, 1]$.

$$ {}^{C}_{0.8}\mathfrak{D}^{1.2}_{0^+}\mu(0) + {}^{C}_{0.8}\mathfrak{D}^{1.2}_{0^+}\mu(0.4) = \sum_{j=1}^{2}\left(\frac{2j+3}{10}\right){}^{R}_{0.8}\mathfrak{J}^{3j/10}_{0^+}\mu(1), $$

$$ {}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(0) + {}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(0.4) = \sum_{j=1}^{2}\left(\frac{12-5j}{10}\right){}^{R}_{0.8}\mathfrak{J}^{3j/10}_{0^+}\left[{}^{C}_{0.8}\mathfrak{D}^{2}_{0^+}\mu(1)\right]. \tag{120} $$

Here $\varsigma = 2.5$, $q = 0.8$, $\omega = 3.8$, $\zeta = 0.4$, $\varrho = 1.2$, $\alpha_j = (2j+3)/10$, $\beta_j = (12-5j)/10$, $\gamma_j = (12-4j)/10$, $\sigma_j = 3j/10$, and $j = 1, 2$. From the given data, we obtain $W_1 \approx 0.676686276 \neq 0$, $W_7 \approx 1.814092676 \neq 0$, and $W_8 \approx 1.431872331 \neq 0$. We consider the functions

$$ G\left(r, \mu(r), {}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right) = \frac{4r-1}{re^{2r}+4} + \frac{9\cos(\pi/3)}{2e^{r}+6} \cdot \frac{|\mu(r)|}{|\mu(r)|+3} + \frac{10\sin(\pi/6)}{(2r+3)^2+2e^{3r}} \cdot \frac{\left|{}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right|}{\left|{}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)\right|+2}. \tag{121} $$

For u_i, $v_i \in \mathbb{R}$, and $r \in \mathcal{O}$, we can find that

$$ |G(r, u_1, v_1) - G(r, u_2, v_2)| \leq \frac{3}{8}|u_1 - u_2| + \frac{5}{11}|v_1 - v_2|. \tag{122} $$

The assumption $(\mathcal{H}_1)$ is satisfied under the values $\mathbb{L}_1 = 3/8$ and $\mathbb{L}_2 = 5/11$. Thus,

$$ \left(\mathbb{L}_1 + \frac{\mathbb{L}_2}{\Gamma_q(\omega+1)}\right)\Lambda \approx 0.8324696807 < 1. \tag{123} $$

All assumptions of Theorem 9 are valid. Then the Cap-q-difference FBVP (120) has a unique solution on $[0, 1]$. Moreover,

$$ C_G = \frac{\Lambda}{1 - \left(\mathbb{L}_1 + \left(\mathbb{L}_2/\Gamma_q(\omega+1)\right)\right)\Lambda} \approx 11.85782552 > 0. \tag{124} $$

By the conclusions of Theorem 18, the Cap-q-difference FBVP (120) is both Ulam–Hyers and also generalized Ulam-Hyers stable on $[0, 1]$. (ii) Set $G(r, \mu(r), {}^{R}_{0.8}\mathfrak{J}^{3.8}_{0^+}\mu(r)) = r^{\lambda}$.

By using the property of integral (16) and setting $\lambda = 2.8$, the implicit solution of the Cap-q-difference FBVP (120) is given by

$$ \begin{aligned} \mu(r) = {} & \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+1)} r^{\lambda+\varsigma} + \frac{\Theta_1(r)}{W_1 W_8} \cdot \left[-\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+1)}\zeta^{\lambda+\varsigma} + \sum_{j=1}^{k}\frac{\alpha_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j+1)}\right] \\ & + \frac{\Theta_2(r)}{W_8}\left[\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma-\rho+1)}\zeta^{\lambda+\varsigma-\varrho} - \sum_{j=1}^{k}\frac{\beta_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j+1)}\right] \\ & + \frac{\Theta_3(r)}{W_1 W_7 W_8}\left[-\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma-1)}\zeta^{\lambda+\varsigma-2} + \sum_{j=1}^{k}\frac{\gamma_j\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\varsigma+\sigma_j-1)}\right]. \end{aligned} \tag{125} $$

Figure 1 displays the solution of the Cap-q-difference FBVP (120) involving various values of $\varsigma = 2.78, 2.80, \cdots, 2.90$ and $q = 0.56, 0.60, \cdots, 0.80$.

Example 2. Consider the coupled system of nonlinear Cap-q-difference FBVP under the conditions

$$ {}^{C}_{0.7}\mathfrak{D}^{2.8}_{0^+}\mu(r) = G_1\left(r, \vartheta(r), {}^{R}_{0.7}\mathfrak{J}^{1.7}_{0^+}\vartheta(r)\right), \quad (r \in \mathcal{O}), $$

$$ {}^{C}_{0.7}\mathfrak{D}^{2.9}_{0^+}\vartheta(r) = G_2\left(r, \mu(r), {}^{R}_{0.7}\mathfrak{J}^{2.3}_{0^+}\mu(r)\right), $$

$$ \mu(0) + \mu(0.3) = \sum_{j=1}^{2}\left(\frac{4j}{10}\right){}^{R}_{0.7}\mathfrak{J}^{5j-3/10}_{0^+}\mu(1), $$

$$\vartheta(0)+\vartheta(0.3)=\sum_{j=1}^{k}\left(\frac{12-5j}{10}\right){}_{0.7}^{R}\mathfrak{J}_{0^+}^{3j-1/10}\vartheta(1),$$

$${}_{0.7}^{C}\mathfrak{D}_{0^+}^{1.8}\mu(0)+{}_{0.7}^{C}\mathfrak{D}_{0^+}^{1.8}\mu(0.3)=\sum_{j=1}^{2}\left(\frac{7-2j}{10}\right){}_{0.7}^{R}\mathfrak{J}_{0^+}^{5j-3/10}\mu(1),$$

$${}_{0.7}^{C}\mathfrak{D}_{0^+}^{1.4}\vartheta(0)+{}_{0.7}^{C}\mathfrak{D}_{0^+}^{1.4}\vartheta(0.3)=\sum_{j=1}^{2}\left(\frac{10-4j}{10}\right){}_{0.7}^{R}\mathfrak{J}_{0^+}^{3j-1/10}\vartheta(1),$$

$${}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\mu(0)+{}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\mu(0.3)=\sum_{j=1}^{2}\left(\frac{10-3j}{10}\right){}_{0.7}^{R}\mathfrak{J}_{0^+}^{5j-3/10}\left[{}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\mu(1)\right],$$

$${}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\vartheta(0)+{}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\vartheta(0.3)=\sum_{j=1}^{2}\left(\frac{8-3j}{10}\right){}_{0.7}^{R}\mathfrak{J}_{0^+}^{3j-1/10}\left[{}_{0.7}^{C}\mathfrak{D}_{0^+}^{2}\vartheta(1)\right]. \tag{126}$$

Here $\varsigma_1=2.8$, $\varsigma_2=2.9$, $q=0.7$, $\omega_1=1.7$, $\omega_2=2.3$, $\zeta=0.3$, $\rho=1.8$, $\rho=1.4$, $\alpha_j=4j/10$, $\beta_j=(7-2j)/10$, $\gamma_j=(10-3j)/10$, $\phi_j=(12-5j)/10$, $\varphi_j=(10-4j)/10$, $\eta_j=(8-3j)/10$, $\sigma_j=(5j-3)/10$, $\delta_j=(3j-1)/10$, and $j=1,2$. From all the given data, we obtain $W_1\approx 0.705064917\neq 0$, $W_7\approx 1.385967560\neq 0$, $W_8\approx 1.029770834\neq 0$, $\bar{W}_1\approx 1.026846802\neq 0$, $\bar{W}_7\approx 2.110974612\neq 0$, and $\bar{W}_8\approx 1.174518052\neq 0$. We consider the functions

$$\begin{aligned}G_1\left(r,\vartheta(r),{}_{0.7}^{R}\mathfrak{J}_{0^+}^{1.7}\vartheta(r)\right)&=3r^2-2r+1+\frac{r+1}{\sin^2(r)+6}\cdot\frac{|\vartheta(r)|}{|\vartheta(r)|+3}+\frac{2\cos(r)}{(3r+4)^2}\cdot\frac{\left|{}_{0.7}^{R}\mathfrak{J}_{0^+}^{1.7}\vartheta(r)\right|}{\left|{}_{0.7}^{R}\mathfrak{J}_{0^+}^{1.7}\vartheta(r)\right|+1},\\ G_2\left(r,\mu(r),{}_{0.7}^{R}\mathfrak{J}_{0^+}^{2.3}\mu(r)\right)&=re^{2r}-3r+\frac{(2r+\sin(r))}{3e^r+4}\cdot\frac{|\mu(r)|}{|\mu(r)|+1}+\frac{r}{\ln(2r+1)+3}\cdot\frac{\left|{}_{0.7}^{R}\mathfrak{J}_{0^+}^{2.3}\mu(r)\right|}{\left|{}_{0.7}^{R}\mathfrak{J}_{0^+}^{2.3}\mu(r)\right|+2}.\end{aligned} \tag{127}$$

For u_i, v_i, $\bar{u}_i$, $\bar{v}_i\in\mathbb{R}$, and $r\in\mathcal{O}$, we can find that

$$\begin{aligned}|G_1(r,u_1,v_1)-G(r,u_2,v_2)|&\leq\frac{1}{9}|u_1-u_2|+\frac{1}{8}|v_1-v_2|,\\ |G_2(r,\bar{u}_1,\bar{v}_1)-G(r,\bar{u}_2,\bar{v}_2)|&\leq\frac{3}{7}|\bar{u}_1-\bar{u}_2|+\frac{1}{3}|\bar{v}_1-\bar{v}_2|.\end{aligned} \tag{128}$$

The assumption $(\mathscr{H}_4)$ is satisfied with $\mathscr{L}_1=1/9$, $\mathscr{L}_2=1/9$, $\mathscr{K}_1=3/7$, and $\mathscr{K}_2=1/3$. Hence, $(\mathscr{L}_1+(\mathscr{L}_2/\Gamma_q(\omega_1+1)))\Lambda_1\approx 0.6937912556<1$ and $(\mathscr{K}_1+(\mathscr{K}_2/\Gamma_q(\omega_2+1)))\Lambda_1\approx 0.8947974715<1$. All assumptions of Theorem 21 are satisfied. Then the coupled system of nonlinear Cap-q-difference FBVPs (126) has a unique solution on $[0,1]$.

6. Conclusion

In this paper, a new category of nonlinear Caputo quantum boundary problems and its relevant generalized coupled q-system involving fractional quantum operators was discussed. We presented new q-difference equations and system in which we dealt with q-integro-sum-difference bundary conditions. Some qualitative aspects of solutions such as the existence, uniqueness, and different classes of stabilities of Ulam-Hyers type were investigated for both given q-Cap-difference problems. The results were examined with some examples. As a new idea in the next papers, we aim to extend our method for similar generalized coupled systems under the newly introduced generalized (p,q)-operators (postquantum operators).

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

The first and third authors would like to thank Azarbaijan Shahid Madani University.

References

[1] K. Shah, M. Sher, A. Ali et al., "On degree theory for non-monotone type fractional order delay differential equations," *AIMS Mathematics*, vol. 7, no. 5, pp. 9479–9492, 2022.

[2] S. W. Ahmad, M. Sarwar, K. Shah, Eiman, and T. Abdeljawad, "Study of a coupled system with sub-strip and multi-valued boundary conditions via topological degree theory on an infinite domain," *Symmetry*, vol. 14, no. 5, p. 841, 2022.

[3] K. Shah, M. Arfan, A. Ullah, Q. Al-Mdallal, K. J. Ansari, and T. Abdeljawad, "Computational study on the dynamics of fractional order differential equations with applications," *Chaos, Solitons & Fractals*, vol. 157, article 111955, 2022.

[4] Y. Rahmani, M. M. Alizadeh, H. Schuh, J. Wickert, and L. C. Tsai, "Probing vertical coupling effects of thunderstorms on lower ionosphere using GNSS data," *Advances in Space Research*, vol. 66, no. 8, pp. 1967–1976, 2020.

[5] F. H. Jackson, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[6] F. H. Jackson, "XI.—On q-functions and a certain difference operator," *Transactions of the Royal Society Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[7] B. Ahmad, S. K. Ntouyas, and J. Tariboon, *Quantum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities*, World Scientific, Singapore, 2016.

[8] T. Ernst, *A Comprehensive Treatment of* q-*Calculus*, Springer, Basel, Switzerland, 2012.

[9] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[10] M. El-Shahed and F. M. Al-Askar, "Positive solutions for boundary value problem of nonlinear fractional q-difference equation," *International Scholarly Research Notices*, vol. 2011, Article ID 385459, 12 pages, 2011.

[11] J. R. Graef and L. Kong, "Positive solutions for a class of higher order boundary value problems with fractional q-derivatives," *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9682–9689, 2012.

[12] J. Alzabut, B. Mohammadaliee, and M. E. Samei, "Solutions of two fractional q-integro-differential equations under sum and integral boundary value conditions on a time scale," *Applied Mathematics and Computation*, vol. 2020, no. 1, p. 304, 2020.

[13] R. I. Butt, T. Abdeljawad, M. A. Alqudah, and M. ur Rehman, "Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach," *Journal of Inequalities and Applications*, vol. 2019, Article ID 305, 3 pages, 2019.

[14] S. Etemad, S. K. Ntouyas, and B. Ahmad, "Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders," *Mathematics*, vol. 7, no. 8, p. 659, 2019.

[15] S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, and Y. M. Chu, "On multi-step methods for singular fractional q-integro-differential equations," *Open Mathematics*, vol. 19, no. 1, pp. 1378–1405, 2021.

[16] R. Ouncharoen, N. Patanarapeelert, and T. Sitthiwirattham, "Nonlocal q-symmetric integral boundary value problem for sequential q-symmetric integro-difference equations," *Mathematics*, vol. 6, no. 11, p. 218, 2018.

[17] J. Wang, C. Yu, B. Zhang, and S. Wang, "Positive solutions for eigenvalue problems of fractional q-difference equation with ϕ -Laplacian," *Advances in Difference Equations*, vol. 2021, Article ID 499, 5 pages, 2021.

[18] N. Patanarapeelert and T. Sitthiwirattham, "On four-point fractional q-integro-difference boundary value problems involving separate nonlinearity and arbitrary fraction order," *Boundary Value Problems*, vol. 2018, Article ID 41, 20 pages, 2018.

[19] S. Abbas, M. Benchohra, and J. R. Graef, "Oscillation and non-oscillation results for the Caputo fractional q-difference equations and inclusions," *Journal of Mathematical Sciences*, vol. 258, no. 5, pp. 577–593, 2021.

[20] B. Ahmad, J. J. Nieto, A. Alsaedi, and H. Al-Hutami, "Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions," *Filomat*, vol. 28, no. 8, pp. 1719–1736, 2014.

[21] S. Etemad, M. Ettefagh, and S. Rezapour, "On the existence of solutions for nonlinear fractional q-difference equations with q -integral boundary conditions," *Journal of Advanced Mathematical Studies*, vol. 8, no. 2, pp. 265–285, 2015.

[22] S. K. Ntouyas and M. E. Samei, "Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus," *Advances in Difference Equations*, vol. 2019, Article ID 475, 20 pages, 2019.

[23] N. D. Phuong, F. M. Sakar, S. Etemad, and S. Rezapour, "A novel fractional structure of a multi-order quantum multi-integro-differential problem," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 633, p. 23, 2020.

[24] Z. Ali, A. Zada, and K. Shah, "Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem," *Boundary Value Problems*, vol. 2018, Article ID 175, 6 pages, 2018.

[25] A. Khan, K. Shah, Y. Li, and T. S. Khan, "Ulam type stability for a coupled systems of boundary value problems of nonlinear fractional differential equations," *Journal of Function Spaces*, vol. 2017, Article ID 3046013, 8 pages, 2017.

[26] P. M. Rajkovic, S. D. Marinkovic, and M. S. Stankovic, "Fractional integrals and derivatives in q-calculus," *Applicable Analysis and Discrete Mathematics*, vol. 1, no. 1, pp. 311–323, 2007.

[27] C. R. Adams, "The general theory of a class of linear partial q -difference equations," *Transactions of the American Mathematical Society*, vol. 26, no. 3, pp. 283–312, 1924.

[28] R. A. C. Ferreira, "Positive solutions for a class of boundary value problems with fractional q-differences," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 367–373, 2011.

[29] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, USA, 2003.

[30] S. Li, L. Shu, X. B. Shu, and F. Xu, "Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays," *Stochastics*, vol. 91, no. 6, pp. 857–872, 2019.

[31] Y. Guo, X.-B. Shu, Y. Li, and F. Xu, "The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$," *Boundary Value Problems*, vol. 2019, Article ID 59, 8 pages, 2019.

[32] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[33] I. A. Rus, "Ulam stabilities of ordinary differential equations in a Banach space," *Carpathian Journal of Mathmatics*, vol. 26, no. 1, pp. 103–107, 2010.