Existence of Solutions for Nonlinear Integral Equations in Tempered Sequence Spaces via Generalized Darbo-Type Theorem

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1. Introduction and Preliminaries

Darbo [1] constructed the fixed point theorem, and later, researchers called this widely studied theorem by his name, that is, “Darbo fixed point theorem” wherein he enforced the technique of measure of noncompactness (shortly, MNC) while Kuratowski [2] was the first who described the idea of MNC. Many researchers are employing Darbo’s theorem to demonstrate the existence or solvability of several functional equations (linear or nonlinear) in conjunction with different kinds of Banach sequence spaces or simply called Banach spaces. Recently, the infinite system of several kinds of integral equations was considered by Banas and Lecko [3], Mursaleen et al. [4, 5], and Mohiuddine et al. [6] to obtain the existence of solutions in the framework of Banach spaces, namely, the spaces $c_0$, $c$, $\ell_p$, and $\ell_1$ of null, convergent, absolutely $p$-summable, and absolutely summable sequences in conjunction with the Darbo-type theorem. The reader can refer to the recent monographs [7, 8] on the normed/paranormed sequence spaces and related topics.

The integral equations play a significant contribution in diverse branches of science and engineering as well as this theory is applicable in several real life problems such as gas kinetic theory, neutron transportation, and radiation [9]. Most recently, the researchers used different kinds of integral equations (infinite system) (see [10–12]) to demonstrate existence of solutions by means of the notion of MNC, i.e., in $\ell_p$ [13] and in Banach space [14–16].

Suppose that $\mathcal{G}$ is a Banach space, and suppose also that $B(\theta, \tilde{r}) = \{ x \in \mathcal{G} : \| x - \theta \| \leq \tilde{r} \}$ is a closed ball. If $\mathcal{X}(\neq \emptyset) \subseteq \mathcal{G}$, then its closure and convex closure, respectively, will write by symbols $\overline{\mathcal{X}}$ and $\text{Conv}\mathcal{X}$. Further, $\mathcal{M}_\mathcal{G}$ will be used to denote the family of bounded (nonempty) subsets of $\mathcal{G}$ as well as its subfamily, $\mathcal{H}_\mathcal{G}$, which consists of all relatively compact sets. The MNC is defined in [17] (see also [18]) as follows.

Definition 1. A mapping $\mathcal{F} : \mathcal{M}_\mathcal{G} \longrightarrow \mathbb{R}_+([0,\infty))$ is called MNC in $\mathcal{G}$ if
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tered sequence space. Inspired by these constructions, very classical spaces and they de
tinuous mappings from \( I \)

\[
\rho \cdot \equiv \alpha \left( I_{\infty} \right)
\]

Note that \( J_{\infty} = \bigcap_{j=1}^{\infty} J_j \) for any \( j \), we infer that \( \mathcal{G}(J_{\infty}) \) holds. Since \( \mathcal{G}(J_{\infty}) \subseteq \mathcal{G}(J_j) \)

Banas and Krajewsk [19] proposed the generalization of classical spaces \( c_0, c, \) and \( \ell_{\infty} \) with the help of tempering sequence \( \alpha = (\alpha_i)_{i=1}^{\infty} \). The tempering sequence means that \( \alpha_i \) is positive for any \( i \in \mathbb{N} \) and \( (\alpha_i) \) is nonincreasing, and they defined \( c_0, c, \) and \( \ell_{\infty} \), which are called the tempered sequence space. Inspired by these constructions, very recently, Rebbani et al. [20] defined the tempered space \( \ell_p^\alpha \) as follows:

\[
\mathbb{L} = \left\{ \rho = (\rho_n)_{n=1}^{\infty} \in \mathcal{W} : \sum_{n=1}^{\infty} \alpha_n |\rho_n|^p < \infty \text{ for } 1 \leq p < \infty \right\},
\]

where \( \mathcal{W} \) is the space of real or complex sequences, or simply, we shall write \( \mathbb{L} = \ell_p^\alpha \). Clearly, \( \ell_p^\alpha \) is a Banach space endowed with

\[
\|\rho\|_{\ell_p^\alpha} = \left( \sum_{n=1}^{\infty} \alpha_n |\rho_n|^p \right)^{1/p}.
\]

In case of \( \alpha_n = 1 \) for all \( n \in \mathbb{N} \), the tempered space \( \ell_p^\alpha \) coincides with \( \ell_p \), and, in addition, if \( p = 1 \), \( \ell_p^\alpha \) coincides with \( \ell_1 \). In the same paper, they gave the Hausdorff MNC \( \chi_{\ell_p^\alpha} \) for a nonempty bounded set \( \mathbb{R}^n \) of \( \ell_p^\alpha \) by

\[
\chi_{\ell_p^\alpha}(B) = \lim_{n \to \infty} \left[ \sup_{j \in \mathbb{B}} \left( \sum_{k \in \mathbb{N}} \alpha_k |j_k|^p \right)^{1/p} \right].
\]

We will use \( C(I, \ell_p^\alpha) \) to denote the collection of all continuous mappings from \( I = [0, a] \) \( (a > 0) \) to \( \ell_p^\alpha \), and \( C(I, \ell_p^\alpha) \) is a Banach space with the norm

\[
\|\rho\|_{C(I, \ell_p^\alpha)} = \sup_{s \in I} |\rho(s)|_{\ell_p^\alpha},
\]

where \( \rho(s) = (\rho_n(s))_{n=1}^{\infty} \in C(I, \ell_p^\alpha) \). For any nonempty bounded set \( \mathbb{R}^n \) of \( C(I, \ell_p^\alpha) \) and for \( s \in I \), one defines \( \mathbb{R}^n(s) = \{ \rho(s) \}_{n=1}^{\infty} \in C(I, \ell_p^\alpha) \) and hence, its MNC is given by

\[
\chi_{\mathbb{R}^n}(t, c) = \sup_{s \in \mathbb{R}^n} \chi_{\ell_p^\alpha}(E^n(s)).
\]

Recall the theorem given in [1] as follows:

**Theorem 2.** Suppose that \( F \) is a nonempty, closed, bounded, and convex subset of \( \mathbb{E} \), and suppose also that \( \mathcal{G} : F \to \mathbb{R}^n \) is a continuous mapping, and there exists \( \kappa \in [0, 1) \) satisfying

\[
\mathcal{G}(\mathbb{E}) \leq \kappa \mathcal{G}(\Lambda), \Lambda \subseteq \mathbb{F}.
\]

Then, \( \mathcal{G} \) has a fixed point.

**2. Dorbo-Type Fixed Point Theorems**

In order to discuss our Dorbo-type theorems, we first recall the set of functions which has been recently used in [13] as follows: Consider the function \( M : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that

\begin{align*}
(1) \quad & \max \{ \theta_1, \theta_2 \} \leq M(\theta_1, \theta_2) \text{ for } \theta_1, \theta_2 \geq 0 \\
(2) \quad & M \text{ is continuous and nondecreasing} \\
(3) \quad & M(\theta_1 + \theta_1, v_1 + v_2) \leq M(\theta_1, v_1) + M(\theta_2, v_2)
\end{align*}

hold. We will denote the collection of such functions by \( M \). The example of aforesaid kind of function is \( M(\theta_1, \theta_2) = \theta_1 + \theta_2 \).

**Theorem 3.** Consider a Banach space \( \mathbb{E} \), a nonempty, closed, bounded convex set \( D \subseteq \mathbb{E} \), and an arbitrary MNC \( \mathcal{G} \). Also, consider a continuous mapping \( \mathbb{T} : D \to D \) satisfying the inequality

\[
\alpha[M(\mathcal{G}(\mathbb{T}x), \gamma(\mathcal{G}(\mathbb{T}x)))] \\
\leq \alpha[M(\mathcal{G}(x), \gamma(\mathcal{G}(x)))] - \beta[M(\mathcal{G}(x), \gamma(\mathcal{G}(x)))]
\]

for any \( x \in \mathbb{E} \) \( (x \neq \mathcal{G}) \) \( \subseteq D \), where \( M \in M \) and \( \alpha, \beta : \mathbb{R} \to \mathbb{R} \) are functions such that \( \alpha, \gamma \) are continuous on \( \mathbb{R} \), and \( \beta \) is lower semicontinuous which satisfies the relations

\[
\beta(0) = 0 \text{ and } \beta(x) > 0 \text{ (} x > 0 \).
\]

Then, \( \mathbb{T} \) has at least one fixed point in \( D \).

**Proof.** Consider a sequence \( \{D_n\}_{n=1}^{\infty} \) such that \( D_1 \subseteq D = D_n \subseteq \mathcal{G}(\mathbb{T}D_n) \) for \( n \in \mathbb{N} \). One can find that \( \mathbb{T}D_1 = D_1 \), \( D_2 = \mathcal{G}(\mathbb{T}D_1) \subseteq D = D_1 \). We obtain in a similar way that \( D_1 \supseteq D_2 \supseteq D_1 \supseteq \cdots \supseteq D_2 \supseteq D_{n+1} \supseteq \cdots \). If there exists \( n_0 \in \mathbb{N} \) satisfying \( \mathcal{G}(D_{n_0}) = 0 \), then \( D_n \) is a compact set. With a view of Schauder theorem [21], \( \mathbb{T} \) has a fixed point in \( D \subseteq \mathbb{E} \).
Further, assume that $\mathcal{S}(D_n) > 0$ for $n \in \mathbb{N}$. Clearly, $\{\mathcal{S}(D_n)\}_{n=1}^{\infty}$ is nonnegative, decreasing, and bounded below sequence. Therefore, $\{\mathcal{S}(D_n)\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n \to \infty} \mathcal{S}(D_n) = r \geq 0, \text{ say.} \quad (10)$$

Inequality (7) gives

$$a[M(\mathcal{S}(D_{n+1}), \gamma(\mathcal{S}(D_{n+1}))) = a[M(\mathcal{S}(\text{Conv}TD_n), \gamma(\mathcal{S}(\text{Conv}TD_n)))] = a[M(\mathcal{S}(TD_n), \gamma(\mathcal{S}(TD_n)))] \leq a[M(\mathcal{S}(D_n), \gamma(\mathcal{S}(D_n)))] - \beta[M(\mathcal{S}(D_n), \gamma(\mathcal{S}(D_n)))]]. \quad (11)$$

If possible, assume $r > 0$. Letting $\limsup_{n \to \infty}$ in the last inequality, one obtains

$$\limsup_{n \to \infty} a[M(\mathcal{S}(D_{n+1}), \gamma(\mathcal{S}(D_{n+1}))) \leq \limsup_{n \to \infty} a[M(\mathcal{S}(D_n), \gamma(\mathcal{S}(D_n)))] - \limsup_{n \to \infty} \beta[M(\mathcal{S}(D_n), \gamma(\mathcal{S}(D_n)))] \quad (12)$$

which yields

$$a[M(r, \gamma(r))] \leq a[M(r, \gamma(r))] - \beta[M(r, \gamma(r))]. \quad (13)$$

It follows from the inequality (13) that

$$\beta[M(r, \gamma(r))] \leq 0. \quad (14)$$

Consequently, we get $\beta[M(r, \gamma(r))] = 0$. So, $\gamma(r) = r = 0$. Therefore, we have

$$\lim_{n \to \infty} \mathcal{S}(D_n) = 0. \quad (15)$$

Using the fact $D_n \supseteq D_{n+1}$ and Definition 1, we fairly have

$$D_{\infty} = \bigcap_{n=1}^{\infty} D_n \subseteq D, \quad (16)$$

which is nonempty, convex, closed subset of $D$ and $D_{\infty}$ is $\mathcal{T}$ invariant. By taking into account Schauder theorem [21], we conclude that (9) holds.

**Theorem 4.** Consider a Banach space $\mathcal{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathcal{E}$, and an arbitrary MNC $\mathcal{S}$. Also, consider a continuous mapping $\mathcal{T} : D \longrightarrow D$ satisfying the inequality

$$a[\mathcal{S}(\mathcal{T}X) + \gamma(\mathcal{S}(\mathcal{T}X))] \leq a[\mathcal{S}(X) + \gamma(\mathcal{S}(X))] - \beta[\mathcal{S}(X) + \gamma(\mathcal{S}(X))] \quad (17)$$

for $\forall (\neq \emptyset) \subseteq D$, where $\alpha, \beta, \gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are functions such that $\alpha, \gamma$ are continuous on $\mathbb{R}_+$ and $\beta$ is lower semicontinuous satisfies relation (8). Then, (9) holds.

**Proof.** This result can be obtained by considering the function $M(v_1, v_2) = v_1 + v_2$ in Theorem 3. \qed

**Theorem 5.** Consider a Banach space $\mathcal{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathcal{E}$, and an arbitrary MNC $\mathcal{S}$. Also, consider a continuous mapping $\mathcal{T} : D \longrightarrow D$ satisfying the inequality

$$M(\mathcal{S}(\mathcal{T}X), \gamma(\mathcal{S}(\mathcal{T}X))) \leq \eta[M(\mathcal{S}(X), \gamma(\mathcal{S}(X)))] \quad (18)$$

where $\gamma, \eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two functions such that $\gamma$ is continuous and $\eta$ is nondecreasing satisfying

$$\lim_{n \to \infty} \eta^n(x) = 0 \quad (x \geq 0). \quad (19)$$

Then, (9) holds.

**Proof.** Consider $\{D_n\}_{n=1}^{\infty}$ such that

$$D_1 = D \text{ and } D_{n+1} = \text{Conv}(TD_n) \quad (n \in \mathbb{N}). \quad (20)$$

Then, we see that

$$TD_{n+1} = TD_n \subseteq D_{n+1} \text{ and } D_2 = \text{Conv}(TD_1) \subseteq D = D_1. \quad (21)$$

Continuing in this way, we obtain

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots \supseteq D_n \supseteq D_{n+1} \supseteq \cdots. \quad (22)$$

If there exists $n_0 \in \mathbb{N}$ satisfying the condition $\mathcal{S}(D_{n_0}) = 0$, then the set $D_{n_0}$ is compact. By taking into account Theorem [21], we conclude that (9) holds. \footnote{\textcopyright 2023 Journal of Function Spaces}

We now assume $\mathcal{S}(D_n) > 0$ $(n \in \mathbb{N})$. Consequently, a sequence $\{\mathcal{S}(D_n)\}_{n=1}^{\infty}$ is decreasing and bounded below. Thus, $\{\mathcal{S}(D_n)\}_{n=1}^{\infty}$ is convergent and so

$$\lim_{n \to \infty} \mathcal{S}(D_n) = r \geq 0, \text{ say.} \quad (23)$$

With a view of (18), one writes

$$M(\mathcal{S}(D_{n+1}), \gamma(\mathcal{S}(D_{n+1}))) = M(\mathcal{S}(\text{Conv}TD_n), \gamma(\mathcal{S}(\text{Conv}TD_n))) = M(\mathcal{S}(TD_n), \gamma(\mathcal{S}(TD_n))) \leq \eta[M(\mathcal{S}(D_n), \gamma(\mathcal{S}(D_n)))] \leq \eta^2[M(\mathcal{S}(D_{n-1}), \gamma(\mathcal{S}(D_{n-1}))))] \cdots \leq \eta^n[M(\mathcal{S}(D), \gamma(\mathcal{S}(D)))] \quad (24)$$
Suppose that \( r > 0 \) (if possible). We obtain by letting \( n \to \infty \) together with (19) and (23) in the inequality (24) that

\[
\eta_n[M(\mathcal{G}(D), \gamma(\mathcal{G}(D))) \to 0, \quad (25)
\]

which yields

\[
M(r, \gamma(r)) = 0. \quad (26)
\]

We therefore have \( \gamma(r) = r = 0 \), so \( \lim_{n \to \infty} \mathcal{G}(D_n) = 0 \). With the help of (22), we obtain nonempty, convex, closed set \( D_\infty \subseteq D \) which is \( T \) invariant. Hence, by Schauder theorem [21], we reach to the desired result.

**Theorem 6.** Consider a Banach space \( \mathbb{E} \), a nonempty, closed, bounded convex set \( D \subseteq \mathbb{E} \), and an arbitrary MNC \( \mathcal{G} \). Also, consider a continuous mapping \( T : D \to D \) satisfies the inequality

\[
M(\mathcal{G}(\mathbb{E}X), \gamma(\mathcal{G}(\mathbb{E}X))) \leq kM(\mathcal{G}(\mathbb{E}X), \gamma(\mathcal{G}(\mathbb{E}X))) \quad (0 \leq k < 1, \mathbb{E}(\neq \emptyset) \subseteq D, M \in \mathbb{M}),
\]

(27)

where a function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous. Then, (9) holds.

**Proof.** This can be easily obtained by considering

\[
\eta(\tau) = k\tau \quad (0 \leq k < 1, \forall \tau \geq 0).
\]

in Theorem 5, above. \( \square \)

**Theorem 7.** Consider a Banach space \( \mathbb{E} \), a nonempty, closed, bounded convex set \( D \subseteq \mathbb{E} \), and an arbitrary MNC \( \mathcal{G} \). Also, consider a continuous mapping \( T : D \to D \) having the property

\[
\mathcal{G}(\mathbb{E}X) + \gamma(\mathcal{G}(\mathbb{E}X)) \leq k(\mathcal{G}(\mathbb{E}X) + \gamma(\mathcal{G}(\mathbb{E}X))) \quad (0 \leq k < 1, \mathbb{E}(\neq \emptyset) \subseteq D),
\]

(29)

where \( \gamma \) is a continuous function. Then, (9) holds.

**Proof.** By using the function \( M(x, y) = x + y \), the proof is obtained as an immediate consequence of Theorem 6. \( \square \)

### 3. Existence of Solutions for Integral Equation

We are studying the existence of solutions for an infinite system of the nonlinear integral equation which is considered as follows:

\[
\Omega_n(\xi) = \mathcal{F}_n\left( \xi, \Omega(\xi), \int_0^\xi G_n(\xi, s, \Omega(s))ds \right) \quad (n \in \mathbb{N}), \quad (30)
\]

where \( \Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty \) \( \xi \in I = [0, a], a > 0 \).

To discuss the result of this section, our assumptions are as below:

(1) For \( n \in \mathbb{N} \), the functions \( \mathcal{F}_n : I \times C(I, \ell_p^p) \times \mathbb{R} \to \mathbb{R} \) are continuous with

\[
\sum_{n=1}^\infty \alpha_n^p \left| \mathcal{F}_n(\xi, \Omega, 0) \right|^p \to 0 \quad (\forall \xi \in I), \quad (31)
\]

where

\[
\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty \in C(I, \ell_p^p) \quad \text{and} \quad \Omega_n(\xi) = 0 \quad (\forall n \in \mathbb{N}, t \in I). \quad (32)
\]

Moreover, these exist continuous functions \( A_n, B_n : I \to \mathbb{R}_+ \) such that the inequality

\[
|\mathcal{F}_n(\xi, \Omega, p) - \mathcal{F}_n(\xi, \bar{\Omega}, p)|^p 
\leq A_n(\xi)|\Omega_n(\xi) - \bar{\Omega}_n(\xi)|^p + B_n(\xi)|p - \bar{p}|^p
\]

holds, where \( \bar{\Omega}(\xi) = (\bar{\Omega}_n(\xi))_{n=1}^\infty \in C(I, \ell_p^p) \).

(2) For \( n \in \mathbb{N} \), the functions \( G_n : I \times I \times C(I, \ell_p^p) \to \mathbb{R} \) are continuous. Also, there exists \( L_k \) satisfying

\[
L_k = \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^k \left| \alpha_n^p B_n(\xi) \int_0^\xi G_n(\xi, s, \Omega(s))ds \right|^p : \xi \in I \right\}. \quad (34)
\]

Further,

\[
\sup_{n \in \mathbb{N}} L_n = L \quad \text{and} \quad \lim_{n \to \infty} L_k = 0 \quad (35)
\]

(3) Define an operator \( H \) on \( I \times C(I, \ell_p^p) \) to \( C(I, \ell_p^p) \) as follows

\[
\langle \xi, \Omega(\xi) \rangle \to \langle H\Omega(\xi) \rangle
\]

\[
= \left( \mathcal{F}_n \left( \xi, \Omega(\xi), \int_0^\xi G_n(\xi, s, \Omega(s))ds \right) \right)_{n=1}^\infty. \quad (36)
\]

(4) Let

\[
\sup \{ A_n(\xi) : \xi \in I, n \in \mathbb{N} \} = \hat{A}, \quad (37)
\]

such that \( 0 < \frac{1}{2\hat{A}^{1/p}} < 1 \) and

\[
\bar{B} = \sup \left\{ \sum_{n=1}^\infty \alpha_n^p B_n(\xi) : \xi \in I \right\}. \quad (38)
\]
**Theorem 8.** Under assumptions (1)–(4), the system
\[
\Omega_n(\xi) = F_n\left(\xi, \Omega(\xi), \int_0^\xi G_n(\xi, s, \Omega(s))ds\right)
\]
(39)
has at least one solution in \(C(I, \ell^p_p)\), where
\[
\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty, \quad \xi \in I = [0, a], \quad a > 0.
\]
(40)

*Proof.* For arbitrary fixed \(\xi \in I\),
\[
\|\Omega(\xi)\|_{\ell^p_p}^p = \sum_{n=1}^\infty a_n^p \left\|F_n\left(\xi, \Omega(\xi), \int_0^\xi G_n(\xi, s, \Omega(s))ds\right)\right\|^p
\]
(41)
which yields
\[
\|\Omega(\xi)\|^p_{\ell^p_p} \leq \frac{2pL}{1 - 2pA} = r^p, \text{ say.}
\]
(42)
It follows from (42) that
\[
\|\Omega(\xi)\|_{\ell^p_p} \leq r,
\]
(43)
and hence,
\[
\|\Omega\|_{C(I, \ell^p_p)} \leq r.
\]
(44)

Let us define nonempty set
\[
B = \{\Omega(\xi) \in C(I, \ell^p_p) : \|\Omega\|_{C(I, \ell^p_p)} \leq r, \xi \in I\},
\]
(45)
which is closed, bounded, and convex subset of \(C(I, \ell^p_p)\). By assumption (3) and for arbitrary fixed \(\xi \in I\), we write
\[
(H\Omega)(\xi) = \left\{ (H_n \Omega)(\xi) \right\}_{n=1}^\infty
\]
(46)
Also,
\[
\sum_{n=1}^\infty a_n^p \|H_n \Omega(\xi)\|^p
\]
(47)
Hence, \((H\Omega)(\xi) \in \ell^p_p\).

Since
\[
\|\Omega - \Omega\|_{C(I, \ell^p_p)} < \frac{\epsilon}{(2A)^{1/p}} = \delta.
\]
(48)
we have that \(H\) maps \(B\) into \(B\). We are now claiming that \(H\) is continuous on \(B\). For this, suppose \(\epsilon > 0\) and \(\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty, \bar{\Omega}(\xi) = (\bar{\Omega}_n(\xi))_{n=1}^\infty \in B\) satisfying
\[
\|\Omega - \bar{\Omega}\|_{C(I, \ell^p_p)} < \frac{\epsilon}{(2A)^{1/p}} = \delta.
\]
(49)
For arbitrary fixed \(\xi \in I\),
\[
\|H_n \Omega(\xi) - H_n \bar{\Omega}(\xi)\|^p
\]
(50)
and so,
\[
\int_0^\xi |G_n(\xi, s, \Omega(s)) - G_n(\xi, s, \bar{\Omega}(s))|ds < \frac{\epsilon}{2^{1/p} (B + 1)^{1/p} a},
\]
(51)
and so,
\[
\int_0^\xi |G_n(\xi, s, \Omega(s)) - G_n(\xi, s, \bar{\Omega}(s))|ds < \frac{\epsilon}{2^{1/p} (B + 1)^{1/p} a},
\]
(52)
It follows from (50) and (52) that
\[\sum_{n=1}^{\infty} a_n^p \left| (H_n \omega)(\xi) - (H_n \bar{\omega})(\xi) \right|^p \leq A \sum_{n=1}^{\infty} a_n^p \left| \Omega_n(\xi) - \bar{\Omega}_n(\xi) \right|^p + \frac{e^p}{2(B + 1)} \sum_{n=1}^{\infty} a_n^p B_n(\xi) < \frac{A e^p}{\lambda} + \frac{e^p B}{2(B + 1)} < e^p,\]
which gives
\[\| (H \omega)(\xi) - (H \bar{\omega})(\xi) \|_{L_p^p}^p < e^p. \]
Therefore,
\[\| H \omega - H \bar{\omega} \|_{C(I, L_p^p)} < \epsilon \quad \text{whenever} \quad \| \omega - \bar{\omega} \|_{C(I, L_p^p)} < \delta. \quad (55)\]

Hence, \( H \) is continuous on \( B \).

Now, for arbitrary fixed \( \xi \in I \) and \( B(\Omega) = \{ \Omega(\xi) : \Omega(\xi) \in B \} \), we write
\[X_{\epsilon^p}(H(B(\xi))) = \lim_{n \to \infty} \sup_{\Omega \in B(\xi)} \left( \sum_{k=0}^{\infty} a_k^p \Omega_k(\xi) \right)^{1/p} \]
\[= \lim_{n \to \infty} \sup_{\Omega \in B(\xi)} \left[ \sum_{k=0}^{\infty} a_k^p \left| F_n(\xi, \Omega(\xi), s, \Omega(s)) ds \right|^p \right]^{1/p} \leq \lim_{n \to \infty} \sup_{\Omega \in B(\xi)} \left[ \sum_{k=0}^{\infty} a_k^p \left| A_k(\Omega_k(\xi) + L_k) \right|^p \right]^{1/p}, \]
for \( \Omega(\xi) \in B(\xi) \), \( \Omega \in B(\xi) \).

Therefore,
\[X_{\epsilon^p}(H(B(\xi))) \leq 2A^{1/p} X_{\epsilon^p}(B(\xi)). \quad (57)\]

Operating \( \sup_{\xi \in I} \) on both sides of (57), we obtain
\[\sup_{\xi \in I} X_{\epsilon^p}(H(B(\xi))) \leq 2A^{1/p} \sup_{\xi \in I} X_{\epsilon^p}(B(\xi)). \quad (58)\]

We thus have
\[X_{\epsilon^p}(H(B)) \leq 2A^{1/p} X_{\epsilon^p}(B). \quad (59)\]

As \( 0 < 2A^{1/p} < 1 \), so applying Theorem 7 for \( \gamma = 0 \) gives that \( H \) has at least one fixed point on \( B \subseteq C(I, L_p^p) \), i.e., the considered system admits a solution in \( C(I, L_p^p) \).

**Example 1.** In order to demonstrate Theorem 8, we consider an infinite system of integral equation as follows:
\[\Omega_n(\xi) = \frac{\xi^2 \Omega_n(\xi)}{6(1 + \xi)^n} + \frac{1}{n^2} \int_0^\xi \cos \left( \Omega_n(s) \right) ds, \quad (60)\]
for \( \xi \in [0, 1] = I \) and \( n \in \mathbb{N} \). For this demonstration, write
\[F_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) = \frac{\xi^2 \Omega_n(\xi)}{6(1 + \xi)^n} + \frac{p_n(\Omega(\xi))}{n^3}, \]
\[p_n(\Omega(\xi)) = \frac{\xi^4}{5 + \sin \left( \sum_{j=1}^{\infty} \Omega_j(s) \right)} ds, \quad (61)\]
\[G_n(\xi, s, \Omega(\xi)) = \frac{\cos \left( \Omega_n(s) \right)}{5 + \sin \left( \sum_{j=1}^{\infty} \Omega_j(s) \right)}. \]
Further, take \( a = 1 \) and let \( \alpha_n = 1/n, n \in \mathbb{N} \). If \( \Omega(\xi) \in L_p^p \) for some fixed \( \xi \in I \), then
\[\sum_{n=1}^{\infty} a_n^p |F_n(\xi, \Omega(\xi), p_n(\Omega(\xi)))| < \infty, \quad (62)\]
as the series
\[\sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \text{converges for } p \geq 1. \quad (63)\]

Therefore, for arbitrary fixed \( \xi \in I \), one has
\[\{ F_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) \}_{n=1}^{\infty} \in L_p^p, \quad (64)\]
and hence,
\[\{ F_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) \}_{n=1}^{\infty} \in C\left( I, L_p^p \right). \quad (65)\]
Let \( \Omega(\xi) = (\Omega_n(\xi))_{n=1}^{\infty} \in C(I, \ell_p^\alpha) \). Then,
\[
\|F_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) - F_n(\xi, \Omega(\xi), p_n(\Omega(\xi)))\|_p \leq \frac{1}{3p} |\Omega_n(\xi) - \Omega_n(\xi)|^p + 2^{p}|p_n(\Omega(\xi)) - p_n(\Omega(\xi))|^p.
\]
(66)

Here
\[
A_n(\xi) = \frac{1}{3p} \quad \text{and} \quad B_n(\xi) = 2^p.
\]
(67)

Also,
\[
\hat{A} = \frac{1}{3p} \Rightarrow 2^{\frac{1}{1/p}} = \frac{2}{3} < 1,
\]
(68)

\[
\sum_{n=1}^{\infty} \alpha_n^p |F_n(\xi, \Omega(\xi), 0)|^p \rightarrow 0 \quad (\forall \xi \in I).
\]
Again,
\[
\sum_{n=k}^{\infty} \alpha_n^p B_n(\xi) |p_n(\Omega(\xi))|^p \leq 2^p \sum_{n=k}^{\infty} \frac{1}{n^p},
\]
(69)
\[
L_k \leq \sup \left\{ 2^p \sum_{n=k}^{\infty} \frac{1}{n^p} : \xi \in I \right\}.
\]

We can find that
\[
L_k \rightarrow 0 (k \rightarrow \infty) \quad \text{and} \quad L = 2^p \sum_{n=1}^{\infty} \frac{1}{n^p}.
\]
(70)

Moreover, we have
\[
\sum_{n \geq k} \alpha_n^p B_n(\xi) = 2^p \sum_{n=k}^{\infty} \frac{1}{n^p},
\]
(71)
\[
\tilde{B} = \sup \left\{ 2^p \sum_{n=1}^{\infty} \frac{1}{n^p} : \xi \in I \right\} = 2^p \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.
\]

The functions \( F_n \) and \( G_n \) are continuous for all \( n \in \mathbb{N} \) as well as the conditions (1)–(4) are fulfilled so with a view of Theorem 8, we reach to our conclusion that the considered system (60) admits a solution in \( C(I, \ell_p^\alpha) \).

4. Concluding Remarks

In this work, we linked three different disciplines such as the concept of measure of noncompactness (MNC), the theory of existence of solutions for functional equations, and the Banach space theory, particularly, in tempered sequence spaces. We first discussed some generalized Dorbo-type fixed point theorems by considering the arbitrary MNC and then discussed the existence of solutions for nonlinear integral equation (infinite system) by taking aforesaid newly investigated Dorbo-type theorem in tempered sequence spaces.

Finally, we constructed an illustrative example by taking an integral equation to validate our result.

It is worth noting to the reader that one can obtain the results of Section 2 by taking into account another suitable function instead of \( M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and consider two dimensional integral (or fraction integral) equation to extend the results of Section 3.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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