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Research Article

Global Existence and Blow-Up of Solutions to a Parabolic Nonlocal Equation Arising in a Theory of Thermal Explosion

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Focusing on the physical context of the thermal explosion model, this paper investigates a semilinear parabolic equation $\begin{cases} u_t = \Delta u + a\int_\Omega u^p dx, & (x,t) \in Q_T, \\ n \cdot \nabla u + g(u)u = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$ with nonlocal sources under nonlinear heat-loss boundary conditions, where $a, p > 0$ is constant, $Q_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$, and Ω is a bounded region in $R^N, N \geq 1$ with a smooth boundary $\partial\Omega$. First, we prove a comparison principle for some kinds of semilinear parabolic equations under nonlinear boundary conditions; using it, we show a new theorem of subsupersolutions. Secondly, based on the new method of subsupersolutions, the existence of global solutions and blow-up solutions is presented for different values of p. Finally, the blow-up rate for solutions is estimated also.

1. Introduction

This paper studies the following semilinear parabolic equations under nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u + a\int_\Omega u^p dx, & (x,t) \in Q_T, \\ n \cdot \nabla u + g(u)u = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where $a, p > 0$ is constant, $Q_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$, and Ω is a bounded region in $R^N, N \geq 1$ with a smooth boundary $\partial\Omega$, n is outward unit normal vector of S_T, initial value $u_0(x)$ is nonnegative continuous function, satisfying assumption (H1) (see below), and $|\Omega|$ denotes Lebusgue measure of Ω.

This equation can be used to describe thermal explosion or spontaneous combustion problems (see [1–3]). It differs from the classical Dirichlet boundary conditions discussed in most of the literature (see [3–9]). For examples, in [5, 7], the authors considered the following equation:

$$u_t = f(u)\left(\Delta u + a\int_\Omega u dx\right), (x,t) \in Q_T, \quad (2)$$

under Dirichlet boundary conditions, where a is positive constant. And they proved the existence of global solution and showed that all the blow-up solutions are blow up globally if f satisfies $\int_0^\infty 1/f(s)ds = \infty$. Furthermore, authors gave the blow-up rate in special cases as follows:

$$c_1(T^* - t)^{-1/p} \leq \max_{x\in\bar{\Omega}} u(x,t) \leq C_1(T^* - t)^{-1/p}, \quad (3)$$

where c_1, C_1 are positive constants and $f(u) = u^p, 0 < p < 1$. In [8], Li and Xie studied global existence of the following equation:

$$u_t - \Delta u^m = au^p\int_\Omega u^q dx, (x,t) \in Q_T, \quad (4)$$

with Dirichlet boundary conditions, where $a > 0, m > 1, p, q \geq 0$. They obtained that there exists a global positive classical solution if $p + q \leq m$ and when $p + q > m$, and the solution blows up in finite time if the initial value u_0 is sufficiently large. Then, the blow-up rate was given as follows:

$$C_1(T^* - t)^{-1/(p+q+1)} \leq \max_{x\in\bar{\Omega}} u(x,t) \leq C_2(T^* - t)^{-1/(p+q+1)}, \tag{5}$$

where C_1, C_2 are positive constants and T^* is the blow-up time of $u(x,t)$.

In [10], the authors investigated the parabolic superquadratic diffusive Hamilton-Jacobi equations as follows:

$$u_t - \Delta u = |\nabla u|^p, \text{in } \Omega \times (0,\infty), \tag{6}$$

with Dirichlet boundary condition, where $p > 2$. They studied the gradient blow-up (GBU) solutions which are defined as

$$T < \infty \Rightarrow \lim_{t\longrightarrow T^-} \|\nabla u\|_\infty = \infty, \tag{7}$$

where T is the existence time of the unique maximal classical solution. And it was showed that in the singular region, the normal derivatives u_ν and $u_{\nu\nu}$, which satisfy $u_{\nu\nu} \sim -|u_\nu|^p$, play a dominant role.

Moreover, some Fujita type results for parabolic inequalities are also studied. In [11], authors studied the quasilinear parabolic inequalities with weights and showed the existence of Fujita type exponents. And in [12], it investigated the nonexistence of nonnegative solutions of a class of quasilinear parabolic inequalities featuring nonlocal terms.

There are also some interesting results on the behaviour and stability for perturbed nonlinear impulsive differential systems (see [13–19]). And the stability of stochastic differential equations with impluses is studied in [20, 21].

In this paper, we will show the existence of global solution and the blow-up property of problem (1).

Now some assumptions are listed below.

(H1) $u_0 \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})(0 < \alpha < 1), u_0(x) \geq 0, \partial u_0/\partial n < 0$

(H2) $g > 0$ and satisfies the local Lipschitz condition

In our paper, we use the method of subsupersolutions (see [22–25]). Since the there exist nonlinear boundary conditions and nonlocal term, we list the definitions of super- and subsolutions for our problem as follows.

Definition 1. $\bar{u}(x,t) \in C^{2,1}(\bar{Q}_T)$ is called a a supersolution to equation (1) if it satisfies that

$$\begin{cases} \bar{u}_t \geq \Delta\bar{u} + a\int_\Omega \bar{u}^p dx, & (x,t) \in Q_T, \\ n\cdot\nabla\bar{u} + g(\bar{u})\bar{u} \geq 0, & (x,t) \in S_T, \\ \bar{u}(x,0) \geq u_0(x), & x \in \Omega. \end{cases} \tag{8}$$

$\underline{u}(x,t) \in C^{2,1}(\bar{Q}_T)$ is called a subsolution to equation (1) if it satisfies that

$$\begin{cases} \underline{u}_t \leq \Delta\underline{u} + a\int_\Omega \underline{u}^p dx, & (x,t) \in Q_T, \\ n\cdot\nabla\underline{u} + g(\underline{u})\underline{u} \leq 0, & (x,t) \in S_T, \\ \underline{u}(x,0) \leq u_0(x), & x \in \Omega. \end{cases} \tag{9}$$

Blow-up and global existence solutions are defined as follows.

Definition 2. The solution u of the problem (1) blows up in finite time if there exists a positive real number $T^* < \infty$, such that

$$\lim_{t\longrightarrow T^{*-}} \sup_{x\in\bar{\Omega}} |u(x,t)| = +\infty. \tag{10}$$

And the solution u of the problem (1) exists globally if for any $t \in (0,+\infty)$,

$$\sup_{x\in\bar{\Omega}} |u(x,t)| < +\infty. \tag{11}$$

Theorem 3 states the problem of local existence of the solution to equation (1) and is the main conclusion of this paper.

Theorem 3. *Suppose $\underline{u}, \bar{u} \geq 0$ are the sub- and supersolutions to equation (1), respectively, and $\underline{u} \leq \bar{u}$. If $u_0(x)$ satisfies assumption (H1) and the function g satisfies assumption (H2), then there exists $\hat{u} \in [\underline{u}, \bar{u}] \cap W_p^{2,1}(Q_T)$, which is the solution to equation (1).*

The following two theorems show that whether the solution to equation (1) exists globally or blows up in finite time is related to constant p.

Theorem 4. *Suppose assumptions (H1) and (H2) hold, and the equation (1) satisfies one of the following conditions.*

(i) $0 < p \leq 1$

(ii) $p > 1$, and the initial value $u_0(x)$ is sufficiently small

Then, the solution $\hat{u}$ of this equation exists globally.

Theorem 5. *Assume (H1) and (H2). If $p > 1$ and the initial value $u_0(x)$ is sufficiently large, then the solution $\hat{u}$ to equation (1) blows up in finite time.*

And the blow-up rate of the equation is given by Theorem 6.

Theorem 6. *Assume (H1)–(H3) (see below). Then, there exists a solution $u(x,t)$ blowing up at $T^* < \infty$. Specifically, there exist constants C_1, C_2 such that*

$$C_1(T^* - t)^{-1/(p-1)} \leq u(x,t) \leq C_2(T^* - t)^{-1/(p-1)}. \tag{12}$$

Remark 7. See Definition 2 for the description of global existence and blow-up solutions.

This paper is organized as follows. In Section 2, the local existence theory of solutions to equation (1) is established and Theorem 3 is proved. In Section 3, the conditions for the global existence of the solution are discussed and Theorem 4 is proved. In Section 4, the conclusions related to the blow-up solution are obtained and Theorem 5 is proved. In Section 5, the blow-up rate of the blow-up solution to equation (1) is further discussed and Theorem 6 is proved.

2. Proof of Theorem 3

In this section, the local existence of the solution to equation (1) is proved by using the fixed-point theorem and monotone iterative technique (see [26–29]).

First, the following lemma is present, which is proved according to [2].

Lemma 8. *Suppose that assumptions (H1) and (H2) hold. Let* $w(x,t)\in C^{2+\alpha}(Q_T)\cap C(\bar{Q}_T)$ *and satisfy*

$$\begin{cases} w_t - d\Delta w \ge c_1 w + c_3\int_\Omega c_2 w dx, & (x,t)\in Q_T,\\ n\cdot\nabla w + g(w)w \ge 0, & (x,t)\in S_T,\\ w(x,0)\ge 0, & x\in\Omega, \end{cases} \tag{13}$$

where $c_i(x,t), i=1,2,3$ *are continuous and bounded functions in* Q_T, $c_2, c_3\ge 0, d(x,t)\ge 0, (x,t)\in Q_T$. *Then,* $w(x,t)\ge 0, (x,t)\in Q_T$.

Proof 1. Let $\bar{c}_i=\sup_{Q_T} c_i(x,t), i=1,2,3$ and $v=e^{-\lambda t}w$, where $\lambda=\bar{c}_1+\bar{c}_2\bar{c}_3|\Omega|+1$. Then, the first equation in equation (13) can be deduced to

$$v_t+\lambda v-d\Delta v\ge c_1 v + c_3\int_\Omega c_2 v dx. \tag{14}$$

Hence,

$$v_t - d\Delta v + (\lambda - c_1)v \ge c_3\int_\Omega c_2 v dx. \tag{15}$$

□

Assume by contradiction that $v<0$ at some points $(x,t)\in Q_T$, so there must be a negative minimum value of v due to continuity, denoted as $v_0=v(x_0,t_0)$. The following two cases are discussed.

(i) If $x_0\in\partial\Omega$, then

$$n\cdot\nabla v_0 + g(v_0)v_0\ge 0. \tag{16}$$

At this point, we have $n\cdot\nabla v_0\ge -g(v_0)v_0>0$, which is contradictory to $n\cdot\nabla v_0<0$.

(ii) If $x_0\in\Omega^\circ$, consider the values of each function at (x_0,t_0). Then,

$$(\lambda-c_1)v_0\ge -d\Delta v_0+(\lambda-c_1)v_0\ge c_3\int_\Omega c_2 v_0 dx\ge c_3\bar{c}_2|\Omega|v_0. \tag{17}$$

It yields $\lambda-c_1\le c_3\bar{c}_2|\Omega|$, contradicting $\lambda\ge c_1+c_3\bar{c}_2|\Omega|$.

Combining (i) and (ii), there is no negative minimum value of v; thus, v is nonnegative. So $v=e^{-\lambda t}w\ge 0$, i.e., $w\ge 0$. Lemma 8 is proved.

Suppose that the assumptions of Theorem 3 hold. Consider the following auxiliary problem

$$\begin{cases} v_t-\Delta v+v=u+a\int_\Omega u^p dx, & (x,t)\in Q_T,\\ n\cdot\nabla v+Kv=G_k(u), & (x,t)\in S_T,\\ v(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{18}$$

where K and $G_k(u)$ satisfy the following rule. Let $G(u)=-g(u)u$. We have that $G(u)$ is Lipschitz continuous on the interval $[\underline{u},\bar{u}]$, which implies that for any $u_1\ge u_2$ given, there exists a fixed positive real number K such that

$$|G(u_1)-G(u_2)|\le K||u_1-u_2||. \tag{19}$$

Thus,

$$G(u_1)-G(u_2)\ge -K(u_1-u_2). \tag{20}$$

Let $G_k(u)=G(u)+Ku$. Then, the function $G_k(u)$ is increasing under this definition.

The auxiliary problem (12) is a third boundary value problem. It is clear that there exists a unique solution v to it, due to Theorem 3.4.7 in [9]. Define the nonlinear operator $T:[\underline{u},\bar{u}]\mapsto[\underline{u},\bar{u}]$ such that $v=Tu$ and construct the following sequences

$$\begin{aligned} &u_1=T\bar{u}, \quad u_2=Tu_1, \quad \cdots, \quad u_n=Tu_{n-1}, \quad \cdots,\\ &v_1=T\underline{u}, \quad v_2=Tv_1, \quad \cdots, \quad v_n=Tv_{n-1}, \quad \cdots. \end{aligned} \tag{21}$$

It can be proved that operator T is increasing. The proof is as follows. For any $y_1,y_2\in[\underline{u},\bar{u}]$, $\underline{u}\le y_1\le y_2\le\bar{u}$, let $z_1=Ty_1, z_2=Ty_2, w=z_2-z_1$. And

$$\begin{cases} w_t-\Delta w+w=a\int_\Omega (y_2^p-y_1^p)dx+(y_2-y_1)\ge 0, & (x,t)\in Q_T,\\ n\cdot\nabla w+Kw=G_k(y_2)-G_k(y_1)\ge 0, & (x,t)\in S_T,\\ w(x,0)=0\ge 0, & x\in\Omega. \end{cases} \tag{22}$$

Applying Lemma 8, where $c_1 = -1, c_2 = c_3 = 0, d = 1$, we have $w = z_2 - z_1 \geq 0$, i.e., $z_2 \geq z_1$. Letting $w = v_1 - \underline{u}$, the above equation is transformed into

$$\begin{cases} w_t - \Delta w + w \geq a\int_\Omega (\underline{u}^p - \underline{u}^p)dx + (\underline{u} - \underline{u}) = 0, & (x,t) \in Q_T, \\ n \cdot \nabla w + Kw \geq G_k(\underline{u}) - G_k(\underline{u}) = 0, & (x,t) \in S_T, \\ w(x,0) = 0 \geq 0, & x \in \Omega, \end{cases} \tag{23}$$

from which we deduce to $v_1 \geq \underline{u}$. The same procedure may be easily adapted to obtain $\bar{u} \geq u_1$. Thus,

$$\underline{u} \leq v_1 = T\underline{u} \leq u_1 = T\bar{u} \leq \bar{u}. \tag{24}$$

By mathematical induction on n, the above sequence (21) exhibits the following comparative relationship

$$\underline{u} \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq u_n \leq u_{n-1} \leq \cdots \leq u_1 \leq \bar{u}, \tag{25}$$

which shows that the sequences $\{u_n\}, \{v_n\}$ are increasing and bounded. So limits

$$\hat{u} = \lim_{n \longrightarrow \infty} u_n \quad \hat{v} = \lim_{n \longrightarrow \infty} v_n \tag{26}$$

exist. And $\hat{u} = T\hat{u}, \hat{v} = T\hat{v}$. Considering the compactness of the nonlinear operator T and $|\Omega| < \infty$, we know that $\hat{u}$, $\hat{v} \in [\underline{u}, \bar{u}] \cap W_p^{2,1}(Q_T)$ is the solution to the auxiliary problem, so as to the problem (1). The local existence of the solution to equation (1), i.e., Theorem 3, is proved.

3. Proof of Theorem 4

In this section, the proof of the global results of solution to equation (1) is given.

Case 1. Combining assumptions (H1) and (H2) and Definition 1, $u(x,t) = 0$ satisfies

$$\begin{cases} u_t = 0 = \Delta u + a\int_\Omega u^p dx, & (x,t) \in Q_T, \\ n \cdot \nabla u + g(u)u = n\nabla u \leq 0, & (x,t) \in S_T, \\ u(x,0) = 0 \leq u_0(x), & x \in \Omega. \end{cases} \tag{27}$$

Therefore, $u(x,t) = 0$ is a subsolution to equation (1). According to Theorem 3, we need to determine a globally existing supersolution. Set φ as the unique solution of the ellipse problem

$$\begin{cases} -\Delta\varphi = 1, & x \in \Omega, \\ n \cdot \nabla\varphi = 0, & x \in \partial\Omega. \end{cases} \tag{28}$$

Let $\phi = M\varphi$ where $M > 0$ is a constant. Obviously, on the boundary, we have

$$n \cdot \nabla\phi + g(\phi)\phi = Mg(\phi)\varphi \geq 0. \tag{29}$$

And the initial value $\phi_0 = \phi \geq 0$ is

$$\begin{aligned} \phi_t - \Delta\phi - a\int_\Omega \phi^p dx &= -M\Delta\varphi - aM^p\int_\Omega \varphi^p dx \\ &= M - aM^p\int_\Omega \varphi^p dx. \end{aligned} \tag{30}$$

Let equation (30) ≥0. Then, ϕ is a supersolution to equation (1) and satisfies $\phi \geq 0$. So,

$$M^{1-p} \geq a\int_\Omega \varphi^p dx. \tag{31}$$

When p is fixed, $\int_\Omega \varphi^p dx$ is a constant. Set $\mu = \int_\Omega \varphi^p dx$.

(1) In case of $0 < p < 1$, equation (31) can be transformed into

$$M \geq a^{1/(1-p)}\mu^{1/(1-p)}. \tag{32}$$

At this time, let N is a sufficiently large constant such that $N\varphi \geq u_0$. Then, we take $M = a^{1/(1-p)}\mu^{1/(1-p)} + N$, which can guarantee that ϕ is a supersolution to equation (1) and the global existence of the solution u.

(2) In case of $p > 1$, equation (31) can be transformed into

$$M \leq a^{1/(1-p)}\mu^{1/(1-p)}. \tag{33}$$

To ensure that ϕ is still the supersolution to equation (1), it needs to satisfy

$$u_0(x) \leq \phi = M\varphi, x \in \Omega. \tag{34}$$

Without loss of generality, we can take $M = a^{1/(1-p)}\mu^{1/(1-p)}$ such that $u_0(x) \leq M\varphi = a^{1/(1-p)}\mu^{1/(1-p)}\varphi$, that is, when $u_0(x)$ is sufficiently small, the solution u to equation (1) exists globally.

Case 2. In case of $p = 1$, the form of equation (1) is as follows:

$$\begin{cases} u_t = \Delta u + a\int_\Omega u dx, & (x,t) \in Q_T, \\ n \cdot \nabla u + g(u)u = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{35}$$

Let $b > a|\Omega|$, $\delta > \|u_0\|_\infty$, and $z(t)$ be the solution to the following Cauchy problem

$$\begin{cases} \dfrac{dz}{dt} = bz, \\ z(0) = \delta, \end{cases} \quad (36)$$

where $t \in (0, T)$ and the solution is $z(t) = \delta e^{bt}$. Then, we have

$$\begin{cases} z_t = bz > a|\Omega|z = \Delta z + \int_\Omega z dx, & (x, t) \in Q_T, \\ n \cdot \nabla z + g(z)z = g(z)z \geq 0, & (x, t) \in S_T, \\ z(0) = \delta > u_0(x), & x \in \Omega. \end{cases} \quad (37)$$

This means that when $p = 1$, for any given a, $z(t)$ is a supersolution to equation (1), and $z(t)$ exists globally. Thus, the solution to equation (1) exists globally.

Combined with Cases 1 and 2, Theorem 4 is proved.

4. Proof of Theorem 5

The above theorem states that the classic solution of equation (1) exists globally when $0 < p \leq 1$. In this section, we will get the blow-up results of solutions to equation (1) when $p > 1$ and prove Theorem 5.

Given assumptions (H1) and (H2) and $u_0(x) > \max\{M\varphi, \delta_0\}$, where M, φ are defined in Section 3 and $\delta_0 > 0$ is a fixed constant, let ψ be the solution of the following eigenvalue problem

$$\begin{cases} \Delta\psi + \lambda\psi = 0, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \quad (38)$$

We normalize ψ, i.e., $\|\psi\|_\infty = 1$, and λ denotes the first eigenvalue of the problem. Let $h(t)$ be the solution to the Cauchy problem below

$$\begin{cases} \dfrac{dh}{dt} = -\lambda h(t) + ah^p(t)\int_\Omega \psi^p dx, \\ h(0) = \delta_0. \end{cases} \quad (39)$$

It can be seen that the solution $h(t)$ of this equation blows up in finite time T^* under the condition of $p > 1$. Let

$$\underline{v}(x, t) = h(t)\psi(x). \quad (40)$$

Equations below state that $\underline{v}$, as defined above, is a subsolution to problem (1).

$$\underline{v}_t(x, t) = h'(t)\psi(x) = -\lambda\psi h(t) + ah^p\psi\int_\Omega \psi^p dx, \quad (41)$$

$$\begin{aligned} \Delta\underline{v} + a\int_\Omega \underline{v}^p dx &= h\Delta\psi + ah^p\int_\Omega \psi^p dx \\ &= -\lambda\psi h + ah^p\int_\Omega \psi^p dx \\ &\geq -\lambda\psi h + ah^p\psi\int_\Omega \psi^p dx = \underline{v}_t. \end{aligned} \quad (42)$$

Consider the boundary and initial value conditions

$$\begin{cases} n \cdot \underline{v} + g(\underline{v})\underline{v} = 0, & (x, t) \in S_T, \\ \underline{v}(x, 0) = \delta_0\psi \leq \delta_0 \leq u_0(x), & (x, t) \in \Omega. \end{cases} \quad (43)$$

Hence, $\underline{v}$ is a subsolution to problem (1), when $p > 1$. According to equations (42) and (43), set $w = u - \underline{v}$. Considering mean value theorem, we have

$$\begin{cases} w_t - \Delta w \geq a\int_\Omega (u^p - \underline{v}^p)dx = a\int_\Omega p\xi^{p-1}w dx, & (x, t) \in Q_T, \\ n \cdot \nabla w + g(w)w = n \cdot \nabla u + g(u)u = 0, & (x, t) \in S_T, \\ w(x, 0) = u_0(x) - \underline{v}(x, 0) \geq u_0(x) - \delta_0 \geq 0, & x \in \Omega, \end{cases} \quad (44)$$

where ξ is a nonnegative function between $\underline{v}$ and u. Applying Lemma 8 with $d = 1, c_1 = 0, c_2 = \xi^{p-1}, c_3 = ap$, $w \geq 0$, i.e., $u \geq \underline{v}$ is obtained. Since $h(t)$ blows up in finite time, so does $\underline{v}$. Therefore, when $p > 1$, the solution u to equation (1) blows up in finite time, which means equation (1) has at least one solution that blows up in finite time, when $p > 1$ and $u_0(x)$ is sufficiently large. Theorem 5 is proved.

5. Proof of Theorem 6

In this section, we show the blow-up rate of the blow-up solution to equation (1) near its blow-up time.

Suppose that the solution u of equation (1) blows up in finite time T^* and the assumptions (H1) and (H2) hold. We need the following assumption on the boundary condition:

(H3) There exists a constant $\gamma > 0$ such that $\inf g(u) \geq \gamma$

Let function $U(t) = \sup_{x\in\bar{\Omega}}|u(x, t)|$, where $u(x, t)$ is a blow-up solution to equation (1). The following lemma is given according to [8, 30, 31].

Lemma 9. *Suppose equation (1) satisfies the assumptions (H1) and (H2), and there exists a positive real number C_1 such that*

$$U(t) \geq C_1(T^* - t)^{-1/(p-1)}, t \in (0, T^*). \quad (45)$$

The following provides an upper bound for the solution $u(x,t)$ to equation (1). Let $z(x,t)$ be the solution of the following auxiliary problem

$$\begin{cases} z_t = z^q\left(\Delta z + a\int_\Omega z^p dx\right), & (x,t)\in\Omega\times(0,T^{**}),\\ n\cdot\nabla z + \gamma z = 0, & (x,t)\in\partial\Omega\times(0,T^{**}),\\ z(x,0) = \max\{c_0, u_0\}, & x\in\Omega, \end{cases} \tag{46}$$

where u_0 is stated in Theorem 5, $q\in(0,1)$, and $c_0>0$ is a fixed constant. Obviously $z(x,0)\ge u_0$, according to [8] Theorem 3.1 (the equation discussed in it is a subsolution to equation (46)), $z(x,t)$ blows up in a finite time, denoted as T^{**}.

Let the function $J(x,t) = z_t - \delta z^{p+q}$, where $\delta>0$. We need to prove that $J\ge 0$. According to [8]. Since $p/(2p+q-1)+(p+q-1)/(2p+q-1)=1$, applying Young's inequality yields

$$\begin{aligned} z^{p+q-1}\left(\int_\Omega z^{2p+q-1}dx\right)^{p/(2p+q-1)} &\le \frac{p+q-1}{2p+q-1}\left(\theta z^{p+q-1}\right)^{(2p+q-1)/(p+q-1)}\\ &\quad + \frac{p}{2p+q-1}\left(\theta^{-(2p+q-1)/p}\right)\int_\Omega z^{2p+q-1}dx, \end{aligned} \tag{47}$$

where $\theta = ((p+q)/(2p+q-1))^{p/(2p+q-1)}|\Omega|^{p(p+q-1)/(2p+q-1)^2}$. Then, from Holder's inequality, we have

$$\int_\Omega z^p dx \le |\Omega|^{(p+q-1)/(2p+q-1)}\left(\int_\Omega z^{2p+q-1}dx\right)^{p/(2p+q-1)}. \tag{48}$$

Combining equations (47) and (48) yields

$$\begin{aligned} &J_t - z^q\Delta J - 2q\delta z^{p+q-1}J - apz^q\int_\Omega z^{p-1}Jdx\\ &= qz^{-1}J^2 + \delta(p+q)(p+q-1)z^{p+2q-2}\|\nabla z\|^2 + q\delta^2 z^{2p+2q-1}\\ &\quad + ap\delta z^q\int_\Omega z^{2p+q-1}dx - a\delta(p+q)z^{p+2q-1}\int_\Omega z^p dx\\ &\ge q\delta^2 z^{2p+2q-1} + ap\delta z^q\int_\Omega z^{2p+q-1}dx\\ &\quad - a\delta(p+q)z^{p+2q-1}\int_\Omega z^p dx\\ &\ge \delta\left(q\delta - a(p+q-1)\theta^{(2p+q-1)^2/[p(p+q-1)]}\right)z^{2p+2q-1}\\ &= q\delta(\delta-\delta_1)z^{2p+2q-1}\ge 0. \end{aligned} \tag{49}$$

The boundary condition leads to

$$\begin{aligned} n\cdot\nabla J + \gamma J &= n\cdot\nabla z_t + \gamma z_t - \delta\left(n\cdot\nabla z^{p+q} + \gamma z^{p+q}\right)\\ &= -\delta\left(n\cdot\nabla z^{p+q} + \gamma z^{p+q}\right)\\ &= -\delta\gamma z^{p+q}(1-p-q)\ge 0. \end{aligned} \tag{50}$$

And the initial value condition leads to

$$J(x,0) = z_t(x,0) + \delta z^{p+q}(x,0) = \delta z^{p+q}(x,0)\ge \delta c_0^{p+q}\ge 0. \tag{51}$$

For any $\varepsilon>0$, applying Lemma 8 on $\Omega\times(0,T^{**}-\varepsilon]$, where $d=z^q, c_1 = 2q\delta z^{p+q-1}, c_2 = z^{p-1}, c_3 = apz^q$, we have $J(x,t)\ge 0$. Considering the arbitrary of ε, $\lim_{\varepsilon\longrightarrow 0} J(x,t)\ge 0$, i.e., $z_t\ge\delta z^{p+q}\ge 0$ can be obtained. Then, there exists a constant $\tau\in(0,T^{**})$ such that $z\ge 1$, when $t\ge\tau$. So we have $z^{p+q}\ge z^p$, i.e., $z_t\ge\delta z^p$. Inegrating this equation over (t,T^{**}),

$$z(x,t)\le C_2(T^{**}-t)^{-1/(p-1)}, \tag{52}$$

where $C_2 = (\delta p)^{-1/(p-1)}$. Set $w = z-u$. According to equation (46), there is $z(x,0)\ge u(x,0)$, i.e., $w(x,0)\ge 0$. On the boundary, we have $n\nabla z + \gamma z = n\nabla u + g(u)u \ge n\nabla u + \gamma u$, i.e., $n\nabla w + \gamma w\ge 0$. Considering $z\ge 1$ and mean value theorem, we obtain

$$w_t + \Delta w \ge a\int_\Omega (z^p - u^p)dx = a\int_\Omega p\eta^{p-1}w dx, \tag{53}$$

where η is a nonnegative function between z and u, $(x,t)\in\Omega\times(\tau,T^{**})$. Applying Lemma 8 with $d=1, c_1=0, c_2=\eta^{p-1}$, $c_3 = ap$, $w\ge 0$ is obtained. So $z\ge u$ in (τ,T^{**}). Combined with Lemma 9, there exists solution u to equation (1) satisfying

$$C_1(T^*-t)^{-1/(p-1)}\le u(x,t)\le C_2(T^*-t)^{-1/(p-1)}, \tag{54}$$

where T^* is the blow-up time of solution $u(x,t)$. Theorem 6 is proved.

Data Availability

No data were used to support the study.

Conflicts of Interest

No conflicts of interest exist.

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References

[1] P. V. Gordon, E. Ko, and R. Shivaji, "Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion," *Nonlinear Analysis: Real World Applications*, vol. 15, pp. 51–57, 2014.

[2] C. V. Pao, "Blowing-up of solution for a nonlocal reaction-diffusion problem in combustion theory," *Journal of Mathematical Analysis and Applications*, vol. 166, no. 2, pp. 591–600, 1992.

[3] P. Souplet, "Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source," *Journal of Difference Equations*, vol. 153, no. 2, 406 pages, 1999.

[4] J. M. Chadam, A. Peirce, and H. Yin, "The blowup property of solutions to some diffusion equations with localized nonlinear reactions," *Journal of Mathematical Analysis and Applications*, vol. 169, no. 2, pp. 313–328, 1992.

[5] Y. Chen and H. Gao, "Asymptotic blow-up behavior for a nonlocal degenerate parabolic equation," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 852–863, 2007.

[6] Y. Chen and M. Wang, "A class of nonlocal and degenerate quasilinear parabolic system not in divergence form," *Nonlinear Analysis: Theory Methods & Applications*, vol. 71, no. 7-8, pp. 3530–3537, 2009.

[7] W. Deng, Y. Li, and C. Xie, "Existence and nonexistence of global solutions of some nonlocal degenerate parabolic equations," *Applied Mathematics Letters*, vol. 16, no. 5, pp. 803–808, 2003.

[8] F. Li and C. Xie, "Global existence and blow-up for a nonlinear porous medium equation," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 185–192, 2003.

[9] Q. Ye and Z. Li, *Introduction to Reaction Diffusion Equations*, Science Press, Beijing, 1990.

[10] R. Filippucci, P. Pucci, and P. Souplet, "A Liouville-type theorem in a half-space and its applications to the gradient blow-up behavior for superquadratic diffusive Hamilton-Jacobi equations," *Communications in Partial Differential Equations*, vol. 45, no. 4, pp. 321–349, 2020.

[11] R. Filippucci and S. Lombardi, "Fujita type results for parabolic inequalities with gradient terms," *Journal of Difference Equations*, vol. 268, no. 5, 1910 pages, 2020.

[12] R. Filippucci and M. Ghergu, "Fujita type results for quasilinear parabolic inequalities with nonlocal terms," 2021, http://arxiv.org/abs/2105.06130.

[13] F. Inkmann, "Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions," *Indiana University Mathematics Journal*, vol. 31, no. 2, pp. 213–221, 1982.

[14] X. Li, T. Caraballo, R. Rakkiyappan, and X. Han, "On the stability of impulsive functional differential equations with infinite delays," *Mathematicsl Methods in the Applied Sciences*, vol. 38, no. 14, pp. 3130–3140, 2015.

[15] X. Li, D. O'Regan, and H. Akca, "Global exponential stabilization of impulsive neural networks with unbounded continuously distributed delays," *IMA Journal of Applied Mathematics*, vol. 80, no. 1, pp. 85–99, 2015.

[16] X. Li, J. Shen, H. Akca, and R. Rakkiyappan, "LMI-based stability for singularly perturbed nonlinear impulsive differential systems with delays of small parameter," *Applied Mathematics and Computation*, vol. 250, pp. 798–804, 2015.

[17] X. Li, J. Shen, and R. Rakkiyappan, "Persistent impulsive effects on stability of functional differential equations with finite or infinite delay," *Applied Mathematics and Computation*, vol. 329, pp. 14–22, 2018.

[18] D. Yang, X. Li, and J. Qiu, "Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback," *Nonlinear Analysis: Hybrid Systems*, vol. 32, pp. 294–305, 2019.

[19] D. Yang, X. Li, J. Shen, and Z. Zhou, "State-dependent switching control of delayed switched systems with stable and unstable modes," *Mathematicsl Methods in the Applied Sciences*, vol. 41, no. 16, pp. 6968–6983, 2018.

[20] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[21] L. Shu, X. Shu, Q. Zhu et al., "Existence and exponential stability of mild solutions for second-order neutral stochastic functional differential equation with random impulses," *Journal of Applied Analysis & Computation*, vol. 11, no. 1, pp. 59–80, 2021.

[22] H. Cheng and R. Yuan, "Existence and asymptotic stability of traveling fronts for nonlocal monostable evolution equations," *Discrete and Continuous Dynamical Systems-B*, vol. 22, no. 7, pp. 3007–3022, 2017.

[23] H. Cheng and R. Yuan, "Existence and stability of traveling waves for Leslie-Gower predator-prey system with nonlocal diffusion," *Discrete and Continuous Dynamical Systems-A*, vol. 37, no. 10, pp. 5433–5454, 2017.

[24] H. Cheng and R. Yuan, "Traveling waves of some Holling-Tanner predator-prey system with nonlocal diffusion," *Applied Mathematics and Computation*, vol. 338, no. 1, pp. 12–24, 2018.

[25] H. Cheng and R. Yuan, "The stability of the equilibria of the Allen-Cahn equation with fractional diffusion," *Applicable Analysis*, vol. 98, no. 3, pp. 600–610, 2019.

[26] Y. Liu and D. O'Regan, "Controllability of impulsive functional differential systems with nonlocal conditions," *Electronic Journal of Differential Equations*, vol. 194, pp. 1–10, 2013.

[27] Y. Liu and H. Yu, "Bifurcation of positive solutions for a class of boundary value problems of fractional differential inclusions," *Abstract and Applied Analysis*, vol. 2013, Article ID 942831, 8 pages, 2013.

[28] Y. Liu, C. Kou, and R. Jin, "Positive solutions using bifurcation techniques for boundary value problems of fractional differential equations," *Abstract and Applied Analysis*, vol. 2013, no. 1, Article ID 162418, 79 pages, 2013.

[29] Y. Liu, "Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations," *Journal of Nonlinear Science and Application*, vol. 9, pp. 340–353, 2015.

[30] Z. Cui, Z. Yang, and R. Zhang, "Blow-up of solutions for nonlinear parabolic equation with nonlocal source and nonlocal boundary condition," *Applied Mathematics and Computation*, vol. 224, pp. 1–8, 2013.

[31] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, NJ, 1964.