1. Introduction

This paper studies the following semilinear parabolic equations under nonlinear boundary conditions

\[
\begin{align*}
\mathbf{u}_t &= \Delta \mathbf{u} + a \int_{\Omega} \mathbf{u}^p d\mathbf{x}, \\
\mathbf{n} \cdot \nabla \mathbf{u} + g(u)\mathbf{u} &= 0, \\
\mathbf{u}(\mathbf{x}, 0) &= u_0(\mathbf{x}),
\end{align*}
\]

(1)

where \(a, p > 0\) is constant, \(Q_T = \Omega \times (0, T], S_T = \partial \Omega \times (0, T]\), and \(\Omega\) is a bounded region in \(\mathbb{R}^N, N \geq 1\) with a smooth boundary \(\partial \Omega\). \(\mathbf{n}\) is outward unit normal vector of \(S_T\), initial value \(u_0(\mathbf{x})\) is nonnegative continuous function, satisfying assumption (H1) (see below), and \(|\Omega|\) denotes Lebesgue measure of \(\Omega\).

This equation can be used to describe thermal explosion or spontaneous combustion problems (see [1–3]). It differs from the classical Dirichlet boundary conditions discussed in most of the literature (see [3–9]). For examples, in [5, 7], the authors considered the following equation:

\[
\mathbf{u}_t = f(u) \left( \Delta \mathbf{u} + a \int_{\Omega} \mathbf{u} d\mathbf{x} \right), (\mathbf{x}, t) \in Q_T,
\]

(2)

under Dirichlet boundary conditions, where \(a\) is positive constant. And they proved the existence of global solution and showed that all the blow-up solutions are blow up globally if \(f\) satisfies \(\int_1^\infty 1/f(s) ds = \infty\). Furthermore, authors gave the blow-up rate in special cases as follows:

\[
c_1 (T^* - t)^{-1/p} \leq \max_{x \in \Omega} u(x, t) \leq C_1 (T^* - t)^{-1/p},
\]

(3)

where \(c_1, C_1\) are positive constants and \(f(u) = u^p, 0 < p < 1\). In [8], Li and Xie studied global existence of the following equation:

\[
\mathbf{u}_t - \Delta \mathbf{u}^m = a \mathbf{u} \int_{\Omega} \mathbf{u}^q d\mathbf{x}, (\mathbf{x}, t) \in Q_T.
\]

(4)
with Dirichlet boundary conditions, where \( a > 0, m > 1, p, q \geq 0 \). They obtained that there exists a global positive classical solution if \( p + q \leq m \) and when \( p + q > m \), and the solution blows up in finite time if the initial value \( u_0 \) is sufficiently large. Then, the blow-up rate was given as follows:

\[
C_1(T^* - t)^{-1/(p+q+1)} \leq \max_{x \in \Omega} u(x, t) \leq C_2(T^* - t)^{-1/(p+q+1)},
\]

where \( C_1, C_2 \) are positive constants and \( T^* \) is the blow-up time of \( u(x, t) \).

In [10], the authors investigated the parabolic superquadratic diffusive Hamilton-Jacobi equations as follows:

\[
u_t - \Delta u = |\nabla u|^p, \text{in } \Omega \times (0, \infty),
\]

with Dirichlet boundary condition, where \( p > 2 \). They studied the gradient blow-up (GBU) solutions which are defined as

\[
T < \infty \Rightarrow \lim_{t \to T} \|\nabla u\|_{\infty} = \infty,
\]

where \( T \) is the existence time of the unique maximal classical solution. And it was shown that in the singular region, the normal derivatives \( u_{\nu} \) and \( u_{\nu\nu} \), which satisfy \( u_{\nu\nu} \sim -|u_{\nu}|^p \), play a dominant role.

Moreover, some Fujita type results for parabolic inequalities are also studied. In [11], authors studied the quasilinear parabolic inequalities with weights and showed the existence of Fujita type exponents. And in [12], it investigated the nonexistence of nonnegative solutions of a class of quasilinear parabolic inequalities featuring nonlocal terms.

There are also some interesting results on the behaviour and stability for perturbed nonlinear impulsive differential systems (see [13–19]). And the stability of stochastic differential equations with impulses is studied in [20, 21].

In this paper, we will show the existence of global solution and the blow-up property of problem (1).

Now some assumptions are listed below.

(H1) \( u_0 \in C^{2,a}(\Omega) \cap C(\Omega)(0 < a < 1), u_0(x) \geq 0, \partial u_0/\partial n < 0 \)

(H2) \( g > 0 \) and satisfies the local Lipschitz condition.

In our paper, we use the method of supersolutions (see [22–25]). Since the there exist nonlinear boundary conditions and nonlocal term, we list the definitions of supersolutions and subsolutions for our problem as follows.

**Definition 1.** \( \bar{u}(x, t) \in C^{2,1}(\bar{Q}_T) \) is called a a supersolution to equation (1) if it satisfies that

\[
\begin{align*}
\bar{u}_t - \Delta \bar{u} + a &\int_{\Omega} \bar{u}^p dx, \quad (x, t) \in Q_T, \\
n \cdot \nabla \bar{u} + g(\bar{u}) &\bar{u} \geq 0, \quad (x, t) \in S_T, \\
\bar{u}(x, 0) &\geq u_0(x), \quad x \in \Omega.
\end{align*}
\]

\( u(x, t) \in C^{2,1}(\bar{Q}_T) \) is called a subsolution to equation (1) if it satisfies that

\[
\begin{align*}
\bar{u}_t - \Delta \bar{u} + a &\int_{\Omega} \bar{u}^p dx, \quad (x, t) \in Q_T, \\
n \cdot \nabla \bar{u} + g(\bar{u}) &\bar{u} \leq 0, \quad (x, t) \in S_T, \\
\bar{u}(x, 0) &\leq u_0(x), \quad x \in \Omega.
\end{align*}
\]

Blow-up and global existence solutions are defined as follows.

**Definition 2.** The solution \( u \) of the problem (1) blows up in finite time if there exists a positive real number \( T^* < \infty \), such that

\[
\lim_{t \to T^*} \sup_{x \in \Omega} |u(x, t)| = +\infty.
\]

And the solution \( u \) of the problem (1) exists globally if for any \( t \in (0, +\infty) \),

\[
\sup_{x \in \Omega} |u(x, t)| < +\infty.
\]

Theorem 3 states the problem of local existence of the solution to equation (1) and is the main conclusion of this paper.

**Theorem 3.** Suppose \( \bar{u}, \tilde{u} \geq 0 \) are the sub- and supersolutions to equation (1), respectively, and \( \bar{u} \leq \tilde{u} \). If \( u_0(x) \) satisfies assumption (H1) and the function \( g \) satisfies assumption (H2), then there exists \( \tilde{u} \in [\bar{u}, \tilde{u}] \cap W^{2,1}_p(Q_T) \), which is the solution to equation (1).

The following two theorems show that whether the solution to equation (1) exists globally or blows up in finite time is related to constant \( p \).

**Theorem 4.** Suppose assumptions (H1) and (H2) hold, and the equation (1) satisfies one of the following conditions.

(i) \( 0 < p \leq 1 \)

(ii) \( p > 1 \), and the initial value \( u_0(x) \) is sufficiently small.

Then, the solution \( \tilde{u} \) of this equation exists globally.

**Theorem 5.** Assume (H1) and (H2). If \( p > 1 \) and the initial value \( u_0(x) \) is sufficiently large, then the solution \( \tilde{u} \) to equation (1) blows up in finite time.

And the blow-up rate of the equation is given by Theorem 6.

**Theorem 6.** Assume (H1)–(H3) (see below). Then, there exists a solution \( u(x, t) \) blowing up at \( T^* < \infty \). Specifically, there exist constants \( C_1, C_2 \) such that

\[
C_1(T^* - t)^{-\frac{1}{2}} \leq u(x, t) \leq C_2(T^* - t)^{-\frac{1}{2}}.
\]
Remark 7. See Definition 2 for the description of global existence and blow-up solutions.

This paper is organized as follows. In Section 2, the local existence theory of solutions to equation (1) is established and Theorem 3 is proved. In Section 3, the conditions for the global existence of the solution are discussed and Theorem 4 is proved. In Section 4, the conclusions related to the blow-up solution are obtained and Theorem 5 is proved. In Section 5, the blow-up rate of the blow-up solution to equation (1) is further discussed and Theorem 6 is proved.

2. Proof of Theorem 3

In this section, the local existence of the solution to equation (1) is proved by using the fixed-point theorem and monotone iterative technique (see [26–29]).

First, the following lemma is present, which is proved according to [2].

Lemma 8. Suppose that assumptions (H1) and (H2) hold. Let $w(x, t) ∈ C^{2+α}(Ω_T) ∩ C(Q_T)$ and satisfy

$$\begin{cases}
    w_t - dΔw ≥ c_1 w + c_3 \int_{Ω} cz_2 dx, & (x, t) ∈ Q_T, \\
    n · ∇w + g(w)w ≥ 0, & (x, t) ∈ S_T, \\
    w(x, 0) ≥ 0, & x ∈ Ω,
\end{cases}
$$

where $c_i(x, t), i = 1, 2, 3$ are continuous and bounded functions in $Q_T$, $c_0 ≥ 0, d(x, t) ≥ 0, (x, t) ∈ Q_T$. Then, $w(x, t) ≥ 0, (x, t) ∈ Q_T$.

Proof. Let $c_i(\bar{c}_i, x, t)$, $i = 1, 2, 3$ and $v = e^{-λ}w$, where $λ = \bar{c}_1 + c_3 |Ω| + 1$. Then, the first equation in equation (13) can be deduced to

$$v_t + λv - dΔv ≥ c_1 v + c_3 \int_{Ω} cz_2 dx.$$

Hence,

$$v_t - dΔv + (λ - c_1)v ≥ c_3 \int_{Ω} cz_2 dx.$$

Assume by contradiction that $v < 0$ at some points $(x, t) ∈ Q_T$, so there must be a negative minimum value of $v$ due to continuity, denoted as $v_0 = v(x_0, t_0)$. The following two cases are discussed.

(i) If $x_0 ∈ ∂Ω$, then

$$n · ∇v_0 + g(v_0)v_0 ≥ 0.$$

At this point, we have $n · ∇v_0 ≥ -g(v_0)v_0 > 0$, which is contradictory to $n · ∇v_0 < 0$.

(ii) If $x_0 ∈ Ω^*$, consider the values of each function at $(x_0, t_0)$. Then,

$$(λ - c_1)v_0 ≥ -dΔv_0 + (λ - c_1)v_0 ≥ c_3 \int_{Ω} cz_2 dx ≥ c_3 \bar{c}_2 |Ω|v_0,$$

(17)

It yields $λ - c_1 ≤ c_3 \bar{c}_2 |Ω|$, contradicting $λ ≥ c_1 + c_3 |Ω|$. Combining (i) and (ii), there is no negative minimum value of $v$; thus, $v$ is nonnegative. So $v = e^{-λ}w ≥ 0$, i.e., $w ≥ 0$. Lemma 8 is proved.

Suppose that the assumptions of Theorem 3 hold. Consider the following auxiliary problem

$$\begin{cases}
    v_t - Δv + v = u + a \int_{Ω} u^p dx, & (x, t) ∈ Q_T, \\
    n · ∇v + Kv = G_k(u), & (x, t) ∈ S_T, \\
    v(x, 0) = u_0(x), & x ∈ Ω,
\end{cases}
$$

where $K$ and $G_k(u)$ satisfy the following rule. Let $G(u) = -g(u)u$. We have that $G(u)$ is Lipschitz continuous on the interval $[a, b]$, which implies that for any $u_1 ≥ u_2$ given, there exists a fixed positive real number $K$ such that

$$|G(u_1) - G(u_2)| ≤ K|u_1 - u_2|.$$

Thus,

$$G(u_1) - G(u_2) ≥ -K(u_1 - u_2).$$

Let $G_k(u) = G(u) + Ku$. Then, the function $G_k(u)$ is increasing under this definition.

The auxiliary problem (12) is a third boundary value problem. It is clear that there exists a unique solution $v$ to it, due to Theorem 3.4.7 in [9]. Define the nonlinear operator $T : [u, \bar{u}] → [u, \bar{u}]$ such that $v = Tu$ and construct the following sequences

$$\begin{align*}
    &u_1 = Tu, \quad u_2 = Tu_1, \quad \ldots, \quad u_n = Tu_{n-1}, \quad \ldots, \\
    &v_1 = Tu, \quad v_2 = Tv_1, \quad \ldots, \quad v_n = Tv_{n-1}, \quad \ldots.
\end{align*}
$$

(21)

It can be proved that operator $T$ is increasing. The proof is as follows. For any $y_1, y_2 ∈ [u, \bar{u}]$, $u ≤ y_1 ≤ y_2 ≤ \bar{u}$, let $\bar{z}_1 = Ty_1, z_2 = Ty_2, w = z_2 - z_1$. And

$$\begin{cases}
    w_t - Δw + w = a \int_{Ω} (y_2^p - y_1^p) dx + (y_2 - y_1) ≥ 0, & (x, t) ∈ Q_T, \\
    n · ∇w + Kw = G_k(y_2) - G_k(y_1) ≥ 0, & (x, t) ∈ S_T, \\
    w(x, 0) = 0 ≥ 0, & x ∈ Ω.
\end{cases}
$$

(22)
Applying Lemma 8, where \(c_1 = -1, c_2 = c_3 = 0, d = 1\), we have \(w = z_2 - z_1 \geq 0\), i.e., \(z_2 \geq z_1\). Letting \(w = v_1 - u\), the above equation is transformed into

\[
\begin{aligned}
w_t - \Delta w + w &\geq a \left( \int_\Omega \left( w^p - u^p \right) dx + (u - u) \right) = 0, & (x, t) &\in Q_T, \\
n \cdot \nabla w + Kw &\geq G_k(u) - G_k(u) = 0, & (x, t) &\in S_T, \\
\end{aligned}
\]

(23)

from which we deduce to \(v_1 \geq u\). The same procedure may be easily adapted to obtain \(\bar{u} \geq u_1\). Thus,

\[
u \leq v_1 = Tu \leq u_1 = T\bar{u} \leq \bar{u}.
\]

(24)

By mathematical induction on \(n\), the above sequence (21) exhibits the following comparative relationship

\[
u \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq u_n \leq u_{n-1} \leq \cdots \leq u_1 \leq \bar{u},
\]

(25)

which shows that the sequences \(\{u_n\}, \{v_n\}\) are increasing and bounded. So limits

\[
\bar{u} = \lim_{n \to \infty} u_n, \quad \bar{v} = \lim_{n \to \infty} v_n.
\]

(26)

exist. And \(\bar{u} = Tu, \bar{v} = T\bar{v}\). Considering the compactness of the nonlinear operator \(T\) and \(|\Omega| < \infty\), we know that \(\bar{u}, \bar{v} \in [u, \bar{u}] \cap W^{2,1}(Q_T)\) is the solution to the auxiliary problem, so as to the problem (1). The local existence of the solution to equation (1), i.e., Theorem 3, is proved.

### 3. Proof of Theorem 4

In this section, the proof of the global results of solution to equation (1) is given.

**Case 1.** Combining assumptions (H1) and (H2) and Definition 1, \(u(x, t) = 0\) satisfies

\[
\begin{aligned}
u_t = 0 = \Delta u + a \int_\Omega u^p dx, & \quad (x, t) \in Q_T, \\
n \cdot \nabla u + g(u)u &\leq n\nabla u \leq 0, & \quad (x, t) \in S_T, \\
u(x, 0) &\leq u_0(x), & \quad x \in \Omega.
\end{aligned}
\]

(27)

Therefore, \(u_t = 0\) is a subsolution to equation (1). According to Theorem 3, we need to determine a globally existing supersolution. Set \(\varphi\) as the unique solution of the ellipse problem

\[
\begin{aligned}
-\Delta \varphi = 1, & \quad x \in \Omega, \\
n \cdot \nabla \varphi = 0, & \quad x \in \partial\Omega.
\end{aligned}
\]

(28)

Let \(\varphi = M\varphi\) where \(M > 0\) is a constant. Obviously, on the boundary, we have

\[
\begin{aligned}
n \cdot \nabla \varphi + g(\varphi)\varphi = M g(\varphi) \varphi \geq 0.
\end{aligned}
\]

(29)

And the initial value \(\varphi_0 = \varphi \geq 0\) is

\[
\begin{aligned}
\phi_t = \Delta \phi - a \int_\Omega \varphi^p dx = -M\Delta \phi - aM^p \int_\Omega \varphi^p dx = M - aM^p \int_\Omega \varphi^p dx.
\end{aligned}
\]

(30)

Let equation (30) \(\geq 0\). Then, \(\varphi\) is a supersetion to equation (1) and satisfies \(\varphi \geq 0\). So,

\[
M^{1-p} \geq a \int_\Omega \varphi^p dx.
\]

(31)

When \(p\) is fixed, \(\int_\Omega \varphi^p dx\) is a constant. Set \(\mu = \int_\Omega \varphi^p dx\).

(1) In case of \(0 < p < 1\), equation (31) can be transformed into

\[
M \geq a^{1/(1-p)} \mu^{1/(1-p)}.
\]

(32)

At this time, let \(N\) is a sufficiently large constant such that \(N\varphi \geq u_0\). Then, we take \(M = a^{1/(1-p)} \mu^{1/(1-p)} + N\), which can guarantee that \(\varphi\) is a supersotution to equation (1) and the global existence of the solution \(u\).

(2) In case of \(p > 1\), equation (31) can be transformed into

\[
M \leq a^{1/(1-p)} \mu^{1/(1-p)}.
\]

(33)

To ensure that \(\varphi\) is still the supersolution to equation (1), it needs to satisfy

\[
u_0(x) \leq \varphi = M\varphi, x \in \Omega.
\]

(34)

Without loss of generality, we can take \(M = a^{1/(1-p)} \mu^{1/(1-p)}\) such that \(u_0(x) \leq M\varphi = a^{1/(1-p)} \mu^{1/(1-p)} \varphi\), that is, when \(u_0(x)\) is sufficiently small, the solution \(u\) to equation (1) exists globally.

**Case 2.** In case of \(p = 1\), the form of equation (1) is as follows:

\[
\begin{aligned}
u_t = \Delta u + a \int_\Omega u^p dx, & \quad (x, t) \in Q_T, \\
n \cdot \nabla u + g(u)u &\leq n\nabla u \leq 0, & \quad (x, t) \in S_T, \\
u(x, 0) = u_0(x), & \quad x \in \Omega.
\end{aligned}
\]

(35)
Let \( b > a|\Omega|, \delta > \|u_0\|_{C^0} \), and \( z(t) \) be the solution to the following Cauchy problem

\[
\begin{aligned}
\frac{dz}{dt} &= bz, \\
\kappa = \delta, \\
\end{aligned}
\]

(36)

where \( t \in (0, T) \) and the solution is \( t = e^{\delta t} \). Then, we have

\[ z_{t} = bz > \alpha|\Omega|z = \Delta z + \int_{\Omega} z \, dx, \quad (x, t) \in Q_T, \]

\[ n \cdot \nabla z + g(z)z = g(z)z \geq 0, \quad (x, t) \in S_T, \]

(37)

This means that when \( p = 1 \), for any given \( a, z(t) \) is a supersolution to equation (1), and \( z(t) \) exists globally. Thus, the solution to equation (1) exists globally.

Combined with Cases 1 and 2, Theorem 4 is proved.

4. Proof of Theorem 5

The above theorem states that the classic solution of equation (1) exists globally when \( 0 < p \leq 1 \). In this section, we will get the blow-up results of solutions to equation (1) when \( p > 1 \) and prove Theorem 5.

Given assumptions (H1) and (H2) and \( u_0(x) > \max \{ M_p, \delta_0 \} \), where \( M, \varphi \) are defined in Section 3 and \( \delta_0 > 0 \) is a fixed constant, let \( \psi \) be the solution of the following eigenvalue problem

\[ \begin{aligned}
\Delta \psi + \lambda \psi &= 0, \quad x \in \Omega, \\
\psi &= 0, \quad x \in \partial \Omega.
\end{aligned} \]

(38)

We normalize \( \psi \), i.e., \( \|\psi\|_{C^0} = 1 \), and \( \lambda \) denotes the first eigenvalue of the problem. Let \( h(t) \) be the solution to the Cauchy problem below

\[ \begin{aligned}
\frac{dh}{dt} &= -\lambda h(t) + ah^p(t), \quad x \in \Omega, \\
h(0) &= \delta_0.
\end{aligned} \]

(39)

It can be seen that the solution \( h(t) \) of this equation blows up in finite time \( T^* \) under the condition of \( p > 1 \). Let

\[ \psi(x, t) = h(t)\psi(x). \]

(40)

Equations below state that \( \psi \), as defined above, is a supersolution to problem (1).

\[ \begin{aligned}
\psi_{t}(x, t) &= h'(t)\psi(x) + \lambda \lambda h(t) + ah^p \int_{\Omega} \psi^p dx, \\
\end{aligned} \]

(41)

Consider the boundary and initial value conditions

\[ \begin{aligned}
n \cdot \psi + g(\psi)\psi &= 0, \quad (x, t) \in S_T, \\
\psi(x, 0) &= \delta_0 \psi \leq \delta_0 \leq u_0(x), \quad (x, t) \in \Omega.
\end{aligned} \]

(43)

Hence, \( \psi \) is a subsolution to problem (1), when \( p > 1 \). According to equations (42) and (43), set \( w = u - \psi \). Considering mean value theorem, we have

\[ \begin{aligned}
\frac{w_t - \Delta w}{\Delta w} &\geq \frac{a\int_{\Omega} (u^p - \psi^p)dx}{\int_{\Omega} \psi^p dx}, \quad (x, t) \in Q_T, \\
\end{aligned} \]

\[ \begin{aligned}
n \cdot \nabla w + g(w)w &= n \cdot \nabla u + g(u)u = 0, \quad (x, t) \in S_T, \\
w(x, 0) &= u_0(x) \geq \psi(x, 0) \geq u_0(x) - \delta_0 \geq 0, \quad x \in \Omega,
\end{aligned} \]

(44)

where \( \xi \) is a nonnegative function between \( \psi \) and \( u \). Applying Lemma 8 with \( d = 1, c_1 = 0, c_1 = \xi^{p-1}, c_2 = ap, w \geq 0 \), i.e., \( u \geq \psi \) is obtained. Since \( h(t) \) blows up in finite time, so does \( \psi \). Therefore, when \( p > 1 \), the solution \( u \) to equation (1) blows up in finite time, which means equation (1) has at least one solution that blows up in finite time, when \( p > 1 \) and \( u_0(x) \) is sufficiently large. Theorem 5 is proved.

5. Proof of Theorem 6

In this section, we show the blow-up rate of the blow-up solution to equation (1) near its blow-up time.

Suppose that the solution \( u \) of equation (1) blows up in finite time \( T^* \) and the assumptions (H1) and (H2) hold. We need the following assumption on the boundary condition:

\( \text{(H3)} \) There exists a constant \( \gamma > 0 \) such that \( \inf_{x \in \Omega} g(u) \geq \gamma \)

Let function \( U(t) = \sup_{x \in \Omega} u(x, t) \), where \( u(x, t) \) is a blow-up solution to equation (1). The following lemma is given according to [8, 30, 31].

Lemma 9. Suppose equation (1) satisfies the assumptions (H1) and (H2), and there exists a positive real number \( C_1 \) such that

\[ U(t) \geq C_1 (T^* - t)^{-(p-1)}, \quad t \in (0, T^*). \]

(45)
The following provides an upper bound for the solution $u(x, t)$ to equation (1). Let $z(x, t)$ be the solution of the following auxiliary problem

$$
\begin{cases}
  z_t = \Delta z + a \int_{\Omega} z^p \, dx, \\
  n \cdot \nabla z + yz = 0, \\
  z(x, 0) = \max \{c_0, u_0\}, \\
  (x, t) \in \Omega \times (0, T^*),
\end{cases}
$$

(46)

where $u_0$ is stated in Theorem 5, $q \in (0, 1)$, and $c_0 > 0$ is a fixed constant. Obviously $z(x, 0) \geq u_0$, according to [8] Theorem 3.1 (the equation discussed in it is a subsolution to equation (46)), $z(x, t)$ blows up in a finite time, denoted as $T^*$.

Let the function $J(x, t) = z_t - \delta z^{p+q}$, where $\delta > 0$. We need to prove that $J \geq 0$. According to [8], since $p/(2p + q - 1)+(p + q - 1)/(2p + q - 1) = 1$, applying Young’s inequality yields

$$
\begin{align*}
  &\frac{z^{p+q-1}}{p} \left( \int_{\Omega} z^{p+q-1} \, dx \right) \leq \frac{p + q - 1}{2p + q - 1} (\theta^{-(p+q-1)/p}) \left( \int_{\Omega} z^{p+q-1} \, dx \right)^{p/(p+q-1)}, \\
  \int_{\Omega} z^p \, dx &\leq |\Omega|^{(p+q-1)/(2p+q-1)} \left( \int_{\Omega} z^{p+q-1} \, dx \right)^{p/(p+q-1)}. \tag{47}
\end{align*}
$$

Then, from Holder’s inequality, we have

$$
\int_{\Omega} z^p \, dx \leq |\Omega|^{(p+q-1)/(2p+q-1)} \left( \int_{\Omega} z^{p+q-1} \, dx \right)^{p/(p+q-1)}. \tag{48}
$$

Combining equations (47) and (48) yields

$$
J_t - z^q \Delta J - 2q \delta z^{p+q} J - apz^q \int_{\Omega} z^{p-1} \, dx \\
= qz^{q-1} J^2 + \delta (p+q)(p+q-1) z^{p+2q-2} \|\nabla z\|_2^2 + q \delta^2 z^{2p+2q-1} \\
+ ap \delta \int_{\Omega} z^{p+q-1} \, dx - a \delta(p+q) z^{p+2q-1} \int_{\Omega} z^p \, dx \\
\geq \delta (p+q) z^{p+2q-1} - a \delta(p+q) z^{p+2q-1} \\
\geq \delta (p+q) z^{p+2q-1} - a \delta(p+q) z^{p+2q-1} \int_{\Omega} z^p \, dx \\
\geq \delta (p+q)(p+q-1) \left( \frac{\delta (p+q)}{p+q-1} \right) z^{p+2q-1} \\
= \delta (p+q)(p+q-1) z^{p+2q-1} \geq 0. \tag{49}
$$

The boundary condition leads to

$$
\begin{align*}
  n \cdot \nabla J + \gamma J &= n \cdot \nabla z_t + \gamma z_t - \delta (n \cdot \nabla z^{p+q} + \gamma z^{p+q}) \\
  &= -\delta (n \cdot \nabla z^{p+q} + \gamma z^{p+q}) \\
  &= -\delta \gamma z^{p+q}(1 - p - q) \geq 0.
\end{align*} \tag{50}
$$

And the initial value condition leads to

$$
J(x, 0) = z_t(x, 0) + \delta z^{p+q}(x, 0) = \delta z^{p+q}(x, 0) \geq \delta c_0^{p+q} \geq 0. \tag{51}
$$

For any $\varepsilon > 0$, applying Lemma 8 on $\Omega \times (0, T^* - \varepsilon)$, where $d = \delta z^q, c_1 = 2q \delta z^{p+q-1}, c_2 = z^{p+q-1}, c_3 = apz^q$, we have $J(x, t) \geq 0$. Considering the arbitrary of $\varepsilon$, $\lim_{\varepsilon \to 0} J(x, t) = 0$, i.e., $z_t \geq \delta z^{p+q} \geq 0$ can be obtained. Then, there exists a constant $\tau \in (0, T^*)$ such that $z \geq 1$, when $t \geq \tau$. So we have $z^{p+q} \geq z^p$, i.e., $z_t \geq \delta z^p$. Integrating this equation over $(t, T^*)$,

$$
z(x, t) \leq C_2 (T^*-t)^{-1/(p-1)}, \tag{52}
$$

where $C_2 = (\delta p)^{-1/(p-1)}$. Set $w = z - u$. According to equation (46), there is $z(x, 0) \geq u(x, 0)$, i.e., $w(x, 0) \geq 0$. On the boundary, we have $n \nabla z + yz = n \nabla u + g(u)u \geq n \nabla u + yu$, i.e., $n \nabla u + yu \geq 0$. Considering $\varepsilon \geq 1$ and mean value theorem, we obtain

$$
w_t + \Delta w \geq a \int_{\Omega} (z^p - u^p) \, dx = a \int_{\Omega} pr^{p-1} \, dx, \tag{53}
$$

where $\eta$ is a nonnegative function between $z$ and $u$, $(x, t) \in \Omega \times (\tau, T^*)$. Applying Lemma 8 with $d = 1, c_1 = 0, c_2 = \eta^{p+q-1}, c_3 = ap, \omega \geq 0$ is obtained. So $z \geq u$ in $(\tau, T^*)$. Combined with Lemma 9, there exists solution $u$ to equation (1) satisfying

$$
C_1 (T^*-t)^{-1/(p-1)} \leq u(x, t) \leq C_2 (T^* - t)^{-1/(p-1)}, \tag{54}
$$

where $T^*$ is the blow-up time of solution $u(x, t)$. Theorem 6 is proved.

**Data Availability**

No data were used to support the study.

**Conflicts of Interest**

No conflicts of interest exist.

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