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Research Article

Weighted Composition Operators from H^∞ to (α, m)-Bloch Space on Cartan-Hartogs Domain of the First Type

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Let Y_I be nonhomogeneous Cartan-Hartogs domain of the first type, ϕ a holomorphic self-map, and ψ a fixed holomorphic function on Y_I. We study the weighted composition operator $\psi C_\phi f = \psi(f \circ \phi)$ for a function f holomorphic on Y_I. Our main results generalize both cases of the unit ploydisc and the unit ball obtained by Li and Stević (Li 2007 and Li 2008). Firstly, we obtain two crucial inequalities on Y_I; furthermore, the boundedness and compactness of operator ψC_ϕ from the space H^∞ of all bounded holomorphic functions to the (α, m)-Bloch space $\mathscr{B}^{(\alpha,m)}$ on Y_I are investigated.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and $H(\Omega)$ be the set of all holomorphic functions on Ω. Let A, B be complex Banach spaces on Ω, let ψ be a fixed holomorphic function on Ω, and let ϕ be a holomorphic self-map of Ω. The weighted composition operator $\psi C_\phi : A \longrightarrow B$ with the multiplication symbol ψ and the composition symbol ϕ is defined by

$$\psi C_\phi f = \psi(f \circ \phi), \tag{1}$$

for a function f holomorphic on A. It should be mentioned that this operator can be regarded as a generalization of a multiplication operator and a composition operator on various Banach spaces; one can see [1] and reference within for more information on composition operators.

Our primary objects of study in this article are bounded and compact weighted composition operators from the space H^∞ of all bounded holomorphic functions to the (α, m)-Bloch space $\mathscr{B}^{(\alpha,m)}$ on the Cartan-Hartogs domain of the first type, which is defined by Yin [2]. In the work of [3], Cartan first split the irreducible bounded symmetric domains into four types of Cartan domains and two exceptional domains whose complex dimensions are 16 and 27, respectively. Based on this pioneering work, Yin [2] constructed the Hua domains in the theory of several complex variables, which mainly contain the Cartan-Hartogs domains, Cartan-Egg domains, Hua domains, generalized Hua domains, and Hua construction. The Cartan-Hartogs domain of the first type is defined as follows:

$$\mathrm{Y}_\mathrm{I}(N, m, n; K) := \left\{ W \in \mathbb{C}^N, Z \in \mathfrak{R}_\mathrm{I}(m, n) : |W|^{2K} < \det\left(I - Z\bar{Z}'\right) \right\}, K > 0, \tag{2}$$

where

$$\mathfrak{R}_\mathrm{I}(m, n) := \left\{ Z \in \mathbb{C}^{m\times n} : I - Z\bar{Z}' > 0 \right\} \tag{3}$$

is the Cartan domain of the first type, $\bar{Z}'$ denotes the conjugate transpose of Z, det denotes the determinant of a square matrix, N, m, n are some positive integers, and K is a positive real number. In particular, when $m = 1$, $W = 0$, and $K = 1$, the Cartan-Hartogs domain of the first type turns to be the case of the unit ball; it is obvious that the unit ball is a specific case of the Hua domain. In [4], they verified that the Hua domain is not a homogeneous domain or a Reinhardt domain unless a ball. For simplicity, the Cartan-Hartogs domain of the first type is characterized as Y_I. Moreover, throughout this paper, we only consider the case of $N = 1$ for convenience. However, we would like to mention that all of results obtained in this work can be extended to the case of $N \geq 1$ naturally.

In [5], Ohno investigated the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space $\mathscr{B}$ in the open unit disc. In the setting of the unit ball, Du and Li [6] study the boundedness and compactness of weighted composition operators from H^∞ to the Bloch space $\mathscr{B}$, whose norm is defined by the radial derivative $\mathscr{R}f(z)$. Li and Stević [7] gave another characterization for the boundedness and compactness of weighted composition operators from H^∞ to the α-Bloch space $\mathscr{B}^\alpha$, whose norm is defined by the gradient $\nabla f(z)$. Actually, these two norms are equivalent (see [8] for details). In the setting of the unit polydisc, Li and Stević [9, 10] presented some necessary and sufficient conditions for the composition operators and weighted composition operators between H^∞ and α-Bloch space $\mathscr{B}^\alpha$ to be bounded and compact. Besides, there are various interesting works in the literature concerning the operators from the Bloch-type space with the normal weight μ or the logarithmic weight to H^∞ in the unit disc, unit ball, or polydisc (cf. [6, 11–15]).

Allen and Colonna [16] investigated the boundedness and compactness of the weighted composition operators from H^∞ to the Bloch space $\mathscr{B}$ in the bounded homogeneous domain. In the case of the infinite dimensional bounded symmetric domains, Hamada in [17] studied the bounded weighted composition operators from H^∞ to the Bloch space $\mathscr{B}$ on the infinite dimensional bounded symmetric domain, which is realized as the open unit ball of a JB^*-triple in [18].

However, in the setting of the Hua domain, the related works only focus on the composition operators between the classic Bloch spaces, the Bloch-type space equipped with the special weight α or the normal weight μ (see, e.g., [19–23]).

In the present paper, motivated by [7, 9], we characterize the boundedness and compactness of weighted composition operators from H^∞ to (α, m)-Bloch space on the Cartan-Hartogs domain of the first type. The remainder of this article is organized as follows. In Section 2, we collect background materials necessary for the understanding of the statements of our main results. In Section 3, two important inequalities on the Cartan-Hartogs domain of the first type are derived. The first one, let $K \geq 1$ and $\alpha \geq m$, for a holomorphic function f in the Cartan-Hartogs domain of the first type, there exists a constant $C > 0$ such that

$$\|f\|_{\mathscr{B}^{(\alpha,m)}} \leq \|f\|_{\mathscr{B}^{(\alpha,m)}} \leq C\|f\|_\infty. \tag{4}$$

The second inequality is that, for $(Z, W), (X, Y) \in Y_I$, we have

$$\begin{aligned} 2\left|\det\left(I - Z\bar{X}'\right)^{1/m} - \langle W, Y\rangle^{K/m}\right| &\geq \left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right] \\ &\quad + \left[\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}\right], \end{aligned} \tag{5}$$

which is in a position to derive the Hua inequality (see [24]). Using these two inequalities and constructing some test functions on Y_I, Section 4 is devoted to studying the boundedness of the weighted composition operator $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$, and in Section 5, the compactness of the weighted composition operator $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is also derived.

Throughout the rest of the paper, C denotes some constants which may change from line to line.

2. Preliminaries

In this section, before we state the main results, we would like to collect some notations and crucial lemmas in order to prove the main results.

Definition 1. We use $H^\infty = H^\infty(Y_I)$ to denote the space of all bounded holomorphic functions on Y_I. The space H^∞ is a Banach algebra under the following supremum norm $\|\cdot\|_\infty$:

$$\|f\|_\infty := \sup_{(Z,W)\in Y_I} |f(Z, W)| < +\infty, \quad \text{for all } f \in H(Y_I). \tag{6}$$

For a holomorphic function f, the complex gradient of f at (Z, W) will be denoted by $\nabla f(Z, W)$, that is

$$\nabla f(Z, W) = \left(\frac{\partial f(Z, W)}{\partial z_{11}}, \frac{\partial f(Z, W)}{\partial z_{12}}, \cdots, \frac{\partial f(Z, W)}{\partial z_{mn}}, \frac{\partial f(Z, W)}{\partial W}\right). \tag{7}$$

Definition 2. Let $\alpha > 0$. The (α, m) -Bloch space $\mathscr{B}^{(\alpha,m)} = \mathscr{B}^{(\alpha,m)}(Y_I)$ consists of all holimorphic functions on $H(Y_I)$ satisfying

$$\sup_{(Z,W)\in Y_I} \left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^\alpha |\nabla f(Z, W)| < +\infty. \tag{8}$$

If we equip the norm

$$\begin{aligned} \|f\|_{\mathscr{B}^{(\alpha,m)}} := |f(0, 0)| + \sup_{(Z,W)\in Y_I} \\ \cdot \left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^\alpha |\nabla f(Z, W)|, \end{aligned} \tag{9}$$

it is clear that the (α, m)-Bloch space $\mathscr{B}^{(\alpha,m)}$ becomes a

Banach space under the norm $\|\cdot\|_{\mathscr{B}^{(\alpha,m)}}$ which can be proved in a standard way.

For more information on H^∞ and the Bloch-type space, we refer to [25, 26] and references therein.

Lemma 3 (see [27], Theorem 3.3.1) (Hadamard). *Let $A = (a_{ij}) \geq 0$ be an $n \times n$ Hermitian matrix. Then,*

$$\det A \leq \prod_{i=1}^{n} a_{ii}, \tag{10}$$

and " = " holds if and only if A is a diagonal matrix.

Lemma 4 (see [28]). *If $x_k \geq -1$, x_k keep the same sign and $n \geq 2$, then,*

$$\prod_{k=1}^{n}(1 + x_k) > 1 + \sum_{k=1}^{n} x_k. \tag{11}$$

Remark 5. When $n = 1$ or $x_k = 0(k = 2,\cdots,n)$, we get $\prod_{k=1}^{n}(1 + x_k) = 1 + \sum_{k=1}^{n} x_k$.

Lemma 6 (see [26], Proposition 5.1). *Let $\mathbb{D}$ be the unit disc on $\mathbb{C}^n$. $H^\infty(\mathbb{D}) \subset \mathscr{B}(\mathbb{D})$. Moreover,*

$$\left(1-|z|^2\right)\left|f'(z)\right| \leq \|f\|_\infty \tag{12}$$

for all $z \in \mathbb{D}$ and $f \in H^\infty$.

This lemma shows that any bounded analytic function on $\mathbb{D}$ is in the Bloch space. We will generalize this lemma to the Cartan-Hartogs domain of the first type in Section 3.

Lemma 7 (see [29], Appendix: Theorem 1.1). *Let*

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix} \tag{13}$$

be an $m \times n$ matrix $(m \leq n)$; then, there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} V \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0), \tag{14}$$

where $\lambda_1{}^2, \cdots, \lambda_m{}^2$ are characteristic values of $Z\bar{Z}'$.

Lemma 8 (see [29], Theorem 3.1.1). *Let*

$$\begin{aligned} \Lambda_1 &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0), \\ \Lambda_2 &= \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mu_m \end{pmatrix} \quad (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0), \end{aligned} \tag{15}$$

satisfying

$$\lambda_j \mu_k < 1 (j, k = 1,\cdots,m). \tag{16}$$

Then, there is an arranged square matrix P such that

$$\inf_{U\bar{U}'=I, V\bar{V}'=I} \left|\det\left(I - \Lambda_1 U \Lambda_2 \bar{U}' V\right)\right| = \left|\det\left(I - \Lambda_1 P \Lambda_2 P'\right)\right|, \tag{17}$$

and the minimum value is obtained when $U = P$ and $V = I$.

Lemma 9 (see [28]). *Let $a_k, b_k \geq 0$; then,*

$$\left[\prod_{k=1}^{n}(a_k + b_k)\right]^{1/n} \geq \left(\prod_{k=1}^{n} a_k\right)^{1/n} + \left(\prod_{k=1}^{n} b_k\right)^{1/n}. \tag{18}$$

Lemma 10. *Let K be a compact subset of Y_I and f be a holomorphic function on $\mathscr{B}^{(\alpha,m)}(Y_I)$. Then for every $(Z, W) \in K$, there exists a constant $C(K) > 0$ such that*

$$|f(Z, W)| \leq C(K)\|f\|_{\mathscr{B}^{(\alpha,m)}}. \tag{19}$$

Proof. This lemma is a special case of Lemma 2.4 in [23] by taking $\mu(|(Z, W)|) = [\det(I - Z\bar{Z}')^{1/m} - |W|^{2K/m}]^\alpha$; here, we omit the details. □

Lemma 11. *Let $\phi = (\phi_{11}, \phi_{12} \cdots \phi_{mn}, \phi_{mn+1})$ be a holomorphic self-map of Y_I and ψ a holomorphic function on Y_I. The weighted composition operator $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is compact if and only if ψC_ϕ is bounded and for any bounded sequence $\{f_k\}_{k\geq 1}$ in $H^\infty(Y_I)$ converging to 0 uniformly on compact subsets of Y_I, $\lim_{k\longrightarrow\infty}\|\psi C_\phi f_k\|_{\mathscr{B}^{(\alpha,m)}} = 0$.*

Proof. Suppose that $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is compact. Let $\{f_k\}_{k\geq 1}$ be a bounded sequence in $H^\infty(Y_I)$ with $\sup_{k\geq 1}\|f_k\|_\infty \leq M < \infty$, and $f_k \longrightarrow 0$ uniformly on compact subsets of Y_I as $k \longrightarrow \infty$. By the definition of compactness of ψC_ϕ, the sequence $\{\psi C_\phi f_k\}_{k\geq 1}$ has a subsequence

$\{\psi C_\phi f_{k_j}\}_{j\geq 1}$ converging to $f \in \mathscr{B}^{(\alpha,m)}(Y_I)$. By (19), there exists a constant $C(K) > 0$ such that

$$\left|\left(\psi C_\phi f_{k_j}\right)(Z, W) - f(Z, W)\right| \leq C(K)\left\|\psi C_\phi f_{k_j} - f\right\|_{\mathscr{B}^{(\alpha,m)}}, \tag{20}$$

for every $(Z, W) \in Y_I$. It follows that $(\psi C_\phi f_{k_j})(Z, W) - f(Z, W) \longrightarrow 0$ uniformly on compact subsets of Y_I as $j \longrightarrow \infty$. Therefore, $f_{k_j} \longrightarrow 0$ uniformly on compact subsets of Y_I as $j \longrightarrow \infty$. Owing to the definition of ψC_ϕ, we obtain $f \equiv 0$; thus, $\lim_{k\longrightarrow\infty}\|\psi C_\phi f_k\|_{\mathscr{B}^{(\alpha,m)}} = 0$.

Conversely, suppose that $\{h_k\}_{k\geq 1}$ is a sequence in the ball $\mathbb{B}(0, M) \subset H^\infty(Y_I)$; then $\|h_k\|_\infty \leq M < \infty$. It is obvious that $\{h_k\}_{k\geq 1}$ is uniformly bounded on compact subsets of Y_I. By Montel's theorem, we know that $\{h_k\}_{k\geq 1}$ has a subsequence $\{h_{k_j}\}_{j\geq 1}$ converges to $h \in H(Y_I)$ uniformly on Y_I. Moveover, $h \in H^\infty(Y_I)$ and $\|h\|_\infty \leq M$. Hence the sequence $\{h_{k_j} - h\}_{j\geq 1}$ is such that $\|h_{k_j} - h\|_\infty \leq 2M$ and $h_{k_j} - h \longrightarrow 0$ uniformly on compact subsets of Y_I. We Following from the hypothesis implies that

$$\lim_{j\longrightarrow\infty}\left\|\psi C_\phi\left(h_{k_j} - h\right)\right\|_{\mathscr{B}^{(\alpha,m)}} = \lim_{j\longrightarrow\infty}\left\|\psi C_\phi h_{k_j} - \psi C_\phi h\right\|_{\mathscr{B}^{(\alpha,m)}} = 0, \tag{21}$$

which yields that the set $\psi C_\phi(\mathbb{B}(0, M))$ is relatively compact. □

3. Two Important Inequalities

In this section, we obtain two important inequalities on Y_I, which are essential in proving our main results. We remark that two inequalities below seem to be known in the unit ball, but we need to prove them correct on the Cartan-Hartogs domain of the first type.

Theorem 12. *Let $K \geq 1$ and $\alpha \geq m$. There exists a positive constant C independent of f such that*

$$\|f\|_{\mathscr{B}^{(\alpha,m)}} \leq \|f\|_{\mathscr{B}^{(m,m)}} \leq C\|f\|_\infty, \tag{22}$$

for all $(Z, W) \in Y_I$ and $f \in H^\infty$.

Proof. Since $\alpha \geq m$ and the definition of $\mathscr{B}^{(\alpha,m)}(Y_I)$, we have $\|f\|_{\mathscr{B}^{(\alpha,m)}} \leq \|f\|_{\mathscr{B}^{(m,m)}}$. For each $(Z, W) \in Y_I$, let

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix}. \tag{23}$$

In view of (10) and (11), we have

$$\begin{aligned} \det\left(I - Z\bar{Z}'\right) &\leq \left[1 - \left(|z_{11}|^2 + |z_{12}|^2 + \cdots + |z_{1n}|^2\right)\right] \\ &\quad \times \left[1 - \left(|z_{21}|^2 + |z_{22}|^2 + \cdots + |z_{2n}|^2\right)\right] \\ &\quad \times \cdots \times \left[1 - \left(|z_{m1}|^2 + |z_{m2}|^2 + \cdots + |z_{mn}|^2\right)\right] \\ &\leq \left(1 - |z_{11}|^2\right)\cdots\left(1 - |z_{1n}|^2\right)\left(1 - |z_{21}|^2\right)\cdots \\ &\quad \cdot \left(1 - |z_{m1}|^2\right)\cdots\left(1 - |z_{mn}|^2\right). \end{aligned} \tag{24}$$

Due to $\sqrt{a^2 + b^2} \leq a + b\,(a \geq 0, b \geq 0)$, it leads to

$$\begin{aligned} |\nabla f(Z, W)| &= \left\{\left|\frac{\partial f}{\partial z_{11}}(Z, W)\right|^2 + \left|\frac{\partial f}{\partial z_{12}}(Z, W)\right|^2 \right. \\ &\quad \left. + \cdots + \left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right|^2 + \left|\frac{\partial f}{\partial W}(Z, W)\right|^2\right\}^{1/2} \\ &\leq \left|\frac{\partial f}{\partial z_{11}}(Z, W)\right| + \left|\frac{\partial f}{\partial z_{12}}(Z, W)\right| \\ &\quad + \cdots + \left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right| + \left|\frac{\partial f}{\partial W}(Z, W)\right|. \end{aligned} \tag{25}$$

Since $|W|^{2K} < \det(I - Z\bar{Z}')$, it is easy to see that $|W| < 1$. Moreover, let $a = \det(I - Z\bar{Z}')$, $b = |W|^{2K}$; we have $0 \leq b < a < 1$. Making use of the following inequality $(a^{1/m} - b^{1/m})^m \leq a - b$, it suffices to obtain

$$\left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^m \leq \det\left(I - Z\bar{Z}'\right) - |W|^{2K}. \tag{26}$$

In fact, to prove $(a^{1/m} - b^{1/m})^m \leq a - b$, we can consider $a^{1/m} \leq (a - b)^{1/m} + b^{1/m}$. Let $c = (a - b)^{1/m}$, $d = b^{1/m}$; it follows that we should prove $c^m + d^m \leq (c + d)^m$, which obviously holds. Moreover, according to Lemma 6, it leads to

$$\begin{aligned} &\left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^m |\nabla f(Z, W)| \leq \left[\det\left(I - Z\bar{Z}'\right) - |W|^{2K}\right]|\nabla f(Z, W)| \\ &\leq \left[\left(1 - |z_{11}|^2\right)\left(1 - |z_{12}|^2\right)\cdots\left(1 - |z_{mn}|^2\right) - |W|^{2K}\right] \\ &\quad \times \left(\left|\frac{\partial f}{\partial z_{11}}(Z, W)\right| + \left|\frac{\partial f}{\partial z_{12}}(Z, W)\right| + \cdots + \left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right| + \left|\frac{\partial f}{\partial W}(Z, W)\right|\right) \\ &\leq \left[\left(1 - |z_{11}|^2\right)\left(1 - |z_{12}|^2\right)\cdots\left(1 - |z_{mn}|^2\right) - \left(1 - |z_{11}|^2\right)\left(1 - |z_{12}|^2\right)\cdots\left(1 - |z_{mn}|^2\right)|W|^{2K}\right] \\ &\quad \times \left(\left|\frac{\partial f}{\partial z_{11}}(Z, W)\right| + \left|\frac{\partial f}{\partial z_{12}}(Z, W)\right| + \cdots + \left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right| + \left|\frac{\partial f}{\partial W}(Z, W)\right|\right) \\ &\leq \left[\left(1 - |z_{11}|^2\right)\left(1 - |z_{12}|^2\right)\cdots\left(1 - |z_{mn}|^2\right)\left(1 - \left|W^K\right|^2\right)\right] \\ &\quad \times \left(\left|\frac{\partial f}{\partial z_{11}}(Z, W)\right| + \left|\frac{\partial f}{\partial z_{12}}(Z, W)\right| + \cdots + \left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right| + \left|\frac{\partial f}{\partial W}(Z, W)\right|\right) \\ &\leq \left(1 - |z_{11}|^2\right)\left|\frac{\partial f}{\partial z_{11}}(Z, W)\right| + \left(1 - |z_{12}|^2\right)\left|\frac{\partial f}{\partial z_{12}}(Z, W)\right| + \cdots + \left(1 - |z_{mn}|^2\right)\left|\frac{\partial f}{\partial z_{mn}}(Z, W)\right| \\ &\quad + \left(1 - \left|W^K\right|^2\right)\left|\frac{\partial f}{\partial W}(Z, W)\right| \leq mn\|f\|_\infty + \left(1 - \left|W^K\right|^2\right)\left|\frac{\partial f(Z, W)}{\partial W^K} \cdot \frac{\partial W^K}{\partial W}\right| \\ &\leq mn\|f\|_\infty + \|f\|_\infty \cdot K|W|^{K-1} \leq (mn + K)\|f\|_\infty, \end{aligned} \tag{27}$$

which gives the desired estimate. □

Remark 13. When the target is the unit ball in $\mathbb{C}^n$, let $m=1$, $W=0$, and $K=1$; we have the inequality $(1-|Z|^2)\,|\nabla f(Z)|\leq (n+1)\|f\|_\infty$, which arrives at the same conclusion in ([7], Lemma 3).

Theorem 14. *Let $Z, X \in \mathbb{C}^{m\times n}$, $W, Y \in \mathbb{C}^N$ and $K>0$. If $I-Z\bar{Z}'>0$, $I-X\bar{X}'>0$, $|W|^{2K}<\det(I-Z\bar{Z}')$ and $|Y|^{2K}<\det(I-X\bar{X}')$. Then, the following inequality holds*

$$\begin{aligned} 2\left|\det\left(I-Z\bar{X}'\right)^{1/m}-\langle W,Y\rangle^{K/m}\right| &\geq \left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right] \\ &+\left[\det\left(I-X\bar{X}'\right)^{1/m}-|Y|^{2K/m}\right], \end{aligned} \tag{28}$$

and " = " holds if and only if $(Z,W)=(X,Y)$.

Proof. When $m=n$, since $Z_1, X_1 \in \mathfrak{R}_1(m,n)$, applying Lemma 7, there exist $m\times m$ unitary matrixes U_1, U_2 and $n\times n$ unitary matrixes V_1, V_2 such that

$$\begin{aligned} Z_1 &= U_1\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_m\end{pmatrix}V_1 = U_1\Lambda_1V_1 \quad (1>\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0),\\ X_1 &= U_2\begin{pmatrix}\mu_1 & 0 & \cdots & 0\\ 0 & \mu_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mu_m\end{pmatrix}V_2 = U_2\Lambda_2V_2 \quad (1>\mu_1\geq\mu_2\geq\cdots\geq\mu_m\geq 0). \end{aligned} \tag{29}$$

Then, it turns out to

$$\begin{aligned} \det\left(I-Z_1\bar{X}_1'\right) &= \det\left(I-U_1\Lambda_1V_1\bar{V}_2'\bar{\Lambda}_2'\bar{U}_2'\right)\\ &= \det\left(U_1\bar{U}_1'-U_1\Lambda_1V_1\bar{V}_2'\bar{\Lambda}_2'\bar{U}_2'\right)\\ &= \det U_1 \det\left(\bar{U}_1'-\Lambda_1V_1\bar{V}_2'\bar{\Lambda}_2'\bar{U}_2'\right)\\ &= \det\left(I-\Lambda_1V_1\bar{V}_2'\bar{\Lambda}_2'\bar{U}_2'U_1\right)\\ &= \det\left(I-\Lambda_1V_1\bar{V}_2'\bar{\Lambda}_2'V_2\bar{V}_1'\bar{U}_2'U_1\right), \end{aligned} \tag{30}$$

and according to Lemma 8, there exists an arrange square matrix P such that

$$\left|\det\left(I-Z_1\bar{X}_1'\right)\right| \geq \left|\det\left(I-\Lambda_1P\Lambda_2P'\right)\right| = \prod_{i=1}^m\left(1-\lambda_i\mu_{k_i}\right). \tag{31}$$

Hence, using (18), we have

$$\begin{aligned} 2\left|\det\left(I-Z_1\bar{X}_1'\right)^{1/m}\right| &\geq 2\left[\prod_{i=1}^m\left(1-\lambda_i\mu_{k_i}\right)\right]^{1/m}\\ &= \left[2^m\prod_{i=1}^m\left(1-\lambda_i\mu_{k_i}\right)\right]^{1/m} = \left[\prod_{i=1}^m\left(2-2\lambda_i\mu_{k_i}\right)\right]^{1/m}\\ &\geq \left\{\prod_{i=1}^m\left[\left(1-\lambda_i^2\right)+\left(1-\mu_{k_i}^2\right)\right]\right\}^{1/m} \geq \left[\prod_{i=1}^m\left(1-\lambda_i^2\right)\right]^{1/m}\\ &+\left[\prod_{i=1}^m\left(1-\mu_{k_i}^2\right)\right]^{1/m} = \det\left(I-Z_1\bar{Z}_1'\right)^{1/m}+\det\left(I-X_1\bar{X}_1'\right)^{1/m}, \end{aligned} \tag{32}$$

where k_i is the rearrangement of i. Moreover, referring to the condition of equalities for (31) and (32), we obtain the inequality

$$2\left|\det\left(I-Z_1\bar{X}_1'\right)^{1/m}\right| \geq \det\left(I-Z_1\bar{Z}_1'\right)^{1/m}+\det\left(I-X_1\bar{X}_1'\right)^{1/m}, \tag{33}$$

which becomes an equality if and only if $Z_1=X_1$.

When $m<n$, there exists an unitary matrix $U^{(n)}$ such that

$$Z=\left(Z_1{}^{(m)},0\right)U,\quad X=\left(X_1{}^{(m)},X_2\right)U. \tag{34}$$

By (32), we obtain

$$\begin{aligned} 2\left|\det\left(I-Z\bar{X}'\right)^{1/m}\right| &= 2\left|\det\left(I-Z_1\bar{X}_1'\right)^{1/m}\right|\\ &\geq \det\left(I-Z_1\bar{Z}_1'\right)^{1/m}+\det\left(I-X_1\bar{X}_1'\right)^{1/m}\\ &\geq \det\left(I-Z_1\bar{Z}_1'\right)^{1/m}+\det\left(I-X_1\bar{X}_1'-X_2\bar{X}_2'\right)^{1/m}\\ &= \det\left(I-Z\bar{Z}'\right)^{1/m}+\det\left(I-X\bar{X}'\right)^{1/m}. \end{aligned} \tag{35}$$

Thus, the inequality

$$2\left|\det\left(I-Z\bar{X}'\right)^{1/m}\right| \geq \det\left(I-Z\bar{Z}'\right)^{1/m}+\det\left(I-X\bar{X}'\right)^{1/m} \tag{36}$$

holds when $m\leq n$, and " = " holds if and only if $Z=X$. By the inequality of arithmetic and geometric means, we have

$$2|W|^{K/m}\left|\bar{Y}'\right|^{K/m} \leq |W|^{2K/m}+|Y|^{2K/m}, \tag{37}$$

and the equality holds if and only if $|W|=|Y|$. Therefore, combining (36) with (37) gives that

$$
\begin{aligned}
2\left|\det\left(I-Z\bar{X}'\right)^{1/m}-\langle W,Y\rangle^{K/m}\right| &\geq 2\left|\det\left(I-Z\bar{X}'\right)^{1/m}\right| \\
&\quad -2\left|\langle W,Y\rangle^{K/m}\right| \geq 2\left|\det\left(I-Z\bar{X}'\right)^{1/m}\right| \\
&\quad -2|W|^{K/m}\left|\bar{Y}'\right|^{K/m} \geq \det\left(I-Z\bar{Z}'\right)^{1/m} \\
&\quad +\det\left(I-X\bar{X}'\right)^{1/m}-|W|^{2K/m}-|Y|^{2K/m} \\
&= \left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]+\left[\det\left(I-X\bar{X}'\right)^{1/m}-|Y|^{2K/m}\right].
\end{aligned} \tag{38}
$$

The first inequality becomes an equality if and only if $\det\left(I-Z\bar{X}'\right)^{1/m}\cdot\overline{\langle W,Y\rangle^{K/m}}' \geq 0$, and the second inequality becomes an equality if and only if $W=0$, $Y=0$, or $W=kY$ $(k>0)$, which implies "=" holds only when $W=Y$ in (38). Hence, in this case, there is equality in (38) if and only if $(Z,W)=(X,Y)$. □

Corollary 15. *Let $Z,X\in\mathbb{C}^{m\times n}$, $W,Y\in\mathbb{C}^{N}$, and $K>0$. If $I-Z\bar{Z}'>0$, $I-X\bar{X}'>0$, $|W|^{2K}<\det\left(I-Z\bar{Z}'\right)$, and $|Y|^{2K}<\det\left(I-X\bar{X}'\right)$. Then, the following inequality holds*

$$
\begin{aligned}
&\left|\det\left(I-Z\bar{X}'\right)^{1/m}-\langle W,Y\rangle^{K/m}\right|^2 \\
&\geq \left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]\left[\det\left(I-X\bar{X}'\right)^{1/m}-|Y|^{2K/m}\right].
\end{aligned} \tag{39}
$$

Proof. This proof only follows the elementary inequality $((a+b)/2)\geq\sqrt{ab}(a\geq 0, b\geq 0)$; here, we omit the details. □

Corollary 16. *Let $Z,X\in\mathbb{C}^{m\times n}$. If $I-Z\bar{Z}'>0$ and $I-X\bar{X}'>0$, then,*

$$
2\left|\det\left(I-Z\bar{X}'\right)^{1/m}\right| \geq \det\left(I-Z\bar{Z}'\right)^{1/m}+\det\left(I-X\bar{X}'\right)^{1/m}. \tag{40}
$$

Proof. Substituting $W=0$ and $Y=0$ into (28) leads to this inequality. □

Remark 17. Since$((a+b)/2)\geq\sqrt{ab}(a\geq 0, b\geq 0)$, we get

$$
\begin{aligned}
\left|\det\left(I-Z\bar{X}'\right)\right|^2 &= \left|\left(I-Z\bar{Z}'\right)^{1/m}\right|^{2m} \\
&\geq \left[\frac{1}{2}\det\left(I-Z\bar{Z}'\right)^{1/m}+\frac{1}{2}\det\left(I-X\bar{X}'\right)^{1/m}\right]^{2m} \\
&\geq \left[\det\left(I-Z\bar{Z}'\right)^{1/m}\det\left(I-X\bar{X}'\right)^{1/m}\right]^{m} \\
&= \det\left(I-Z\bar{Z}'\right)\det\left(I-X\bar{X}'\right),
\end{aligned} \tag{41}
$$

which yields the Hua inequality discovered by Hua Loo-Keng in [24].

4. Boundedness of $\psi C_\phi : H^\infty \longrightarrow \mathscr{B}^{(\alpha,m)}$

In this section, we characterize the bounded weighted composition operator in the case $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$. The following theorem describes such properties.

We will begin by introducing some notations. Let $\phi=(\phi_{11},\phi_{12}\cdots\phi_{mn},\phi_{mn+1})$ be a holomorphic self-map of Y_I, denoting

$$
D\phi(Z,W)=\begin{pmatrix}
\dfrac{\partial\phi_{11}(Z,W)}{\partial z_{11}} & \cdots & \dfrac{\partial\phi_{11}(Z,W)}{\partial z_{mn}} & \dfrac{\partial\phi_{11}(Z,W)}{\partial W} \\
\vdots & \ddots & \vdots & \vdots \\
\dfrac{\partial\phi_{mn}(Z,W)}{\partial z_{11}} & \cdots & \dfrac{\partial\phi_{mn}(Z,W)}{\partial z_{mn}} & \dfrac{\partial\phi_{mn}(Z,W)}{\partial W} \\
\dfrac{\partial\phi_{mn+1}(Z,W)}{\partial z_{11}} & \cdots & \dfrac{\partial\phi_{mn+1}(Z,W)}{\partial z_{mn}} & \dfrac{\partial\phi_{mn+1}(Z,W)}{\partial W}
\end{pmatrix}. \tag{42}
$$

Theorem 18. *For $K\geq 1$ and $\alpha\geq m$, let $\phi=(\phi_{11},\phi_{12},\cdots,\phi_{mn},\phi_{mn+1})$ be a holomorphic self-map of Y_I, ψ a holomorphic function on Y_I, and $(Z_\phi,W_\phi)=\phi(Z,W)$. If*

$$
\begin{aligned}
&\psi\in\mathscr{B}^{(\alpha,m)}, \\
&\sup_{(Z,W)\in Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}}|\psi(Z,W)||D\phi(Z,W)|<\infty,
\end{aligned} \tag{43}
$$

then the weighted composition operator $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is bounded.

Conversely, if the weighted composition operator $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is bounded, then,

$$
\begin{aligned}
&\psi\in\mathscr{B}^{(\alpha,m)}, \\
&\sup_{(Z,W)\in Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}}|\psi(Z,W)|G(Z,W)<\infty,
\end{aligned} \tag{44}
$$

where

$$
\begin{aligned}
G(Z,W)=\Bigg\{&\sum_{\substack{1\leq k\leq m\\ 1\leq l\leq n}}\Bigg|\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n}}\frac{1}{m}\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}tr\left[\left(I-Z_\phi\bar{Z}_\phi'\right)^{-1}I_{uv}\bar{Z}_\phi'\right] \\
&\times\frac{\partial\phi_{uv}}{\partial z_{kl}}(Z,W)+\frac{K}{m}\left|W_\phi\right|^{(2K/m)-2}\bar{W}_\phi'\frac{\partial\phi_{mn+1}}{\partial z_{kl}}(Z,W)\Bigg|^2 \\
&+\Bigg|\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n}}\frac{1}{m}\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}tr\left[\left(I-Z_\phi\bar{Z}_\phi'\right)^{-1}I_{uv}\bar{Z}_\phi'\right] \\
&\times\frac{\partial\phi_{uv}}{\partial W}(Z,W)+\frac{K}{m}\left|W_\phi\right|^{(2K/m)-2}\bar{W}_\phi'\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\Bigg|^2\Bigg\}^{1/2}.
\end{aligned} \tag{45}
$$

Proof. Assume that (43) holds. There exists a positive constant C_1 such that

$$\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_{\phi}\bar{Z}_{\phi}'\right)^{1/m}-\left|W_{\phi}\right|^{2K/m}\right]^{m}}|\psi(Z,W)||D\phi(Z,W)|\le C_1, \tag{46}$$

for all $(Z,W)\in Y_I$ and $(Z_\phi,W_\phi)=\phi(Z,W)\in Y_I$. Firstly, we know that

$$\begin{aligned}\nabla\left(C_\phi f\right)(Z,W)=&\Bigg(\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial z_{11}}(Z,W)\\&+\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial z_{11}}(Z,W),\cdots,\\&\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial z_{mn}}(Z,W)\\&+\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial z_{mn}}(Z,W),\\&\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial W}(Z,W)\\&+\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\Bigg).\end{aligned} \tag{47}$$

Therefore, it leads to

$$\begin{aligned}&\left|\nabla\left(C_\phi f\right)(Z,W)\right|^2\\&=\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial z_{kl}}(Z,W)+\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial z_{kl}}(Z,W)\right|^2\\&\quad+\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial W}(Z,W)+\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\right|^2\\&\le2\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial z_{kl}}(Z,W)\right|^2\\&\quad+2\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial z_{kl}}(Z,W)\right|^2\\&\quad+2\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\frac{\partial\phi_{uv}}{\partial W}(Z,W)\right|^2+2\left|\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\right|^2\\&\le2\left[\sum_{\substack{1\le u\le m\\1\le v\le n}}\left|\frac{\partial f}{\partial Y_{uv}}(\phi(Z,W))\right|^2+\left|\frac{\partial f}{\partial Y_{mn+1}}(\phi(Z,W))\right|^2\right]\\&\quad\times\left[\sum_{\substack{1\le k\le m\\1\le l\le n}}\sum_{\substack{1\le u\le m\\1\le v\le n}}\left|\frac{\partial\phi_{uv}}{\partial z_{kl}}(Z,W)\right|^2+\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\frac{\partial\phi_{mn+1}}{\partial z_{kl}}(Z,W)\right|^2\right.\\&\quad\left.+\sum_{\substack{1\le u\le m\\1\le v\le n}}\left|\frac{\partial\phi_{uv}}{\partial W}(Z,W)\right|^2+\left|\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\right|^2\right]=2|\nabla f(\phi(Z,W))|^2|D\phi(Z,W)|^2.\end{aligned} \tag{48}$$

Namely,

$$\left|\nabla\left(C_\phi f\right)(Z,W)\right|\le\sqrt{2}|\nabla f(\phi(Z,W))||D\phi(Z,W)|. \tag{49}$$

For a function $f\in H^\infty(Y_I)$, we obtain the following estimate

$$\begin{aligned}&\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\left|\nabla\left(\psi C_\phi f\right)(Z,W)\right|\\&=\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\\&\quad\cdot\left|\nabla\psi(Z,W)\cdot\left(C_\phi f\right)(Z,W)+\psi(Z,W)\cdot\nabla\left(C_\phi f\right)(Z,W)\right|\\&\le\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)|\left|\left(C_\phi f\right)(Z,W)\right|\\&\quad+\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\psi(Z,W)|\left|\nabla\left(C_\phi f\right)(Z,W)\right|\\&\le\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)||f(\phi(Z,W))|\\&\quad+\sqrt{2}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\psi(Z,W)||\nabla f(\phi(Z,W))|\\&\quad\cdot|D\phi(Z,W)|\le\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)|\\&\quad\cdot|f(\phi(Z,W))|+\sqrt{2}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}}\\&\quad\cdot|\psi(Z,W)||D\phi(Z,W)|\times\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}\\&\quad\cdot|\nabla f(\phi(Z,W))|\|\psi\|_{\mathscr{B}^{(\alpha,m)}}\|f\|_\infty+\sqrt{2}C_1\|f\|_{\mathscr{B}^{(m,m)}}.\end{aligned} \tag{50}$$

Since $\psi\in\mathscr{B}^{(\alpha,m)}$ and (22), it leads to

$$\begin{aligned}&\|\psi C_\phi f\|_{\mathscr{B}^{(\alpha,m)}}=\left|\left(\psi C_\phi f\right)(0,0)\right|+\sup_{(Z,W)\in Y_I}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\\&\quad\cdot\left|\nabla\left(\psi C_\phi f\right)(Z,W)\right|\le|\psi(0,0)||f(\phi(0,0))|+\|\psi\|_{\mathscr{B}^{(\alpha,m)}}\|f\|_\infty\\&\quad+\sqrt{2}C_1\|f\|_{\mathscr{B}^{(m,m)}}\le|\psi(0,0)|\|f\|_\infty+\|\psi\|_{\mathscr{B}^{(\alpha,m)}}\|f\|_\infty+\sqrt{2}CC_1\|f\|_\infty\\&\le C_2\|f\|_\infty,\end{aligned} \tag{51}$$

which implies that $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is bounded.

Conversely, assume that $\psi C_\phi : H^\infty(Y_I) \longrightarrow \mathscr{B}^{(\alpha,m)}(Y_I)$ is bounded. It follows that there exists a positive constant C such that

$$\left\|\psi C_\phi f\right\|_{\mathscr{B}^{(\alpha,m)}} \le C\|f\|_\infty. \tag{52}$$

Let $f \equiv 1$; we have $\|\psi\|_{\mathscr{B}^{(\alpha,m)}} \le C$, which implies $\psi \in \mathscr{B}^{(\alpha,m)}$. For $(X, Y) \in Y_I$, define a test function $f_{(X,Y)} \in H(Y_I)$ by

$$f_{(X,Y)}(Z, W) := \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\det\left(I - Z\bar{X}'\right)^{1/m} - \langle W, Y\rangle^{K/m}}. \tag{53}$$

From (28), it follows that

$$\begin{aligned}
\left|f_{(X,Y)}(Z, W)\right| &= \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left|\det\left(I - Z\bar{X}'\right)^{1/m} - \langle W, Y\rangle^{K/m}\right|} \\
&\le \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{(1/2)\left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right] + (1/2)\left[\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}\right]} \\
&\le \frac{2\left[\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}\right]}{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}} = 2,
\end{aligned} \tag{54}$$

which implies $f_{(X,Y)} \in H^\infty(Y_I)$ and $\|f_{(X,Y)}\|_\infty \le 2$.

For the test function f, we have

$$\begin{aligned}
\frac{\partial f_{(X,Y)}}{\partial Y_{uv}}(\phi(Z, W)) &= \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right]^2} \cdot \mathscr{F}_{uv}\left(Z_\phi, X\right), \\
\frac{\partial f_{(X,Y)}}{\partial Y_{mn+1}}(\phi(Z, W)) &= \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{2K/m}\right]^2} \cdot \frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}',
\end{aligned} \tag{55}$$

where $\mathscr{F}_{uv}(Z_\phi, X) = (1/m)\det\left(I - Z_\phi\bar{X}'\right)^{1/m}\operatorname{tr}[(I - Z_\phi\bar{X}')^{-1} I_{uv}\bar{X}']$. It leads to

$$\begin{aligned}
&\left|\nabla\left(C_\phi f_{(X,Y)}\right)(Z, W)\right| \\
&= \left\{\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f_{(X,Y)}}{\partial Y_{uv}}(\phi(Z, W))\frac{\partial \phi_{uv}}{\partial z_{kl}}(Z, W) + \frac{\partial f_{(X,Y)}}{\partial Y_{mn+1}}(\phi(Z, W))\frac{\partial \phi_{mn+1}}{\partial z_{kl}}(Z, W)\right|^2\right. \\
&\quad \left. + \left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\partial f_{(X,Y)}}{\partial Y_{uv}}(\phi(Z, W))\frac{\partial \phi_{uv}}{\partial W}(Z, W) + \frac{\partial f_{(X,Y)}}{\partial Y_{mn+1}}(\phi(Z, W))\frac{\partial \phi_{mn+1}}{\partial W}(Z, W)\right|^2\right\}^{1/2} \\
&= \left\{\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right]^2}\cdot\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial z_{kl}}(Z, W)\right.\right. \\
&\quad \left. + \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right]^2}\cdot\frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial z_{kl}}(Z, W)\right|^2 \\
&\quad + \left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right]^2}\cdot\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial W}(Z, W)\right. \\
&\quad \left.\left. + \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left[\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right]^2}\cdot\frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial W}(Z, W)\right|^2\right\}^{1/2} \\
&= \frac{\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}}{\left|\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right|^2} \\
&\quad \times\left\{\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial z_{kl}}(Z, W) + \frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial z_{kl}}(Z, W)\right|^2\right. \\
&\quad \left. + \left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial W}(Z, W) + \frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial W}(Z, W)\right|^2\right\}^{1/2}.
\end{aligned} \tag{56}$$

Then, it follows that

$$\begin{aligned}
&\left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^\alpha\left|\nabla\left(C_\phi f_{(X,Y)}\right)(Z, W)\right| \\
&= \frac{\left[\det\left(I - Z\bar{Z}'\right)^{1/m} - |W|^{2K/m}\right]^\alpha\left[\det\left(I - X\bar{X}'\right)^{1/m} - |Y|^{2K/m}\right]}{\left|\det\left(I - Z_\phi\bar{X}'\right)^{1/m} - \left\langle W_\phi, Y\right\rangle^{K/m}\right|^2} \\
&\quad \times\left\{\sum_{\substack{1\le k\le m\\1\le l\le n}}\left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial z_{kl}}(Z, W) + \frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial z_{kl}}(Z, W)\right|^2\right. \\
&\quad \left. + \left|\sum_{\substack{1\le u\le m\\1\le v\le n}}\mathscr{F}_{uv}\left(Z_\phi, X\right)\frac{\partial \phi_{uv}}{\partial W}(Z, W) + \frac{K}{m}\left\langle W_\phi, Y\right\rangle^{(K/m)-1}\bar{Y}'\frac{\partial \phi_{mn+1}}{\partial W}(Z, W)\right|^2\right\}^{1/2}.
\end{aligned} \tag{57}$$

Let

$$(X, Y) = \left(Z_\phi, W_\phi\right) = \phi(Z, W). \tag{58}$$

Since $f(\phi(Z,W))=1$, (52) and (57), we obtain

$$\begin{aligned} 2C &\geq \left\|\psi C_\phi f\right\|_{\beta^{(\alpha,m)}} \geq \left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{(2K)/m}\right]^\alpha\left|\nabla\left(\psi C_\phi f\right)(Z,W)\right| = \\ &\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha\left|\psi(Z,W)\cdot\nabla\left(C_\phi f\right)(Z,W)+\nabla\psi(Z,W)\cdot\left(C_\phi f\right)(Z,W)\right| \\ &\geq\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha\left|\psi(Z,W)\cdot\nabla\left(C_\phi f\right)(Z,W)\right| \\ &-\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha\left|\nabla\psi(Z,W)\cdot f(\phi(Z,W))\right| \\ &\geq\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}}|\psi(Z,W)| \\ &\times\left\{\sum_{\substack{1\leq k<m\\1\leq l\leq n}}\left|\sum_{\substack{1\leq u<m\\1\leq v<n}}\mathscr{F}_{uv}\left(Z_\phi,Z_\phi\right)\frac{\partial\phi_{uv}}{\partial z_{kl}}(Z,W)+\frac{K}{m}\left|W_\phi\right|^{\frac{2K}{m}-2}\overline{W_\phi'}\frac{\partial\phi_{mn+1}}{\partial z_{kl}}(Z,W)\right|^2\right. \\ &\left.+\left|\sum_{\substack{1\leq u<m\\1\leq v<n}}\mathscr{F}_{uv}\left(Z_\phi,Z_\phi\right)\frac{\partial\phi_{uv}}{\partial W}(Z,W)+\frac{K}{m}\left|W_\phi\right|^{(2K/m)-2}\overline{W_\phi'}\frac{\partial\phi_{mn+1}}{\partial W}(Z,W)\right|^2\right\}^{1/2} \\ &-\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha|\nabla\psi(Z,W)| \\ &\geq\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}}|\psi(Z,W)|G(Z,W) \\ &-\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha|\nabla\psi(Z,W)|. \end{aligned} \tag{59}$$

Since $\psi\in\mathscr{B}^{(\alpha,m)}$, we obtain

$$\sup_{(Z,W)\in Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\det\left(I-Z_\phi\bar{Z}_\phi{}'\right)^{1/m}-\left|W_\phi\right|^{2K/m}}|\psi(Z,W)|G(Z,W)<\infty. \tag{60}$$

The proof is completed. □

Remark 19. Let $m=1$, $W=0$ and $K=1$, we obtain the following results in the case of the unit ball $\mathbb{B}=\{Z\in\mathbb{C}^n : |Z|^2<1\}$. Let $\alpha=1$. If

$$\begin{gathered}\psi\in\mathscr{B},\\ \sup_{Z\in\mathbb{B}}\frac{1-|Z|^2}{1-|\phi(Z)|^2}|\psi(Z)||D\phi(Z)|<\infty,\end{gathered} \tag{61}$$

then the weighted composition operator $\psi C_\phi : H^\infty(\mathbb{B})\longrightarrow\mathscr{B}(\mathbb{B})$ is bounded. Conversely, the weighted composition operator $\psi C_\phi : H^\infty(\mathbb{B})\longrightarrow\mathscr{B}(\mathbb{B})$ is bounded, then

$$\begin{gathered}\psi\in\mathscr{B}\\ \sup_{Z\in\mathbb{B}}\frac{|\psi(Z)|\left(1-|Z|^2\right)}{1-|\phi(Z)|^2}\left|D\phi(Z)'\phi(\bar{Z})'\right|<\infty,\end{gathered} \tag{62}$$

where

$$|D\phi(Z)|=\left(\sum_{k,l=1}^{n}\left|\frac{\partial\phi_l(Z)}{\partial Z_k}\right|^2\right)^{1/2}. \tag{63}$$

Li and Stević investigated the boundedness of this weighted composition operator in [7], which is as the same as the above results; therefore, our main results cover and substantially improve the work of [7].

5. Compactness of $\psi C_\phi : H^\infty\longrightarrow\mathscr{B}^{(\alpha,m)}$

In this section, we characterize the compact weighted composition operator $\psi C_\phi : H^\infty(\mathrm{Y_I})\longrightarrow\mathscr{B}^{(\alpha,m)}(\mathrm{Y_I})$.

Theorem 20. *For $K\geq1$ and $\alpha\geq m$, let $\phi=(\phi_{11},\phi_{12}\cdots\phi_{mn},\phi_{mn+1})$ be a holomorphic self-map of Y_I, ψ a holomorphic function on Y_I, and $(Z_\phi,W_\phi)=\phi(Z,W)$. If*

$$\begin{gathered}\lim_{\phi(Z,W)\longrightarrow\partial Y_I}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha|\nabla\psi(Z,W)|=0,\\ \lim_{\phi(Z,W)\longrightarrow\partial Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\left[\det\left(I-Z_\phi\bar{Z}_\phi{}'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^m}|\psi(Z,W)||D\phi(Z,W)|=0,\end{gathered} \tag{64}$$

then the weighted composition operator $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is compact.

Conversely, if the weighted composition operator $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is compact, then

$$\begin{gathered}\lim_{\phi(Z,W)\longrightarrow\partial Y_I}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha|\nabla\psi(Z,W)|=0,\\ \lim_{\phi(Z,W)\longrightarrow\partial Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\det\left(I-Z_\phi\bar{Z}_\phi{}'\right)^{1/m}-\left|W_\phi\right|^{2K/m}}|\psi(Z,W)|G(Z,W)=0,\end{gathered} \tag{65}$$

where $G(Z,W)$ is the same as (45).

Proof. Suppose that (64) holds. We have

$$\sup_{(Z,W)\in Y_I}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^\alpha}{\left[\det\left(I-Z_\phi\bar{Z}_\phi{}'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^m}|\psi(Z,W)||D\phi(Z,W)|<\infty. \tag{66}$$

Following from Theorem 18, we obtain that $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is bounded. Let $\{f_k\}_{k\geq1}$ be a bounded sequence, and f_k converges to 0 uniformly on compact subsets of Y_I. Let $M:=\sup_{k\geq1}\|f_k\|_\infty$. By the assumptions, for any $\varepsilon>0$, there is a constant $\delta\in(0,1)$ such that

$$\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)|<\varepsilon, \quad (67)$$

$$\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}}|\psi(Z,W)||D\phi(Z,W)|<\varepsilon, \quad (68)$$

whenever $\operatorname{dist}(\phi(Z,W),\partial Y_I)<\delta$. Taking (49), (67), (68), and Theorem 12 into account, it turns out that

$$\begin{aligned}
&\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\left|\nabla\left(\psi C_\phi f_k\right)(Z,W)\right| \\
&=\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha} \\
&\quad\cdot\left|\nabla\psi(Z,W)\cdot\left(C_\phi f_k\right)(Z,W)+\psi(Z,W)\cdot\nabla\left(C_\phi f_k\right)(Z,W)\right| \\
&=\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\left|\nabla\psi(Z,W)\cdot\left(C_\phi f_k\right)(Z,W)\right| \\
&\quad+\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\left|\psi(Z,W)\cdot\nabla\left(C_\phi f_k\right)(Z,W)\right| \\
&\le\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)||f_k(\phi(Z,W))| \\
&\quad+\sqrt{2}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}} \\
&\quad\cdot|\psi(Z,W)||D\phi(Z,W)|\times\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m} \\
&\quad\cdot|\nabla f_k(\phi(Z,W))|\le\varepsilon|f_k(\phi(Z,W))|+\sqrt{2}\varepsilon\|f_k\|_{\mathscr{B}^{(m,m)}} \\
&\le\left(1+\sqrt{2}C\right)\varepsilon\|f_k\|_\infty\le\left(1+\sqrt{2}C\right)M\varepsilon.
\end{aligned} \quad (69)$$

In addition, we set

$$E_\delta:=\{\operatorname{dist}(\phi(Z,W),\partial Y_I)\ge\delta\}. \quad (70)$$

Note that E_δ is a compact subset of Y_I. For ε defined in (67), it leads to $f_k\longrightarrow 0$ uniformly on E_δ as $k\longrightarrow\infty$. Cauchy's estimate gives that $|\nabla f_k|\longrightarrow 0$ as $k\longrightarrow\infty$ on compact subsets, in particular on $\phi(E_\delta)$. Hence, as $k\longrightarrow\infty$, by (49) we obtain

$$\begin{aligned}
&\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}\left|\nabla\left(\psi C_\phi f_k\right)(Z,W)\right| \\
&\le\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}|\nabla\psi(Z,W)||f_k(\phi(Z,W))| \\
&\quad+\sqrt{2}\frac{\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha}}{\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}}|\psi(Z,W)||D\phi(Z,W)| \\
&\quad\times\left[\det\left(I-Z_\phi\bar{Z}_\phi'\right)^{1/m}-\left|W_\phi\right|^{2K/m}\right]^{m}|\nabla f_k(\phi(Z,W))|\longrightarrow 0.
\end{aligned} \quad (71)$$

According to the two inequalities (69) and (71), as $k\longrightarrow\infty$, we have

$$\begin{aligned}
&\left\|\psi C_\phi f_k\right\|_{\mathscr{B}^{(\alpha,m)}} \\
&=\left|\left(\psi C_\phi f_k\right)(0,0)\right|+\sup_{(Z,W)\in Y_I}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha} \\
&\quad\cdot\left|\nabla\left(\psi C_\phi f_k\right)(Z,W)\right| \\
&=|\psi(0,0)\cdot f_k(\phi(0,0))|+\sup_{(Z,W)\in Y_I}\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]^{\alpha} \\
&\quad\cdot\left|\nabla\left(\psi C_\phi f_k\right)(Z,W)\right|\longrightarrow 0.
\end{aligned} \quad (72)$$

Consequently, making use of Lemma 11, we get $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is compact.

Conversely, suppose that $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is compact. Let $\{(X^i,Y^i)\}_{i\ge1}=\{\phi(Z^i,W^i)\}_{i\ge1}$ be a sequence on Y_I such that $\phi(Z^i,W^i)\longrightarrow\partial Y_I$, as $i\longrightarrow\infty$. If the sequence is nonexistent, conditions (c) and (d) obviously hold. Moreover, let us introduce a test function sequence $\{f_i\}_{i\ge1}$:

$$f_i(Z,W):=\frac{\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}}{\det\left(I-Z\bar{X}^{i\prime}\right)^{1/m}-\left\langle W,Y^i\right\rangle^{K/m}}. \quad (73)$$

The proof of Theorem 12 gives $f_i\in H^\infty$ and $\|f_i\|_\infty\le 2$. Due to (28), it gives that

$$\begin{aligned}
|f_i(Z,W)|&=\frac{\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}}{\left|\det\left(I-Z\bar{X}^{i\prime}\right)^{1/m}-\left\langle W,Y^i\right\rangle^{K/m}\right|} \\
&\le\frac{\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}}{(1/2)\left[\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}\right]+(1/2)\left[\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}\right]} \\
&\le\frac{2\left[\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}\right]}{\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}}.
\end{aligned} \quad (74)$$

Taking $i\longrightarrow\infty$, we have $(X^i,Y^i)\longrightarrow\partial Y_I$. This implies $\det\left(I-X^i\bar{X}^{i\prime}\right)^{1/m}-\left|Y^i\right|^{2K/m}\longrightarrow 0$, as $i\longrightarrow\infty$. Let E be a compact subset of Y_I. For $(Z,W)\in E$, it is easy to see that $\det\left(I-Z\bar{Z}'\right)^{1/m}-|W|^{2K/m}$ has a positive lower bound. Hence, we obtain $f_i\longrightarrow 0$ uniformly on all compact subsets of Y_I, as $i\longrightarrow\infty$.

Since $\psi C_\phi : H^\infty(Y_I)\longrightarrow\mathscr{B}^{(\alpha,m)}(Y_I)$ is compact, according to Lemma 11, we have

$$\lim_{k\longrightarrow\infty}\|\psi C_\phi f_i\|_{\mathscr{B}^{(\alpha,m)}}=0. \quad (75)$$

For the test function f_i, we have

$$\frac{\partial f_i}{\partial Y_{uv}}(\phi(Z,W)) = \frac{\det\left(I - X^i\bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\left[\det\left(I - Z_\phi \bar{X}^{i\prime}\right)^{1/m} - \left\langle W_\phi, Y^i\right\rangle^{K/m}\right]^2} \cdot \mathscr{F}_{uv}\left(Z_\phi, X^i\right),$$

$$\frac{\partial f_i}{\partial Y_{mn+1}}(\phi(Z,W)) = \frac{\det\left(I - X^i\bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\left[\det\left(I - Z_\phi \bar{X}^{i\prime}\right)^{1/m} - \left\langle W_\phi, Y^i\right\rangle^{K/m}\right]^2} \cdot \frac{K}{m}\left\langle W_\phi, Y^i\right\rangle^{(K/m)-1}\bar{Y}^{i\prime}, \tag{76}$$

where $\mathscr{F}_{uv}(Z_\phi, X^i) = (1/m)\det(I - Z_\phi\bar{X}^{i\prime})^{1/m}\operatorname{tr}[(I - Z_\phi\bar{X}^{i\prime})^{-1} I_{uv}\bar{X}^{i\prime}]$. Thus,

$$\begin{aligned}\left|\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right| = {} & \frac{\det\left(I - X^i\bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\left|\det\left(I - Z^i_\phi \bar{X}^{i\prime}\right) - \left\langle W^i_\phi, Y^i\right\rangle^{K}\right|^2} \\ & \times \left\{ \sum_{\substack{1\le k\le m\\1\le l\le n}} \left| \sum_{\substack{1\le u\le m\\1\le v\le n}} \mathscr{F}_{uv}\left(Z^i_\phi, X^i\right)\frac{\partial\phi_{uv}}{\partial z^i_{kl}}\left(Z^i, W^i\right) \right.\right. \\ & \left. + \frac{K}{m}\left\langle W^i_\phi, Y^i\right\rangle^{(K/m)-1}\bar{Y}^{i\prime}\frac{\partial\phi_{mn+1}}{\partial z^i_{kl}}\left(Z^i, W^i\right)\right|^2 \\ & + \left| \sum_{\substack{1\le u\le m\\1\le v\le n}} \mathscr{F}_{uv}\left(Z^i_\phi, X^i\right)\frac{\partial\phi_{uv}}{\partial W^i}\left(Z^i, W^i\right) \right. \\ & \left.\left. + \frac{K}{m}\left\langle W^i_\phi, Y^i\right\rangle^{(K/m)-1}\bar{Y}^{i\prime}\frac{\partial\phi_{mn+1}}{\partial W^i}\left(Z^i, W^i\right)\right|^2\right\}^{1/2},\end{aligned} \tag{77}$$

and we have

$$\begin{aligned}\left|\nabla\left(\psi C_\phi f_i\right)\left(Z^i, W^i\right)\right| &= \left|\nabla\psi\left(Z^i, W^i\right)\cdot\left(C_\phi f_i\right)\left(Z^i, W^i\right) + \psi\left(Z^i, W^i\right)\cdot\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right| \\ &= \left|\nabla\psi\left(Z^i, W^i\right)\cdot f_i\left(\phi\left(Z^i, W^i\right)\right) + \psi\left(Z^i, W^i\right)\cdot\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right|.\end{aligned} \tag{78}$$

Let

$$\left(X^i, Y^i\right) = \left(Z^i_\phi, W^i_\phi\right) = \phi\left(Z^i, W^i\right). \tag{79}$$

Since $f_i(\phi(Z^i, W^i)) = 1$ and (78), we obtain that

$$\begin{aligned}\left\|\psi C_\phi f_i\right\|_{\mathscr{B}^{(\alpha,m)}} \ge {} & \left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \left|\nabla\left(\psi C_\phi f_i\right)\left(Z^i, W^i\right)\right| \\ = {} & \left|\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \nabla\psi\left(Z^i, W^i\right)\right. \\ & \left. + \left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \psi\left(Z^i, W^i\right)\cdot\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right| \\ \ge {} & \left|\left|\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \nabla\psi\left(Z^i, W^i\right)\right|\right. \\ & \left. - \left|\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \psi\left(Z^i, W^i\right)\cdot\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right|\right| \\ = {} & \left|\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \left|\nabla\psi\left(Z^i, W^i\right)\right|\right. \\ & \left. - \left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \left|\psi\left(Z^i, W^i\right)\right|\left|\nabla\left(C_\phi f_i\right)\left(Z^i, W^i\right)\right|\right| \\ = {} & \left|\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{\frac{2K}{m}}\right]^\alpha \left|\nabla\psi\left(Z^i, W^i\right)\right|\right. \\ & \left. \cdot - \frac{\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha}{\det\left(I - Z^i_\phi\bar{Z}^{i\prime}_\phi\right)^{1/m} - \left|W^i_\phi\right|^{2K/m}}\right|\left|\psi\left(Z^i, W^i\right)\right| G\left(Z^i, W^i\right)\Big|,\end{aligned} \tag{80}$$

where

$$\begin{aligned}G\left(Z^i, W^i\right) = {} & \left\{ \sum_{\substack{1\le k\le m\\1\le l\le n}} \left| \sum_{\substack{1\le u\le m\\1\le v\le n}} \mathscr{F}_{uv}\left(Z^i_\phi, Z^i_\phi\right)\frac{\partial\phi_{uv}}{\partial z^i_{kl}}\left(Z^i, W^i\right) \right.\right. \\ & \left. + \frac{K}{m}\left|W^i_\phi\right|^{(2K/m)-2}\bar{W}^{i\prime}_\phi\frac{\partial\phi_{mn+1}}{\partial z^i_{kl}}\left(Z^i, W^i\right)\right|^2 \\ & + \left| \sum_{\substack{1\le u\le m\\1\le v\le n}} \mathscr{F}_{uv}\left(Z^i_\phi, Z^i_\phi\right)\frac{\partial\phi_{uv}}{\partial W^i}\left(Z^i, W^i\right) \right. \\ & \left.\left. + \frac{K}{m}\left|W^i_\phi\right|^{(2K/m)-2}\bar{W}^{i\prime}_\phi\frac{\partial\phi_{mn+1}}{\partial W^i}\left(Z^i, W^i\right)\right|^2\right\}^{1/2}.\end{aligned} \tag{81}$$

So we get

$$\begin{aligned}& \lim_{\phi\left(Z^i, W^i\right)\longrightarrow\partial\mathrm{Y}_\mathrm{I}} \frac{\left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha}{\det\left(I - Z^i_\phi\bar{Z}^{i\prime}_\phi\right)^{1/m} - \left|W^i_\phi\right|^{2K/m}} \cdot \left|\psi\left(Z^i, W^i\right)\right| G\left(Z^i, W^i\right) \\ & = \lim_{\phi\left(Z^i, W^i\right)\longrightarrow\partial\mathrm{Y}_\mathrm{I}} \left[\det\left(I - Z^i\bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha \left|\nabla\psi\left(Z^i, W^i\right)\right|,\end{aligned} \tag{82}$$

if one of these two limits exists.

Next, let

$$g_i(Z,W) := \frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}} - \left\{\frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}}\right\}^{1/2}. \tag{83}$$

for a sequence $\{(Z^i, W^i)\}_{i\geq 1}$ in Y_I such that $\phi(Z^i, W^i) \longrightarrow \partial Y_I$, as $i \longrightarrow \infty$. Then,

$$\begin{aligned} |g_i(Z,W)| &= \left|\frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}} + \left\{\frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}}\right\}^{1/2}\right. \\ &\quad \cdot \left| \leq \left|\frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}}\right| + \left\{\frac{\det\left(I - X^i \bar{X}^{i\prime}\right)^{1/m} - \left|Y^i\right|^{2K/m}}{\det\left(I - Z\bar{X}^{i\prime}\right)^{1/m} - \left\langle W, Y^i\right\rangle^{K/m}}\right\}^{1/2}\right. \\ &= |f_i(Z,W)| + |f_i(Z,W)|^{1/2}, \end{aligned} \tag{84}$$

It is easy to obtain $\{g_i\}_{i\geq 1}$ is a bounded sequence in H^∞ and $g_i \longrightarrow 0$ uniformly on every compact subset of Y_I. Moreover, we notice that $g_i(\phi(Z^i, W^i)) = 0$ and

$$\nabla g_i\left(\phi\left(Z^i, W^i\right)\right) = \frac{G\left(Z^i, W^i\right)}{2\left[\det\left(I - Z^i_\phi \bar{Z}^{i\prime}_\phi\right)^{1/m} - \left|W^i_\phi\right|^{2K/m}\right]}. \tag{85}$$

By the similar method as above,

$$\begin{aligned} 0 &\longleftarrow \left\|\psi C_\phi g_i\right\|_{\mathscr{B}^{(\alpha,m)}} \\ &\geq \left[\det\left(I - Z^i \bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha \left|\nabla\left(\psi C_\phi g_i\right)\left(Z^i, W^i\right)\right| \\ &= \left[\det\left(I - Z^i \bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha \left|\nabla\psi\left(Z^i, W^i\right)\right. \\ &\quad \left.\cdot \left(C_\phi g_i\right)\left(Z^i, W^i\right) + \psi\left(Z^i, W^i\right) \cdot \nabla\left(C_\phi g_i\right)\left(Z^i, W^i\right)\right| \\ &= \left[\det\left(I - Z^i \bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha \\ &\quad \cdot \left|0 + \psi\left(Z^i, W^i\right) \cdot \frac{G\left(Z^i, W^i\right)}{2\left[\det\left(I - Z^i_\phi \bar{Z}^{i\prime}_\phi\right)^{1/m} - \left|W^i_\phi\right|^{2K/m}\right]}\right| \\ &= \frac{\left[\det\left(I - Z^i \bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha}{2\left[\det\left(I - Z^i_\phi \bar{Z}^{i\prime}_\phi\right)^{1/m} - \left|W^i_\phi\right|^{2K/m}\right]} \left|\psi\left(Z^i, W^i\right)\right| G\left(Z^i, W^i\right). \end{aligned} \tag{86}$$

And by (82),

$$\lim_{\phi(Z^i, W^i) \longrightarrow \partial Y_I} \left[\det\left(I - Z^i \bar{Z}^{i\prime}\right)^{1/m} - \left|W^i\right|^{2K/m}\right]^\alpha \left|\nabla\psi\left(Z^i, W^i\right)\right| = 0. \tag{87}$$

All of the proofs are complete. □

Remark 21. Let $m = 1$, $W = 0$, and $K = 1$; we get the following results in the case of the unit ball $\mathbb{B} = \{Z \in \mathbb{C}^n : |Z|^2 < 1\}$. Let $\alpha = 1$. If

$$\begin{aligned} &\lim_{|Z| \longrightarrow 1} \left(1 - |Z|^2\right) |\nabla\psi(Z)| = 0, \\ &\lim_{|Z| \longrightarrow 1} \frac{|\psi(Z)| \left(1 - |Z|^2\right)}{1 - |\phi(Z)|^2} |D\phi(Z)| = 0, \end{aligned} \tag{88}$$

then, the weighted composition operator $\psi C_\phi : H^\infty(\mathbb{B}) \longrightarrow \mathscr{B}(\mathbb{B})$ is compact. Conversely, the weighted composition operator $\psi C_\phi : H^\infty(\mathbb{B}) \longrightarrow \mathscr{B}(\mathbb{B})$ is compact; then,

$$\begin{aligned} &\lim_{\phi(Z) \longrightarrow 1} \left(1 - |Z|^2\right) |\nabla\psi(Z)| = 0, \\ &\lim_{\phi(Z) \longrightarrow 1} \frac{|\psi(Z)| \left(1 - |Z|^2\right)}{1 - |\phi(Z)|^2} \left|D\phi(Z)' \phi(\bar{Z})'\right| = 0, \end{aligned} \tag{89}$$

where

$$|D\phi(Z)| = \left(\sum_{k,l=1}^{n} \left|\frac{\partial\phi_l(Z)}{\partial Z_k}\right|^2\right)^{1/2}. \tag{90}$$

It turns out to be the same as the results obtained by Li and Stević in [7].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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