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Research Article

An Application of Pascal Distribution Series on Certain Analytic Functions Associated with Stirling Numbers and Sălăgean Operator

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In the present paper, we will observe that the Sălăgean differential operator can be written in terms of Stirling numbers. Furthermore, we find a necessary and sufficient condition and inclusion relation for Pascal distribution series to be in the class $\mathbb{P}_k(\lambda, \alpha)$ of analytic functions with negative coefficients defined by the Sălăgean differential operator. Also, we consider an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

1. Preliminaries

Special functions are used in many applications of physics, engineering, and applied mathematics and statistics. Special polynomials have a close connection with number theory, and one of the most important sets of special numbers is the class of Stirling numbers (of the first and second kind), introduced in 1730 by the Scottish mathematician James Stirling.

In combinatorics, a Stirling number of the second kind (or Stirling partition number) is the number of ways to partition a set of k objects into j nonempty subsets and is denoted by $S(k, j)$ or by $b_{k,j}$ as used in this paper. These numbers occur in the field of mathematics called combinatorics and the study of partitions. In this paper, we will observe that the Sălăgean differential operator D^k can be written in terms of Stirling numbers.

Let $\mathbb{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Furthermore, let $\mathcal{T}$ be a subclass of $\mathbb{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{D}. \quad (2)$$

For a function $f(z)$ in $\mathbb{A}$, we define

$$D^0 f(z) = f(z), \quad (3)$$

$$D^1 f(z) = z f'(z), \quad (4)$$

and in general, we have

$$D^k f(z) = z\left(D^{k-1} f(z)\right)', \quad k \in \mathbb{N}. \quad (5)$$

The differential operator D^k was introduced by Sălăgean [1].

We note that

$$D^k f(z) = z\left(D^{k-1} f(z)\right)' = b_{k,1} z f'(z) + b_{k,2} z^2 f''(z) + b_{k,3} z^3 f'''(z) + \cdots + b_{k,k} z^k f^{(k)}(z) = \sum_{j=1}^{k} b_{k,j} z^j f^{(j)}(z), \quad (k \in \mathbb{N}), \tag{6}$$

where

$$b_{k,j} = j b_{k-1,j} + b_{k-1,j-1}, \text{ and } b_{k,1} = b_{k,k} = 1. \tag{7}$$

For example,

(i) If $k = 2$, we have

$$D^2 f(z) = z\left(D^1 f(z)\right)' = z f'(z) + z^2 f''(z) = b_{21} z f'(z) + b_{22} z^2 f''(z), \tag{8}$$

where

$$b_{21} = b_{22} = 1. \tag{9}$$

(ii) If $k = 3$, we have

$$D^3 f(z) = z\left(D^2 f(z)\right)' = z f'(z) + 3z^2 f''(z) + z^3 f'''(z) = b_{3,1} z f'(z) + b_{3,2} z^2 f''(z) + b_{3,3} z^3 f'''(z), \tag{10}$$

where

$$\left.\begin{aligned} b_{3,2} &= 2b_{2,2} + b_{2,1} = 3, \\ b_{3,1} &= b_{3,3} = 1 \end{aligned}\right]. \tag{11}$$

(iii) If $k = 4$, we have

$$D^4 f(z) = z\left(D^3 f(z)\right)' = z f'(z) + 7z^2 f''(z) + 6z^3 f'''(z) + z^4 f^{(4)}(z) = b_{4,1} z f'(z) + b_{4,2} z^2 f''(z) + b_{4,3} z^3 f'''(z) + b_{4,4} z^4 f^{(4)}(z), \tag{12}$$

where

$$\begin{aligned} b_{4,2} &= 2b_{3,2} + b_{3,1} = 7, \\ b_{4,3} &= 3b_{3,3} + b_{3,2} = 6, \\ b_{4,1} &= b_{4,4} = 1. \end{aligned} \tag{13}$$

(iv) If $k = 5$, we have

$$D^5 f(z) = z\left(D^4 f(z)\right)' = z f'(z) + 15z^2 f''(z) + 25z^3 f'''(z) + 10z^4 f^{(4)}(z) + z^5 f^{(5)}(z) = b_{5,1} z f'(z) + b_{5,2} z^2 f''(z) + b_{5,3} z^3 f'''(z) + b_{5,4} z^4 f^{(4)}(z) + b_{5,5} z^5 f^{(5)}(z), \tag{14}$$

where

$$\begin{aligned} b_{5,2} &= 2b_{4,2} + b_{4,1} = 15, \\ b_{5,3} &= 3b_{4,3} + b_{4,2} = 25, \\ b_{5,4} &= 4b_{4,4} + b_{4,3} = 10, \\ b_{5,1} &= b_{5,5} = 1. \end{aligned} \tag{15}$$

Table 1 represents the coefficients b_{kj} of $z^k f^{(k)}(z)$.

Table 1 (see [2]) shows the first few possibilities for Stirling numbers of the second kind. Also, from this table, we note that:

(1) $b_{k,j} = j b_{k-1,j} + b_{k-1,j-1}$
(2) $b_{k,k-1} = (k(k-1)/2) = \binom{k}{2}$
(3) $b_{k,2} = 2^{k-1} - 1$
(4) $b_{k,3} = (1/6)(3^k - 3.2^k + 3)$
(5) $b_{k,1} = b_{k,k} = 1$
(6) $b_{k,j} = 0$, when $j > k$.
(7) $b_{p,j} \equiv 0 \pmod{p}$ iff $1 < j < p$, where p is a prime number.

Furthermore, for $k = 2, 3, 4, 5$, we observe that

Table 1: Stirling numbers of the second kind.

	$zf'(z)$	$z^2f''(z)$	$z^3f'''(z)$	$z^4f^{(4)}(z)$	$z^5f^{(5)}(z)$	$z^6f^{(6)}(z)$	$z^7f^{(7)}(z)$	$\cdots$	$z^kf^{(k)}(z)$
$D^1f(z)$	1	0	0	0	0	0	0	0	0
$D^2f(z)$	1	1	0	0	0	0	0	0	0
$D^3f(z)$	1	3	1	0	0	0	0	0	0
$D^4f(z)$	1	7	6	1	0	0	0	0	0
$D^5f(z)$	1	15	25	10	1	0	0	0	0
$D^6f(z)$	1	31	90	65	15	1	0	0	0
$D^7f(z)$	1	63	301	350	140	21	1	0	0
$\vdots$	1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0
$D^kf(z)$	1	$b_{k,2}$	$b_{k,3}$	$b_{k,4}$	$b_{k,5}$	$b_{k,6}$	$b_{k,7}$	$\cdots$	1

$$\begin{aligned} D^2f(z) &= z^2f''(z) + zf'(z), \\ z + \sum_{n=2}^{\infty} n^2 a_n z^n &= z + \sum_{n=2}^{\infty} [n(n-1) + n] a_n z^n, \end{aligned} \tag{16}$$

$$\begin{aligned} D^3f(z) &= z^3f'''(z) + 3z^2f''(z) + zf'(z), \\ z + \sum_{n=2}^{\infty} n^3 a_n z^n &= z + \sum_{n=2}^{\infty} [n(n-1)(n-2) \\ &\quad + 3n(n-1) + n] a_n z^n, \end{aligned} \tag{17}$$

$$\begin{aligned} D^4f(z) &= z^4f^{(4)}(z) + 6z^3f'''(z) + 7z^2f''(z) + zf'(z), \\ z + \sum_{n=2}^{\infty} n^4 a_n z^n &= z + \sum_{n=2}^{\infty} [n(n-1)(n-2)(n-3) + 6n(n-1)(n-2) + 7n(n-1) + n]. \end{aligned} \tag{18}$$

and

$$\begin{aligned} D^5f(z) &= z^5f^{(5)}(z) + 15z^4f^{(4)}(z) + 25z^3f'''(z) + 10z^2f''(z) + zf'(z), \\ z + \sum_{n=2}^{\infty} n^5 a_n z^n &= z + \sum_{n=2}^{\infty} [n(n-1)(n-2)(n-3)(n-4) + 10n(n-1)(n-2)(n-3) \\ &\quad + 25n(n-1)(n-2) + 15n(n-1) + n] a_n z^n. \end{aligned} \tag{19}$$

From (16)–(19), we conclude that

$$\begin{aligned} n &= (n-1) + 1 = b_{2,2}(n-1) + b_{2,1}, \\ n^2 &= (n-1)(n-2) + 3(n-1) + 1 = b_{3,3}(n-1)(n-2) + b_{3,2}(n-1) + b_{3,1}, \\ n^3 &= (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1 \\ &= b_{4,4}(n-1)(n-2)(n-3) + b_{4,3}(n-1)(n-2) + b_{4,2}(n-1) + b_{4,1}, \end{aligned} \tag{20}$$

$$\begin{aligned} n^4 &= (n-1)(n-2)(n-3)(n-4) + 10(n-1)(n-2)(n-3) + 25(n-1)(n-2) + 15(n-1) + 1 \\ &= b_{5,5}(n-1)(n-2)(n-3)(n-4) + b_{5,4}(n-1)(n-2)(n-3) + b_{5,3}(n-1)(n-2) + b_{5,2}(n-1) + b_{5,1}. \end{aligned} \tag{21}$$

In general, we have

$$n^k = b_{k+1,1} + b_{k+1,2}(n-1) + b_{k+1,3}(n-1)(n-2) + \cdots + b_{k+1,k+1}(n-1)(n-2)(n-3)\cdots(n-k)$$
$$= b_{k+1,1} + \sum_{j=1}^{k} b_{k+1,j+1}(n-1)(n-2)(n-3)\cdots(n-j), \quad k = 1,2,3,\cdots. \tag{22}$$

For functions $f \in \mathbb{A}$ given by (1) and $g \in \mathbb{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we recall that the well-known *Hadamard product* of f and g is given by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}. \tag{23}$$

For $\epsilon \in \mathbb{C}/\{0\}$ and $-1 \le \mathfrak{D} < \mathfrak{C} \le 1$, we say that a function $f \in \mathbb{A}$ is in the class $\mathscr{R}^{\epsilon}(\mathfrak{C}, \mathfrak{D})$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(\mathfrak{C} - \mathfrak{D})\varepsilon - \mathfrak{D}[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{D}. \tag{24}$$

The function class $\mathscr{R}^{\epsilon}(\mathfrak{C}, \mathfrak{D})$ was introduced by Dixit and Pal [3].

With the help of the differential operator D^k, we say that a function $f(z)$ belonging to $\mathbb{A}$ is said to be in the class $\mathcal{Q}(\lambda, \alpha)$ if and only if

$$\Re\left\{ \frac{(1-\lambda)z\left(D^k f(z)\right)' + \lambda z\left(D^{k+1} f(z)\right)'}{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{25}$$

for some $\alpha (0 \le \alpha < 1)$, $\lambda (0 \le \lambda \le 1)$, and for all z in $\mathbb{D}$.

Furthermore, we define the class $\mathbb{P}_k(\lambda, \alpha)$ by

$$\mathbb{P}_k(\lambda, \alpha) = \mathcal{Q}(\lambda, \alpha) \cap \mathcal{T}. \tag{26}$$

The class $\mathbb{P}_{\mathcal{T}}(\lambda, \alpha)$ was introduced and studied by Aouf and Srivatava [4].

We note that, by specializing the parameters k and λ, we obtain the following subclasses:

(i) $\mathbb{P}_0(0, \alpha) = \mathcal{T}^*(\alpha)$ and $\mathbb{P}_0(1, \alpha) = \mathcal{C}(\alpha)$, where $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ represent the classes of starlike functions of order α and convex functions of order α with negative coefficients, respectively, introduced and studied by Silverman [5]

(ii) $\mathbb{P}_k(1, \alpha) = \mathcal{C}_k(\alpha)$ (see [4]), where $\mathcal{C}_k(\alpha)$ represents the class of functions $f(z) \in \mathcal{T}$ satisfying the inequality

$$\Re\left\{ \frac{z\left(D^{k+1} f(z)\right)'}{D^{k+1} f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0). \tag{27}$$

(iii) $\mathbb{P}_k(0, \alpha) = \mathcal{T}_k^*(\alpha)$ (see [4]), where $\mathcal{T}_k^*(\alpha)$ represents the class of functions $f(z) \in \mathcal{T}$ satisfying the inequality

$$\Re\left\{ \frac{z\left(D^k f(z)\right)'}{D^k f(z)} \right\} > \alpha, \quad (k \in \mathbb{N}_0). \tag{28}$$

In statistics and probability, distributions of random variables play a basic role and are used extensively to describe and model a lot of real-life phenomenon; they describe the distribution of the probabilities over the values of the random variable. In recent years, many researchers have examined some important features in the geometric function theory, such as coefficient estimates, inclusion relations, and conditions of being in some known classes, using different probability distributions such as the Poisson, Pascal, Borel, Mittag–Leffler-type Poisson distribution, etc.(see, for example, [6–10]).

The probability density function of a discrete random variable X which follows the Pascal distribution is given by

$$Prob(X = r) = \binom{r+m-1}{m-1} \sigma^r (1-\sigma)^m, \quad r = 0,1,2,3,\ldots. \tag{29}$$

Very recently, El-Deeb et al. [11] introduced a power series whose coefficients are probabilities of the Pascal distribution

$$\Lambda_{\sigma}^{m}(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \sigma^{n-1} (1-\sigma)^m z^n, \quad z \in \mathbb{D} \tag{30}$$

where $m \ge 1$ and $0 \le \sigma \le 1$ and we note that, by a ratio test, the radius of convergence of above series is infinity. We also define the series

$$\Upsilon_{\sigma}^{m}(z) = 2z - \Lambda_{\sigma}^{m}(z)$$
$$= z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \sigma^{n-1} (1-\sigma)^m z^n, \quad z \in \mathbb{D}. \tag{31}$$

Now, we considered the linear operator

$$\mathcal{J}_\sigma^m(z): \mathbb{A} \longrightarrow \mathbb{A} \tag{32}$$

defined by the Hadamard product

$$\begin{aligned}\mathcal{J}_\sigma^m f(z) &= \Lambda_\sigma^m(z) * f(z)\\ &= z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \sigma^{n-1}(1-\sigma)^m a_n z^n, \quad z \in \mathbb{D}.\end{aligned} \tag{33}$$

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions, generalized Bessel functions, Struve functions, Poisson distribution series, and Pascal distribution series (see, for example, [12], [13–15], [7–9, 16–23], [24]), we determine a necessary and sufficient condition for $\Upsilon_\sigma^m(z)$ to be in our class $\mathbb{P}_k(\lambda, \alpha)$. Furthermore, we give sufficient conditions for $\mathcal{J}_\sigma^m(\mathscr{R}^\epsilon(\mathfrak{C}, \mathfrak{D})) \subset \mathbb{P}_k(\lambda, \alpha)$. Finally, we give conditions for the integral operator $\mathscr{G}_\sigma^m f(z) = \int_0^z (Y_\sigma^m(t)/t)\mathrm{d}t$ belonging to the class $\mathbb{P}_k(\lambda, \alpha)$.

The following results will be required in our investigation.

Lemma 1 (see [4]). *Let the function $f(z)$ be defined by (2). Then, $f(z) \in \mathbb{P}_k(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n^k (n-\alpha)(1+(n-1)\lambda)|a_n| \le 1-\alpha, \quad z \in \mathbb{D}. \tag{34}$$

The result (34) is sharp.

Lemma 2 (see [3]). *If $f \in \mathscr{R}^\epsilon(\mathfrak{C}, \mathfrak{D})$ is of the form (1), then*

$$|a_n| \le (\mathfrak{C} - \mathfrak{D}) \frac{|\epsilon|}{n}, \quad n \in \mathbb{N} - \{1\}. \tag{35}$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + (\mathfrak{C} - \mathfrak{D}) \frac{\epsilon t^{n-1}}{1 + \mathfrak{D} t^{n-1}}\right) \mathrm{d}t, \quad (z \in \mathbb{D}; n \in \mathbb{N} - \{1\}). \tag{36}$$

2. Necessary and Sufficient Conditions

By simple calculations, we derive the following relations:

$$\begin{aligned}
\sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \sigma^{n-1} &= \frac{1}{(1-\sigma)^m} - 1,\\
\sum_{n=5}^{\infty} (n-1)(n-2)(n-3)(n-4) \binom{n+m-2}{m-1} \sigma^{n-1} &= \frac{24\sigma^4 \binom{m+3}{m-1}}{(1-\sigma)^{m+4}},\\
\sum_{n=4}^{\infty} (n-1)(n-2)(n-3) \binom{n+m-2}{m-1} \sigma^{n-1} &= \frac{6\sigma^3 \binom{m+2}{m-1}}{(1-\sigma)^{m+3}},\\
\sum_{n=3}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} \sigma^{n-1} &= \frac{2\sigma^2 \binom{m+1}{m-1}}{(1-\sigma)^{m+2}},\\
\sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} \sigma^{n-1} &= \frac{\sigma \binom{m}{m-1}}{(1-\sigma)^{m+1}},
\end{aligned} \tag{37}$$

and, in general, for $s = 1, 2, 3, \ldots$, we have

$$\sum_{n=s+1}^{\infty} (n-1)(n-2)(n-3)\cdots(n-s)\binom{n+m-2}{m-1}\sigma^{n-1} = s!\sigma^{s}\frac{\binom{m+s-1}{m-1}}{(1-\sigma)^{m+s}}. \tag{38}$$

Unless otherwise mentioned, we shall assume in this paper that $0 \le \alpha < 1$, $0 \le \lambda \le 1$ $m \ge 1$, and $0 \le \sigma < 1$.

First of all, with the help of Lemma 1, we obtain the following necessary and sufficient condition for $\Upsilon_{\sigma}^{m}(z)$ to be in $\mathbb{P}_k(\lambda, \alpha)$.

Theorem 1. *Let $k \ge 1$. Then, $\Upsilon_{\sigma}^{m}(z) \in \mathbb{P}_k(\lambda, \alpha)$ if and only if*

$$\begin{aligned}
&\sum_{j=1}^{k}\left(\lambda b_{k+3,j+1} + (1-\lambda-\alpha\lambda)b_{k+2,j+1} + \alpha(\lambda-1)b_{k+1,j+1}\right)j!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{m+j}}\\
&\quad+\left(\lambda b_{k+3,k+2} + (1-\lambda-\alpha\lambda)\right)(k+1)!\sigma^{k+1}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{m+k+1}}\\
&\quad+\lambda(k+2)!\sigma^{k+2}\frac{\binom{m+k+1}{m-1}}{(1-\sigma)^{m+k+2}}\\
&\quad\le 1-\alpha.
\end{aligned} \tag{39}$$

Proof. In view of Lemma 1, we only need to show that $Q \le 1-\alpha$, where

$$Q = \sum_{n=2}^{\infty} n^{k}(n-\alpha)(1+(n-1)\lambda)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}. \tag{40}$$

Using (22) and (38), we have

$$\begin{aligned}
Q &= \sum_{n=2}^{\infty}\left[\lambda n^{k+2} + (1-\lambda-\alpha\lambda)n^{k+1} - \alpha(1-\lambda)n^{k}\right]\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}\\
&= \sum_{n=2}^{\infty}\Bigg[\lambda\left(b_{k+3,1} + \sum_{j=1}^{k+2} b_{k+3,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\\
&\quad+(1-\lambda-\alpha\lambda)\left(b_{k+2,1} + \sum_{j=1}^{k+1} b_{k+2,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\\
&\quad+\alpha(\lambda-1)\left(b_{k+1,1} + \sum_{j=1}^{k} b_{k+1,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\Bigg]
\end{aligned}$$

$$\times\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$=\sum_{n=2}^{\infty}\lambda b_{k+3,1}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\lambda b_{k+3,k+3}(n-1)(n-2)(n-3)\cdots(n-(k+2))\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\lambda b_{k+3,k+2}(n-1)(n-2)(n-3)\cdots(n-(k+1))\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{k}\lambda b_{k+3,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}(1-\lambda-\alpha\lambda)b_{k+2,1}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}(1-\lambda-\alpha\lambda)b_{k+2,k+2}(n-1)(n-2)(n-3)\cdots(n-(k+1))\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{k}(1-\lambda-\alpha\lambda)b_{k+2,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\alpha(\lambda-1)b_{k+1,1}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{k}\alpha(\lambda-1)b_{k+1,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$=\left(\lambda b_{k+3,1}+(1-\lambda-\alpha\lambda)b_{k+2,1}+\alpha(\lambda-1)b_{k+1,1}\right)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\lambda b_{k+3,k+3}\sum_{n=k+2}^{\infty}(n-1)(n-2)(n-3)\cdots(n-(k+2))\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\left(\lambda b_{k+3,k+2}+(1-\lambda-\alpha\lambda)b_{k+2,k+2}\right)\sum_{n=k+1}^{\infty}(n-1)(n-2)(n-3)\cdots(n-(k+1))$$

$$\times\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}$$

$$+\left(\sum_{j=1}^{k}\left(\lambda b_{k+3,j+1}+(1-\lambda-\alpha\lambda)b_{k+2,j+1}+\alpha(\lambda-1)b_{k+1,j+1}\right)\right)$$

$$\times\left(\sum_{n=j+1}^{\infty}(n-1)(n-2)(n-3)\cdots(n-j)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}\right)$$

$$
\begin{aligned}
&= (1-\alpha)\left(1-(1-\sigma)^m\right) \\
&+ \lambda (k+2)!\sigma^{k+2} \frac{\begin{pmatrix} m+k+1 \\ m-1 \end{pmatrix}}{(1-\sigma)^{k+2}} \\
&+ \left(\lambda b_{k+3,k+2} + (1-\lambda-\alpha\lambda)\right)(k+1)!\sigma^{k+1} \frac{\begin{pmatrix} m+k \\ m-1 \end{pmatrix}}{(1-\sigma)^{k+1}} \\
&+ \sum_{j=1}^{k} \left(\lambda b_{k+3,j+1} + (1-\lambda-\alpha\lambda) b_{k+2,j+1} + \alpha(\lambda-1) b_{k+1,j+1}\right) j!\sigma^j \frac{\begin{pmatrix} m+j-1 \\ m-1 \end{pmatrix}}{(1-\sigma)^j}.
\end{aligned} \tag{41}
$$

Therefore, we see that the last expression is bounded above by $1-\alpha$ if (39) is satisfied. □

3. Inclusion Properties

Making use of Lemma 2, we will study the action of the Pascal distribution series on the class $\mathbb{P}_k(\lambda, \alpha)$.

Theorem 2. *Let $k \geq 2$ and $f \in \mathscr{R}^{\epsilon}(\mathfrak{C}, \mathfrak{D})$. Then, $\mathscr{I}_{\sigma}^{m} f(z) \in \mathbb{P}_k(\lambda, \alpha)$ if*

$$
(\mathfrak{C}-\mathfrak{D})|\epsilon| \left[\begin{aligned}
&\sum_{j=1}^{k-1} \left[\lambda b_{k+2,j+1} + (1-\lambda-\alpha\lambda) b_{k+1,j+1} + \alpha(\lambda-1) b_{k,j+1}\right](j)!\sigma^j \frac{\begin{pmatrix} m+j-1 \\ m-1 \end{pmatrix}}{(1-\sigma)^j} + \lambda(k+1)!\sigma^{(k+1)} \frac{\begin{pmatrix} m+k \\ m-1 \end{pmatrix}}{(1-\sigma)^{(k+1)}} + \left(\lambda b_{k+2,k+1} + (1-\lambda-\alpha\lambda)\right)(k)!\sigma^k \frac{\begin{pmatrix} m+k-1 \\ m-1 \end{pmatrix}}{(1-\sigma)^k} \\
&+ (1-\alpha)\left(1-(1-\sigma)^m\right) \sum_{j=1}^{k-1} \left[\lambda b_{k+2,j+1} + (1-\lambda-\alpha\lambda) b_{k+1,j+1} + \alpha(\lambda-1) b_{k,j+1}\right](j)!\sigma^j \frac{\begin{pmatrix} m+j-1 \\ m-1 \end{pmatrix}}{(1-\sigma)^j} \\
&+ \lambda(k+1)!\sigma^{(k+1)} \frac{\begin{pmatrix} m+k \\ m-1 \end{pmatrix}}{(1-\sigma)^{(k+1)}} + \left(\lambda b_{k+2,k+1} + (1-\lambda-\alpha\lambda)\right)(k)!\sigma^k \frac{\begin{pmatrix} m+k-1 \\ m-1 \end{pmatrix}}{(1-\sigma)^k} + (1-\alpha)\left(1-(1-\sigma)^m\right)
\end{aligned} \right]
\leq 1-\alpha. \tag{42}
$$

Proof. In view of Lemma 1, it suffices to show that $L \leq 1-\alpha$, where

$$L=\sum_{n=2}^{\infty} n^{k}(n-\alpha)(1+(n-1)\lambda)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}\left|a_{n}\right|. \tag{43}$$

Applying Lemma 2, we find from equations (22) and (38) that

$$\begin{aligned}
L \leq\; & (\mathfrak{C}-\mathfrak{D})|\epsilon|\left[\sum_{n=2}^{\infty}\left(\lambda n^{k+1}+(1-\lambda-\alpha\lambda)n^{k}-\alpha(1-\lambda)n^{k-1}\right)\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}\right] \\
=\; & (\mathfrak{C}-\mathfrak{D})|\epsilon|(1-t\sigma)^{m}\left[\sum_{n=2}^{\infty}\left[\lambda\left(b_{k+2,1}+\sum_{j=1}^{k+1} b_{k+2,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\right.\right. \\
& +(1-\lambda-\alpha\lambda)\left(b_{k+1,1}+\sum_{j=1}^{k} b_{k+1,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right) \\
& \left.\left.+\alpha(\lambda-1)\left(b_{k,1}+\sum_{j=1}^{k-1} b_{k,j+1}(n-1)(n-2)(n-3)\cdots(n-j)\right)\right]\binom{n+m-2}{m-1}\sigma^{n-1}\right] \\
=\; & (\mathfrak{C}-\mathfrak{D})|\epsilon|(1-\sigma)^{m}\left[\sum_{n=2}^{\infty}\left(\lambda b_{k+2,1}+(1-\lambda-\alpha\lambda)b_{k+1,1}+\alpha(\lambda-1)b_{k,1}\right)\binom{n+m-2}{m-1}\sigma^{n-1}\right. \\
& +\lambda b_{k+2,k+2}\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\cdots(n-(k+1))\binom{n+m-2}{m-1}\sigma^{n-1} \\
& +\lambda b_{k+2,k+1}\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\cdots(n-k)\binom{n+m-2}{m-1}\sigma^{n-1} \\
& +(1-\lambda-\alpha\lambda)b_{k+1,k+1}\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\cdots(n-k)\binom{n+m-2}{m-1}\sigma^{n-1} \\
& +\left(\lambda\sum_{j=1}^{k-1} b_{k+2,j+1}+(1-\lambda-\alpha\lambda)\sum_{j=1}^{k-1} b_{k+1,j+1}+\alpha(\lambda-1)\sum_{j=1}^{k-1} b_{k,j+1}\right) \\
& \left.\times\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\cdots(n-j)\binom{n+m-2}{m-1}\sigma^{n-1}\right] \\
=\; & (\mathfrak{C}-\mathfrak{D})|\epsilon|\left[\left(\lambda b_{k+2,1}+(1-\lambda-\alpha\lambda)b_{k+1,1}+\alpha(\lambda-1)b_{k,1}\right)\left(1-(1-\sigma)^{m}\right)\right. \\
& +\lambda b_{k+2,k+2}(k+1)!\sigma^{(k+1)}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{(k+1)}}+\lambda b_{k+2,k+1}(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{k}} \\
& +\lambda\sum_{j=1}^{k-1} b_{k+2,j+1}(j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}+(1-\lambda-\alpha\lambda)b_{k+1,k+1}(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{k}}
\end{aligned}$$

$$+(1-\lambda-\alpha\lambda)\sum_{j=1}^{k-1} b_{k+1,j+1}(j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}+\alpha(\lambda-1)\sum_{j=1}^{k-1} b_{k,j+1}(j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}\Bigg]$$

$$=(\mathfrak{C}-\mathfrak{D})|\epsilon|\Bigg[\sum_{j=1}^{k-1}\left[\lambda b_{k+2,j+1}+(1-\lambda-\alpha\lambda)b_{k+1,j+1}+\alpha(\lambda-1)b_{k,j+1}\right](j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}$$
$$+\lambda(k+1)!\sigma^{(k+1)}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{(k+1)}}+\left(\lambda b_{k+2,k+1}+(1-\lambda-\alpha\lambda)\right)(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{k}}$$
$$+(1-\alpha)\left(1-(1-\sigma)^{m}\right)\Bigg]. \quad (44)$$

However, this last expression is bounded by $1-\alpha$, if (42) holds. This completes the proof of Theorem 2. □

4. An Integral Operator

In this section, we consider the integral operator $\mathscr{G}_{\sigma}^{m}$ defined by

$$\mathscr{G}_{\sigma}^{m} f(z)=\int_{0}^{z}\frac{\Upsilon_{\sigma}^{m}(t)}{t}dt. \quad (45)$$

Theorem 3. *Let $k\geq 2$. Then, the integral operator $\mathscr{G}_{\sigma}^{m} f(z)$ defined by (45) is in the class $\mathbb{P}_{k}(\lambda,\alpha)$ if and only if*

$$\sum_{j=1}^{k-1}\left[\lambda b_{k+2,j+1}+(1-\lambda-\alpha\lambda)b_{k+1,j+1}+\alpha(\lambda-1)b_{k,j+1}\right](j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{m+j}}$$
$$+\lambda(k+1)!\sigma^{(k+1)}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{m+k+1}}+\left(\lambda b_{k+2,k+1}+(1-\lambda-\alpha\lambda)\right)(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{m+k}}$$
$$\leq 1-\alpha. \quad (46)$$

Proof. From definitions (31) and (45), it is easily verified that

$$\mathscr{G}_{\sigma}^{m} f(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}\frac{z^{n}}{n}. \quad (47)$$

Then, by Lemma 1, we need only to show that $H\leq 1-\alpha$, where

$$H=\sum_{n=2}^{\infty} n^{k}(n-\alpha)(1+(n-1)\lambda)\times\frac{1}{n}\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}. \quad (48)$$

or equivalently,

$$H=\sum_{n=2}^{\infty}\left(\lambda n^{k+1}+(1-\lambda-\alpha\lambda)n^{k}+\alpha(\lambda-1)n^{k-1}\right)\cdot\binom{n+m-2}{m-1}\sigma^{n-1}(1-\sigma)^{m}. \quad (49)$$

The remaining part of the proof of Theorem 3 is similar to that of Theorem 2, and so, we omit the details. □

5. Corollaries and Consequences

By specializing the parameter $\lambda=1$ in Theorems 1–3, we obtain the following corollaries.

$$\sum_{j=1}^{k-1}\left(b_{k+2,j+1}-\alpha b_{k+1,j+1}\right)j!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}$$

$$+\left(b_{k+2,k+1}-\alpha b_{k+1,k+1}\right)(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{k}}+b_{k+2,k+2}(k+1)!\sigma^{(k+1)}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{(k+1)}} \tag{52}$$

$$+\left(b_{k+2,1}-\alpha b_{k+1,1}\right)\left(1-t(1-\sigma)^{m}\right)\Big]$$

$$\leq 1-\alpha.$$

Corollary 1. *Let $k\geq 1$. Then, $\Upsilon_{\sigma}^{m}(z)\in\mathscr{C}_{k}(\alpha)$ if and only if*

$$\sum_{j=1}^{k}\left(b_{k+3,j+1}-\alpha b_{k+2,j+1}\right)j!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{m+j}}$$

$$+\left(b_{k+3,k+2}-\alpha\right)(k+1)!\sigma^{k+1}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{m+k+1}}+(k+2)!\sigma^{k+2}\frac{\binom{m+k+1}{m-1}}{(1-\sigma)^{m+k+2}} \tag{50}$$

$$\leq 1-\alpha.$$

Corollary 2. *Let $k\geq 2$ and $f\in\mathscr{R}^{\epsilon}(\mathfrak{C},\mathfrak{D})$. Then, $\mathscr{I}_{\sigma}^{m}f(z)\in\mathscr{C}_{k}(\alpha)$ if*

$$(\mathfrak{C}-\mathfrak{D})|\epsilon|\left[\sum_{j=1}^{k-1}\left(b_{k+2,j+1}-\alpha b_{k+1,j+1}\right)(j)!\sigma^{j}\frac{\binom{m+j-1}{m-1}}{(1-\sigma)^{j}}\right.$$

$$+\left(b_{k+2,k+1}-\alpha\right)(k)!\sigma^{k}\frac{\binom{m+k-1}{m-1}}{(1-\sigma)^{k}}+(k+1)!\sigma^{(k+1)}\frac{\binom{m+k}{m-1}}{(1-\sigma)^{(k+1)}} \tag{51}$$

$$+(1-\alpha)\left(1-(1-\sigma)^{m}\right)\Big]$$

$$\leq 1-\alpha.$$

Corollary 3. *Let $k\geq 2$. Then, the integral operator $\mathscr{G}_{\sigma}^{m}f(z)$ defined by (45) is in the class $\mathscr{C}_{k}(\alpha)$ if and only if*

By specializing the parameter $k=2$ in Theorems 1–3, we obtain the following corollaries.

Corollary 4. *The series* $\Upsilon_\sigma^m(z) \in \mathbb{P}_2(\lambda,\alpha)$ *if and only if*

$$\begin{aligned}&(8\lambda-4\alpha\lambda-3\alpha+7)\frac{\sigma\binom{m}{m-1}}{(1-\sigma)^{m+1}}+2(19\lambda-5\alpha\lambda-\alpha+6)\frac{\sigma^2\binom{m+1}{m-1}}{(1-\sigma)^{m+2}}\\&+6(9\lambda-\alpha\lambda+1)\sigma^3\frac{\binom{m+2}{m-1}}{(1-\sigma)^{m+3}}+24\lambda\sigma^4\frac{\binom{m+3}{m-1}}{(1-\sigma)^{m+4}}\\&\le 1-\alpha.\end{aligned} \tag{53}$$

Corollary 5. *Let* $f \in \mathscr{R}^\epsilon(\mathfrak{C},\mathfrak{D})$. *Then,* $\mathscr{I}_\sigma^m f(z) \in \mathbb{P}_2(\lambda,\alpha)$ *if*

$$\begin{aligned}&(\mathfrak{C}-\mathfrak{D})|\epsilon|\Bigg[(4\lambda-2\alpha\lambda+3-\alpha)\sigma\frac{\binom{m}{m-1}}{1-\sigma}\\&+2(5\lambda-\alpha\lambda+1)\sigma^2\frac{\binom{m+1}{m-1}}{(1-\sigma)^2}+6\lambda\sigma^3\frac{\binom{m+2}{m-1}}{(1-\sigma)^3}\\&+(1-\alpha)\left(1-(1-\sigma)^m\right)\Bigg]\\&\le 1-\alpha.\end{aligned} \tag{54}$$

Corollary 6. *The integral operator* $\mathscr{G}_\sigma^m f(z)$ *defined by (45) is in the class* $\mathbb{P}_2(\lambda,\alpha)$ *if and only if*

$$\begin{aligned}&(4\lambda-2\alpha\lambda+3-\alpha)\sigma\frac{\binom{m}{m-1}}{1-\sigma}+2(5\lambda-\alpha\lambda+1)\sigma^2\frac{\binom{m+1}{m-1}}{(1-\sigma)^2}+6\lambda\sigma^3\frac{\binom{m+2}{m-1}}{(1-\sigma)^3}\\&\le 1-\alpha.\end{aligned} \tag{55}$$

Remark 1. Using relation (22) and Lemma 1, we can obtain new necessary and sufficient conditions and inclusion relations for the Pascal distribution series to be in the class $\mathbb{P}_k(\lambda,\alpha)$ for $k=3,4,\ldots$.

6. Conclusions

The Sălăgean differential operator plays an important role in the geometric function theory. Several authors have used this operator to define and consider the properties of certain known and new classes of analytic univalent functions (see, for example, [25, 26]). In the present paper, and due to the earlier works (see, for example, [11, 16, 18]), we find a necessary and sufficient condition and inclusion relation for the Pascal distribution series to be in the class $\mathbb{P}_k(\lambda,\alpha)$ of analytic functions associated with the Stirling numbers and Sălăgean differential operator. Furthermore, we consider an integral operator related to the Pascal distribution series. Some interesting corollaries and applications of the results are also discussed. Making use of the relation (22) could inspire researchers to find new necessary and sufficient conditions and inclusion relations for the Pascal distribution series to be in different classes of analytic functions with negative coefficients defined by the Sălăgean differential operator.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author read and approved the final manuscript.

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