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Research Article

Study of Nonlocal Boundary Value Problem for the Fredholm–Volterra Integro-Differential Equation

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In this paper, the existence and uniqueness of the Fredholm–Volterra integro-differential equation with the nonlocal condition will be studied. Also, we study the continuous dependence of the initial data. The numerical solution of the problem will be studied using the central difference approximations and trapezoidal rule to transform the Volterra–Fredholm integro-differential equation into a system of algebraic equations which can be solved together to get the solution. Finally, we solve some examples numerically to show the accuracy of the proposed method.

1. Introduction

Recently, some researchers were interested in studying the existence and uniqueness of different types of integro-differential equation with the different conditions. El-Sayed et al. studied the existence of solutions to some integro-differential equations with infinite point and integral conditions, and they have also studied some properties of these solutions [1–4]. There are also many authors interested in studying the numerical solution for integral and integro-differential equations. Mirzaee and Piroozfar used modified Simpson's quadrature rule for solving linear Fredholm integral equations of the second kind [5]. Rahman et al. solved the system of linear Volterra Integral equations of the second kind using Simpson's quadrature rule [6]. Garba and Bichi studied the numerical solution for first-order Fredholm integro-differential equation using finite difference-composite Simpson method [7]. Ibrahim et al. studied the existence of a unique solution to nonlinear Fredholm integro-differential equation of the second order, and they introduced the exact solution using the direct computation method, introduced numerical solution using the combination of the finite difference method with the composite Simpson method to transform the Fredholm integro-differential equation into a system of nonlinear algebraic equations, and also computed the error estimation for the scheme to show the accuracy of the presented method [8]. Pandey used the finite difference method and the composite trapezoidal quadrature method to solve the Fredholm integro-differential equation [9]. Saadati et al. solved the linear Volterra and Fredholm integro-differential equation using the combination of the trapezoidal rule and the finite difference method and compared it with the variational iteration method (VIM). The result of comparison shows that VIM is better than the trapezoidal method [10]. Ishak and Norazura Ahmed obtained the numerical solution for the first-order Volterra integro-differential equation using the trapezoidal method and compared the results with the Euler method. The results of comparisons show that the trapezoidal method is better than the Euler method [11]. Raftari used the homotopy perturbation method (HPM) and the finite difference method to solve the Volterra integro-differential equation of the first order. The results of applying these methods demonstrate the validity and applicability of these techniques.

In this paper, we study the nonlocal boundary value problem for the Fredholm–Volterra integro-differential equation:

$$u''(x) = F\left(x, u(x), \int_a^b f(x,t,u'(t))(x,t,u'(t))\mathrm{d}t, \int_a^x g(x,t,u'(t))\mathrm{d}t\right), \quad x \in [a,b], \tag{1}$$

with the nonlocal condition

$$\begin{aligned}\sum_{j=0}^{m} a_j u(\tau_j) &= \mu_0,\\ u'(a) &= \rho_0,\\ a_j &\geq 0,\\ \tau_j &\in [a,b].\end{aligned} \tag{2}$$

We study the existence of solution $u(x) \in C[a,b]$. We study the continuous dependence of the unique solution on μ_0 and on the nonlocal parameter a_j.

As applications, the nonlocal problem of the Fredholm–Volterra integro-differential equation (1) with the integral condition

$$\int_a^b u(s)\mathrm{d}\phi(s) = \mu_0 \tag{3}$$

will be studied.

This paper is organized as follows. In Section 2, we discuss the integral representation. We discuss the existence of solution and the nonlocal integral condition in Section 3. We discuss the uniqueness of the solution in Section 4. In Section 5, we discuss the continuous dependence on μ_0 and a_j. In Section 6, we present the methodology of numerical technique and numerical examples. Section 7 gives the conclusion.

2. Integral Representation

Consider nonlocal problems (1) and (2) with the following assumptions:

(1) F: $[a,b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies Caratheodory condition, i.e., F is measurable in x for any $\xi, \alpha, \gamma \in \mathbb{R}$ and continuous for almost all $x \in [a,b]$. There exist a function $M_1(x) \in L^1[a,b]$ and a positive constant $C_1 > 0$, such that

$$|F(x,\xi,\alpha,\gamma)| \leq M_1(x) + C_1|\xi| + C_1|\alpha| + C_1|\gamma|. \tag{4}$$

(2) f: $[a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Caratheodory condition, i.e., f is measurable in x for any $v(t) \in \mathbb{R}$ and continuous for almost all $x \in [a,b]$. There exist a function $M_2(x,t) \in L^1[a,b]$ and a positive constant $C_2 > 0$, such that

$$|f(x,t,v(t))| \leq M_2(x,t) + C_2|v(t)|. \tag{5}$$

(3) g: $[a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Caratheodory condition. There exist a function $M_3(x,t) \in L^1[a,b]$ and a positive constant $C_3 > 0$, such that

$$|g(x,t,v(t))| \leq M_3(x,t) + C_3|v(t)|. \tag{6}$$

(4)

$$\begin{aligned}\sup_{x\in[a,b]} \int_a^x M_1(\theta)\mathrm{d}\theta &\leq N_1,\\ \sup_{\theta\in[a,b]} \int_a^b M_2(\theta,t)\mathrm{d}t &\leq N_2,\\ \sup_{\theta\in[a,b]} \int_a^\theta M_3(\theta,t)\mathrm{d}t &\leq N_3.\end{aligned} \tag{7}$$

(5) $(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2) < 1$.

Lemma 1. *Let $\beta = \sum_{j=0}^{m} a_j \neq 0$, and we can represent the solution of nonlocal problems (1) and (2), if it exists by the integral equation*

$$u(x) = \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^x v(s)\mathrm{d}s, \tag{8}$$

where

$$v(x) = \rho_0 + \int_a^x F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^\theta v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^\theta g(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta. \tag{9}$$

Proof. Integrating both sides of (1), we get

$$u'(x) = u'(a) + \int_a^x F\left(\theta, u(\theta), \int_a^b f(\theta, t, u'(t))dt, \int_a^\theta g(\theta, t, u'(t))dt\right)d\theta, \quad x \in [a,b]. \tag{10}$$

Let $u'(x) = v(x)$ in (10), and we obtain

$$v(x) = \rho_0 + \int_a^x F\left(\theta, u(\theta), \int_a^b f(\theta, t, v(t))dt, \int_a^\theta g(\theta, t, v(t))dt\right)d\theta, \quad x \in [a,b], \tag{11}$$

where

$$u(x) = u(a) + \int_a^x v(s)ds, \quad x \in [a,b], \tag{12}$$

and using nonlocal condition (2), we get

$$\sum_{j=0}^m a_j u(\tau_j) = u(a)\sum_{j=0}^m a_j + \sum_{j=0}^m a_j \int_a^{\tau_j} v(s)ds, \tag{13}$$

and then

$$u(a) = \beta^{-1}\left[\mu_0 - \sum_{j=0}^m a_j \int_a^{\tau_j} v(s)ds\right]. \tag{14}$$

We obtain (8) and (9) from (11), (12), and (14). This completes the proof. □

3. Existence of Solution

Definition 1. By a solution of Fredholm–Volterra integral equation (9), we mean a function $u(x) \in C[a,b]$ that satisfies (5).

Theorem 1. *Let the assumptions (1)–(5) hold. Then, Fredholm–Volterra integral equation (9) has at least one solution $u(x) \in C[a,b]$.*

Proof. Define the operator E associated with integral equation (9) by

$$Ev(x) = \rho_0 + \int_a^x F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^m a_j \int_a^{\tau_j} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b f(\theta, t, v(t))dt, \int_a^\theta g(\theta, t, v(t))dt\right)d\theta. \tag{15}$$

Let $Q_r = \{v(x) \in \mathbb{R}: \|v\|_C \le r\}$, where $r = (|\rho_0| + N_1 + C_1 b\beta^{-1}|\mu_0| + C_1 bN_2 + C_1 bN_3)/(1 - (2C_1 b^2 + C_1 C_2 b^2 + C_1 C_3 b^2))$.

Then, we have that for $v(x) \in Q_r$,

$$\begin{aligned}
\|Ev(x)\|_C &\le |\rho_0| + \int_a^x \left|F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^m a_j \int_a^{\tau_j} v(s)ds\right] + \int_a^\theta v(s)ds, \int_a^b f(\theta, t, v(t))dt, \int_a^\theta g(\theta, t, v(t))dt\right)d\theta\right|, \\
&\le |\rho_0| + \int_a^x \left[M_1(\theta) + C_1\beta^{-1}\left|\mu_0 - \sum_{j=0}^m a_j \int_a^{\tau_j} v(s)ds\right| + C_1\int_a^\theta |v(s)|ds + C_1\int_a^b f|(\theta, t, v(t))|dt + C_1\int_a^\theta g|(\theta, t, v(t))|dt\right]d\theta \\
&\le |\rho_0| + N_1 + \int_a^x \left[\begin{array}{c} C_1\beta^{-1}|\mu_0| + C_1\beta^{-1}\sum_{j=0}^m a_j \int_a^{\tau_j} |v(s)|ds + C_1\int_a^\theta |v(s)|ds \\ +C_1\int_a^b |M_2(\theta, t)|dt + C_1 C_2\int_a^b |v(t)|dt + C_1\int_a^\theta |M_3(\theta, t)|dt + C_1 C_3\int_a^\theta |v(t)|dt \end{array}\right]d\theta \\
&\le |\rho_0| + N_1 + \int_a^x \left[C_1\beta^{-1}|\mu_0| + C_1 b\|v\| + C_1 b\|v\| + C_1 N_2 + C_1 C_2 b\|v\| + C_1 N_3 + C_1 C_3 b\|v\|\right]d\theta \\
&\le |\rho_0| + N_1 + C_1 b\beta^{-1}|\mu_0| + 2C_1 b^2 r + C_1 bN_2 + C_1 C_2 b^2 r + C_1 bN_3 + C_1 C_3 b^2 r = r.
\end{aligned} \tag{16}$$

This proves that $E: Q_r \longrightarrow Q_r$ and the class of functions $Ev(x)$ is uniformly bounded in Q_r.

Now, let $x_1, x_2 \in [a,b]$ such that $|x_2 - x_1| < \delta$; then,

$$\begin{aligned}
|Ev(x_2) - Ev(x_1)| &= \Bigg|\rho_0 + \int_a^{x_2} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta \\
&\quad -\rho_0 - \int_a^{x_1} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)d\theta\Bigg| \\
&= \Bigg|\int_a^{x_1} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta \\
&\quad + \int_{x_1}^{x_2} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)d\theta \\
&\quad - \int_a^{x_1} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)d\theta\Bigg| \\
&\le \int_{x_1}^{x_2} \left|F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\right)\right|d\theta \\
&\le \int_{x_1}^{x_2} M_1(\theta)\mathrm{d}\theta + \left(C_1\beta^{-1}\mu_0 + 2C_1 br + C_1 N_2 + C_1 C_2 br + C_1 N_3 + C_1 C_3 br\right)\delta.
\end{aligned} \tag{17}$$

This means that the class of functions $Ev(x)$ is equicontinuous in Q_r. □

Let $v_n(x) \in Q_r$, $v_n(x) \longrightarrow v(x)\,(n \longrightarrow \infty)$; then, from the continuity of the three functions F, f, and g, we obtain $F(x, \xi_n, \alpha_n, \gamma_n) \longrightarrow F(x, \xi, \alpha, \gamma)$, $f(x,t,v_n(t)) \longrightarrow f(x,t,v(t))$ and $g(x,t,v_n(t)) \longrightarrow g(x,t,v(t))$ as $n \longrightarrow \infty$. Also,

$$\lim_{n\longrightarrow\infty} Ev_n(x) = \lim_{n\longrightarrow\infty}\left[\rho_0 + \int_a^{x} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v_n(s)\mathrm{d}s\right] + \int_a^{\theta} v_n(s)\mathrm{d}s, \int_a^b f(\theta,t,v_n(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v_n(t))\mathrm{d}t\right)\mathrm{d}\theta\right]. \tag{18}$$

Using assumptions (1)–(3) and Lebesgue dominated convergence theorem [13], we obtain

$$\lim_{n\longrightarrow\infty} Ev_n(x) = \rho_0 + \int_a^{x} \lim_{n\longrightarrow\infty} F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v_n(s)\mathrm{d}s\right] + \int_a^{\theta} v_n(s)\mathrm{d}s, \int_a^b f(\theta,t,v_n(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v_n(t))\mathrm{d}t\right)\mathrm{d}\theta = Ev(x). \tag{19}$$

Then, $Ev_n(x) \longrightarrow Ev(x)$ as $n \longrightarrow \infty$. This means that the operator E is continuous in Q_r. Then, by Schauder fixed point theorem [14], there exists at least one solution $v(x) \in C[a,b]$ of integral equation (9). Thus, based on Lemma 1, nonlocal problems (1) and (2) possess a solution $u(x) \in C[a,b]$.

3.1. Nonlocal Integral Condition. Let $v(x) \in C[a,b]$ be the solution of integral equation (9). Let $a_j = \phi(x_j) - \phi(x_{j-1})$, ϕ be increasing function, $\tau_j \in (x_{j-1}, x_j)$, and $a = x_0 < x_1 < x_2 < \ldots < x_N = b$; then, as $m \longrightarrow \infty$, nonlocal condition (2) will be

$$\sum_{j=0}^{m}\left(\phi\left(x_j\right)-\phi\left(x_{j-1}\right)\right)u\left(\tau_j\right)=\mu_0, \tag{20}$$

$$\lim_{m\longrightarrow\infty}\sum_{j=0}^{m}\left(\phi\left(x_j\right)-\phi\left(x_{j-1}\right)\right)u\left(\tau_j\right)=\int_a^b u(s)\mathrm{d}(s)=\mu_0. \tag{21}$$

Theorem 2. *Let the assumptions (1)–(5) hold; then, nonlocal problems (1) and (3) have at least one solution given by*

$$u(x)=\frac{1}{\phi(b)-\phi(a)}\left(\mu_0-\int_a^b\int_a^\theta v(s)\mathrm{d}s\mathrm{d}\phi(\theta)\right)+\int_a^x v(s)\mathrm{d}s, \tag{22}$$

where

$$v(x)=\rho_0+\int_a^x F\left(\theta,\frac{1}{\phi(b)-\phi(a)}\left(\mu_0-\int_a^b\int_a^\theta v(s)\mathrm{d}s\mathrm{d}\phi(\theta)\right)+\int_a^\theta v(s)\mathrm{d}s,\int_a^b f(\theta,t,v(t))dt\ ,\int_a^\theta g(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta\Bigg). \tag{23}$$

Proof. As $m\longrightarrow\infty$, the solution of nonlocal problems (1) and (3) will be

$$\begin{aligned}u(x)&=\lim_{m\longrightarrow\infty}\left[\beta^{-1}\left[\mu_0-\sum_{j=0}^{m}a_j\int_a^{\tau_j}v(s)\mathrm{d}s\right]+\int_a^x v(s)\mathrm{d}s\right]\\&=\frac{1}{\phi(b)-\phi(a)}\left[\mu_0-\lim_{m\longrightarrow\infty}\sum_{j=0}^{m}\int_a^{\tau_j}v(s)\mathrm{d}s\left(\phi\left(x_j\right)-\phi\left(x_{j-1}\right)\right)\right]+\int_a^x v(s)\mathrm{d}s\\&=\frac{1}{\phi(b)-\phi(a)}\left[\mu_0-\int_a^b\int_a^\theta v(s)\mathrm{d}s\mathrm{d}\phi(\theta)\right]+\int_a^x v(s)\mathrm{d}s,\end{aligned} \tag{24}$$

where

$$v(x)=\rho_0+\int_a^x F\left(\theta,\frac{1}{\phi(b)-\phi(a)}\left(\mu_0-\int_a^b\int_a^\theta v(s)\mathrm{d}s\mathrm{d}\phi(\theta)\right)+\int_a^\theta v(s)\mathrm{d}s,\int_a^b f(\theta,t,v(t))\mathrm{d}t,\int_a^\theta g(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta. \tag{25}$$

□

4. Uniqueness of the Solution

Let F, f, and g satisfy the following assumptions:

(i) F: $[a,b]\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is measurable in x for any $\xi,\alpha,\gamma\in\mathbb{R}$ and satisfies the Lipschitz condition

$$\begin{aligned}&\left|F(x,\xi,\alpha,\gamma)-F\left(x,v,\alpha_1,\gamma_1\right)\right|\\&\quad\le C_1|\xi-v|+C_1\left|\alpha-\alpha_1\right|+C_1\left|\gamma-\gamma_1\right|.\end{aligned} \tag{26}$$

(ii) f: $[a,b]\times[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is measurable in x for any $v(t)\in\mathbb{R}$ and satisfies the Lipschitz condition

$$|f(x,t,v(t))-f(x,t,w(t))|\le C_2|v(t)-w(t)|. \tag{27}$$

(iii) g: $[a,b]\times[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is measurable in x for any $v(t)\in\mathbb{R}$ and satisfies the Lipschitz condition

$$|g(x,t,v(t))-g(x,t,w(t))|\le C_3|v(t)-w(t)|. \tag{28}$$

Theorem 3. *Let the assumptions $(i)-(iii)$ hold; then, the solution of Fredholm–Volterra integral equation (9) is unique.*

Proof. Let $v(x),w(x)$ be two solutions of Fredholm–Volterra integral *equation* (9); then,

$$|v(x)-w(x)|\le\int_a^x|F\left(\theta,\beta^{-1}\left[\mu_0-\sum_{j=0}^{m}a_j\int_a^{\tau_j}v(s)\mathrm{d}s\right]+\int_a^\theta v(s)\mathrm{d}s,\int_a^b f(\theta,t,v(t))\mathrm{d}t,\int_a^\theta f(\theta,t,v(t))\mathrm{d}t\right)\mathrm{d}\theta$$

$$- F\left(\theta,\beta^{-1}\left[\mu_0-\sum_{j=0}^{m}a_j\int_a^{\tau_j}w(s)\mathrm{d}s\right]+\int_a^{\theta}w(s)\mathrm{d}s,\int_a^{b}f(\theta,t,w(t))\mathrm{d}t,\int_a^{\theta}g(\theta,t,w(t))\mathrm{d}t\right)|d\theta$$
$$\le \int_a^x\left[C_1\left|\beta^{-1}\sum_{j=0}^{m}a_j\int_a^{\tau_j}w(s)-v(s)\mathrm{d}s+\int_a^{\theta}v(s)-w(s)\mathrm{d}s\right|\right.$$
$$+C_1\left|\int_a^b(f(\theta,t,v(t))-f(\theta,t,w(t)))dt\right|+C_1\left|\int_a^{\theta}(g(\theta,t,v(t))-g(\theta,t,w(t)))\mathrm{d}t\right|$$
$$\le C_1\int_a^x\left[\beta^{-1}\sum_{j=0}^{m}a_j\int_a^{\tau_j}|w(s)-v(s)|\mathrm{d}s+\int_a^{\theta}|w(s)-v(s)|\mathrm{d}s\right.$$
$$\left.+\int_a^b|f(\theta,t,v(t))-f(\theta,t,w(t))|\mathrm{d}t+\int_a^{\theta}\int_a^b|g(\theta,t,v(t))-g(\theta,t,w(t))|\mathrm{d}t\right]\mathrm{d}\theta$$
$$\le C_1\|w-v\|b^2+C_1\|w-v\|b^2+C_1\int_a^x\int_a^b C_2|v(t)-w(t)|\mathrm{d}t\mathrm{d}\theta$$
$$+C_1\int_a^x\int_a^{\theta}C_3|v(t)-w(t)|\mathrm{d}t\mathrm{d}\theta$$
$$\le 2C_1\|w-v\|b^2+C_1C_2b^2\|w-v\|+C_1C_3b^2\|w-v\|$$
$$\le\left(2C_1b^2+C_1C_2b^2+C_1C_3b^2\right)\|w-v\|. \tag{29}$$

Hence,

$$\left[1-\left(2C_1b^2+C_1C_2b^2+C_1C_3b^2\right)\right]\|w-v\|\le 0. \tag{30}$$

Since $2C_1b^2+C_1C_2b^2+C_1C_3b^2<1$, then $w(x)=v(x)$ and the solution of Fredholm–Volterra integral equation (9) is unique. Thus, based on Lemma 1, nonlocal problems (1) and (2) possess a unique solution $u(x)\in C[a,b]$. □

5. Continuous Dependence

5.1. Continuous Dependence on μ_0

Definition 2. The solution$u(x)\in C[a,b]$of nonlocal Fredholm–Volterra problems (1) and (2) *depends continuously onμ_0, if*

$$\forall\varepsilon>0,\quad \exists\quad \delta(\varepsilon)\quad s.t\quad \left|\mu_0-\mu_0^*\right|<\delta\Rightarrow\left\|u-u^*\right\|<\varepsilon, \tag{31}$$

where u^* is the solution of the nonlocal problem

$$u^{*\prime\prime}(x)=F\left(x,u^*(x),\int_a^b f\left(x,t,u^{*'}(t)\right)\mathrm{d}t,\int_a^x g\left(x,t,u^{*'}(t)\right)\mathrm{d}t\right),\quad x\in[a,b], \tag{32}$$

with the nonlocal condition

$$\sum_{j=0}^{m}a_ju^*\left(\tau_j\right)=\mu_0^*,\quad u^{*'}(a)=\rho_0,$$
$$a_j\geqslant 0,\quad \tau_j\in[a,b]. \tag{33}$$

Theorem 4. *Let the assumptions (1)–(5) of Theorem 1hold; then, the solution of nonlocal Fredholm–Volterra problems (1) and (2) depends continuously onμ_0.*

Proof. Let $u(x)$, $u^*(x)$ be two solutions of nonlocal Fredholm–Volterra problems (1) and (2) *and* (23)–(33), respectively. Then,

$$\left|v(x)-v^*(x)\right|=|\int_a^x\left[F\left(\theta,\beta^{-1}\left[\mu_0-\sum_{j=0}^{m}a_j\int_a^{\tau_j}v(s)ds\right]+\int_a^{\theta}v(s)\mathrm{d}s,\int_a^b f(\theta,t,v(t))\mathrm{d}t,\int_a^{\theta}g(\theta,t,v(t))\mathrm{d}t\right)\right.$$
$$\left.-F\left(\theta,\beta^{-1}\left[\mu_0^*-\sum_{j=0}^{m}a_j\int_a^{\tau_j}v^*(s)\mathrm{d}s\right]+\int_a^{\theta}v^*(s)ds\ ,\int_a^b f(\theta,t,v^*(t))\mathrm{d}t,\int_a^{\theta}g(\theta,t,v^*(t))\mathrm{d}t\right)\right]\mathrm{d}\theta|$$

$$
\begin{aligned}
&\leq \int_a^x |F\left(\theta, \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta, t, v(t))\mathrm{d}t, \int_a^{\theta} g(\theta, t, v(t))\mathrm{d}t\right) \\
&\quad - F\left(\theta, \beta^{-1}\left[\mu_0^* - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v^*(s)\mathrm{d}s\right] + \int_a^{\theta} v^*(s)\mathrm{d}s, \int_a^b f(\theta, t, v^*(t))\mathrm{d}t, \int_a^{\theta} g(\theta, t, v^*(t))\mathrm{d}t\right)|\mathrm{d}\theta \\
&\leq \int_a^x \Bigg[C_1\left|\beta^{-1}(\mu_0 - \mu_0^*) + \beta^{-1}\sum_{j=0}^{m} a_j \int_a^{\tau_j}(v^*(s) - v(s))\mathrm{d}s + \int_a^{\theta} v(s) - v^*(s)\mathrm{d}s\right| \\
&\quad + C_1\left|\int_a^b (f(\theta, t, v(t)) - f(\theta, t, v^*(t)))\mathrm{d}t\right| + C_1\left|\int_a^{\theta}(g(\theta, t, v(t)) - g(\theta, t, v^*(t)))\mathrm{d}t\right|\Bigg]\mathrm{d}\theta \\
&\leq \int_a^x \Bigg[C_1\beta^{-1}|\mu_0 - \mu_0^*| + C_1\beta^{-1}\sum_{j=0}^{m} a_j \int_a^{\tau_j}|v^*(s) - v(s)|\mathrm{d}s + C_1\int_a^{\theta}|v(s) - v^*(s)|\mathrm{d}s \\
&\quad + C_1\int_a^b |f(\theta, t, v(t)) - f(\theta, t, v^*(t))|\mathrm{d}t + C_1\int_a^{\theta}|g(\theta, t, v(t)) - g(\theta, t, v^*(t))|\mathrm{d}t\Bigg]\mathrm{d}\theta \\
&\leq C_1\beta^{-1}|\mu_0 - \mu_0^*|b + C_1\|v - v^*\|b^2 + C_1\|v - v^*\|b^2 \\
&\quad + C_1\int_a^x\int_a^b C_2|v(t) - v^*(t)|\mathrm{d}t + C_1\int_a^x\int_a^{\theta} C_3|v(t) - v^*(t)|\mathrm{d}t\mathrm{d}\theta \\
&\leq C_1 b\beta^{-1}\delta + 2C_1\|v - v^*\|b^2 + C_1C_2b^2\|v - v^*\| + C_1C_3b^2\|v - v^*\|.
\end{aligned} \tag{34}
$$

Hence,

$$\|v - v^*\| \leq \frac{C_1 b\beta^{-1}\delta}{1 - \left(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2\right)}. \tag{35}$$

Since

$$|u(x) - u^*(x)| = \beta^{-1}\left[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\right] + \int_a^x v(s)\mathrm{d}s - \beta^{-1}\left[\mu_0^* - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v^*(s)\mathrm{d}s\right] + \int_a^x v^*(s)\mathrm{d}s\beta^{-1}|\mu_0 - \mu_0^*| + 2b\|v - v^*\|, \tag{36}$$

then

$$\|u - u^*\| \leq \beta^{-1}\delta + \frac{2C_1b^2\beta^{-1}\delta}{1 - \left(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2\right)} = \varepsilon. \tag{37}$$

Therefore, the solution of nonlocal Fredholm–Volterra problems (1) and (2) depends continuously on μ_0. □

5.2. Continuous Dependence on a_j

Definition 3. The solution $u(x) \in C[a, b]$ *of nonlocal Fredholm–Volterra problems* (1) and (2) *depends continuously on* a_j, *if*

$$\forall \varepsilon > 0, \quad \exists \quad \delta(\varepsilon) \quad s.t \quad \left|a_j - a_j^*\right| < \delta \Rightarrow \|u - u^*\| < \varepsilon, \tag{38}$$

where $u^*(x)$ is the solution of the nonlocal problem

$$u^{*\prime\prime}(x) = F\left(x, u^*(x), \int_a^b f\left(x, t, u^{*\prime}(t)\right)\mathrm{d}t, \int_a^x g\left(x, t, u^{*\prime}(t)\right)\mathrm{d}t\right), \quad x \in [a, b], \tag{39}$$

with the nonlocal condition

$$\sum_{j=0}^{m} a_j^* u^*(\tau_j) = \mu_0, \quad u^{*'}(a) = \rho_0, \quad a_j \geqslant 0, \quad \tau_j \in [a,b]. \tag{40}$$

Theorem 5. *Let the assumptions (1)–(5) ofTheorem 1hold; then, the solution of nonlocal problems (1) and (2) depends continuously on a_j.*

Proof. Let $\beta^* = \sum_{j=0}^{m} a_j^* \neq 0$ and $v(x)$, $v^*(x)$ be two solutions of nonlocal Fredholm–Volterra problems (1) and (2) *and* (39)–(40), respectively. Then,

$$\begin{aligned}
|v(x) - v^*(x)| \leq & \int_a^x \Bigg| F\Bigg(\theta, \beta^{-1}\Bigg[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\Bigg] + \int_a^{\theta} v(s)\mathrm{d}s, \int_a^b f(\theta,t,v(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v(t))\mathrm{d}t\Bigg) \\
& -F\Bigg(\theta, \beta^{*-1}\Bigg[\mu_0 - \sum_{j=0}^{m} a_j^* \int_a^{\tau_j} v^*(s)\mathrm{d}s\Bigg] + \int_a^{\theta} v^*(s)\mathrm{d}s, \int_a^b f(\theta,t,v^*(t))\mathrm{d}t, \int_a^{\theta} g(\theta,t,v^*(t))\mathrm{d}t\Bigg)\Bigg|\mathrm{d}\theta \\
\leq & \int_a^x \Bigg[C_1|\beta^{-1}(\mu_0) - \beta^{*-1}(\mu_0) + \beta^{*-1}\sum_{j=0}^{m} a_j^* \int_a^{\tau_j} v^*(s)\mathrm{d}s - \beta^{-1}\sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s \\
& + \int_a^{\theta} v(s)\mathrm{d}s - \int_a^{\theta} v^*(s)\mathrm{d}s| + C_1\Bigg|\int_a^b (f(\theta,t,v(t)) - f(\theta,t,v^*(t)))\mathrm{d}t\Bigg| \\
& + C_1\Bigg|\int_a^{\theta} (g(\theta,t,v(t)) - g(\theta,t,v^*(t)))\mathrm{d}t\Bigg|\Bigg]\mathrm{d}\theta \\
\leq & \int_a^x \Bigg[C_1\left|\beta^{-1}(\mu_0) - \beta^{*^{-1}}(\mu_0)\right| + C_1\beta^{*^{-1}}\sum_{j=0}^{m} a_j^* \int_a^{\tau_j} \left|v^*(s) - v(s)\right|\mathrm{d}s \\
& + C_1\beta^{*-1}\Bigg(\sum_{j=0}^{m}\left|a_j^* - a_j\right|\Bigg)\int_a^{\tau_j} |v(s)|\mathrm{d}s + C_1\beta^{-1}\beta^{*^{-1}}\sum_{j=0}^{m}\left|a_j - a_j^*\right|\sum_{j=0}^{m} a_j \int_a^{\tau_j} |v(s)|\mathrm{d}s \\
& + C_1\int_a^{\theta}\left|v(s) - v^*(s)\right|\mathrm{d}s + C_1\int_a^b \left|f(\theta,t,v(t)) - f(\theta,t,v^*(t))\right|\mathrm{d}t \\
& + C_1\int_a^{\theta}\left|g(\theta,t,v(t)) - g(\theta,t,v^*(t))\right|\mathrm{d}t\Bigg]\mathrm{d}\theta \\
\leq & C_1\beta^{-1}\beta^{*^{-1}} m\delta\mu_0 + C_1\left\|v - v^*\right\|b^2 + C_1\beta^{*^{-1}} m\delta\|v\|b^2 + C_1\beta^{*^{-1}} m\delta\|v\|b^2 \\
& + C_1\left\|v - v^*\right\|b^2 + C_1C_2b^2\left\|v - v^*\right\| + C_1C_3b^2\left\|v - v^*\right\| \\
\leq & C_1\beta^{-1}\beta^{*^{-1}} m\delta\mu_0 + 2C_1\beta^{*^{-1}} m\delta\|v\|b^2 + \left(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2\right)\left\|v - v^*\right\|.
\end{aligned} \tag{41}$$

Hence,

$$\left\|v - v^*\right\| \leq \frac{C_1\beta^{-1}\beta^{*^{-1}} m\delta\mu_0 + 2C_1\beta^{*^{-1}} m\delta\|v\|b^2}{\left[1 - \left(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2\right)\right]}. \tag{42}$$

Since

$$\begin{aligned}
\left|u(x) - u^*(x)\right| = & \Bigg|\frac{1}{\sum_{j=0}^{m} a_j}\Bigg[\mu_0 - \sum_{j=0}^{m} a_j \int_a^{\tau_j} v(s)\mathrm{d}s\Bigg] + \int_a^x v(s)\mathrm{d}s - \frac{1}{\sum_{j=0}^{m} a_j^*} \\
& \Bigg[\mu_0 - \sum_{j=0}^{m} a_j^* \int_a^{\tau_j} v^*(s)\mathrm{d}s\Bigg] + \int_a^x v^*(s)\mathrm{d}s\Bigg| \leq \frac{m\delta\left|\mu_0\right|}{\beta\beta^*} + 2\,m\delta b\beta^{*^{-1}} r + 2b\left\|v - v^*\right\|,
\end{aligned} \tag{43}$$

then

$$\|u-u^*\| \le \frac{m\delta|\mu_0|}{\beta\beta^*} + 2m\delta b\beta^{*-1}r + 2b \frac{C_1\beta^{-1}\beta^{*-1}m\delta\mu_0 + 2C_1\beta^{*-1}m\delta\|v\|b^2}{\left[1-\left(2C_1b^2 + C_1C_2b^2 + C_1C_3b^2\right)\right]} = \epsilon. \tag{44}$$

Thus, the solution of nonlocal Fredholm–Volterra problems (1) and (2) depends continuously on a_j. □

6. Methodology of Numerical Technique

In this section, we wish to determine the numerical solution of equation (1). We divide the domain $[a,x]$ and $[a,b]$ of equation (1) into N finite points as $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = x = b$. We use uniform step length $h = ((b-a)/N) = ((x_i - a)/i)$, $i \ge 1$, as $x_j = a + jh = t_j$, $j = 0, 1, 2, \ldots, N$. Then, we use the trapezoidal rule to approximate the integral parts of (1) as follows [10]:

$$\int_a^b k(x_i,t_j)u'(t_j)\mathrm{d}t \simeq \frac{h}{2}\left[k(x_i,t_0)u'(t_0) + 2\sum_{j=1}^{N-1} k(x_i,t_j)u'(t_j) + k(x_i,t_N)u'(t_N)\right], \tag{45}$$

$$\int_a^{x_i} K(x_i,t_j)u'(t_j)\mathrm{d}t \simeq \frac{h_i}{2}\left[K(x_i,t_0)u'(t_0) + 2\sum_{j=1}^{N-1} K(x_i,t_j)u'(t_j) + K(x_i,t_N)u'(t_N)\right], \tag{46}$$

where $K(x_i,t_j) = 0$ for $t_j \le x_i$, $j \ge 1$.

Then, we use central difference approximations to approximate the derivative parts of (1) as

$$u_i'' \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad u_i' \simeq \frac{u_{i+1} - u_{i-1}}{2h}, \tag{47}$$

where $u_i'' = u''(x_i)$, $u_i' = u'(x_i)$.

6.1. Numerical Examples. Now, we apply Theorem 1 on some examples of the nonlocal Fredholm–Volterra integro-differential equation and we solve it numerically by using the finite difference-trapezoidal method. The results obtained are tabulated in Tables 1–4, and all results for these examples are obtained by using Wolfram Mathematica.

Example 1. Consider the equation

$$\begin{aligned} u''(x) - \frac{1}{8}u^2(x) &= \frac{1}{240}\left(-2x^5 - 6x^2\sin(2x) + 9x\sin^2(x) + 3\sin^3(x)\cos(x)\right) \\ &\quad - \frac{1}{48}x\left(\frac{3}{2} - e^{-\sin(1)}\right) - \frac{1}{8}\cos^2(x) - \cos(x) + \frac{1}{48}\int_0^1\left(tx + x\cos(t)e^{u'(t)}\right)\mathrm{d}t \\ &\quad + \frac{1}{40}\int_0^x\left(tx + \sin(x)u'^2(t)\right)\mathrm{d}t, u(0.4) + u(0.6) = 1.746, u'(0) = 0. \end{aligned} \tag{48}$$

The exact solution of this problem is $u(x) = \cos(x)$.

Firstly, we apply the assumptions of Theorem 1 to prove that this example has a continuous solution:

$$\begin{aligned} &F\left(x, u(x), \int_a^b f(x,t,u'(t))\mathrm{d}t, \int_a^x g(x,t,u'(t))\mathrm{d}t\right) \\ &= \frac{1}{240}\left(-2x^5 - 6x^2\sin(2x) + 9x\sin^2(x) + 3\sin^3(x)\cos(x)\right) - \frac{1}{48}x\left(\frac{3}{2} - e^{-\sin(1)}\right) - \frac{1}{8}\cos^2(x) - \cos(x) \\ &\quad + \frac{1}{8}u^2(x) + \frac{1}{48}\int_0^1\left(tx + x\cos(t)e^{u'(t)}\right)\mathrm{d}t + \frac{1}{40}\int_0^x\left(tx + \sin(x)u'^2(t)\right)\mathrm{d}t. \end{aligned} \tag{49}$$

Table 1: The exact and numerical solutions of example 1.

x_i	Numerical solution	Exact solution	Absolute error
0.0	1.00007	1.00000	7.1195 *E*-5
0.1	0.99507	0.99500	6.7119 *E*-5
0.2	0.98012	0.98007	5.5076 *E*-5
0.3	0.95537	0.95534	3.6189 *E*-5
0.4	0.92108	0.92106	1.3545 *E*-5
0.5	0.87758	0.87758	6.6020 *E*-6
0.6	0.82532	0.82534	1.3545 *E*-5
0.7	0.76485	0.76484	9.2025 *E*-6
0.8	0.69679	0.69671	8.5149 *E*-5
0.9	0.62186	0.62161	2.4584 *E*-4
1.0	0.54083	0.54030	5.3145 *E*-4

Table 2: The exact and numerical solutions of example 2.

x_i	Approximate solution	Exact solution	Absolute error
−1.0	−0.845787	−0.841471	4.31629 *E*-3
−0.8	−0.720890	−0.717356	3.53409 *E*-3
−0.6	−0.567342	−0.564642	2.69934 *E*-3
−0.4	−0.391242	−0.389418	1.82331 *E*-3
−0.2	−0.199588	−0.198669	9.18883 *E*-4
0.0	0.000000	0.000000	0.00000
0.2	0.199588	0.198669	9.18883 *E*-4
0.4	0.391242	0.389418	1.82331 *E*-3
0.6	0.567342	0.564642	2.69934 *E*-3
0.8	0.720890	0.717356	3.53409 *E*-3
1.0	0.845787	0.841471	4.31629 *E*-3

Table 3: The exact and numerical solutions of example 3.

x_i	Approximate solution	Exact solution	Absolute error
0.0	0.99967	1.00000	3.20617 *E*-4
0.1	1.00468	1.00500	3.24861 *E*-4
0.2	1.01972	1.02007	3.49559 *E*-4
0.3	1.04493	1.04534	4.06892 *E*-4
0.4	1.08056	1.08107	5.09384 *E*-4
0.5	1.12696	1.12763	6.70039 *E*-4
0.6	1.18456	1.18547	9.02486 *E*-4
0.7	1.25395	1.25517	1.22112 *E*-3
0.8	1.33579	1.33743	1.64126 *E*-3
0.9	1.43091	1.43309	2.17931 *E*-3
1.0	1.54023	1.54308	2.85299 *E*-3

Table 4: The exact and numerical solutions of example 4.

x_i	Approximate solution	Exact solution	Absolute error
0.0	1.00094	1.	9.4143 *E*-4
0.1	1.10594	1.10517	7.7156 *E*-4
0.2	1.222	1.2214	5.9439 *E*-4
0.3	1.35027	1.34986	4.0876 *E*-4
0.4	1.49204	1.49182	2.1332 *E*-4
0.5	1.64873	1.64872	6.5459 *E*-6
0.6	1.82191	1.82212	2.1332 *E*-4
0.7	2.0133	2.01375	4.4832 *E*-4
0.8	2.22484	2.22554	7.0085 *E*-4
0.9	2.45863	2.4596	9.7372 *E*-4
1.0	2.71701	2.71828	1.2703 *E*-3

Then,

$$\begin{aligned}&\left|F\left(x,u(x),\int_a^b f(x,t,u'(t))\mathrm{d}t,\int_a^x g(x,t,u'(t))\mathrm{d}t\right)\right|\leq\\&\left|\frac{1}{240}\left(-2x^5-6x^2\sin(2x)+9x\sin^2(x)+3\sin^3(x)\cos(x)\right)-\frac{1}{48}x\left(\frac{3}{2}-e^{-\sin(1)}\right)-\frac{1}{8}\cos^2(x)-\cos(x)\right|\\&+\frac{1}{8}\left|u^2(x)\right|+\frac{1}{8}\int_0^1\frac{1}{6}\left|tx+x\cos(t)e^{u'(t)}\right|\mathrm{d}t+\frac{1}{8}\int_0^x\frac{1}{5}\left|tx+\sin(x)u'^2(t)\right|\mathrm{d}t,\end{aligned}\tag{50}$$

and also

$$\begin{aligned}\left|f(x,t,u'(t)\mathrm{d}t)\right|&\leq\frac{1}{6}(xt)+\frac{1}{6}\left|e^{u'(t)}\right|,\\\left|g(x,t,u'(t)\mathrm{d}t)\right|&\leq\frac{1}{5}(xt)+\frac{1}{5}\left|u'^2(t)\right|,\end{aligned}\tag{51}$$

where $M_1(x)=(1/240)(-2x^5-6x^2\sin(2x)+9x\sin^2(x)+3\sin^3(x)\cos(x))-(1/48)x((3/2)-e^{-\sin(1)})-(1/8)\cos^2(x)-\cos(x)\in L^1[a,b]$, $M_2(x,t)=(1/6)(xt)\in L^1[a,b]$, $M_3(x,t)=(1/5)(xt)\in L^1[a,b]$, $C_1=(1/8)$, $C_2=(1/6)$, $C_3=(1/5)$, $b=1$; then, $2C_1b^2+C_1C_2b^2+C_1C_3b^2=(2/8)+(1/48)+(1/40)=(71/240)<1$. It is clear that the assumptions (1)–(5) of Theorem 1 hold; therefore, the given nonlocal problem has a continuous solution.

Now, we use the finite difference-trapezoidal method with $N=10$ to find the numerical solution of this problem. Table 1 and Figure 1 give the comparison between the numerical and exact solutions of this problem.

Through our observation of Table 1, the interval $[0,1]$ was divided into 10 subintervals of equal length. We obtain solutions at the endpoints of subintervals and show that the method used is effective, and this is evident from the absolute error that was calculated for the difference between the numerical and real solutions. Also, by looking at Figure 1, we find that the numerical solution and the real solution are very close, which means that the numerical solutions are good.

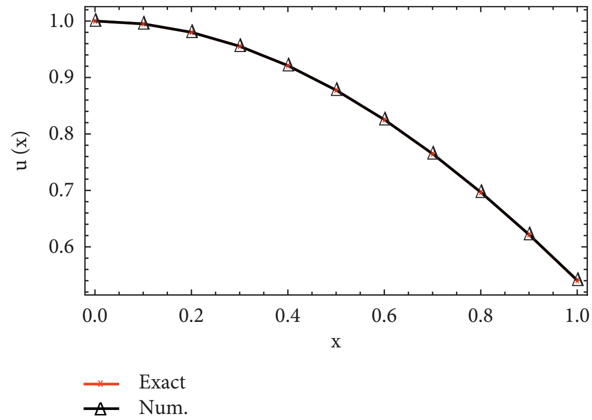

Figure 1: Comparison between the numerical and exact solutions of example 1.

Example 2. Consider the equation

$$u''(x) - \frac{1}{7}u(x) = \frac{1}{56}\left(-\frac{1}{2}\left(x^2 + 2\sin(x) - 1\right)\cos(x) - \sin(x)(x\sin(x) - \sin(1) - \cos(1))\right) - \frac{8\sin(x)}{7} + \frac{1}{70}\int_{-1}^{1}(tx + \sin(xt)u'(t))dt + \frac{1}{56}\int_{-1}^{x}(t\cos(x) + t\sin(x)u'(t))dt, \quad u(-1) + u(1) = 0,\ u'(-1) = \cos(-1). \tag{52}$$

The exact solution of this problem is $u(x) = \sin(x)$.

Firstly, we apply the assumptions of Theorem 1 to prove that this example has a continuous solution:

$$F\left(x, u(x), \int_a^b f(x,t,u'(t))dt, \int_a^x g(x,t,u'(t))dt\right) = \frac{1}{56}\left(-\frac{1}{2}\left(x^2 + 2\sin(x) - 1\right)\cos(x) - \sin(x)(x\sin(x) - \sin(1) - \cos(1))\right) - \frac{8\sin(x)}{7} + \frac{1}{7}u(x) + \frac{1}{70}\int_{-1}^{1}(tx + \sin(xt)u'(t))dt + \frac{1}{56}\int_{-1}^{x}(t\cos(x) + t\sin(x)u'(t))dt. \tag{53}$$

Then,

$$\left|F\left(x, u(x), \int_a^b f(x,t,u'(t))dt, \int_a^x g(x,t,u'(t))dt\right)\right| \leq \left|\frac{1}{56}\left(-\frac{1}{2}\left(x^2 + 2\sin(x) - 1\right)\cos(x) - \sin(x)(x\sin(x) - \sin(1) - \cos(1))\right) - \frac{8\sin(x)}{7}\right| + \frac{1}{7}|u(x)| + \frac{1}{7}\int_{-1}^{1}\frac{1}{10}|tx + \sin(xt)u'(t)|dt + \frac{1}{7}\int_{-1}^{x}\frac{1}{8}|t\cos(x) + t\sin(x)u'(t)|dt, \tag{54}$$

and also

$$\left|f\left(x,t,u'(t)\mathrm{d}t\right)\right| \le \frac{1}{10}|tx| + \frac{1}{10}\left|u'(t)\right|,$$
$$\left|g\left(x,t,u'(t)\mathrm{d}t\right)\right| \le \frac{1}{8}|t\cos(x)| + \frac{1}{8}\left|u'(t)\right|, \tag{55}$$

where

$$\begin{aligned}
M_1(x) &= \frac{1}{56}\left(-\frac{1}{2}\left(x^2 + 2\sin(x) - 1\right)\cos(x) - \sin(x)\left(x\sin(x) - \sin(1) - \cos(1)\right)\right) - \frac{8\sin(x)}{7} \in L^1[a,b],\\
M_2(x,t) &= \frac{1}{10}(tx) \in L^1[a,b],\\
M_3(x,t) &= \frac{1}{8}(t\cos(x)) \in L^1[a,b],\\
C_1 &= \frac{1}{7},\\
C_2 &= \frac{1}{10},\\
C_3 &= \frac{1}{8},\\
b &= 1.
\end{aligned} \tag{56}$$

Then, $2C_1b^2 + C_1C_2b^2 + C_1C_3b^2 = (2/7) + (1/70) + (1/56) = (89/280) < 1$. It is clear that the assumptions (1)–(5) of Theorem 1 hold; therefore, the given nonlocal problem has a continuous solution.

Now, we use the finite difference-trapezoidal method with $N = 10$ to find the numerical solution of this problem. Table 2 and Figure 2 give the comparison between the numerical and exact solutions of this problem.

Through our observation of Table 2, the interval $[-1, 1]$ was divided into 10 subintervals of equal length. We obtain solutions at the endpoints of subintervals and show that the method used is effective, and this is evident from the absolute error that was calculated for the difference between the numerical and real solutions. Also, by looking at Figure 2, we find that the numerical solution and the real solution are very close, which means that the numerical solutions are good.

Example 3. Consider the equation

$$\begin{aligned}
u''(x) - \frac{1}{12}u(x) &= \frac{1}{60}\left(-\frac{x^3}{2} - \sinh(x)\left(x\cosh(x) - \sinh(x)\right)\right) + \frac{11\cosh(x)}{12} + \frac{1}{84}\left(-\frac{\sinh(x)}{2} - \frac{\cosh(x)}{e}\right)\\
&\quad + \frac{1}{84}\int_0^1 \left(t\sinh(x) + t\cosh(x)u'(t)\right)\mathrm{d}t + \frac{1}{60}\int_0^x \left(tx + t\sinh(x)u'(t)\right)\mathrm{d}t,\\
\int_0^1 u(x)\mathrm{d}x &= \sinh(1), \quad u'(0) = 0.
\end{aligned} \tag{57}$$

The exact solution of this problem is $u(x) = \cosh(x)$.

Firstly, we apply the assumptions of Theorem 1 to prove that this example has a continuous solution:

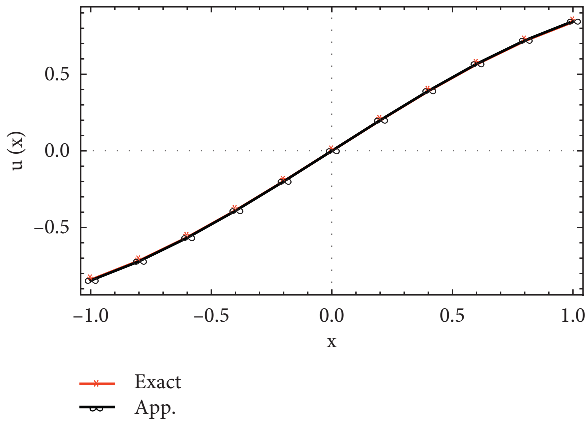

Figure 2: Comparison between the numerical and exact solutions of example 2.

$$F\left(x,u(x),\int_a^b f(x,t,u'(t))dt,\int_a^x g(x,t,u'(t))dt\right)=\frac{1}{60}\left(-\frac{x^3}{2}-\sinh(x)(x\cosh(x)-\sinh(x))\right)+\frac{11\cosh(x)}{12}+\frac{1}{84}\left(-\frac{\sinh(x)}{2}-\frac{\cosh(x)}{e}\right)+\frac{1}{12}u(x)+\frac{1}{84}\int_0^1(t\sinh(x)+t\cosh(x)u'(t))dt+\frac{1}{60}\int_0^x(tx+t\sinh(x)u'(t))dt. \quad (58)$$

Then,

$$\left|F\left(x,u(x),\int_a^b f(x,t,u'(t))dt,\int_a^x g(x,t,u'(t))dt\right)\right|\le\left|\frac{1}{60}\left(-\frac{x^3}{2}-\sinh(x)(x\cosh(x)-\sinh(x))\right)+\frac{11\cosh(x)}{12}+\frac{1}{84}\left(-\frac{\sinh(x)}{2}-\frac{\cosh(x)}{e}\right)\right|+\frac{1}{12}|u(x)|+\frac{1}{12}\int_0^1\frac{1}{7}\left|t\sinh(x)+t\cosh(x)u'(t)\right|dt+\frac{1}{12}\int_0^x\frac{1}{5}\left|tx+t\sinh(x)u'(t)\right|dt, \quad (59)$$

and also

$$\left|f(x,t,u'(t)dt)\right|\le\frac{1}{7}|t\sinh(x)|+\frac{1}{7}\left|u'(t)\right|,$$
$$\left|g(x,t,u'(t)dt)\right|\le\frac{1}{5}|tx|+\frac{1}{5}\left|u'(t)\right|, \quad (60)$$

where

$$M_1(x)=\frac{1}{60}\left(-\frac{x^3}{2}-\sinh(x)(x\cosh(x)-\sinh(x))\right)+\frac{11\cosh(x)}{12}+\frac{1}{84}\left(-\frac{\sinh(x)}{2}-\frac{\cosh(x)}{e}\right)\in L^1[a,b],$$

$$M_2(x,t)=\frac{1}{7}(t\sinh(x))\in L^1[a,b],$$

$$\begin{aligned} M_3(x,t) &= \frac{1}{5}(tx) \in L^1[a,b],\\ C_1 &= \frac{1}{12},\\ C_2 &= \frac{1}{7},\\ C_3 &= \frac{1}{5},\\ b &= 1. \end{aligned} \tag{61}$$

Then, $2C_1b^2 + C_1C_2b^2 + C_1C_3b^2 = (2/12) + (1/84) + (1/60) = (41/210) < 1$. It is clear that the assumptions (1)–(5) of Theorem 1 hold; therefore, the given nonlocal problem has a continuous solution.

Now, we use the finite difference-trapezoidal method with $N = 10$ to find the numerical solution of this problem. Table 3 and Figure 3 give the comparison between the numerical and exact solutions of this problem.

Through our observation of Table 3, the interval $[0,1]$ was divided into 10 subintervals of equal length. We obtain solutions at the endpoints of subintervals and show that the method used is effective, and this is evident from the absolute error that was calculated for the difference between the numerical and real solutions. Also, by looking at Figure 3, we find that the numerical solution and the real solution are very close, which means that the numerical solutions are good.

Example 4. Consider the equation

$$\begin{aligned} u''(x) - \frac{1}{9}u^2(x) &= -\frac{1}{144}(x + 2e^x - 2)x^2 - \frac{1}{324}\left(e^2x + x + 2\right)x + e^x - \frac{e^{2x}}{9}\\ &\quad + \frac{1}{81}\int_0^1\left(xt + x^2tu'^2(t)\right)\mathrm{d}t + \frac{1}{72}\int_0^x\left(tx + x^2u'(t)\right)\mathrm{d}t, \end{aligned} \tag{62}$$

$$u(0.4) + u(0.6) = 3.31394, \quad u'(0) = 1.$$

The exact solution of this problem is $u(x) = e^x$.

Firstly, we apply the assumptions of Theorem 1 to prove that this example has a continuous solution:

$$\begin{aligned} &F\left(x, u(x), \int_a^b f(x,t,u'(t))\mathrm{d}t, \int_a^x g(x,t,u'(t))\mathrm{d}t\right) =\\ &-\frac{1}{144}(x + 2e^x - 2)x^2 - \frac{1}{324}\left(e^2x + x + 2\right)x + e^x - \frac{e^{2x}}{9}\\ &+\frac{1}{9}u^2(x) + \frac{1}{81}\int_0^1\left(xt + x^2tu'^2(t)\right)\mathrm{d}t + \frac{1}{72}\int_0^x\left(tx + x^2u'(t)\right)\mathrm{d}t. \end{aligned} \tag{63}$$

Then,

$$\begin{aligned} &\left|F\left(x, u(x), \int_a^b f(x,t,u'(t))\mathrm{d}t, \int_a^x g(x,t,u'(t))\mathrm{d}t\right)\right|\\ &\le \left|-\frac{1}{144}(x + 2e^x - 2)x^2 - \frac{1}{324}\left(e^2x + x + 2\right)x + e^x - \frac{e^{2x}}{9}\right|\\ &\quad + \frac{1}{9}\left|u^2(x)\right| + \frac{1}{9}\int_0^1\frac{1}{9}\left|xt + x^2tu'^2(t)\right|\mathrm{d}t + \frac{1}{9}\int_0^x\frac{1}{8}\left|tx + x^2u'(t)\right|\mathrm{d}t, \end{aligned} \tag{64}$$

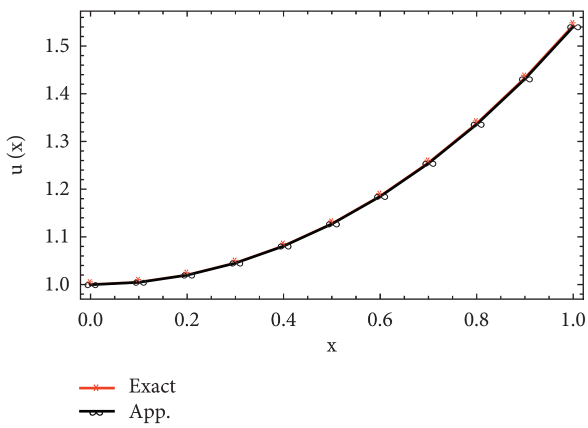

FIGURE 3: Comparison between the numerical and exact solutions of example 3.

and also

$$\left|f\left(x,t,u'(t)\mathrm{d}t\right)\right| \le \frac{1}{9}|tx| + \frac{1}{9}\left|u'(t)\right|,$$
$$\left|g\left(x,t,u'(t)\mathrm{d}t\right)\right| \le \frac{1}{8}|tx| + \frac{1}{8}\left|u'(t)\right|, \quad (65)$$

where

$$M_1(x) = -\frac{1}{144}\left(x + 2e^x - 2\right)x^2 - \frac{1}{324}\left(e^2 x + x + 2\right)x + e^x - \frac{e^{2x}}{9} \in L^1[a,b],$$
$$M_2(x,t) = \frac{1}{9}(tx) \in L^1[a,b],$$
$$M_3(x,t) = \frac{1}{8}(tx) \in L^1[a,b],$$
$$C_1 = \frac{1}{9},$$
$$C_2 = \frac{1}{9},$$
$$C_3 = \frac{1}{8},$$
$$b = 1. \quad (66)$$

Then, $2C_1b^2 + C_1C_2b^2 + C_1C_3b^2 = (2/9) + (1/81) + (1/72) = (161/684) < 1$. It is clear that the assumptions (1)–(5) of Theorem 1 hold; therefore, the given nonlocal problem has a continuous solution.

Now, we use the finite difference-trapezoidal method with $N = 10$ to find the numerical solution of this problem. Table 4 and Figure 4 give the comparison between the numerical and exact solutions of this problem.

Through our observation of Table 4, the interval $[0, 1]$ was divided into 10 subintervals of equal length. We obtain solutions at the endpoints of subintervals and show that the method used is effective, and this is evident from the absolute error that was calculated for the difference between the numerical and real solutions. Also, by looking at Figure 4, we find that the numerical solution and the real solution are very close, which means that the numerical solutions are good.

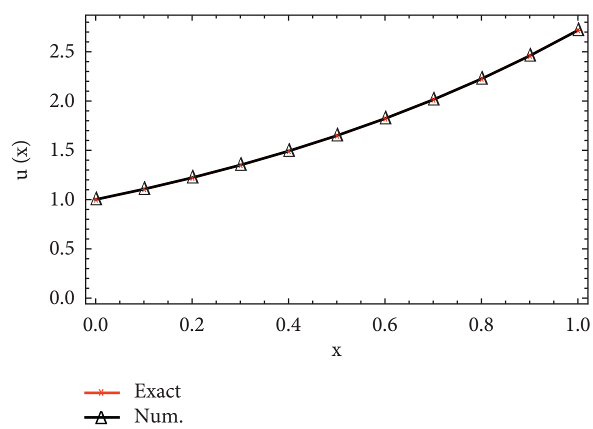

FIGURE 4: Comparison between the numerical and exact solutions of example 4.

7. Conclusion

The existence and uniqueness of the nonlocal boundary value problem for the Fredholm–Volterra integro-differential equation with the nonlocal condition and the integral condition have been studied. The continuous dependence of the solution on μ_0 and a_j has been introduced. Also, we used the central difference approximations and trapezoidal rule to obtain a numerical solution for problems. The error estimation has been derived in this paper. Finally, we solve some numerical examples to illustrate the accuracy of the proposed method [12].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

References

[1] A. M. A. El-Sayed and R. G. Ahmed, "Existence of solutions for a functional integro-differential equation with infinite point and integral conditions," *International Journal of Algorithms, Computing and Mathematics*, vol. 5, no. 4, p. 108, 2019.

[2] A. M. A. El-Sayed and R. Gamal, "Infinite point and Riemann-Stieltjes integral conditions for an integro-differential equation," *Nonlinear Analysis Modelling and Control*, vol. 24, pp. 733–754, 2019.

[3] A. M. A. El-sayed, H. El-Owaidy, and R. Gamal Ahmed, "Solvability of a boundary value problem of self-reference functional differential equation with infinite point and integral conditions," *The Journal of Mathematics and Computer Science*, vol. 21, no. 04, pp. 296–308, 2020.

[4] A. M. A. El-Sayed and R. G. Ahmed, "Solvability of a coupled system of functional integro-differential equations with infinite point and Riemann-Stieltjes integral conditions," *Applied Mathematics and Computation*, vol. 370, Article ID 124918, 2020.

[5] F. Mirzaee and S. Piroozfar, "Numerical solution of linear Fredholm integral equations via modified Simpson's quadrature rule," *Journal of King Saud University Science*, vol. 23, no. 1, pp. 7–10, 2011.

[6] I. Rahman, M. M. Parvez, and S. Ghosh, "A new technique for numerical solution of system of Volterra integral equations of the second kind by Simpson's quadrature rule," *Journal of Computer and Mathematical Sciences*, vol. 8, pp. 332–339, 2017.

[7] B. D. Garba and S. L. Bichi, "On solving linear Fredholm integro-differential equations via finite difference-Simpson's approach," *Malaya Journal of Matematik*, vol. 8, no. 2, pp. 469–472, 2020.

[8] A. A. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, and K. K. Ali, "On the analytical and numerical study for nonlinear Fredholm integro-differential equations," *Appl. Math. Inf. Sci*, vol. 14, pp. 921–929, 2020.

[9] P. K. Pandey, "Numerical solution of Linear Fredholm integro-differential equations by non-standard finite difference method," *Applications and Applied Mathematics*, vol. 10, pp. 1019–1026, 2015.

[10] R. Saadati, B. Raftari, H. Adibi, S. M. Vaezpour, and S. Shakeri, "A comparison between the variational iteration method and Trapezoidal rule for solving linear integro-differential equations," *World Applied Sciences Journal*, vol. 4, pp. 321–325, 2008.

[11] F. Ishak and S. Norazura Ahmed, "Developement of extended Trapeziodal method for numerical solution of Volterra integro-differential equations," *International Journal of Mathematics and Computer Science*, vol. 10, pp. 579–582, 2016.

[12] B. Raftari, "Numerical solutions of the linear Volterra integro-differential equations: Homotopy perturbation method and finite difference method," *World Applied Sciences Journal*, vol. 9, pp. 7–12, 2010.

[13] A. N. Kolomogorov and S. V. Fomin, *Inroductory Real Analysis*, Dover Puble. Inc, Downers Grove, IL, USA, 1975.

[14] K. Goeble and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambirdge Universty Press, Cambridge, UK, 1990.