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Research Article

On a Coupled System of Fractional Differential Equations via the Generalized Proportional Fractional Derivatives

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This work investigates the existence and uniqueness of solutions for a coupled system of fractional differential equations with three-point generalized fractional integral boundary conditions within generalized proportional fractional derivatives of the Riemann-Liouville type. By using the Schauder and Banach fixed point theorems, we study the existence and uniqueness of solutions for the aforesaid system. Finally, we present an example to validate our theoretical outcomes.

1. Introduction

The theory of fractional calculus has become an attractive area of research for mathematicians and physicians because of its fertile aspects in many applications in natural science [1, 2], engineering [3], and many other fields. Moreover, the fractional differential equations have been employed successfully in the modeling of many biological problems, for example, human liver [4], hepatitis B [5–7], mumps virus [8] and methanol detoxification in the human body [9], and other differential models in thermodynamic and physics such as thermostat [10], pantograph [11], diffusion-wave system [12], and dynamical systems [13]. For additional specifics about the theory of fractional calculus and applications, we suggest the books of Kilbas et al. [14], Podlubny [15], and Samko et al. [16]. During the last years, there have exhibited several concepts about fractional derivatives. Here, we point out the most famous kinds including Riemann-Liouville, Liouville-Caputo, generalized Caputo [17], and Hadamard derivatives [18]. This has lead researchers to numerous research papers concerning several fractional operators which were conducted that one can see, for example, in complex plain [19, 20], extended Riemann-Liouville [21], the Mittag-Leer type function [22], the q-derivative [23], the local fractional derivative [24], and in stability result [25–27].

More recently, Jarad et al. [28] constructed a new generalized fractional derivative which is called the generalized proportional fractional derivative. This new fractional operator has the advantage of being well-behaved as it is considered to be a generalization of many of the previously known and widely used fractional operators such as Liouville-Caputo and Riemann-Liouville fractional operators. In detail, fractional differential equations with generalized proportional derivatives have seen significant contributions from an interested researcher. For instance, we refer to works of Abbas and Ragusa and Hristova and Abbas [29, 30] and Khaminsou et al. [31, 32], and the references existing therein.

At the same time, coupled systems of differential equations of fractional order with different boundary conditions have been the focus of many mathematicians. The literature on the topic involves the existence, uniqueness, and stability results. Ahmad and Luca [33] studied a system of nonlinear Caputo fractional differential equations with coupled boundary conditions involving Riemann-Liouville fractional integrals. Baitiche et al. [34] discussed the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the ψ-Caputo fractional derivative with multipoint boundary conditions. Mahmudov et al. [35] investigated existence and uniqueness results for a coupled system of Caputo fractional differential equations with integral boundary conditions.

Some existing frameworks mentioned above encourage us to study the following coupled system of fractional differential equations:

$$\begin{cases} \left({}_a^R\mathscr{D}^{\alpha,\rho}u\right)(t) = \psi_1(t, u(t), v(t)), \\ \left({}_a^R\mathscr{D}^{\beta,\rho}v\right)(t) = \psi_2(t, u(t), v(t)), \\ t \in \mathscr{J} := [a, b], \end{cases} \tag{1}$$

equipped with the generalized fractional integral boundary conditions:

$$\begin{cases} u(\delta_1) = 0, \quad u(b) = ({}_a\mathscr{I}^{\gamma_1,\rho}u)(\mu_1), \\ v(\delta_2) = 0, \quad v(b) = ({}_a\mathscr{I}^{\gamma_2,\rho}u)(\mu_2), \end{cases} \tag{2}$$

where $\rho \in (0, 1]$, ${}_a^R\mathscr{D}^{\alpha,\rho}$ and ${}_a^R\mathscr{D}^{\beta,\rho}$ denote the generalized proportional fractional derivatives of Riemann-Liouville type of order $\alpha, \beta \in (1, 2]$, ${}_a\mathscr{I}^{\gamma_1,\rho}$ and ${}_a\mathscr{I}^{\gamma_2,\rho}$ denote the generalized proportional fractional integrals of order $\gamma_1, \gamma_2 \in (0, 1)$, and $\delta_1, \delta_2, \mu_1, \mu_2 \in (a, b)$ and $\psi_1, \psi_2 : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. In the current work, we establish the existence and uniqueness of solutions of the coupled system (1) and (2) by means of Schauder's and Banach's fixed point theorems.

To the best of our knowledge, there are no contributions considering a coupled system of generalized proportional fractional differential equations with generalized fractional integral boundary conditions.

The paper structure is designed as follows: in Section 2, we collect some essential definitions and lemmas relevant to the generalized proportional fractional derivatives and integrals; in Section 3, we establish the Green function associated with the linear issue of the coupled system (1) and (2), while in Section 4, we prove the main existence and uniqueness results in the current paper; in Section 5, an example is given to validate our theoretical outcomes.

2. Preliminaries

Here, we review some definitions of the generalized proportional fractional derivatives and integrals; see [28, 30, 36].

Definition 1 (see [37]). Take $\rho \in (0, 1]$, let the functions ε_0, $\varepsilon_1 : [0, 1] \times \mathbb{R} \longrightarrow [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have $\lim_{\rho\longrightarrow 0^+}\varepsilon_1(\rho, t) = 1$, $\lim_{\rho\longrightarrow 0^+}\varepsilon_0(\rho, t) = 0$, $\lim_{\rho\longrightarrow 1^-}\varepsilon_1(\rho, t) = 0$, $\lim_{\rho\longrightarrow 1^-}\varepsilon_0(\rho, t) = 1$, $\varepsilon_1(\rho, t)$=0 for $\rho \in [0, 1)$, and $\varepsilon_0(\rho, t)$=0 for $\rho \in (0, 1]$. Then, the amended conformable derivative of order ρ is defined by

$$(\mathscr{D}^\rho v)(t) = \varepsilon_1(\rho, t)v(t) + \varepsilon_0(\rho, t)v'(t). \tag{3}$$

The above amended conformable derivative (3) is said to be a proportional derivative (see [37]). When $\varepsilon_1(\rho, t) = 1 - \rho$ and $\varepsilon_0(\rho, t) = \rho$, (3) takes the form

$$(\mathscr{D}^\rho v)(t) = (1 - \rho)v(t) + \rho v'(t). \tag{4}$$

Note that, $\lim_{\rho\longrightarrow 0^+}(\mathscr{D}^\rho v)(t) = v(t)$ and $\lim_{\rho\longrightarrow 1^-}(\mathscr{D}^\rho v)(t) = v'(t)$.

Remark 2. By using (4) for the function $v(t) = e^t$ and any arbitrary order ρ, it can be easily concluded that $(\mathscr{D}^\rho v)(t) = e^t$.

Example 1. If $v(t) = \sin(t)$, then $(\mathscr{D}^\rho v)(t) = (1 - \rho)\sin(t) + \rho\cos(t)$. One can find the graphs of $\mathscr{D}^\rho \sin(t)$ for different value of $\rho = 0.1,0.5,0.9,1$, in Figure 1. As can be seen from Figure 1, in some points, the value of conformable derivative in this case is independent of ρ, and this can be one of the interesting properties of fractional calculus.

Definition 3 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized proportional fractional integral of the function $v \in L^1(\mathscr{J})$ by $({}_a\mathscr{I}^{0,\rho}v)(t) = v(t)$ and

$$({}_a\mathscr{I}^{\alpha,\rho}v)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{((\rho-1)/\rho)(t-s)}(t - s)^{\alpha-1}v(s)\,ds, t \in \mathscr{J}. \tag{5}$$

Definition 4 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized Caputo-proportional fractional derivative of the function $v \in C^{(n)}(\mathscr{J})$ by $({}_a^C\mathscr{D}^{0,\rho}v)(t) = v(t)$ and

$$\begin{aligned} \left({}_a^C\mathscr{D}^{\alpha,\rho}v\right)(t) &= {}_a\mathscr{I}^{n-\alpha,\rho}(\mathscr{D}^{n,\rho}v)(t) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n - \alpha)} \\ &\quad \cdot \int_a^t e^{((\rho-1)/\rho)(t-s)}(t - s)^{n-\alpha-1}(\mathscr{D}^{n,\rho}v)(s)\,ds, \end{aligned} \tag{6}$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}$, $(\mathscr{D}^{1,\rho}v)(t) = (\mathscr{D}^\rho v)(t) = (1 - \rho)v(t) + \rho v'(t)$, and $(\mathscr{D}^{n,\rho}v)(t) = (\underbrace{\mathscr{D}^\rho\mathscr{D}^\rho \cdots \mathscr{D}^\rho}_{n \text{ times}})v(t)$.

Definition 5 (see [28, 30]). Take $\rho \in (0, 1]$, $\alpha \geq 0$, we define the left generalized proportional fractional derivative of Riemann-Liouville type of the function v by $({}_a\mathscr{D}^{0,\rho}v)(t) = v(t)$ and

$$\begin{aligned} \left({}_a^R\mathscr{D}^{\alpha,\rho}v\right)(t) &= \mathscr{D}^{n,\rho}{}_a\mathscr{I}^{n-\alpha,\rho}v(t) \\ &= \frac{\mathscr{D}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n - \alpha)} \\ &\quad \cdot \int_a^t e^{((\rho-1)/\rho)(t-s)}(t - s)^{n-\alpha-1}v(s)\,ds, \end{aligned} \tag{7}$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}$.

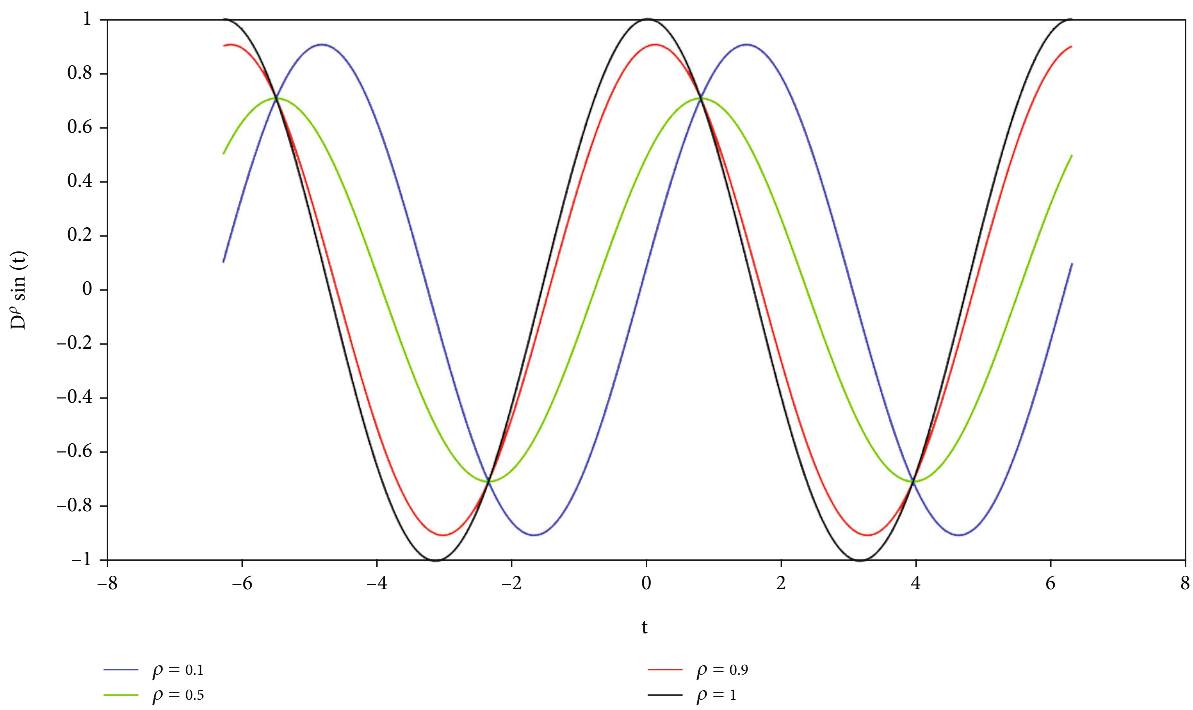

FIGURE 1: The graph of $\mathscr{D}^{\rho}\sin(t)$.

Lemma 6 (see [36]). *If $\rho\in(0,1]$, $\beta>0$, and $\alpha>0$ with $n-1<\alpha\le n$ and $v\in L^1(\mathscr{J})$, we have the following statements:*

$$\left({}_a\mathscr{I}^{\alpha,\rho}e^{\rho-1/\rho\tau}(\tau-a)^{\beta-1}\right)(t)=\frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\beta+\alpha)}e^{((\rho-1)/\rho)t}(t-a)^{\alpha+\beta-1}, \tag{8}$$

$$\left({}_a^R\mathscr{D}^{\alpha,\rho}e^{\rho-1/\rho\tau}(\tau-a)^{\beta-1}\right)(t)=\frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{((\rho-1)/\rho)t}(t-a)^{\beta-1-\alpha}, \tag{9}$$

$${}_a\mathscr{I}^{\alpha,\rho}\left({}_a\mathscr{I}^{\beta,\rho}v\right)(t)={}_a\mathscr{I}^{\beta,\rho}\left({}_a\mathscr{I}^{\alpha,\rho}v\right)(t)=\left({}_a\mathscr{I}^{\alpha+\beta,\rho}v\right)(t), \tag{10}$$

$$\left({}_a^R\mathscr{D}^{\eta,\rho}{}_a\mathscr{I}^{\alpha,\rho}v\right)(t)=\left({}_a\mathscr{I}^{\alpha-\eta,\rho}v\right)(t),\ 0<\eta<\alpha, \tag{11}$$

$$\left({}_a^R\mathscr{D}^{\alpha,\rho}{}_a\mathscr{I}^{\alpha,\rho}v\right)(t)=v(t), \tag{12}$$

$${}_a\mathscr{I}^{\alpha,\rho}\left({}_a^R\mathscr{D}^{\alpha,\rho}v\right)(t)=v(t)-\sum_{k=1}^{n}d_k e^{((\rho-1)/\rho)(t-a)}(t-a)^{\alpha-k}, \tag{13}$$

where $d_k=\left({}_a\mathscr{I}^{k-\alpha,\rho}v\right)(a)/\rho^{\alpha-k}\Gamma(\alpha-k+1)$.

Theorem 7 (Schauder's fixed point theorem) [38]. *Let $\mathscr{U}$ be a closed, convex, and nonempty subset of a Banach space C; let $\mathscr{H}:\mathscr{U}\longrightarrow\mathscr{U}$ be a continuous mapping such that $\mathscr{H}(\mathscr{U})$ is a relatively compact subset of C. Then, $\mathscr{H}$ has at least one fixed point in $\mathscr{U}$.*

3. The Equivalent Integral Equations

Let $\mathscr{C}=C(\mathscr{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathscr{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_{\mathscr{C}}=\max_{t\in\mathscr{J}}|u(t)|, \tag{14}$$

and $\mathbf{C}=\mathscr{C}\times\mathscr{C}$ be the product Banach space with the norm

$$\|(u,v)\|_{\mathbf{C}}=\|u\|_{\mathscr{C}}+\|v\|_{\mathscr{C}}. \tag{15}$$

Definition 8. By a solution of the coupled system (1) and (2), we mean a coupled ordered pair of continuous functions $(u,v)\in C$ that satisfy (1) and (2).

Lemma 9. *Let $\rho\in(0,1]$, $\Lambda_1\Lambda_4=\Lambda_2\Lambda_3$, and $\omega:\mathscr{J}\longrightarrow\mathbb{R}$. Then, the solution of*

$$\begin{cases}\left({}_a^R\mathscr{D}^{\alpha,\rho}u\right)(t)=\omega(t),\ t\in\mathscr{J},\ \alpha\in(1,2],\\ u(\delta_1)=0,\ u(b)=\left({}_a\mathscr{I}^{\gamma_1,\rho}u\right)(\mu_1),\ \delta_1,\mu_1\in(a,b),\ \gamma_1\in(0,1),\end{cases} \tag{16}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) = {} & ({}_a\mathcal{J}^{\alpha,\rho}\omega)(t) + \Lambda_6 (t-a)^{\alpha-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot (\Lambda_2 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) - \Lambda_2 ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_4 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1)) \\ & + \Lambda_5 (t-a)^{\alpha-2} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot (\Lambda_1 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) - \Lambda_1 ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_3 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1)), \end{aligned} \tag{17}$$

where

$$\begin{cases} \Lambda_1 = (\delta_1 - a)^{\alpha-1} e^{((\rho-1)/\rho)(\delta_1 - a)}, \quad \Lambda_2 = (\delta_1 - a)^{\alpha-2} e^{((\rho-1)/\rho)(\delta_1 - a)}, \\ \Lambda_3 = (b-a)^{\alpha-1} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\alpha)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1)} (\mu_1 - a)^{\alpha+\gamma_1-1} e^{((\rho-1)/\rho)(\mu_1-a)}, \\ \Lambda_4 = (b-a)^{\alpha-2} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\alpha-1)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1-1)} (\mu_1 - a)^{\alpha+\gamma_1-2} e^{((\rho-1)/\rho)(\mu_1-a)}, \\ \Lambda_5 = (\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4)^{-1}, \text{ and } \Lambda_6 = (\Lambda_1\Lambda_4 - \Lambda_2\Lambda_3)^{-1}. \end{cases} \tag{18}$$

Proof. By applying the generalized fractional proportional integral ${}_a\mathcal{J}^{\alpha,\rho}(\cdot)$ to both sides of the first equation (16) and using (13) in Lemma 6, one has

$$\begin{aligned} u(t) = {} & ({}_a\mathcal{J}^{\alpha,\rho}\omega)(t) + d_1 e^{((\rho-1)/\rho)(t-a)} (t-a)^{\alpha-1} \\ & + d_2 e^{((\rho-1)/\rho)(t-a)} (t-a)^{\alpha-2}. \end{aligned} \tag{19}$$

Using the boundary condition $u(\delta_1) = 0$ and (19), one has

$$\begin{aligned} & (\delta_1 - a)^{\alpha-1} e^{((\rho-1)/\rho)(\delta_1-a)} d_1 + (\delta_1 - a)^{\alpha-2} e^{((\rho-1)/\rho)(\delta_1-a)} d_2 \\ & \quad = -({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1). \end{aligned} \tag{20}$$

Using (18), the above equation becomes

$$\Lambda_1 d_1 + \Lambda_2 d_2 = -({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1). \tag{21}$$

In the light of (8) and (10) in Lemma 6, the boundary condition $u(b) = ({}_a\mathcal{J}^{\gamma_1,\rho}u)(\mu_1)$ and (19) give

$$\begin{aligned} & ({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) + d_1 e^{((\rho-1)/\rho)(b-a)} (b-a)^{\alpha-1} + d_2 e^{((\rho-1)/\rho)(b-a)} (b-a)^{\alpha-2} \\ & \quad = ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) + d_1 \frac{\Gamma(\alpha)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1)} (\mu_1 - a)^{\alpha+\gamma_1-1} e^{((\rho-1)/\rho)(\mu_1-a)} \\ & \qquad + d_2 \frac{\Gamma(\alpha-1)}{\rho^{\gamma_1}\Gamma(\alpha+\gamma_1-1)} (\mu_1 - a)^{\alpha+\gamma_1-2} e^{((\rho-1)/\rho)(\mu_1-a)}. \end{aligned} \tag{22}$$

Again, using (18), the above equation takes the form

$$\Lambda_3 d_1 + \Lambda_4 d_2 = -({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) + ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1). \tag{23}$$

Therefore, by merging equations (21) and (23), using (18), we get

$$\begin{aligned} d_1 &= \Lambda_6 (\Lambda_2 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) - \Lambda_2 ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_4 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1)), \\ d_2 &= \Lambda_5 (\Lambda_1 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(b) - \Lambda_1 ({}_a\mathcal{J}^{\alpha+\gamma_1,\rho}\omega)(\mu_1) - \Lambda_3 ({}_a\mathcal{J}^{\alpha,\rho}\omega)(\delta_1)). \end{aligned} \tag{24}$$

Thus, by inserting the values of d_1 and d_2 in (19), we obtain (17). The proof is finished. □

By hint of Lemma 9, the solution $(u, v) \in \mathbf{C}$ of the system (1) and (2) is given by

$$\begin{aligned} u(t) = {} & {}_a\mathcal{J}^{\alpha,\rho}\psi_1(t, u(t), v(t)) + \Lambda_6 (t-a)^{\alpha-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot (\Lambda_{2a}\mathcal{J}^{\alpha,\rho}\psi_1(b, u(b), v(b)) \\ & - \Lambda_{2a}\mathcal{J}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1, u(\mu_1), v(\mu_1)) \\ & - \Lambda_{4a}\mathcal{J}^{\alpha,\rho}\psi_1(\delta_1, u(\delta_1), v(\delta_1))) \\ & + \Lambda_5 (t-a)^{\alpha-2} e^{((\rho-1)/\rho)(t-a)} (\Lambda_{1a}\mathcal{J}^{\alpha,\rho}\psi_1(b, u(b), v(b)) \\ & - \Lambda_{1a}\mathcal{J}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1, u(\mu_1), v(\mu_1)) \\ & - \Lambda_{3a}\mathcal{J}^{\alpha,\rho}\psi_1(\delta_1, u(\delta_1), v(\delta_1))), \quad t \in \mathcal{J}, \end{aligned}$$

$$\begin{aligned} v(t) = {} & {}_a\mathcal{J}^{\beta,\rho}\psi_2(t, u(t), v(t)) + \Lambda'_6 (t-a)^{\beta-1} e^{((\rho-1)/\rho)(t-a)} \\ & \cdot \Big(\Lambda'_{2a}\mathcal{J}^{\beta,\rho}\psi_2(b, u(b), v(b)) - \Lambda'_{2a}\mathcal{J}^{\beta+\gamma_2,\rho}\psi_2 \\ & \cdot (\mu_2, u(\mu_2), v(\mu_2)) - \Lambda'_{4a}\mathcal{J}^{\beta,\rho}\psi_2(\delta_2, u(\delta_2), v(\delta_2))\Big) \\ & + \Lambda'_5 (t-a)^{\beta-2} e^{((\rho-1)/\rho)(t-a)} \Big(\Lambda'_{1a}\mathcal{J}^{\beta,\rho}\psi_2(b, u(b), v(b)) \\ & - \Lambda'_{1a}\mathcal{J}^{\beta+\gamma_2,\rho}\psi_2(\mu_2, u(\mu_2), v(\mu_2)) \\ & - \Lambda'_{3a}\mathcal{J}^{\beta,\rho}\psi_2(\delta_2, u(\delta_2), v(\delta_2))\Big), \quad t \in \mathcal{J}, \end{aligned} \tag{25}$$

where

$$\begin{cases} \Lambda'_1 = (\delta_2 - a)^{\beta-1} e^{((\rho-1)/\rho)(\delta_2 - a)}, \quad \Lambda'_2 = (\delta_2 - a)^{\beta-2} e^{((\rho-1)/\rho)(\delta_2 - a)}, \\ \Lambda'_3 = (b-a)^{\beta-1} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\beta)}{\rho^{\gamma_2}\Gamma(\beta+\gamma_2)} (\mu_2 - a)^{\beta+\gamma_2-1} e^{((\rho-1)/\rho)(\mu_2-a)}, \\ \Lambda'_4 = (b-a)^{\beta-2} e^{((\rho-1)/\rho)(b-a)} - \dfrac{\Gamma(\beta-1)}{\rho^{\gamma_2}\Gamma(\beta+\gamma_2-1)} (\mu_2 - a)^{\beta+\gamma_2-2} e^{((\rho-1)/\rho)(\mu_2-a)}, \\ \Lambda'_5 = \left(\Lambda'_2\Lambda'_3 - \Lambda'_1\Lambda'_4\right)^{-1}, \text{ and } \Lambda'_6 = \left(\Lambda'_1\Lambda'_4 - \Lambda'_2\Lambda'_3\right)^{-1}, \Lambda'_1\Lambda'_4 = \Lambda'_2\Lambda'_3. \end{cases} \tag{26}$$

4. Existence and Uniqueness Results

Define the operator $\mathcal{H} : \mathbf{C} \longrightarrow \mathbf{C}$ by

$$(\mathcal{H}(u, v))(t) = \begin{pmatrix} (\mathcal{H}_1(u, v))(t) \\ (\mathcal{H}_2(u, v))(t) \end{pmatrix}, \tag{27}$$

where

$$(\mathcal{H}_1(u,v))(t) = {}_a\mathcal{J}^{\alpha,\rho}\psi_1(t,u(t),v(t)) + \Lambda_6(t-a)^{\alpha-1}e^{((\rho-1)/\rho)(t-a)} \cdot (\Lambda_{2a}\mathcal{J}^{\alpha,\rho}\psi_1(b,u(b),v(b)) - \Lambda_{2a}\mathcal{J}^{\alpha+\gamma_1,\rho}\psi_1 (\mu_1,u(\mu_1),v(\mu_1)) - \Lambda_{4a}\mathcal{J}^{\alpha,\rho}\psi_1(\delta_1,u(\delta_1),v(\delta_1))) + \Lambda_5(t-a)^{\alpha-2}e^{((\rho-1)/\rho)(t-a)}(\Lambda_{1a}\mathcal{J}^{\alpha,\rho}\psi_1(b,u(b),v(b)) - \Lambda_{1a}\mathcal{J}^{\alpha+\gamma_1,\rho}\psi_1(\mu_1,u(\mu_1),v(\mu_1)) - \Lambda_{3a}\mathcal{J}^{\alpha,\rho}\psi_1(\delta_1,u(\delta_1),v(\delta_1))),(t\in\mathcal{J}), \tag{28}$$

$$(\mathcal{H}_2(u,v))(t) = {}_a\mathcal{J}^{\beta,\rho}\psi_2(t,u(t),v(t)) + \Lambda_6'(t-a)^{\beta-1}e^{((\rho-1)/\rho)(t-a)} \cdot \left(\Lambda_{2a}'\mathcal{J}^{\beta,\rho}\psi_2(b,u(b),v(b)) - \Lambda_{2a}'\mathcal{J}^{\beta+\gamma_2,\rho}\psi_2 \cdot (\mu_2,u(\mu_2),v(\mu_2)) - \Lambda_{4a}'\mathcal{J}^{\beta,\rho}\psi_2(\delta_2,u(\delta_2),v(\delta_2))\right) + \Lambda_5'(t-a)^{\beta-2}e^{((\rho-1)/\rho)(t-a)}\left(\Lambda_{1a}'\mathcal{J}^{\beta,\rho}\psi_2(b,u(b),v(b)) - \Lambda_{1a}'\mathcal{J}^{\beta+\gamma_2,\rho}\psi_2(\mu_2,u(\mu_2),v(\mu_2)) - \Lambda_{3a}'\mathcal{J}^{\beta,\rho}\psi_2(\delta_2,u(\delta_2),v(\delta_2))\right), \quad t\in\mathcal{J}. \tag{29}$$

According to Lemma 9, the solution $(u,v)\in\mathbf{C}$ of the coupled system (1) and (2) conforms with the fixed point operator $\mathcal{H}$.

For fulfillment the main results, the following assumptions will be imposed.

(A1) The functions $\psi_1,\psi_2:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous

(A2) There exist nonnegative constants L_1 and L_2 such that

$$|\psi_i(t,u_1,v_1)-\psi_i(t,u_2,v_2)| \le L_i(|u_1-u_2|+|v_1-v_2|), \tag{30}$$

for each $t\in\mathcal{J}$ and $u_i,v_i\in\mathbb{R}, i=1,2$

Further, we set $\psi_i^* = \max_{t\in\mathcal{J}}|\psi_i(t,0,0)|, i=1,2$.

The following notations will be introduced:

$$\Delta_1 := \left[\frac{(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + |\Lambda_6|(b-a)^{\alpha-1}\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) + |\Lambda_5|(b-a)^{\alpha-2}\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\right], \tag{31}$$

$$\Delta_2 := \left[\frac{(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + |\Lambda_6'|(b-a)^{\beta-1}\left(\frac{|\Lambda_2'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_2'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_4'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right) + |\Lambda_5'|(b-a)^{\beta-2}\left(\frac{|\Lambda_1'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_1'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_3'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right)\right]. \tag{32}$$

Theorem 10. *Assume that the assumptions (A1) and (A2) are satisfied. Then, the coupled system (1) and (2) has at least one solution on $\mathcal{J}$.*

Proof. Consider the operator $\mathcal{H}:\mathbf{C}\longrightarrow\mathbf{C}$ as defined in (27). Let us introduce the ball

$$\zeta_\ell = \{(u,v)\in\mathbf{C}: \|(u,v)\|_{\mathbf{C}} \le \ell\}, \tag{33}$$

where ℓ is a positive real number such that

$$\ell \ge \frac{\Delta_1\psi_1^* + \Delta_2\psi_2^*}{1-(\Delta_1L_1+\Delta_2L_2)}, \Delta_1L_1+\Delta_2L_2<1. \tag{34}$$

It is obvious that ζ_ℓ is a closed, bounded, and convex subset of the Banach space $\mathbf{C}$. We shall show that $\mathcal{H}$ achieves the hypothesis of Schauder's fixed point theorem in four steps.

Step 1. $\mathcal{H}$ maps bounded sets into bounded sets in $\mathbf{C}$.

By virtue of (A2) and since $|e^{((\rho-1)/\rho)(t-a)}|<1, \forall t>a$, then for each $t\in\mathcal{J}$ and $(u,v)\in\zeta_\ell$, one has

$$|(\mathcal{H}_1(u,v))(t)| \tag{35}$$

Thus, by using (31), we get

$$\|(\mathcal{H}_1(u,v))\|_{\mathscr{C}} \le \Delta_1(L_1\ell+\psi_1^*). \tag{36}$$

Similarly, we obtain that

$$\|(\mathcal{H}_2(u,v))\|_{\mathscr{C}} \le \Delta_2(L_2\ell+\psi_2^*), \tag{37}$$

where Δ_2 is defined in (32). Hence, we conclude that

$$\|(\mathcal{H}_1(u,v))\|_{\mathscr{C}} + \|(\mathcal{H}_2(u,v))\|_{\mathscr{C}} \le (\Delta_1L_1+\Delta_2L_2)\ell + (\Delta_1\psi_1^*+\Delta_2\psi_2^*) \le \ell, \tag{38}$$

which implies that $\mathcal{H}:\zeta_\ell\longrightarrow\zeta_\ell$

Step 2. $\mathcal{H}$ is continuous.

In view of the assumption (A1), we conclude that $\mathscr{H}_1$ and $\mathscr{H}_2$ are continuous on $\mathscr{J}$. Thus, the operator $\mathscr{H}$ is also continuous

Step 3. $\mathscr{H}(\zeta_\ell)$ is equicontinuous.

Set $\max_{t\in\mathscr{J}}|\psi_i(t,u(t),v(t))| := M_i < \infty, i = 1, 2$. For $t_1, t_2 \in \mathscr{J}$, with $t_1 < t_2$ and $(u,v) \in \zeta_\ell$, we have

$$
\begin{aligned}
|(\mathscr{H}_1(u,v))(t_2) - (\mathscr{H}_1(u,v))(t_1)| \le{}& \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{t_2} e^{((\rho-1)/\rho)(t_2-s)}(t_2-s)^{\alpha-1}\psi_1(s,u(s),v(s))ds - \int_a^{t_1} e^{((\rho-1)/\rho)(t_1-s)}(t_1-s)^{\alpha-1}\psi_1(s,u(s),v(s))ds\right| \\
&+ |\Lambda_6|\left|\left(e^{((\rho-1)/\rho)(t_2-a)}(t_2-a)^{\alpha-1} - e^{((\rho-1)/\rho)(t_1-a)}(t_1-a)^{\alpha-1}\right)\right| \\
&\cdot\left(\frac{|\Lambda_2|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_2|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_4|}{\rho^\alpha\Gamma(\alpha)}\right. \\
&\left.\cdot\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds\right) + |\Lambda_5|\left|\left(e^{((\rho-1)/\rho)(t_2-a)}(t_2-a)^{\alpha-2} - e^{((\rho-1)/\rho)(t_1-a)}(t_1-a)^{\alpha-2}\right)\right| \\
&\cdot\left(\frac{|\Lambda_1|}{\rho^\alpha\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds + \frac{|\Lambda_1|}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1)}\int_a^{\mu_1}(\mu_1-s)^{\alpha+\gamma_1-1}|\psi_1(s,u(s),v(s))|ds\right. \\
&\left.+ \frac{|\Lambda_3|}{\rho^\alpha\Gamma(\alpha)}\int_a^{\delta_1}(\delta_1-s)^{\alpha-1}|\psi_1(s,u(s),v(s))|ds\right) \\
\le{}& \frac{M_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_a^{t_2}\left|e^{((\rho-1)/\rho)(t_2-s)}\right|\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|ds + \int_a^{t_2}\left|e^{((\rho-1)/\rho)(t_2-s)} - e^{((\rho-1)/\rho)(t_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right. \\
&\left.+ \int_{t_1}^{t_2}\left|e^{((\rho-1)/\rho)(t_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right) + |\Lambda_6|M_1\left(\left|e^{((\rho-1)/\rho)(t_2-a)}\right|\left|(t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right|\right. \\
&\left.+ \left|e^{((\rho-1)/\rho)(t_2-a)} - e^{((\rho-1)/\rho)(t_1-a)}\right|\left|(t_1-a)^{\alpha-1}\right|\right) \times \left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
&+ |\Lambda_5|M_1\left(\left|e^{((\rho-1)/\rho)(t_2-a)}\right|\left|(t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right| + \left|e^{((\rho-1)/\rho)(t_2-a)} - e^{((\rho-1)/\rho)(t_1-a)}\right|\left|(t_1-a)^{\alpha-2}\right|\right) \\
&\times\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
\le{}& \frac{M_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_a^{t_2}\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|ds + \int_a^{t_2}\left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_1-s)}\right|\left|(t_1-s)^{\alpha-1}\right|ds\right. \\
&\left.+ \int_{t_1}^{t_2}\left|(t_1-s)^{\alpha-1}ds\right) + |\Lambda_6|M_1\left(\left((t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right) + \left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_2-s)}\right|(t_1-a)^{\alpha-1}\right) \\
&\times\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) + |\Lambda_5|M_1\left(\left((t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right)\right. \\
&\left.+ \left|\frac{\rho-1}{\rho}(t_2-t_1)e^{((\rho-1)/\rho)(\xi_2-s)}\right|(t_1-a)^{\alpha-2}\right) \times\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right),
\end{aligned} \tag{39}
$$

where the mean value theorem is used on the function $e^{((\rho-1)/\rho)t}$ with $\xi_1, \xi_2 \in (t_1, t_2)$.

Thus, we get

$$
\begin{aligned}
&|(\mathscr{H}_1(u,v))(t_2) - (\mathscr{H}_1(u,v))(t_1)| \\
&\quad\le \frac{M_1}{\rho^\alpha\Gamma(\alpha+1)}\left(\left((t_2-a)^\alpha - (t_1-a)^\alpha\right) + (t_2-t_1)(t_1-a)^\alpha\right) \\
&\qquad + |\Lambda_6|M_1\left(\left((t_2-a)^{\alpha-1} - (t_1-a)^{\alpha-1}\right) + (t_2-t_1)(t_1-a)^{\alpha-1}\right) \\
&\qquad \times\left(\frac{|\Lambda_2|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_2|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right) \\
&\qquad + |\Lambda_5|M_1\left(\left((t_2-a)^{\alpha-2} - (t_1-a)^{\alpha-2}\right) + (t_2-t_1)(t_1-a)^{\alpha-2}\right) \\
&\qquad \times\left(\frac{|\Lambda_1|(b-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{|\Lambda_1|(\mu_1-a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1}\Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3|(\delta_1-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right).
\end{aligned} \tag{40}
$$

In an identical way, we obtain that

$$
\begin{aligned}
&|(\mathscr{H}_2(u,v))(t_2) - (\mathscr{H}_2(u,v))(t_1)| \\
&\quad\le \frac{M_2}{\rho^\beta\Gamma(\beta+1)}\left(\left((t_2-a)^\beta - (t_1-a)^\beta\right) + (t_2-t_1)(t_1-a)^\beta\right) \\
&\qquad + |\Lambda_6'|M_2\left(\left((t_2-a)^{\beta-1} - (t_1-a)^{\beta-1}\right) + (t_2-t_1)(t_1-a)^{\beta-1}\right) \\
&\qquad \times\left(\frac{|\Lambda_2'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_2'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_4'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right) \\
&\qquad + |\Lambda_5'|M_1\left(\left((t_2-a)^{\beta-2} - (t_1-a)^{\beta-2}\right) + (t_2-t_1)(t_1-a)^{\beta-2}\right) \\
&\qquad \times\left(\frac{|\Lambda_1'|(b-a)^\beta}{\rho^\beta\Gamma(\beta+1)} + \frac{|\Lambda_1'|(\mu_2-a)^{\beta+\gamma_2}}{\rho^{\beta+\gamma_2}\Gamma(\beta+\gamma_2+1)} + \frac{|\Lambda_3'|(\delta_2-a)^\beta}{\rho^\beta\Gamma(\beta+1)}\right).
\end{aligned} \tag{41}
$$

As $t_2 \longrightarrow t_1$, the R.H.S. of the last two inequalities $\longrightarrow 0$ independently of $(u, v) \in \zeta_\ell$. As a consequence of Steps 1 to 3 and in the light of the Arzelá-Ascoli theorem, we conclude that the operator $\mathscr{H}(\zeta_\ell)$ is relatively compact in $\mathbf{C}$. Hence, in accordance with Schauder's fixed point theorem (Theorem 7), the operator $\mathscr{H}$ has a fixed point and so the coupled system (1) and (2) possesses at least one solution on $\mathscr{J}$. The proof is completed. □

Theorem 11. *Assume that the assumptions (A1) and (A2) are satisfied. If $\Delta_1 L_1 + \Delta_2 L_2 < 1$, then the coupled system (1) and (2) has a unique solution on $\mathscr{J}$.*

Proof. Consider the operator $\mathscr{H}$ as defined in (27). We have to show that $\mathscr{H}$ is a contraction mapping.

For each $t \in \mathscr{J}$ and $(u_1, v_1), (u_2, v_2) \in \mathbf{C}$, one has

$$\begin{aligned}
|(\mathscr{H}_1(u_1, v_1))(t) - (\mathscr{H}_1(u_2, v_2))(t)| \le{}& \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))|\, ds + |\Lambda_6| (t-a)^{\alpha-1} \\
&\cdot \left(\frac{|\Lambda_2|}{\rho^\alpha \Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds + \frac{|\Lambda_2|}{\rho^{\alpha+\gamma_1} \Gamma(\alpha+\gamma_1)} \right. \\
&\cdot \int_a^{\mu_1} (\mu_1 - s)^{\alpha+\gamma_1-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds + \frac{|\Lambda_4|}{\rho^\alpha \Gamma(\alpha)} \\
&\cdot \left. \int_a^{\delta_1} (\delta_1 - s)^{\alpha-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds \right) + |\Lambda_5| (t-a)^{\alpha-2} \\
&\cdot \left(\frac{|\Lambda_1|}{\rho^\alpha \Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds + \frac{|\Lambda_1|}{\rho^{\alpha+\gamma_1} \Gamma(\alpha+\gamma_1)} \right. \\
&\cdot \int_a^{\mu_1} (\mu_1 - s)^{\alpha+\gamma_1-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds + \frac{|\Lambda_3|}{\rho^\alpha \Gamma(\alpha)} \\
&\cdot \left. \int_a^{\delta_1} (\delta_1 - s)^{\alpha-1} |\psi_1(s, u_1(s), v_1(s)) - \psi_1(s, u_2(s), v_2(s))| ds \right) \\
\le{}& \left[\frac{(b-a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + |\Lambda_6| (b-a)^{\alpha-1} \left(\frac{|\Lambda_2| (b-a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{|\Lambda_2| (\mu_1 - a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1} \Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_4| (\delta_1 - a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \right) \right. \\
&\left. + |\Lambda_5| (b-a)^{\alpha-2} \left(\frac{|\Lambda_1| (b-a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{|\Lambda_1| (\mu_1 - a)^{\alpha+\gamma_1}}{\rho^{\alpha+\gamma_1} \Gamma(\alpha+\gamma_1+1)} + \frac{|\Lambda_3| (\delta_1 - a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \right) \right] \times L_1 \left(\|u_1 - u_2\|_{\mathscr{C}} + \|v_1 - v_2\|_{\mathscr{C}} \right).
\end{aligned} \tag{42}$$

Thus, by (31), we get

$$\|(\mathscr{H}_1(u_1, v_1)) - (\mathscr{H}_1(u_2, v_2))\|_{\mathscr{C}} \le \Delta_1 L_1 \left(\|u_1 - u_2\|_{\mathscr{C}} + \|v_1 - v_2\|_{\mathscr{C}} \right). \tag{43}$$

In a similar way, using (32), we get

$$\|(\mathscr{H}_2(u_1, v_1)) - (\mathscr{H}_2(u_2, v_2))\|_{\mathscr{C}} \le \Delta_2 L_2 \left(\|u_1 - u_2\|_{\mathscr{C}} + \|v_1 - v_2\|_{\mathscr{C}} \right). \tag{44}$$

From (43) and (44), we get

$$\|\mathscr{H}(u_1, v_1) - \mathscr{H}(u_2, v_2)\|_{\mathbf{C}} \le (\Delta_1 L_1 + \Delta_2 L_2) \|(u_1, v_1) - (u_2, v_2)\|_{\mathbf{C}}. \tag{45}$$

By virtue of the condition $\Delta_1 L_1 + \Delta_2 L_2 < 1$, we conclude that $\mathscr{H}$ is a contraction mapping.

Hence, with the aid of Banach's fixed point theorem, we deduce that $\mathscr{H}$ has a unique fixed point, and so the coupled system (1) and (2) possesses a solution on $\mathscr{J}$ uniquely. The proof is finished. □

Example 2. Consider the following coupled system of fractional differential equations

$$\begin{cases} \left({}_{0^+}^{R}\mathscr{D}^{3/2,1/2} u \right)(t) = \psi_1(t, u(t), v(t)), \\ \left({}_{0^+}^{R}\mathscr{D}^{5/4,1/2} v \right)(t) = \psi_2(t, u(t), v(t)), \\ t \in [0, 1], \end{cases} \tag{46}$$

with the generalized fractional integral boundary conditions:

$$\begin{cases} u\left(\frac{1}{3}\right) = 0, \quad u(1) = \left({}_a\mathscr{I}^{1/7,1/2} u \right)\left(\frac{1}{5}\right), \\ v\left(\frac{1}{6}\right) = 0, \quad v(1) = \left({}_a\mathscr{I}^{1/9,1/2} u \right)\left(\frac{1}{8}\right). \end{cases} \tag{47}$$

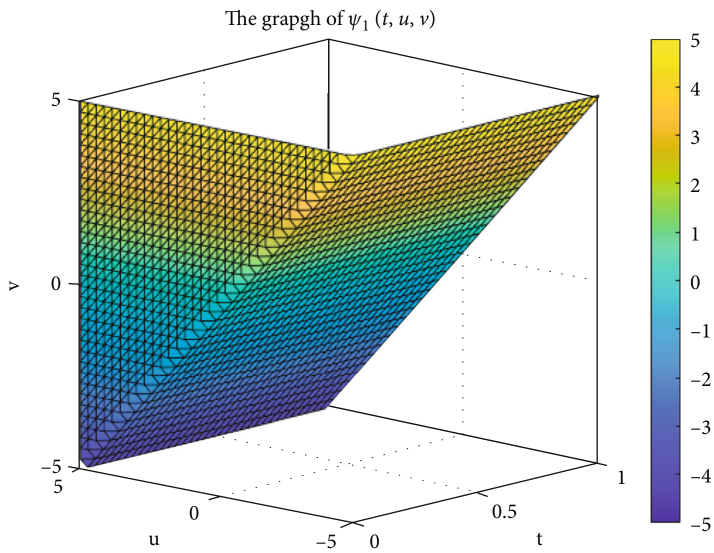

FIGURE 2: The graph of $\psi_1(t, u, v)$.

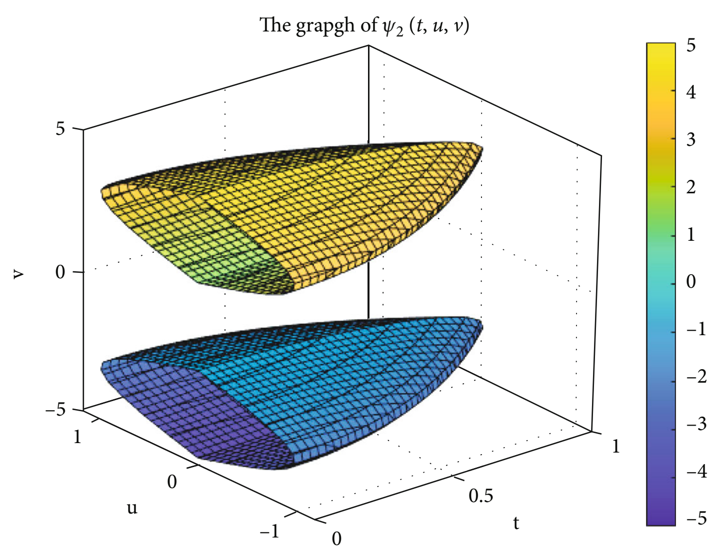

FIGURE 3: The graph of $\psi_2(t, u, v)$.

Here, $\alpha = 3/2, \beta = 5/4, \rho = 1/2, \gamma_1 = 1/7, \gamma_2 = 1/9, \delta_1 = 1/3$ $, \delta_2 = 1/6, \mu_1 = 1/5, \mu_2 = 1/8,$ and $[a, b] = [0, 1]$. Set $\psi_1(t, u, v)$ $= (1/25(t^2 + 2))(\sin^2|u| + (|v|/1 + |v|))$ and $\psi_2(t, u, v) = (1/$ $100)(t^2 + |u| + \cos |v|)$ that their graphs show in Figures 2 and 3.

For each $t \in [0, 1]$ and $(u_1, v_1), (u_2, v_2) \in \mathbf{C}$, one has

$$|\psi_1(t, u_1, v_1) - \psi_1(t, u_2, v_2)| \le \frac{1}{50}(|u_1 - u_2| + |v_1 - v_2|), \tag{48}$$

$$|\psi_2(t, u_1, v_1) - \psi_2(t, u_2, v_2)| \le \frac{1}{100}(|u_1 - u_2| + |v_1 - v_2|), \tag{49}$$

which implies that the assumption (A2) holds true with $L_1 = 1/50$ and $L_2 = 1/100$. We calculate functions in (18), (26), (31), and (32) for $\rho = 1/4$, $\rho = 1/2$, and $\rho = 3/4$ and present their numerical results in Table 1. We have in all three cases:

$$L_1\Delta_1 + L_2\Delta_2 < 1. \tag{50}$$

By virtue of the above discussion, we infer that all the assumptions of Theorems 10 and 11 are satisfied. Consequently, we deduce that the coupled system (46) and (47) has a solution on $[0, 1]$ uniquely.

Table 1: Numerical results for some functions in Example 2.

	$\rho = \frac{1}{4}$	$\rho = \frac{1}{2}$	$\rho = \frac{3}{4}$
Λ_1	0.2124	0.4137	0.5166
Λ_2	0.6372	1.2411	1.5499
Λ_3	0.0394	0.0512	0.0865
Λ_4	-1.2107	-1.6677	-1.2144
Λ_5	0.9680	1.3272	1.6257
Λ_6	-0.9680	-1.3272	-1.6257
Λ_1'	0.3215	0.5409	0.8321
Λ_2'	2.7841	3.2451	3.9521
Λ_3'	-0.0612	-0.0902	1.5934
Λ_4'	-4.1284	-4.9254	-5.6418
Λ_5'	0.2996	0.4217	0.7294
Λ_6'	-0.2996	-0.4217	-0.7294
Δ_1	6.7451	8.0645	10.0121
Δ_2	2.7423	3.1234	4.0096
$L_1\Delta_1 + L_2\Delta_2$	0.1623	0.2225	0.2403

5. Conclusion

As you know, there are many events in nature which we know nothing about those. One of the best ways for better understanding these types of phenomena is studying new notions in the fractional calculus field. In this work, we investigated the existence and uniqueness of solutions for a coupled system of fractional differential equations with three-point generalized fractional integral boundary conditions in the frame of the generalized proportional fractional derivatives of the Riemann-Liouville type which was introduced in 2017 by Jarad et al. In this way, we provided some results under some conditions. To better explain the notion, we gave some figures of some functions. Finally, we provided an illustrated example for our main result.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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