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Research Article

On a Subclass of Analytic Functions That Are Starlike with Respect to a Boundary Point Involving Exponential Function

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In the present exploration, the authors define and inspect a new class of functions that are regular in the unit disc $\mathfrak{D} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, by using an adapted version of the interesting analytic formula offered by Robertson (unexploited) for starlike functions with respect to a boundary point by subordinating to an exponential function. Examples of some new subclasses are presented. Initial coefficient estimates are specified, and the familiar Fekete-Szegö inequality is obtained. Differential subordinations concerning these newly demarcated subclasses are also established.

1. Introduction and Preliminary Results

Let $\mathscr{H}$ be the class comprising of all holomorphic functions in the unit disc $\mathfrak{D} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. Also, let $\mathscr{A}$ signify the subclass of $\mathscr{H}$ entailing of functions $h \in \mathscr{A}$ be of the form

$$h(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n, \quad \varsigma \in \mathfrak{D}, \tag{1}$$

with the normalization $h(0) = h'(0) - 1 = 0$. Denote by $\mathscr{S}$, the subclass of $\mathscr{A}$ comprising univalent functions. Two conversant subclasses of $\mathscr{A}$ are familiarized by Robertson [1], are defined with their analytical description as

$$\begin{aligned} \mathscr{S}^*(\alpha) &:= \left\{ h \in \mathscr{A} : \Re\left(\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right) > \alpha, \quad \varsigma \in \mathfrak{D} \right\}, \\ \mathscr{C}(\alpha) &:= \left\{ h \in \mathscr{A} : \Re\left(1 + \frac{\varsigma h''(\varsigma)}{h'(\varsigma)}\right) > \alpha, \quad \varsigma \in \mathfrak{D} \right\}, \end{aligned} \tag{2}$$

and are correspondingly known as starlike and convex functions of order $\alpha (0 \le \alpha < 1)$. It is well known that $\mathscr{S}^*(\alpha) \subset \mathscr{S}$ and $\mathscr{C}(\alpha) \subset \mathscr{S}$. In interpretation of Alexander's relation, $h \in \mathscr{C}(\alpha) \Leftrightarrow \varsigma h'(\varsigma) \in \mathscr{S}^*(\alpha)$ for $\varsigma \in \mathfrak{D}$. For $\alpha = 0$, the class $\mathscr{S}^* := \mathscr{S}^*(0)$ condenses to the well-known class of normalized starlike univalent functions, and $\mathscr{C} := \mathscr{C}(0)$ reduces to the normalized convex univalent functions.

A function $f \in \mathscr{H}$ is subordinate to $g \in \mathscr{H}$ written as $f \prec g$ if there exists $\omega \in \mathscr{H}$ with $\omega(0) = 0$ and $\omega(\mathfrak{D}) \subset \mathfrak{D}$ such that $f(\varsigma) = g(\omega(\varsigma))$ for every $\varsigma \in \mathfrak{D}$. In precise, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathfrak{D}) \subset g(\mathfrak{D})$.

Let $\mathscr{P}$ symbolize the class of functions $p \in \mathscr{H}$ with the normalization $p(0) = 1$, i.e., of the form

$$p(\varsigma) = 1 + \sum_{n=1}^{\infty} p_n \varsigma^n, \quad \varsigma \in \mathfrak{D}, \tag{3}$$

and such that $\Re p(\varsigma) > 0$ for $\varsigma \in \mathfrak{D}$. Functions in $\mathscr{P}$ are called familiarly as the Carathéodory class of functions. Ma and Minda [2] proposed a appropriate subclass of $\mathscr{P}$ denoted by $\mathscr{P}^*(1)$ comprising of all Φ that is univalent in $\mathfrak{D}$ with

$$\Phi(0)=1\,;\Phi'(0)>0, \tag{4}$$

$\Phi(\mathfrak{D})$ is symmetric with respect to the real axis

(2) Starlike with respect to 1

He also represented the class $\Phi\in\mathscr{P}^*(1)$ by

$$\Phi(\varsigma)=1+\sum_{n=1}^{\infty}B_n\varsigma^n, B_1>0\,;\varsigma\in\mathfrak{D}. \tag{5}$$

The class $\mathscr{P}^*(1)$ plays a vital part in defining generalized form of holomorphic functions. Ma and Minda [2] considered the function $\Phi\in\mathscr{P}^*(1)$ and defined $\mathscr{S}^*(\Phi)$ as the class of all $h\in\mathscr{A}$ such that $\varsigma h'(\varsigma)/h(\varsigma)\prec\Phi(\varsigma)$ for $\varsigma\in\mathfrak{D}$. The above functions defined are called as functions of Ma and Minda kind. Observe that $\mathscr{S}^*(\alpha)=\mathscr{S}^*(\Phi)$ with $\Phi(\varsigma)=(1+(1-2\alpha)\varsigma)/(1-\varsigma), \varsigma\in\mathfrak{D}$.

There are recent articles ([3–6]) where subclasses of $\mathscr{A}$ were defined by using subordination satisfying the relation $\varsigma h'(\varsigma)/h(\varsigma)\prec\Phi(\varsigma)$ for $\varsigma\in\mathfrak{D}$ (see also [7, 8]). In particular, the exponential function $\Phi_e(\varsigma)=e^{\varsigma}\coloneqq\exp(\varsigma)$, an entire function in $\mathbb{C}$ has positive real part in $\mathfrak{D}$, $\Phi_e(0)=1$, $\Phi_e'(0)=1$, and $\Phi_e(\mathfrak{D})=\{w\in\mathbb{C}:|\log w|<1\}$, is symmetric with respect to the real axis and starlike with respect to 1. Further, $\Phi_e\in\mathscr{P}^*(1)$ and therefore, it is now to make a remark that the class

$$\mathscr{S}_e=\left\{f\in\mathscr{A}:\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\prec\Phi_e(\varsigma)=e^{\varsigma},\quad \varsigma\in\mathfrak{D}\right\} \tag{6}$$

is well defined. For an attractive study on starlike functions connected with the exponential function, an individual can refer to Mendiratta et al. [9, 10] (see also the works of [11–13]).

We recall the class of close-to-convex functions denoted by $\mathscr{K}$ introduced and studied by Kaplan [14]. A function $h\in\mathscr{H}$ is called to be close-to-convex if and only if there exist a function $\psi\in\mathscr{C}$ and $\beta\in(-\pi/2,\pi/2)$ such that

$$\mathfrak{R}\left(\frac{e^{i\beta}h'(\varsigma)}{\psi'(\varsigma)}\right)>0,\quad \varsigma\in\mathfrak{D}. \tag{7}$$

Remarking at this time that even though starlikeness of a fixed order has been discussed and well thought-out in detail in countless articles in excess of a elongated stage of period, class of univalent functions $g\in\mathscr{H}$ that maps $\mathfrak{D}$ onto Ω, starlike domain with reverence to a boundary point is still a conception that is not exclusively explored. Robertson [15] recognized this examination and introduced a new subclass

$$\mathscr{G}^*=\left\{g\in\mathscr{H}:\mathfrak{R}\left(e^{i\delta}g(\varsigma)\right)>0\,;\delta\in\mathbb{R};\forall\varsigma\in\mathfrak{D}\right\}, \tag{8}$$

with

$$g(0)=1,\quad g(1)\coloneqq\lim_{r\longrightarrow 1^-}g(r)=0, \tag{9}$$

and maps (univalently) $\mathfrak{D}$ onto a domain starlike with respect to the origin. Presume in addition that the constant function $g\equiv 1\in\mathscr{G}^*$, in addition, Robertson through a conjecture that $\mathscr{G}^*$ coincides with the class $\mathscr{G}$ of all $g\in\mathscr{H}$ of the structure

$$g(\varsigma)=1+\sum_{n=1}^{\infty}\vartheta_n\varsigma^n,\quad \varsigma\in\mathfrak{D}, \tag{10}$$

such that

$$\mathfrak{R}\left(\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{1+\varsigma}{1-\varsigma}\right)>0,\quad \varsigma\in\mathfrak{D}, \tag{11}$$

proving that $\mathscr{G}\subset\mathscr{G}^*$. Definitely, in the same article Robertson shown that if $g\in\mathscr{G}$ and $g\not\equiv 1$, then $g\in\mathscr{K}$ and so univalent in $\mathfrak{D}$. It is importance of citing that (11) was identified by much erstwhile by Styer [16]. This surmise of Robertson that $\mathscr{G}^*$ coincide with the class $\mathscr{G}$ was soon after proved by Lyzzaik [17], where he established that $\mathscr{G}^*\subset\mathscr{G}$.

A different analytical categorization of starlike functions with respect to a boundary point was proposed by Lecko [18] proving the necessity. The sufficiency part of the categorization was afterwards proved by Lecko and Lyzzaik [19] (see [[20], Chapter VII] as well). Encouraged by the article of Robertson [15], Aharanov et al. [21] (see also [22]) investigated about the class of functions that are spirallike with respect to a boundary point. Let

$$\mathrm{P}(\varsigma\,;M)\coloneqq\frac{4\varsigma}{\left(\sqrt{(1-\varsigma)^2+4\varsigma/M}+1-\varsigma\right)^2},\ \sqrt{1}\coloneqq 1,\quad \varsigma\in\mathfrak{D}, \tag{12}$$

be the Pick function. By using the Pick function $\mathrm{P}(\varsigma\,;M)$, the author in [23] considered another closely related class to $\mathscr{G}$, the family $\mathscr{G}(M)$, $M>1$, comprising of all $g\in\mathscr{H}$ of the form (10) such that

$$\mathfrak{R}\left(\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{\varsigma\mathrm{P}'(\varsigma\,;M)}{\mathrm{P}(\varsigma\,;M)}\right)>0,\quad \varsigma\in\mathfrak{D}. \tag{13}$$

In [24], Todorov established a structural formula and coefficient estimates by associating $\mathscr{G}$ with a functional $f(\varsigma)/1-\varsigma$ for $\varsigma\in\mathfrak{D}$. For $g\in\mathscr{H}$ in (10), Obradovič and Owa [25] and Silverman and Silvia [26] separately introduced the classes

$$\mathscr{G}_\alpha=\left\{\mathfrak{R}\left(\frac{\varsigma g'(\varsigma)}{g(\varsigma)}+(1-\alpha)\frac{1+\varsigma}{1-\varsigma}\right)>0,\quad \varsigma\in\mathfrak{D}\right\}, \tag{14}$$

where $\alpha \in [0, 1)$. The authors in [26] confirmed a remarkable fact that for each $\alpha \in [0, 1)$, the class $\mathcal{G}_\alpha$ is a subclass of $\mathcal{G}^*$. Clearly, $\mathcal{G}_{1/2} = \mathcal{G}$ and appealing coefficient inequalities of $\mathcal{G}$ were established in [27].

For $g \in \mathcal{H}$ assumed as in (10) and $-1 < E \leq 1\,; -E < F \leq 1$, Jakubowski and Włodarczyk [28] defined the class $\mathcal{G}(E, F)$ as

$$\Re(J(\varsigma)) > 0, \quad \varsigma \in \mathfrak{D}, \tag{15}$$

where

$$J(\varsigma) = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + E\varsigma}{1 - F\varsigma}. \tag{16}$$

By desirable quality of the initiative proposed in [2], Mohd and Darus in [29] presented a new class $\mathcal{S}_b^*(\Phi)$, where $\Phi \in \mathcal{P}^*(1)$, of all $g \in \mathcal{H}$ of the form (10) such that

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma} \prec \Phi(\varsigma), \quad \varsigma \in \mathfrak{D}. \tag{17}$$

An additional appealing class on the above direction was in recent times analyzed by Lecko et al. [30].

The most important intend of the present article is to illustrate and do a organized inquiry of the function class defined as below.

Definition 1. For $g \in \mathcal{H}$ and as assumed in (10), we let a new class $\mathcal{G}_e$ as

$$\mathcal{G}_e = \left\{ g \in \mathcal{H} : \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D} \right\}. \tag{18}$$

Remark 2. Note that the condition (18) is well defined, for

$$p(\varsigma) := \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma}, \quad \varsigma \in \mathfrak{D} \tag{19}$$

is holomorphic in $\mathfrak{D}$.

Based on the description of the class $\mathcal{G}_e$ and on the analytical characterization of the class $\mathcal{G}^*$ of starlike functions with respect to a boundary point, we can prepare the next result.

2. Representation Theorem and Coefficient Results

Let us start the section with the following representation theorem which in fact offers a handy procedure to build functions in our new class $\mathcal{G}_e$.

Theorem 3. *A function $g \in \mathcal{G}_e$ if and only if there exists $p \in \mathcal{H}$ such that $p \prec \Phi_e$ and*

$$g(\varsigma) = (1 - \varsigma) \exp \left(\frac{1}{2} \int_0^{\varsigma} \frac{p(\zeta) - 1}{\zeta} d\zeta \right), \quad \varsigma \in \mathfrak{D}. \tag{20}$$

Proof. Let us suppose that $g \in \mathcal{G}_e$, then, a function p defined by (19) is holomorphic and satisfies $p \prec \Phi_e$. Also, (19) can be rewritten in the type

$$\frac{2g'(\varsigma)}{g(\varsigma)} + \frac{2}{1 - \varsigma} = \frac{p(\varsigma) - 1}{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{21}$$

This upon integration give

$$\log \frac{(g(\varsigma))^2}{(1 - \varsigma)^2} = \int_0^{\varsigma} \frac{p(\zeta) - 1}{\zeta} d\zeta, \quad \varsigma \in \mathfrak{D}, \quad \log 1 := 0. \tag{22}$$

This in essence gives

$$(g(\varsigma))^2 = (1 - \varsigma)^2 \exp \left(\int_0^{\varsigma} \frac{p(\zeta) - 1}{\zeta} d\zeta \right), \quad \varsigma \in \mathfrak{D}, \tag{23}$$

which imply (20).□ □

Let us presume $p \prec \Phi_e$. By defining a function g as in (20), and by observing that $p(0) = 1$, it is noticeable that g is holomorphic in $\mathfrak{D}$. A working out shows that g satisfies (21); so, (19). Thus, $g \in \mathcal{G}_e$, which ends the confirmation of the theorem.

Let Ψ_e be a holomorphic function which is the solution of the differential equation (see also [[10], p. 367])

$$\frac{\varsigma \Psi_e{}'(\varsigma)}{\Psi_e(\varsigma)} = e^{\varsigma}, \quad \varsigma \in \mathfrak{D}, \quad \Psi_e(0) = 0, \quad \Psi_e{}'(0) = 1, \tag{24}$$

i.e.,

$$\Psi_e(\varsigma) = \varsigma \exp \left(\int_0^{\varsigma} \frac{e^{\zeta} - 1}{\zeta} d\zeta \right) = \varsigma + \varsigma^2 + \frac{3}{4}\varsigma^3 + \frac{17}{36}\varsigma^4 + \cdots, \quad \varsigma \in \mathfrak{D}. \tag{25}$$

Next, we present few examples for the class $\mathcal{G}_e$.

Example 4.

(1) For a specified $A \in \mathbb{R}$ and $\varsigma \in \mathfrak{D}$, let us name

$$p_A(\varsigma) := 1 + A\varsigma,$$
$$g_A(\varsigma) := (1 - \varsigma) \exp \left(\frac{A\varsigma}{2} \right), \quad \varsigma \in \mathfrak{D}. \tag{26}$$

Note down that $g_A \in \mathscr{H}$ with $g_A(0)=1$. Observe that

$$\frac{2\varsigma g_{A'}(\varsigma)}{g_A(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = p_A(\varsigma), \quad \varsigma \in \mathfrak{D}. \tag{27}$$

We finish that $g_A \in \mathscr{G}_e$ for $|A| \le 1 - 1/\mathrm{e}$.

(2) Given $-1 < A \le 1$ and $-A < B < 1$, define

$$w = p_{A,B}(\varsigma) := \frac{1 + A\varsigma}{1 - B\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{28}$$

Then, we identify that $p_{A,B}(\mathfrak{D})$ is an open disk symmetrical with respect to the real axis centered at $(1+AB)/(1-B^2)$ of radius $(A+B)/(1-B^2)$. In particular, for $B=A$, this disk is given by

$$\left| w - \frac{1+A^2}{1-A^2} \right| < \frac{2A}{1-A^2}, \tag{29}$$

with diametric end points $x_L := (1-|A|)/(1+|A|)$ and $x_R := (1+|A|)/(1-|A|)$. Since $x_L \ge 1/\mathrm{e}$ and $x_R \le \mathrm{e}$ iff $|A| \le (\mathrm{e}-1)/(\mathrm{e}+1)$, we perceive that then $p_{A,A} \prec \Phi_e$. As a result, a function $g \in \mathscr{H}$ with $g(0)=1$ defined by

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma} = p_{A,A}(\varsigma), \quad \varsigma \in \mathfrak{D}, \tag{30}$$

i.e., the function

$$g(\varsigma) = \frac{1-\varsigma}{1-A\varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{31}$$

belongs to the class $\mathscr{G}_e$ for $|A| \le (\mathrm{e}-1)/(\mathrm{e}+1)$.

Theorem 5. *Let $0<r<1$. If $g \in \mathscr{G}_e$, then*

(i)

$$\sqrt{\frac{-\Psi_e(-r)}{r}}(1-r) \le |g(\varsigma)| \le \sqrt{\frac{\Psi_e(-r)}{r}}(1+r), \quad |\varsigma| = r. \tag{32}$$

(ii)

$$\left| \arg \frac{g(\varsigma_0)}{(1-\varsigma_0)^2} \right| \le \frac{1}{2} \max_{|\varsigma|=r} \arg \frac{\Psi_e(\varsigma)}{\varsigma}, |\varsigma_0| = r, \quad \arg 1 := 0. \tag{33}$$

Proof. Let $g \in \mathscr{G}_e$.

(i) Describe the function

$$h(\varsigma) := \frac{\varsigma (g(\varsigma))^2}{(1-\varsigma)^2}, \quad \varsigma \in \mathfrak{D}. \tag{34}$$

Obviously, h is a holomorphic function in $\mathfrak{D}$, and an uncomplicated working out yields

$$\frac{\varsigma h'(\varsigma)}{h(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{35}$$

It is straightforward to witness from the above that $g \in \mathscr{G}_e$ if and only if

$$\frac{\varsigma h'(\varsigma)}{h(\varsigma)} \prec \mathrm{e}^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{36}$$

By the result of Corollary $1'$ of [2], we obtain

$$-\Psi_e(-r) \le |h(\varsigma)| \le \Psi_e(r), \quad |\varsigma| = r, \tag{37}$$

i.e., by using (34),

$$-\Psi_e(-r) \le \left| \frac{\varsigma (g(\varsigma))^2}{(1-\varsigma)^2} \right| \le \Psi_e(r), \quad |\varsigma| = r, \tag{38}$$

which gives (32).

(ii) By (36), a function h defined by (34) belongs to $\mathscr{S}^*(\Phi_e)$. Due to Corollary $3'$ of [2], the inequality

$$\left| \arg \frac{h(\varsigma_0)}{\varsigma_0} \right| \le \max_{|\varsigma|=r} \arg \frac{\Psi_e(\varsigma)}{\varsigma}, \quad |\varsigma_0| = r \tag{39}$$

is valid. Using now (34) in turn yields (33).□ □

Next, we ascertain some coefficient results for the class $g \in \mathscr{G}_e$. Let $\mathscr{B} := \{\omega \in \mathscr{H} : |\omega(\varsigma)| \le 1, \varsigma \in \mathfrak{D}\}$ and $\mathscr{B}_0$ be the subclass of $\mathscr{B}$ consisting of functions ω such that $\omega(0)=0$. We comment at this time that the elements of $\mathscr{B}_0$ are termed as Schwarz functions.

We will pertain two lemmas below to prove our main results.

Lemma 6. (see [2]). *If $p \in \mathscr{P}$ is of the form (3), then for $\mu \in \mathbb{C}$,*

$$\left| p_2 - \mu p_1^2 \right| \le 2 \max \{1, |2\mu - 1|\}. \tag{40}$$

In particular, if μ is a real number, then

$$\left| p_2 - \mu p_1^2 \right| \le \begin{cases} -4\mu + 2, & \mu \le 0, \\ 2, & 0 \le \mu \le 1, \\ 4\mu - 2, & \mu \ge 1. \end{cases} \tag{41}$$

When $\mu < 0$ or $\mu > 1$, the equality holds true if and only if $p(\varsigma) = (1+\varsigma)/(1-\varsigma) =: \mathscr{L}(\varsigma), \varsigma \in \mathfrak{D}$, or one of its rotations. If $0 < \mu < 1$, then the equality holds true if and only if $p(\varsigma = \mathscr{L}(\varsigma^2), \varsigma \in \mathfrak{D}$, or one of its rotations. If $\mu = 0$, the equality holds true if and only if

$$p(\varsigma)=\frac{1}{2}(1+\lambda)\mathscr{L}(\varsigma)+\frac{1}{2}(1-\lambda)\mathscr{L}(-\varsigma),\quad \varsigma\in\mathfrak{D},\tag{42}$$

where $0\le\lambda\le1$, or one of its rotations. If $\mu=1$, then the equality holds true if p is a reciprocal of one of the functions such that the equality holds true in the case when $\mu=0$.

Lemma 7. (see [31]). *If $p\in\mathscr{P}$ is of the form (3) and $\beta(2\beta-1)\le\delta\le\beta$, then*

$$\left|p_3-2\beta p_1p_2+\delta p_1^3\right|\le 2.\tag{43}$$

At the moment, we are in a position to state the theorem which give a few better bounds for early coefficients and the Fekete-Szegö inequalities for $f\in\mathscr{G}_e$.

Theorem 8. *If $g\in\mathscr{G}_e$ is of the form (10), then*

$$|\vartheta_1+1|\le\frac{1}{2},\tag{44}$$

$$|\vartheta_1|\le\frac{3}{2},\tag{45}$$

$$\left|2\vartheta_2-\vartheta_1^2+1\right|\le\frac{1}{2},\tag{46}$$

$$|\vartheta_2|\le\frac{3}{4},\tag{47}$$

$$\left|3\vartheta_3-3\vartheta_1\vartheta_2+\vartheta_1^3+1\right|\le\frac{1}{2},\tag{48}$$

and for $\delta\in\mathbb{R}$,

$$\left|\vartheta_2-\delta\vartheta_1^2\right|\le\frac{1}{4}(\max\{1,|\delta-1|\}+2|2\delta-1|+4|\delta|).\tag{49}$$

Inequalities (44), (45), (46), (47), and (48) are sharp.

Proof. In view of (18), there exists $\omega\in\mathscr{B}_0$ such that

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{1+\varsigma}{1-\varsigma}=\Phi_e(\omega(\varsigma))=\exp(\omega(\varsigma)),\quad \varsigma\in\mathfrak{D}.\tag{50}$$

By an application of (10), one can easily obtain with simple computation that

$$\begin{aligned}\frac{2\varsigma g'(\varsigma)}{g(\varsigma)}+\frac{1+\varsigma}{1-\varsigma}&=1+2(\vartheta_1+1)\varsigma+2\left(2\vartheta_2-\vartheta_1^2+1\right)\varsigma^2\\&\quad+2\left(3\vartheta_3-3\vartheta_1\vartheta_2+\vartheta_1^3+1\right)\varsigma^3+\cdots,\quad \varsigma\in\mathfrak{D}.\end{aligned}\tag{51}$$

Define the function p by

$$p(\varsigma)=\frac{1+\omega(\varsigma)}{1-\omega(\varsigma)}=1+p_1\varsigma+p_2\varsigma^2+\cdots,\quad \varsigma\in\mathfrak{D}.\tag{52}$$

Clearly, $p\in\mathscr{P}$. Moreover,

$$\begin{aligned}\omega(\varsigma)=\frac{p(\varsigma)-1}{p(\varsigma)+1}&=\frac{p_1}{2}\varsigma+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\varsigma^2\\&\quad+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)\varsigma^3+\cdots,\quad \varsigma\in\mathfrak{D}.\end{aligned}\tag{53}$$

Hence,

$$\begin{aligned}\exp(\omega(\varsigma))&=1+\omega(\varsigma)+\frac{(\omega(\varsigma))^2}{2}+\frac{(\omega(\varsigma))^3}{6}+\cdots=1+\frac{p_1\varsigma}{2}\\&\quad+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)\varsigma^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)\varsigma^3+\cdots,\quad \varsigma\in\mathfrak{D}.\end{aligned}\tag{54}$$

□ □

Substituting (51) and (54) into (50), by comparing the corresponding coefficients, we obtain

$$2(\vartheta_1+1)=\frac{p_1}{2},\tag{55}$$

$$2\left(2\vartheta_2-\vartheta_1^2+1\right)=\frac{p_2}{2}-\frac{p_1^2}{8},\tag{56}$$

$$2\left(3\vartheta_3-3\vartheta_1\vartheta_2+\vartheta_1^3+1\right)=\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}.\tag{57}$$

Since (e.g., ([[32]], Vol. I, p. 80)),

$$|p_n|\le 2,\quad n\in\mathbb{N}.\tag{58}$$

From (55), we obtain (44). Rewriting (55) as $\vartheta_1=p_1/4-1$, (45) easily follows. Further, (56) together with (40) yields

$$\left|2\left(2\vartheta_2-\vartheta_1^2+1\right)\right|=\left|\frac{p_2}{2}-\frac{p_1^2}{8}\right|\le 1,\tag{59}$$

which proves (46).

Upon applying (55) for ϑ_1 in (56), we get

$$4\vartheta_2=\frac{p_2}{2}-p_1.\tag{60}$$

Hence, by applying (41), we obtain (47).

An application of (43) in (57) gives

$$\left|6\vartheta_3-6\vartheta_1\vartheta_2+2\vartheta_1^3+2\right|=\left|\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right|\le 1,\tag{61}$$

i.e., the inequality (48).

Using (60) and making use of the expression for ϑ_1 and in turn by applying (41) and (58), we get

$$\left|\vartheta_2 - \delta\vartheta_1^2\right| \le \frac{1}{8}\left(\left|p_2 - \frac{\delta}{2}p_1^2\right| + 2|2\delta - 1||p_1| + 8|\delta|\right), \quad \delta \in \mathbb{R}, \tag{62}$$

which leads to the inequality (49).

Equalities in (44) and (45) hold for the function $p = \mathscr{L}$; in (46) for the function $p(\varsigma) = \mathscr{L}(\varsigma^2), \varsigma \in \mathfrak{D}$, in (47) for the function $p(\varsigma) = \mathscr{L}(-\varsigma), \varsigma \in \mathfrak{D}$ and in (48) for the function $p(\varsigma) = \mathscr{L}(\varsigma^3), \varsigma \in \mathfrak{D}$.

3. Differential Subordination Results Involving $\mathscr{G}_e$

In this segment, we derive certain differential subordination result concerning the class $\mathscr{G}_e$.

To demonstrate differential subordination results, we recollect the next lemma (see ([[33]], Theorem 8.4 h, p. 132)).

Q is starlike univalent in $\mathfrak{D}$, or

h is convex univalent in $\mathfrak{D}$

Lemma 9. *Suppose q is univalent in $\mathfrak{D}$,θ and φ be holomorphic in a domain D containing $q(\mathfrak{D})$ with $\varphi(w) \ne 0$ when $w \in q(\mathfrak{D})$. Let $Q(\varsigma) := \varsigma q'(\varsigma)\varphi(q(\varsigma))$ and $h(\varsigma) := \theta(q(\varsigma)) + Q(\varsigma)$ for $\varsigma \in \mathfrak{D}$. Suppose that either*

Assume also that
(iii)

$$\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} > 0, \quad \varsigma \in \mathfrak{D}. \tag{63}$$

If $p \in \mathscr{H}$ with $p(0) = q(0)$,$p(\mathfrak{D}) \subset D$, and

$$\theta(p(\varsigma)) + \varsigma p'(\varsigma)\varphi(p(\varsigma)) \prec \theta(q(\varsigma)) + \varsigma q'(\varsigma)\varphi(q(\varsigma)), \quad \varsigma \in \mathfrak{D}, \tag{64}$$

then $p \prec q$ and q are the best dominant.

Theorem 10. *Let $g \in \mathscr{H}$ and $g(0) = 1$. If g satisfies the subordination condition,*

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma} \prec 1 + \varsigma, \quad \varsigma \in \mathfrak{D}. \tag{65}$$

Then,

$$p(\varsigma) := \frac{(g(\varsigma))^2}{(1 - \varsigma)^2} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{66}$$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\theta(w) := 1, w \in \mathbb{C}$ and $\varphi(w) := 1/w, w \in D$. Note that $\Phi_e(\mathfrak{D}) \subset D$ and θ and φ are holomorphic in D. Thus,

$$Q(\varsigma) := \varsigma\Phi_e'(\varsigma)\varphi(\Phi_e(\varsigma)) = \frac{\varsigma\Phi_e{}'(\varsigma)}{\Phi_e(\varsigma)} = \varsigma, \quad \varsigma \in \mathfrak{D} \tag{67}$$

is well defined and holomorphic. Clearly, Q is a univalent starlike function and so for a function $h(\varsigma) := \theta(\Phi_e(\varsigma)) + Q(\varsigma) = 1 + Q(\varsigma)$,$\varsigma \in \mathfrak{D}$, we achieve

$$\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} = \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} = 1 > 0, \quad \varsigma \in \mathfrak{D}. \tag{68}$$

Hence, for any function p belonging to $\mathscr{H}$ with $p(0) = \Phi_e(0) = 1$ such that $p(\mathfrak{D}) \subset D$, i.e., for p nonvanishing in $\mathfrak{D}$, by applying Lemma 9, we infer that from the subordination

$$1 + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec 1 + \frac{\varsigma\Phi_{e'}(\varsigma)}{\Phi_e(\varsigma)} = 1 + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{69}$$

it follows the subordination $p \prec \Phi_e$.□ □

Next, we at this time take $g \in \mathscr{H}$ with $g(0) = 1$ and $g(\varsigma)$ be nonzero for $\varsigma \in \mathfrak{D}$ satisfying (65). Let a function p be taken as in (66). Then, one can notice that $p(0) = \Phi_e(0) = 1$, $p(\varsigma) \ne 0$, for $\varsigma \in \mathfrak{D}$, and p is holomorphic. Since

$$1 + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{70}$$

from (69), the conclusion (66) follows, which complete the proof.

Theorem 11. *Let $g \in \mathscr{H}$ with $g(0) = 1$. If g satisfies*

$$\frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1 + \varsigma}{1 - \varsigma} \prec e^{\varsigma} + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{71}$$

then

$$p(\varsigma) := \varsigma\left(\frac{g(\varsigma)}{1 - \varsigma}\right)^2\left(\int_0^{\varsigma}\left(\frac{g(\zeta)}{1 - \zeta}\right)^2 d\zeta\right)^{-1} \prec e^{\varsigma}, \quad \varsigma \in \mathfrak{D}. \tag{72}$$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\phi(w) := w, w \in \mathbb{C}$, and $\psi(w) := 1/w, w \in D$. Note that $\Phi_e(\mathfrak{D}) \subset D$ and ϕ and ψ are holomorphic in D. Thus, the function Q defined by (67), i.e., the identity function, is univalent starlike. Hence, for a function $h(\varsigma) := \theta(\Phi_e(\varsigma)) + Q(\varsigma) = \Phi_e(\varsigma) + Q(\varsigma)$,$\varsigma \in \mathfrak{D}$, we obtain

$$\begin{aligned}\mathfrak{R}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} &= \mathfrak{R}\frac{\varsigma\Phi_{e'}(\varsigma)}{Q(\varsigma)} + \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} \\ &= \mathfrak{R}\Phi_e(\varsigma) + \mathfrak{R}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} > 0, \quad \varsigma \in \mathfrak{D}.\end{aligned} \tag{73}$$

Thus, for any function $p \in \mathscr{H}$ with $p(0) = \Phi_e(0) = 1$ such that $p(\mathfrak{D}) \subset D$, i.e., $p(\varsigma) \neq 0$ for $\varsigma \in \mathfrak{D}$, by applying Lemma 9, we deduce that from the subordination

$$p(\varsigma) + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec \Phi_e(\varsigma) + \frac{\varsigma \Phi_{e'}(\varsigma)}{\Phi_e(\varsigma)} = e^{\varsigma} + \varsigma, \quad \varsigma \in \mathfrak{D}, \tag{74}$$

it follows the subordination $p \prec \Phi_e$.□ □

Let now take $g \in \mathscr{H}$ with $g(0) = 1$ and $g(\varsigma) \neq 0$ for $\varsigma \in \mathfrak{D}$ satisfying (65). Define a function p as in (72). We see that

$$\begin{aligned} p(0) &= \lim_{\varsigma \longrightarrow 0} \varsigma \left(\frac{g(\varsigma)}{1-\varsigma} \right)^2 \left(\int_0^{\varsigma} \left(\frac{g(\zeta)}{1-\zeta} \right)^2 d\zeta \right)^{-1} \\ &= (g(0))^2 \lim_{\varsigma \longrightarrow 0} \varsigma \left(\int_0^{\varsigma} \left(\frac{g(\zeta)}{1-\zeta} \right)^2 d\zeta \right)^{-1} = 1 = \Phi_e(0), \end{aligned} \tag{75}$$

$p(\varsigma)$=0 for $\varsigma \in \mathfrak{D}$ and p is holomorphic. Since

$$p(\varsigma) + \frac{\varsigma p'(\varsigma)}{p(\varsigma)} = \frac{2\varsigma g'(\varsigma)}{g(\varsigma)} + \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathfrak{D}, \tag{76}$$

from (74), (71) follows which completes the proof.

Data Availability

No data sets were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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