

Research Article

Bayesian Estimation for the Doubly Censored Topp Leone Distribution using Approximate Methods and Fuzzy Type of Priors

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The Topp Leone distribution (TLD) is a lifetime model having finite support and U-shaped hazard rate; these features distinguish it from the famous lifetime models such as gamma, Weibull, or Log-normal distribution. The Bayesian methods are very much linked to the Fuzzy sets. The Fuzzy priors can be used as prior information in the Bayesian models. This paper considers the posterior analysis of TLD, when the samples are doubly censored. The independent informative priors (IPs) which are very close to the Fuzzy priors have been proposed for the analysis. The symmetric and asymmetric loss functions have also been assumed for the analysis. As the marginal PDs are not available in a closed form, therefore, we have used a Quadrature method (QM), Lindley's approximation (LA), Tierney and Kadane's approximation (TKA), and Gibbs sampler (GS) for the approximate estimation of the parameters. A simulation study has been conducted to assess and compare the performance of various posterior estimators. In addition, a real dataset has been analyzed for the illustration of the applicability of the results obtained in the study. The study suggests that the TKA performs better than its counterparts.

1. Introduction

Topp and Leone [1] introduced a lifetime distribution and named it TLD. Nadarajah and Kotz [2] have discussed that the hazard rate function of the TLD is U-shaped. The distributions with U-shaped hazard rate are used to model the human populations where the death rate is high at the infant age owing to infant diseases and birth defects; thereafter, the death rate remains constant up to thirties, and then, it increases again. Some of the manufactured items also have similar life patterns. The advantage of TLD is that it has only two parameters, while most of the distributions with U-shaped hazard rate have at least three parameters which increases the computational difficulties. The regularity conditions are satisfied by the TLD. The

distribution function of the TLD is in a compact form, due to which it can be very easily applied when the data are censored, unlike the gamma and lognormal distributions. Another feature of the distribution, which distinguishes it from the other lifetime distributions, is that it has a finite support. Due to these distinguishing aspects, the TLD has received much attention from the researchers recently. Al-Zahrani and Alshomrani [3] analyzed the stress strength reliability for the TLD. Genc [4] derived the expressions for single and product moments for the order statistics from the TLD. Genc [5] considered the estimation of $P(X > Y)$ from the TLD. Mir Mostafaei et al. [6] discussed the estimation for moments of order statistics from the TLD. Bayoud [7] estimated the shape parameter of the TLD using the Bayesian and classical methods under type-I censored

samples. Bayoud [8] used progressive type-II censored samples to obtain the Bayesian and classical estimates for the shape parameter of the TLD. The approximate maximum likelihood method, LA, and importance sampling method have been used for the estimation. Mir Mostafaei et al. [9] studied the lower k-record values from the TLD using the Bayesian approach. Reyad and Othman [10] used the Topp Leone genesis to produce and generalization of the Burr-XII distribution and called it Topp-Leone Burr-XII distribution. Different properties and estimation of the proposed model have been discussed. The effectiveness of the proposed model has been demonstrated using three real-life datasets. Rezaei et al. [11] introduced a new family of distributions from the TLD and discussed the mathematical properties of this family of distributions.

The U-shaped hazard rate with a fewer number of parameters and finite support for the TLD distinguishes it from repeatedly used lifetime models such as Weibull, Gamma, and Log-normal distributions. Motivated by these features of the TLD, we planned to estimate this distribution under a Bayesian framework. The reason for the choice of Bayesian estimation is that the results under Bayesian inference are often better than the classical methods even if we do not have sufficient prior information; for example, please see Kundu and Joarder [12]. Further, the lifetime data are often censored. Therefore, we have considered doubly censored samples from the TLD for the estimation. Doubly censoring is used to analyze the duration times between two events. For example, let the ages of certain equipment are not known when the monitoring of the equipment starts and tracking of the equipment is stopped after the predetermined observational period. Then, the ages of the equipment which are still functioning will be doubly censored. The doubly censored samples are very useful in reliability analysis for the products which are already under use. In medical applications, the doubly censored samples usually deal with time between infection and onset of a disease; for example, the infectious diseases like COVID-19 often produce doubly censored data. Although Feroze and Aslam [13] and Feroze and Aslam [14] have considered the analysis of doubly censored samples from the single and mixture of TLD, respectively, these contributions considered only single parametric TLD which restricted the flexibility of the TLD to a great extent. In addition, these contributions did not include any credible intervals.

In addition, the Bayesian methods are very much linked to the Fuzzy sets. The Fuzzy priors can be used as prior information in the Bayesian models. The fuzzy approximations enable users to incorporate more flexible sets of priors and likelihood functions in the Bayesian inference. This method allows authors to bypass the closed form conjugacy link between likelihood function and prior [15]. Stegmaier and Mikut [16] suggested that the fuzzy priors can improve the performance of the Bayesian inference. This paper considers the posterior analysis of TLD, when the samples are doubly censored. The independent informative priors (IPs) which are very close to the Fuzzy priors have been proposed for the analysis. It has been noticed that the expressions for the posterior estimators are not available in the closed form, so we have used four approximation techniques, namely,

QM, LA, TKA, and GS, for the analysis. The assumption of independent gamma priors has been made for both of the parameters; this assumption is not rare; for example, see Kundu and Howlder [17]. Further symmetric and asymmetric loss functions have been considered for the estimation.

The rest of the paper is placed in the following sections. The model and likelihood function have been introduced in Section 2. The Bayesian estimation has been considered in Section 3. The results regarding the simulation study have been reported in Section 4. The real-life example has been presented in Section 5. The concluding remarks have been given in Section 6.

2. The Model

In this section, the TLD and likelihood function under doubly censored samples have been presented.

The probability density function (pdf) of the TLD is

$$f(x) = \frac{\theta_1}{\theta_2} \left(2 - \frac{2x}{\theta_2}\right) \left(\frac{2x}{\theta_2} - \frac{x^2}{\theta_2^2}\right)^{\theta_1 - 1}, \quad 0 < x < \theta_2, \quad \theta_1, \theta_2 > 0, \quad (1)$$

where θ_1 and θ_2 are model parameters from TLD.

The cumulative distribution function for TLD is

$$F(x) = \left(\frac{2x}{\theta_2} - \frac{x^2}{\theta_2^2}\right)^{\theta_1}, \quad 0 < x < \theta_2, \quad \theta_1, \theta_2 > 0. \quad (2)$$

Suppose that a random sample of size “ n ” is selected from the TLD. Further assume the complete information for ordered observations x_r, \dots, x_s was available, and the information regarding the smallest “ $r - 1$ ” and largest “ $n - s$ ” items was incomplete. Therefore, $m = s - r + 1$ observations can only be used for the analysis from the sample of size “ n .” Using the doubly censored sample $x = (x_r, \dots, x_s)$, the likelihood function is

$$L(x|\theta_1, \theta_2) \propto [F(x_r|\theta_1, \theta_2)]^{r-1} [1 - F(x_s|\theta_1, \theta_2)]^{n-s} \prod_{i=r}^s f(x_i|\theta_1, \theta_2). \quad (3)$$

Putting results in (3), we have

$$L(x|\theta_1, \theta_2) \propto \frac{\theta_1^m}{\theta_2^m} \left(\frac{2x_r}{\theta_2} - \frac{x_r^2}{\theta_2^2}\right)^{\theta_1(r-1)} \cdot \left[1 - \left(\frac{2x_s}{\theta_2} - \frac{x_s^2}{\theta_2^2}\right)^{\theta_1}\right]^{n-s} \prod_{i=r}^s \left(2 - \frac{2x_i}{\theta_2}\right) \left(\frac{2x_i}{\theta_2} - \frac{x_i^2}{\theta_2^2}\right)^{\theta_1 - 1}. \quad (4)$$

3. Posterior Estimation

This section reported the posterior estimation for the model parameters assuming doubly censored samples. The gamma priors have been assumed for model parameters. The

posterior estimation has been carried out using two loss functions. Various approximation methods have been suggested for numerical solutions of the estimates.

3.1. Prior and Posterior Distribution (PD). In this subsection, we have assumed IP for the construction of the PD for the parameters θ_1 and θ_2 . Consider the informative gamma priors for the parameters θ_1 and θ_2 as $g_1(\theta_1) \propto \theta_1^{a-1} e^{-b\theta_1}$, $\theta_1 > 0$ and $g_2(\theta_2) \propto \theta_2^{c-1} e^{-d\theta_2}$, $\theta_2 > 0$, respectively. Now, under the assumption of independence the combined IP for the parameters θ_1 and θ_2 is

$$g(\theta_1, \theta_2) \propto \theta_1^{a-1} \theta_2^{c-1} e^{-b\theta_1} e^{-d\theta_2}, \quad \theta_1, \theta_2 > 0. \quad (5)$$

The prior given in (5) is quite close to the Fuzzy priors [15].

From (4) and (5), the joint PD for model parameters θ_1 and θ_2 under IP are

$$g(\theta_1, \theta_2 | x) = \frac{L(x | \theta_1, \theta_2) g(\theta_1, \theta_2)}{\int_0^\infty \int_0^\infty L(x | \theta_1, \theta_2) g(\theta_1, \theta_2) d\theta_1 d\theta_2}. \quad (6)$$

Putting values in (6), we have the PD as

$$L(x | \theta_1, \theta_2) \propto \frac{\theta_1^{m+a-1}}{\theta_2^{m-c+1}} e^{-b\theta_1} e^{-d\theta_2} \left(\frac{2x_r}{\theta_2} - \frac{x_r^2}{\theta_2^2} \right)^{\theta_1(r-1)} \cdot \left[1 - \left(\frac{2x_s}{\theta_2} - \frac{x_s^2}{\theta_2^2} \right)^{\theta_1} \right]^{n-s} \prod_{i=r}^s \left(2 - \frac{2x_i}{\theta_2} \right) \left(\frac{2x_i}{\theta_2} - \frac{x_i^2}{\theta_2^2} \right)^{\theta_1-1}. \quad (7)$$

3.2. Loss Functions. The posterior estimation has been carried out using following two loss functions.

- (1) Squared Error Loss Function (SELF): Legendre [18] and Gauss [19] proposed the SELF that can be defined as $L(\theta_1, \theta_{1,s}) = (\theta_1 - \theta_{1,s})^2$, where θ_1 is the model parameter. Using SELF, the Bayes estimator (BE) for the parameter θ_1 is $\theta_{1,s} = E(\theta_1)$.
- (2) Precautionary Loss Function (PLF): the PLF, introduced by Norstrom [20], has the following expression $L(\theta_{1,p}, \theta_1) = \theta_{1,p}^{-1} (\theta_{1,p} - \theta_1)^2$. Using PLF, the BE is $\theta_{1,p} = [E(\theta_1^2)]^{1/2}$.

3.3. Quadrature Method (QM). It should be noted that the assumed priors are flexible; however, the BEs under SELF and PLF cannot be obtained in an explicit form. This is due to the fact that the ratio of the integrals in the BEs cannot be solved directly. In such situations, the BEs can be obtained numerically using QM which can be used to evaluate an integral numerically.

The BEs for the parameters θ_1 and θ_2 under SELF using the PD based on IP distribution are

$$\begin{aligned} \theta_{1,S} &= \int_0^\infty \int_0^\infty \theta_1 g(\theta_1, \theta_2 | x) d\theta_1 d\theta_2, \\ \theta_{2,S} &= \int_0^\infty \int_0^\infty \theta_2 g(\theta_1, \theta_2 | x) d\theta_1 d\theta_2. \end{aligned} \quad (8)$$

In the Bayesian QM, we choose a set of points between the finite integral in an order to ensure the stability of our uncertainty. Consider the posterior density $g(\theta_1, \theta_2 | x)$, where θ_1 and θ_2 are the parameters. We evaluate this density over a number of the points in the entire range as

$$\int_0^\infty \int_0^\infty g(\theta_1, \theta_2 | x) d\theta_1 d\theta_2 = \sum_{i=0}^m \sum_{j=0}^m w_i g(\theta_{1,i}, \theta_{2,j} | x), \quad (9)$$

where w_i are the increments. The Mathematica software has been used to obtain BEs and associated PRs for the parameters θ_1 and θ_2 using SELF and PLF under IPs.

3.4. Lindley's Approximation (LA). The QM is not suitable in many situations, for example, in the case of functions with singularities; the method is not well suited. In such situations, some other approximation methods, such as LA, can be used. The LA has an important role in Bayesian inference. Using this method, we can compute BEs without performing any complex numerical integration. Hence, if we need to obtain only the BEs, the LA can be efficiently used to serve this purpose. Bayoud [8] used LA to estimate the shape parameter of the TLD considering uncensored data. We have considered the more general and complex case by performing the LA for both parameters of the TLD using censored data.

Assuming a sufficiently large sample, Lindley [21] proposed that any ratio of the integral of the form

$$I(\theta) = E[h(\theta_1, \theta_2)] = \frac{\int_{(\theta_1, \theta_2)} h(\alpha, \beta) e^{l(x|\theta_1, \theta_2) + G(\theta_1, \theta_2)} d(\theta_1, \theta_2)}{\int_{(\theta_1, \theta_2)} e^{l(x|\theta_1, \theta_2) + G(\theta_1, \theta_2)} d(\theta_1, \theta_2)}, \quad (10)$$

where $h(\theta_1, \theta_2)$ is any function of θ_1 or θ_2 , $l(\theta_1, \theta_2 | x)$ is the log-likelihood function, and $G(\theta_1, \theta_2)$ is the logarithmic of joint prior for the parameters θ_1 and θ_2 , can be evaluated as

$$I(\theta) = g(\hat{\theta}_1, \hat{\theta}_2) + (g_1 d_1 + g_2 d_2 + d_3 + d_4) + \frac{1}{2} (A_1 B_1 + A_2 B_2), \quad (11)$$

where $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are MLEs of the parameters θ_1 and θ_2 , respectively,

$$\begin{aligned} B_i &= g_1\sigma_{i1} + g_2\sigma_{i2}, A_i = \sigma_{11}L_{11i} + \sigma_{22}L_{22i} + 2\sigma_{12}L_{12i}, d_i \\ &= P_1\sigma_{i1} + P_2\sigma_{i2}, \quad i = 1, 2, \\ d_3 &= g_{12}\sigma_{12}, d_4 = \frac{1}{2}(g_{11}\sigma_{11} + g_{22}\sigma_{22}), P_i \\ &= \frac{\partial G(\theta)}{\partial \theta_i}, \quad i = 1, 2, \quad \theta = (\theta_1, \theta_2), \\ g_{ij} &= \frac{\partial^2 h(\theta)}{\partial \theta_i \partial \theta_j}, L_{ij} = \frac{\partial^2 l(x|\theta)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, L_{ijk} \\ &= \frac{\partial^3 l(x|\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \quad i, j, k = 1, 2, \end{aligned} \quad (12)$$

and σ_{ij} is the (i, j) th element of the inverse of the matrix $\{L_{ij}\}$, all evaluated at the MLEs of the parameters.

Now, the log-likelihood function from (4) can be obtained as

$$\begin{aligned} l(x|\theta_1, \theta_2) &\propto m \log \theta_1 + m \log \theta_2 + \theta_1(r-1) \log \left(\frac{2x_r}{\theta_2} - \frac{x_r^2}{\theta_2^2} \right) \\ &+ (n-s) \log \left[1 - \left(\frac{2x_s}{\theta_2} - \frac{x_s^2}{\theta_2^2} \right)^{\theta_1} \right] + \sum_{i=r}^s \log \left(2 - 2 \frac{x_i}{\theta_2} \right) \\ &+ (\theta_1 - 1) \sum_{i=r}^s \left(\frac{2x_i}{\theta_2} - \frac{x_i^2}{\theta_2^2} \right). \end{aligned} \quad (13)$$

The maximum likelihood estimates (MLEs) of the parameters θ_1 and θ_2 can be obtained by differentiating (13) with respect to θ_1 and θ_2 and equating to zero, respectively, as

Let $T_i = (2x_i/\theta_2) - (x_i^2/\theta_2^2)$, $T_{i1} = (2x_i^2/\theta_2^3) - (2x_i/\theta_2^2)$, $T_{i2} = -(6x_i^2/\theta_2^4) + (4x_i/\theta_2^3)$, $T_{i3} = 2x_i/(2 - (2x_i/\theta_2))$, $T_r = (2x_r/\theta_2) - (x_r^2/\theta_2^2)$, $T_{r1} = (2x_r^2/\theta_2^3) - (2x_r/\theta_2^2)$, $T_{r2} = -(6x_r^2/\theta_2^4) + (4x_r/\theta_2^3)$, $T_s = (2x_s/\theta_2) - (x_s^2/\theta_2^2)$, $T_{s1} = (2x_s^2/\theta_2^3) - (2x_s/\theta_2^2)$, and $T_{s2} = -(6x_s^2/\theta_2^4) + (4x_s/\theta_2^3)$; then,

$$\frac{m}{\theta_1} + (r-1) \log(T_r) - \frac{(n-s)T_s^{\theta_1} \log(T_s)}{1 - T_s^{\theta_1}} + \sum_{i=r}^s (T_i) = 0, \quad (14)$$

$$\frac{m}{\theta_2} - \frac{(r-1)\theta_1 T_{r1}}{T_r} - \frac{(n-s)\theta_1 T_{s1} T_s^{\theta_1 - 1}}{1 - T_s^{\theta_1}} + (\theta_1 - 1) \sum_{i=r}^s \left(\frac{T_{i1}}{T_i} \right) + \theta_2^{-2} \sum_{i=r}^s T_{i2} = 0. \quad (15)$$

Based on (14) and (15), the approximate MLEs have been obtained using numerical methods. The second order derivatives of the log-likelihood function are presented in the following:

$$L_{11} = -\frac{m}{\theta_1^2} - \frac{2(n-s)T_s^{\theta_1} \log(T_s)}{1 - T_s^{\theta_1}} - \frac{2(n-s)T_s^{2\theta_1} \log(T_s)}{\{1 - T_s^{\theta_1}\}^2}, \quad (16)$$

$$\begin{aligned} L_{12} &= -\frac{(r-1)T_{r1}}{T_r} - \frac{(n-s)T_s^{\theta_1 - 1} T_{s1}}{1 - T_s^{\theta_1}} \\ &\cdot \left[1 - \theta_1 \log(T_s) - \frac{\theta_1 T_s^{\theta_1} \log(T_s)}{1 - T_s^{\theta_1}} \right] + \sum_{i=r}^s \left(\frac{T_{i1}}{T_i} \right), \end{aligned} \quad (17)$$

$$\begin{aligned} L_{22} &= -\frac{(r-1)\theta_1 T_{r1}^2}{T_r^2} - \frac{(r-1)\theta_1 T_{s2}}{T_{r1}} \\ &- \frac{(n-s)\theta_1(\theta_1 - 1)T_s^{\theta_1 - 1} T_{s1}^2}{1 - T_s^{\theta_1}} \\ &- \frac{(n-s)\theta_1 T_{s2} T_s^{\theta_1 - 1}}{1 - T_s^{\theta_1}} \\ &- \frac{(n-s)\theta_1^2 T_{s1}^2 T_s^{2\theta_2 - 2}}{\{1 - T_s^{\theta_1}\}^2} - \frac{m}{\theta_2^2} \\ &+ (\theta_1 - 1) \sum_{i=r}^s \left\{ \left(\frac{T_{i1}}{T_i} \right)^2 + \frac{T_{i2}}{T_i} \right\} \\ &+ \sum_{i=r}^s \{ -T_{i3}^2 \theta_2^{-4} - 2T_{i3} \theta_2^{-3} \}. \end{aligned} \quad (18)$$

Now, ((16)–(18)) have been evaluated at the MLEs of θ_1 and θ_2 .

As the third order derivatives with respect to θ_1 and θ_2 contain long expressions, therefore, they have not been presented in the paper.

Based on the second order derivatives, the matrix $\{L_{ij}\}$ is

$$\{L_{ij}\} = - \begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix}, \text{ and its inverse is } \{L_{ij}\}^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}. \quad (19)$$

Now, using LA, the BEs for the parameters θ_1 and θ_2 under IP using SELF are, respectively, presented as

$$\theta_{1,S} = \widehat{\theta}_1 + \frac{1}{2}(\sigma_{11}A_1 + \sigma_{21}A_2) + \left(\frac{a-1}{\widehat{\theta}_1} - b \right) \sigma_{11} + \left(\frac{c-1}{\widehat{\theta}_2} - d \right) \sigma_{12},$$

$$\theta_{2,S} = \widehat{\theta}_2 + \frac{1}{2}(\sigma_{12}A_1 + \sigma_{22}A_2) + \left(\frac{a-1}{\widehat{\theta}_1} - b \right) \sigma_{21} + \left(\frac{c-1}{\widehat{\theta}_2} - d \right) \sigma_{22}. \quad (20)$$

TABLE 1: BEs and PRs (in parenthesis) under IP for $n = 15$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
QM	SELF	0.64768 (0.02914)	0.66956 (0.03035)	1.23349 (0.06560)	1.28384 (0.07194)	1.83394 (0.10827)	1.86712 (0.13635)	2.40370 (0.14151)	2.60133 (0.20020)
	PLF	0.66356 (0.04433)	0.69243 (0.04526)	1.26583 (0.05187)	1.31651 (0.06418)	1.86667 (0.06010)	1.90783 (0.08316)	2.41512 (0.06918)	2.66793 (0.12064)
LA	SELF	0.64973 (0.02240)	0.74495 (0.02886)	1.29516 (0.05817)	1.37123 (0.07047)	1.85958 (0.08794)	1.96068 (0.11494)	2.56812 (0.10865)	2.72349 (0.15229)
	PLF	0.66361 (0.03086)	0.76528 (0.03297)	1.31925 (0.04230)	1.40722 (0.06305)	1.88491 (0.05499)	1.99844 (0.08133)	2.60632 (0.06315)	2.72845 (0.10016)
TKA	SELF	0.63510 (0.01245)	0.68001 (0.01709)	1.25705 (0.03292)	1.29944 (0.04900)	1.83774 (0.06277)	1.90447 (0.08841)	2.35670 (0.07366)	2.58210 (0.09832)
	PLF	0.63657 (0.01922)	0.69139 (0.02790)	1.27408 (0.02856)	1.33544 (0.04438)	1.85413 (0.04656)	1.94446 (0.07314)	2.38959 (0.06710)	2.64411 (0.09013)
GS	SELF	0.64947 (0.01296)	0.69434 (0.01805)	1.27788 (0.03760)	1.32688 (0.04958)	1.87776 (0.07516)	1.94516 (0.09258)	2.41864 (0.07997)	2.61734 (0.11334)
	PLF	0.64956 (0.02224)	0.69362 (0.02763)	1.28410 (0.03434)	1.34288 (0.04526)	1.88588 (0.06480)	1.95209 (0.07221)	2.42161 (0.08435)	2.63855 (0.11457)

*AM: approximation methods; LF: loss functions.

TABLE 2: BEs and PRs (in parenthesis) under IP for $n = 30$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
QM	SELF	0.61823 (0.02781)	0.63872 (0.02900)	1.18516 (0.06249)	1.22941 (0.06907)	1.75983 (0.10315)	1.79332 (0.13004)	2.29007 (0.13540)	2.49253 (0.19178)
	PLF	0.63730 (0.04260)	0.66016 (0.04312)	1.20812 (0.04984)	1.25961 (0.06117)	1.78651 (0.05729)	1.82135 (0.07930)	2.31782 (0.06627)	2.54941 (0.11521)
LA	SELF	0.61934 (0.02147)	0.71175 (0.02759)	1.24480 (0.05573)	1.31849 (0.06767)	1.77169 (0.08422)	1.88293 (0.10971)	2.46622 (0.10430)	2.60841 (0.14632)
	PLF	0.63261 (0.02966)	0.73417 (0.03155)	1.26621 (0.04064)	1.34813 (0.06008)	1.79613 (0.05249)	1.90438 (0.07753)	2.48949 (0.06057)	2.62091 (0.09587)
TKA	SELF	0.60729 (0.01195)	0.65056 (0.01640)	1.20208 (0.03149)	1.24821 (0.04710)	1.75466 (0.06035)	1.82566 (0.08488)	2.26273 (0.07026)	2.45938 (0.09437)
	PLF	0.61186 (0.01841)	0.65942 (0.02676)	1.21461 (0.02720)	1.27451 (0.04263)	1.77546 (0.04464)	1.86912 (0.06990)	2.29239 (0.06438)	2.52041 (0.08654)
GS	SELF	0.61968 (0.01235)	0.66384 (0.01727)	1.22661 (0.03588)	1.27368 (0.04741)	1.79047 (0.07169)	1.86291 (0.08866)	2.30892 (0.07689)	2.50958 (0.10870)
	PLF	0.62123 (0.02137)	0.66583 (0.02647)	1.23203 (0.03290)	1.27981 (0.04351)	1.80007 (0.06177)	1.87398 (0.06894)	2.32215 (0.08050)	2.52594 (0.10942)

Again, using LA, the BEs for the parameters θ_1 and θ_2 under IP using PLF are, respectively, presented as

$$\theta_{1,P} = \sqrt{\hat{\theta}_1^2 + (2\hat{\theta}_1\sigma_{11} + \sigma_{11}A_1 + \sigma_{21}A_2) + \left(\frac{a-1}{\hat{\theta}_1} - b\right)\sigma_{11} + \left(\frac{c-1}{\hat{\theta}_1} - d\right)\sigma_{12}},$$

$$\theta_{2,P} = \sqrt{\hat{\theta}_2^2 + \frac{1}{2}(2\hat{\theta}_2\sigma_{22} + \sigma_{12}A_1 + \sigma_{22}A_2) + \left(\frac{a-1}{\hat{\theta}_1} - b\right)\sigma_{21} + \left(\frac{c-1}{\hat{\theta}_2} - d\right)\sigma_{22}}. \tag{21}$$

3.5. *Tierney and Kadane's Approximation (TKA)*. The LA requires the evaluation of the third derivatives from the log-likelihood function which at times gets tedious, especially in case of problems with several parameters. This issue can be resolved by considering rather easily computable approximation called TKA. The additional advantage of this approximation is that it has less error as compared to LA. Therefore, in this subsection, we have considered TKA for the approximate Bayes estimation of the parameters from the doubly censored TLD. Consider $Q(\theta_1, \theta_2) = G(\theta_1, \theta_2) + l(x|\theta_1, \theta_2)$, where

TABLE 3: BEs and PRs (in parentheses) under IP for $n = 50$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
QM	SELF	0.57532 (0.01614)	0.60966 (0.01635)	1.09683 (0.03209)	1.18003 (0.03942)	1.63681 (0.05209)	1.77049 (0.07967)	2.20793 (0.08252)	2.36178 (0.09564)
	PLF	0.58730 (0.02676)	0.62306 (0.02693)	1.11020 (0.02902)	1.19838 (0.03719)	1.65177 (0.03212)	1.79506 (0.04939)	2.22522 (0.04130)	2.39332 (0.06390)
LA	SELF	0.60470 (0.01692)	0.67696 (0.01645)	1.16473 (0.03389)	1.28825 (0.04117)	1.67752 (0.04907)	1.84105 (0.05718)	2.34247 (0.06335)	2.45587 (0.07114)
	PLF	0.61736 (0.02358)	0.69769 (0.02173)	1.18319 (0.02669)	1.31516 (0.03928)	1.69751 (0.03240)	1.85778 (0.04667)	2.36014 (0.03867)	2.46160 (0.05611)
TKA	SELF	0.59292 (0.00943)	0.61876 (0.00976)	1.12476 (0.01915)	1.21959 (0.02865)	1.66140 (0.03516)	1.78504 (0.04424)	2.14918 (0.04267)	2.33839 (0.04589)
	PLF	0.59710 (0.01465)	0.62668 (0.01841)	1.13499 (0.01786)	1.24333 (0.02786)	1.67797 (0.02755)	1.82338 (0.04208)	2.17329 (0.04111)	2.37863 (0.05064)
GS	SELF	0.60502 (0.00975)	0.63138 (0.01028)	1.14771 (0.02181)	1.24448 (0.02885)	1.69532 (0.04177)	1.82147 (0.04622)	2.19305 (0.04670)	2.36281 (0.05286)
	PLF	0.60625 (0.01698)	0.63275 (0.01830)	1.15126 (0.02161)	1.24850 (0.02844)	1.70124 (0.03815)	1.82813 (0.04150)	2.20150 (0.05139)	2.37240 (0.06403)

TABLE 4: BEs and PRs (in parentheses) under IP for $n = 100$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
QM	SELF	0.53116 (0.00753)	0.51939 (0.00878)	1.03240 (0.01622)	1.02490 (0.01957)	1.52928 (0.02545)	1.51997 (0.03766)	2.05749 (0.04065)	2.01327 (0.04449)
	PLF	0.53679 (0.01335)	0.52547 (0.01421)	1.03889 (0.01493)	1.03309 (0.01889)	1.53657 (0.01630)	1.53067 (0.02449)	2.06574 (0.02093)	2.02453 (0.03132)
LA	SELF	0.53189 (0.00930)	0.54310 (0.00832)	1.04861 (0.01590)	1.07361 (0.01797)	1.54299 (0.02382)	1.56460 (0.02737)	2.17465 (0.03148)	2.07817 (0.03829)
	PLF	0.54263 (0.01401)	0.55927 (0.01222)	1.06368 (0.01218)	1.09430 (0.01902)	1.55883 (0.01651)	1.57608 (0.02360)	2.18699 (0.01966)	2.07883 (0.02778)
TKA	SELF	0.52153 (0.00519)	0.49642 (0.00494)	1.01262 (0.00898)	1.01639 (0.01251)	1.52816 (0.01706)	1.51701 (0.02117)	1.99523 (0.02121)	1.98892 (0.02123)
	PLF	0.52484 (0.00869)	0.50233 (0.01036)	1.02033 (0.00894)	1.03454 (0.01063)	1.54089 (0.01403)	1.54691 (0.02129)	2.01385 (0.02088)	1.99911 (0.02509)
GS	SELF	0.53217 (0.00537)	0.50654 (0.00517)	1.03329 (0.01021)	1.03714 (0.01260)	1.55935 (0.02027)	1.54797 (0.02211)	2.03594 (0.02320)	1.99941 (0.02446)
	PLF	0.53287 (0.01007)	0.50721 (0.01026)	1.03497 (0.01079)	1.03885 (0.01378)	1.56226 (0.01940)	1.55092 (0.02099)	2.03998 (0.02611)	2.00350 (0.03170)

$G(\theta_1, \theta_2)$ is the logarithmic of the joint IP for the parameters (θ_1, θ_2) given in (5), and $l(x|\theta_1, \theta_2)$ is the logarithmic of likelihood function given in (4).

Further consider $H(\theta_1, \theta_2) = Q(\theta_1, \theta_2)/n$ and $H^*(\theta_1, \theta_2) = [\log h(\theta_1, \theta_2) + Q(\theta_1, \theta_2)]/n$, where $\log h(\theta_1, \theta_2)$ is the logarithmic of the function of the parameter(s) θ_1 or θ_2 . Then, according to Tierney and Kadane [22], the expression $E\{h(\theta_1, \theta_2|x)\}$ using (7) can be reexpressed as

$$E\{h(\theta_1, \theta_2|x)\} = \frac{\int_0^\infty \int_0^\infty e^{nH^*(\theta_1, \theta_2)} d\theta_1 d\theta_2}{\int_0^\infty \int_0^\infty e^{nH(\theta_1, \theta_2)} d\theta_1 d\theta_2}. \quad (22)$$

Now using the Laplace's method, the approximation for $E\{h(\theta_1, \theta_2|x)\}$ can be given as

$$\hat{h}(\theta_1, \theta_2) = \left[\frac{\det \Sigma^*}{\det \Sigma} \right]^{1/2} \exp \left[n \left\{ H^*(\hat{\theta}_1^*, \hat{\theta}_2^*) - H(\hat{\theta}_1, \hat{\theta}_2) \right\} \right], \quad (23)$$

where $(\hat{\theta}_1^*, \hat{\theta}_2^*)$ and $(\hat{\theta}_1, \hat{\theta}_2)$ maximize $H^*(\theta_1, \theta_2)$ and $H(\theta_1, \theta_2)$, respectively, and Σ^* and Σ are the negatives

TABLE 5: BEs and PRs (in parentheses) under IP for $n = 15$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 1.00$	$\theta_1 = 1.00$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.50$	$\theta_1 = 1.50$	$\theta_2 = 1.00$
QM	SELF	0.65652	1.32428	1.20746	0.64358	1.19623	1.82671	1.76093	1.25387
		(0.02945)	(0.05951)	(0.06447)	(0.03584)	(0.06955)	(0.13369)	(0.10466)	(0.09758)
LA	PLF	0.67454	1.35143	1.22964	0.65529	1.21691	1.86714	1.76757	1.28912
		(0.04515)	(0.08908)	(0.05060)	(0.03194)	(0.03859)	(0.08145)	(0.05101)	(0.05852)
TKA	SELF	0.65833	1.46000	1.25266	0.68675	1.21426	1.91243	1.88258	1.30887
		(0.02267)	(0.05638)	(0.05638)	(0.03512)	(0.05683)	(0.11292)	(0.07936)	(0.07382)
GS	PLF	0.67108	1.49895	1.30178	0.70292	1.20834	1.95163	1.90840	1.32043
		(0.03143)	(0.06473)	(0.04179)	(0.03168)	(0.03560)	(0.07969)	(0.04615)	(0.04810)
TKA	SELF	0.64206	1.34211	1.23486	0.64953	1.19139	1.87479	1.73675	1.24200
		(0.01266)	(0.03337)	(0.03259)	(0.02439)	(0.04069)	(0.08627)	(0.05421)	(0.04803)
GS	PLF	0.64322	1.35863	1.23216	0.66817	1.21176	1.89680	1.76458	1.28329
		(0.01960)	(0.05508)	(0.02819)	(0.02225)	(0.03029)	(0.07159)	(0.04904)	(0.04367)
GS	SELF	0.65690	1.36534	1.24235	0.66309	1.22171	1.91403	1.77453	1.25899
		(0.01310)	(0.03557)	(0.03630)	(0.02467)	(0.04913)	(0.09075)	(0.05904)	(0.05447)
GS	PLF	0.66098	1.36422	1.26429	0.66874	1.20844	1.91522	1.78797	1.27028
		(0.02267)	(0.05424)	(0.03358)	(0.02271)	(0.04172)	(0.07098)	(0.06181)	(0.05576)

TABLE 6: BEs and PRs (in parentheses) under IP for $n = 30$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 1.00$	$\theta_1 = 1.00$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.50$	$\theta_1 = 1.50$	$\theta_2 = 1.00$
QM	SELF	0.62668	1.26329	1.16015	0.61629	1.14789	1.75451	1.67768	1.20143
		(0.02811)	(0.05687)	(0.06142)	(0.03441)	(0.06627)	(0.12750)	(0.10015)	(0.09348)
LA	PLF	0.64785	1.28845	1.17357	0.62697	1.16465	1.78251	1.69636	1.23185
		(0.04339)	(0.08487)	(0.04862)	(0.03044)	(0.03679)	(0.07767)	(0.04886)	(0.05588)
TKA	SELF	0.62753	1.39492	1.20395	0.66033	1.15687	1.83660	1.80788	1.25356
		(0.02173)	(0.05390)	(0.05401)	(0.03373)	(0.05442)	(0.10779)	(0.07618)	(0.07093)
GS	PLF	0.63973	1.43801	1.24945	0.67341	1.15143	1.85978	1.82285	1.26839
		(0.03021)	(0.06195)	(0.04015)	(0.03019)	(0.03398)	(0.07596)	(0.04426)	(0.04604)
TKA	SELF	0.61395	1.28398	1.18086	0.62393	1.13753	1.79721	1.66750	1.18297
		(0.01214)	(0.03202)	(0.03117)	(0.02344)	(0.03912)	(0.08283)	(0.05171)	(0.04610)
GS	PLF	0.61825	1.29581	1.17465	0.63768	1.16034	1.82330	1.69281	1.22325
		(0.01877)	(0.05283)	(0.02685)	(0.02137)	(0.02904)	(0.06841)	(0.04705)	(0.04193)
GS	SELF	0.62677	1.30537	1.19250	0.63650	1.16492	1.83310	1.69403	1.20715
		(0.01248)	(0.03404)	(0.03464)	(0.02359)	(0.04686)	(0.08691)	(0.05677)	(0.05223)
GS	PLF	0.63215	1.30956	1.21303	0.63733	1.15345	1.83859	1.71454	1.21606
		(0.02179)	(0.05197)	(0.03217)	(0.02184)	(0.03977)	(0.06776)	(0.05899)	(0.05325)

of the inverse Hessians of $H^*(\theta_1, \theta_2)$ and $H(\theta_1, \theta_2)$ evaluated at $(\hat{\theta}_1^*, \hat{\theta}_2^*)$ and $(\hat{\theta}_1, \hat{\theta}_2)$, respectively.

Here, we have

$$\begin{aligned}
 H(\theta_1, \theta_2) = & \frac{1}{n} \left[k + (m + a - 1) \log \theta_1 + (m + c - 1) \log \theta_2 \right. \\
 & + \theta_1(r - 1) \log(T_r) + (n - s) \log \left\{ 1 - T_s^{\theta_1} \right\} \\
 & \left. + \sum_{i=r}^s \log \left(\frac{T_{i2}}{2x_i} \right) + (\theta_1 - 1) \sum_{i=r}^s T_i - b\theta_1 - d\theta_2 \right],
 \end{aligned}$$

$$\begin{aligned}
 H^*(\theta_1, \theta_2) = & \frac{1}{n} \left[k + \log h(\alpha, \beta) + (m + a - 1) \log \theta_1 \right. \\
 & + (m + c - 1) \log \theta_2 + \theta_1(r - 1) \log(T_r) \\
 & + (n - s) \log \left\{ 1 - T_s^{\theta_1} \right\} + \sum_{i=r}^s \log \left(\frac{T_{i2}}{2x_i} \right) \\
 & \left. + (\theta_1 - 1) \sum_{i=r}^s T_i - b\theta_1 - d\theta_2 \right],
 \end{aligned} \tag{24}$$

TABLE 7: BEs and PRs (in parentheses) under IP for $n = 50$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 1.00$	$\theta_1 = 1.00$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.50$	$\theta_1 = 1.50$	$\theta_2 = 1.00$
QM	SELF	0.58318 (0.01631)	1.20582 (0.03206)	1.07368 (0.03154)	0.59154 (0.01964)	1.06765 (0.03346)	1.73217 (0.07812)	1.61751 (0.06104)	1.13840 (0.04662)
	PLF	0.59702 (0.02725)	1.21604 (0.05300)	1.07845 (0.02831)	0.59650 (0.01851)	1.07682 (0.02063)	1.75678 (0.04837)	1.62859 (0.03045)	1.15643 (0.03099)
LA	SELF	0.61270 (0.01713)	1.32674 (0.03214)	1.12650 (0.03285)	0.64519 (0.02052)	1.09538 (0.03171)	1.79575 (0.05618)	1.71717 (0.04627)	1.18025 (0.03449)
	PLF	0.62431 (0.02402)	1.36656 (0.04267)	1.16752 (0.02637)	0.65694 (0.01974)	1.08821 (0.02098)	1.81427 (0.04573)	1.72814 (0.02826)	1.19129 (0.02695)
TKA	SELF	0.59942 (0.00958)	1.22122 (0.01905)	1.10490 (0.01896)	0.60962 (0.01426)	1.07707 (0.02279)	1.75722 (0.04317)	1.58382 (0.03140)	1.12477 (0.02242)
	PLF	0.60334 (0.01494)	1.23147 (0.03634)	1.09765 (0.01763)	0.62208 (0.01397)	1.09663 (0.01792)	1.77869 (0.04119)	1.60486 (0.03005)	1.15444 (0.02453)
GS	SELF	0.61194 (0.00985)	1.24155 (0.02026)	1.11580 (0.02106)	0.62191 (0.01436)	1.10302 (0.02730)	1.79232 (0.04531)	1.60902 (0.03448)	1.13655 (0.02540)
	PLF	0.61690 (0.01731)	1.24450 (0.03593)	1.13351 (0.02113)	0.62174 (0.01427)	1.09013 (0.02456)	1.79361 (0.04079)	1.62546 (0.03766)	1.14214 (0.03116)

TABLE 8: BEs and PRs (in parentheses) under IP for $n = 100$.

AM	LF	$\theta_1 = 0.50$	$\theta_2 = 1.00$	$\theta_1 = 1.00$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.50$	$\theta_1 = 1.50$	$\theta_2 = 1.00$
QM	SELF	0.53842 (0.00761)	1.02728 (0.01722)	1.01061 (0.01594)	0.51377 (0.00975)	0.99751 (0.01635)	1.48707 (0.03693)	1.50730 (0.03007)	0.97042 (0.02169)
	PLF	0.54567 (0.01360)	1.02558 (0.02797)	1.00918 (0.01456)	0.51422 (0.00940)	1.00171 (0.01047)	1.49803 (0.02399)	1.51187 (0.01543)	0.97824 (0.01519)
LA	SELF	0.53893 (0.00941)	1.06439 (0.01626)	1.01419 (0.01541)	0.53769 (0.00896)	1.00753 (0.01539)	1.52610 (0.02689)	1.59414 (0.02299)	0.99874 (0.01856)
	PLF	0.54873 (0.01427)	1.09544 (0.02400)	1.04960 (0.01203)	0.54662 (0.00956)	0.99931 (0.01069)	1.53917 (0.02312)	1.60136 (0.01437)	1.00605 (0.01334)
TKA	SELF	0.52725 (0.00527)	0.97976 (0.00964)	0.99474 (0.00889)	0.50805 (0.00623)	0.99069 (0.01106)	1.49337 (0.02066)	1.47037 (0.01561)	0.95667 (0.01037)
	PLF	0.53032 (0.00886)	0.98711 (0.02045)	0.98676 (0.00882)	0.51761 (0.00533)	1.00704 (0.00913)	1.50899 (0.02084)	1.48712 (0.01526)	0.97025 (0.01216)
GS	SELF	0.53826 (0.00543)	0.99606 (0.01019)	1.00456 (0.00986)	0.51830 (0.00627)	1.01455 (0.01325)	1.52320 (0.02167)	1.49375 (0.01713)	0.96175 (0.01175)
	PLF	0.54223 (0.01027)	0.99759 (0.02014)	1.01901 (0.01055)	0.51734 (0.00692)	1.00107 (0.01249)	1.52163 (0.02063)	1.50620 (0.01913)	0.96455 (0.01543)

where k is any constant independent of the parameters θ_1 and θ_2 .

$$\frac{\partial H(\theta_1, \theta_2)}{\partial \theta_1} = \frac{1}{n} \left[\frac{m+a-1}{\theta_1} + (r-1) \log(T_r) - \frac{(n-s)T_s^{\theta_1} \log(T_s)}{1-T_s^{\theta_1}} + \sum_{i=r}^s T_i - b \right] = 0, \tag{25}$$

$$\frac{\partial H(\theta_1, \theta_2)}{\partial \theta_2} = \frac{1}{n} \left[\frac{m+c-1}{\theta_2} + \frac{(r-1)\theta_1 T_{r1}}{T_r} - \frac{(n-s)\theta_1 T_{s1} T_s^{\theta_1-1}}{1-T_s^{\theta_1}} + (\theta_1-1) \sum_{i=r}^s \left(\frac{T_{i2}}{T_i} \right) + \theta_2^{-2} \sum_{i=r}^s T_{i3} - d \right] = 0. \tag{26}$$

TABLE 9: Sensitivity analysis of BEs with respect functional form of the prior.

Parametric values	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
Values for hyperparameters	$a = 0.5, b = 1$		$a = 0.5, b = 1.5$		$a = 0.5, b = 0.75$		$a = 1, b = 2$	
QM	0.57532 (0.01614)	0.60966 (0.01635)	0.58490 (0.01641)	0.61804 (0.01665)	0.58456 (0.01637)	0.61697 (0.01652)	0.58041 (0.01625)	0.61082 (0.01640)
	0.58730 (0.02676)	0.62306 (0.02693)	0.59667 (0.02718)	0.63532 (0.02742)	0.59365 (0.02719)	0.63113 (0.02726)	0.59070 (0.02677)	0.62556 (0.02708)
LA	0.60470 (0.01692)	0.67696 (0.01645)	0.61332 (0.01722)	0.69013 (0.01674)	0.61110 (0.01713)	0.68682 (0.01673)	0.60877 (0.01705)	0.68207 (0.01650)
	0.61736 (0.02358)	0.69769 (0.02173)	0.62619 (0.02388)	0.70737 (0.02201)	0.62967 (0.02383)	0.71014 (0.02216)	0.61902 (0.02364)	0.70445 (0.02182)
TKA	0.59292 (0.00943)	0.61876 (0.00976)	0.60332 (0.00953)	0.62678 (0.00994)	0.60210 (0.00955)	0.62524 (0.00987)	0.59360 (0.00951)	0.62082 (0.00977)
	0.59710 (0.01465)	0.62668 (0.01841)	0.60855 (0.01482)	0.63435 (0.01866)	0.60647 (0.01489)	0.63870 (0.01868)	0.59984 (0.01479)	0.62845 (0.01849)
GS	0.60502 (0.00975)	0.63138 (0.01028)	0.61631 (0.00990)	0.63809 (0.01042)	0.61332 (0.00987)	0.64181 (0.01042)	0.60985 (0.00979)	0.63717 (0.01036)
	0.60625 (0.01698)	0.63275 (0.01830)	0.61360 (0.01731)	0.63968 (0.01861)	0.61237 (0.01731)	0.64388 (0.01850)	0.61099 (0.01705)	0.63559 (0.01838)

TABLE 10: ML estimates under different sample sizes.

n	$\theta_1 = 0.50$	$\theta_2 = 0.50$	$\theta_1 = 1.00$	$\theta_2 = 1.00$	$\theta_1 = 1.50$	$\theta_2 = 1.50$	$\theta_1 = 2.00$	$\theta_2 = 2.00$
15	0.67236 (0.03021)	0.69601 (0.03142)	1.28258 (0.06817)	1.33429 (0.07471)	1.90678 (0.11246)	1.93951 (0.14161)	2.49437 (0.14669)	2.69919 (0.20745)
30	0.63965 (0.02865)	0.66035 (0.02998)	1.22963 (0.06462)	1.27257 (0.07139)	1.82424 (0.10682)	1.86098 (0.13486)	2.37228 (0.14024)	2.58149 (0.19852)
50	0.59220 (0.01657)	0.62906 (0.01686)	1.13391 (0.03317)	1.21837 (0.04063)	1.69285 (0.05374)	1.83541 (0.08241)	2.28006 (0.08504)	2.44215 (0.09863)
100	0.54507 (0.00772)	0.53529 (0.00901)	1.06582 (0.01674)	1.05575 (0.02001)	1.57692 (0.02622)	1.56095 (0.03846)	2.11160 (0.04158)	2.07552 (0.04571)

TABLE 11: BEs and PRs (in parentheses) using real dataset-1.

LF	QM		LA		AM		TKA		GS	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
SELF	3.06256 (1.17810)	39.11757 (1.71383)	3.11065 (1.15486)	40.97670 (1.57371)	3.00389 (0.65224)	38.79277 (1.09555)	3.06520 (0.74158)	39.58474 (1.10344)		
PLF	3.08182 (1.08441)	39.43016 (1.65428)	3.15535 (0.88467)	41.76638 (1.46566)	3.02676 (0.64934)	39.48551 (0.93091)	3.07019 (0.68371)	39.65001 (1.00677)		
AIC (SELF)	99.25209		91.57919		80.66709		83.10349			
BIC (self)	102.83561		95.16271		84.25061		86.68701			
AIC (PLF)	97.88159		91.14069		78.57859		78.95319			
BIC (PLF)	101.46511		94.72421		82.16211		82.53671			

Now, $(\hat{\theta}_1, \hat{\theta}_2)$ can be obtained by solving (25) and (26).

The determinant for the negative of the inverse Hessian of $H(\theta_1, \theta_2)$ evaluated at $(\hat{\theta}_1, \hat{\theta}_2)$ is

$$\det \Sigma = (H_{11}H_{22} - H_{12}^2)^{-1}, \quad (27)$$

where $H_{11} = \partial^2 H(\theta_1, \theta_2) / \partial \theta_1^2 |_{\hat{\theta}_1, \hat{\theta}_2}$, $H_{22} = \partial^2 H(\theta_1, \theta_2) / \partial \theta_2^2 |_{\hat{\theta}_1, \hat{\theta}_2}$, and $H_{12} = \partial^2 H(\theta_1, \theta_2) / \partial \theta_1 \partial \theta_2 |_{\hat{\theta}_1, \hat{\theta}_2}$.

The second order derivatives from $H(\theta_1, \theta_2)$ contains lengthy expressions; therefore, they have not been presented here. Similarly, MLEs and the second order derivatives for the $H^*(\theta_1, \theta_2)$ can be obtained. Further, the BEs and the PRs under IP can be obtained by using (23).

3.6. Gibbs Sampler. Unfortunately, by using QM, Lindley's method, and TKA, it is impossible to obtain the highest posterior density (HPD) credible intervals. The HPD credible intervals can be obtained by using the GS. Hence, in this subsection, we have used the GS to obtain the approximate BEs along with HPD credible intervals.

Consider a PD with two parameters θ_1 and θ_2 . Assuming that the full densities $g(\theta_1 | \theta_2, x)$ and $g(\theta_2 | \theta_1, x)$ are extractable, we need to obtain $g(\theta_1 | x)$ and $g(\theta_2 | x)$. Using GS, we start with choosing initial guess values for parameters θ_1 and θ_2 . The said initial values are denoted by θ_{10} and θ_{20} .

Then, we generate random samples from the conditional distributions in the following sequence:

$$\begin{aligned} \theta_{11} &\sim g(\theta_1 | \theta_{20}, x), \\ \theta_{21} &\sim g(\theta_2 | \theta_{11}, x), \\ \theta_{12} &\sim g(\theta_1 | \theta_{21}, x), \\ \theta_{22} &\sim g(\theta_2 | \theta_{12}, x), \\ &\vdots \\ \theta_{1m} &\sim g(\theta_1 | \theta_{2(m-1)}, x), \\ \theta_{2m} &\sim g(\theta_2 | \theta_{1m}, x). \end{aligned} \quad (28)$$

In order to employ the GS for the PD (7), we extract the conditional distribution for θ_1 given θ_2 under IP as

$$\begin{aligned} g_1(\theta_1 | \theta_2, x) &\propto \theta_1^{m+a-1} \exp \\ &\cdot \left[\theta_1 \left\{ (r-1) \log \left(\frac{2x_r}{\theta_2} - \frac{x_r^2}{\theta_2^2} \right) + \sum_{i=r}^s \log \left(\frac{2x_i}{\theta_2} - \frac{x_i^2}{\theta_2^2} \right) \right\} \right] \\ &\cdot \left[1 - \left(\frac{2x_s}{\theta_2} - \frac{x_s^2}{\theta_2^2} \right) \theta_1 \right]^{n-s}. \end{aligned} \quad (29)$$

TABLE 12: BEs and PRs (in parentheses) using real dataset-2.

LF	QM		LA		AM		TKA		GS	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
SELF	0.56486 (0.10653)	3.00998 (0.42577)	0.57480 (0.10317)	3.06395 (0.41086)	0.57424 (0.08898)	3.08697 (0.35404)	0.57364 (0.09519)	3.07860 (0.38078)		
PLF	0.57204 (0.09892)	3.04820 (0.36318)	0.58334 (0.09606)	3.09551 (0.35396)	0.58712 (0.08219)	3.09182 (0.30227)	0.58650 (0.08930)	3.12185 (0.32614)		
AIC (SELF)	34.86567		32.26568		26.28640		29.03856			
BIC (self)	37.66807		35.06807		29.08879		31.84095			
AIC (PLF)	34.24837		31.70037		25.84067		28.53779			
BIC (PLF)	36.74210		34.22010		28.42020		31.08980			

The conditional distribution of the parameter θ_2 given θ_1 under IP is

$$g_2(\theta_2|\theta_1, x) \propto \theta_2^{-m+c-1} \left(\frac{2x_r}{\theta_2} - \frac{x_r^2}{\theta_2^2}\right)^{\theta_1(r-1)} \cdot \left[1 - \left(\frac{2x_s}{\theta_2} - \frac{x_s^2}{\theta_2^2}\right)^{\theta_1}\right]^{n-s} \prod_{i=r}^s \left(2 - \frac{2x_i}{\theta_2}\right) \left(\frac{2x_i}{\theta_2} - \frac{x_i^2}{\theta_2^2}\right)^{\theta_1-1} \quad (30)$$

Now, using the conditional distributions from (29) and (30), the GS can be implemented following the methodology proposed by Pandey and Bandyopadhyay [23] using Winbugs software. The generated samples for the parameters θ_1 and θ_2 can be used for the estimation of the parameters using SELF and PLF. The BE and the posterior risks (PRs) for the parameter θ_1 using SELF and IP can be obtained by using the formulae $\theta_{1,S} = \sum_{i=1}^m \theta_{1,i}/m$ and $\rho(\theta_{1,S}) = \sum_{i=1}^m (\theta_{1,i} - \theta_{1,S})^2$, respectively. Similarly, the BE and PR for the parameter θ_1 using PLF and IP can be calculated by using the formulae $\theta_{1,P} = \sqrt{\sum_{i=1}^m \theta_{1,i}^2/m}$ and $\rho(\theta_{1,P}) = 2\{\sqrt{(\sum_{i=1}^m \theta_{1,i}^2/m)} - (\sum_{i=1}^m \theta_{1,i}^2/m)\}$, respectively. A similar process can be followed to implement GS for the PD under IP and for the PDs for the parameter θ_2 .

4. Simulation Study

As the performance of the BEs cannot be compared analytically, we have carried out a simulation study to compare different estimators numerically. The comparison among the BEs is based on the level of the convergence of the estimates and the amounts of the PRs. The following parametric space has been used for the estimation: $(\theta_1, \theta_2) = \{(0.50, 0.50), (1, 1), (1.50, 1.50), (2, 2)\}$ and $(\theta_1, \theta_2) = \{(0.50, 1), (1, 0.5), (1, 1.50), (1.50, 1)\}$. The samples of sizes $n = 15, 30, 50$, and 100 have been considered. The values for the hyperparameters have been considered in such a way that, for these values, the mean of the concerned prior is equal to the true parametric value. The samples have been assumed

to be 20% censored: 10% from the left and 10% from the right. The test termination point in each sample is assumed to be so that observed the said censoring rate. The results for the simulated samples have been presented in Tables 1–8. The results using parametric space $(\theta_1, \theta_2) = \{(0.50, 0.50), (1, 1, 1.50, 1.50, 2, 2)\}$, using samples of sizes 15, 30, 50, and 100, have been reported in Tables 1–4, respectively. Similarly, the results for sample sizes 15, 30, 50, and 100, using parametric space $(\theta_1, \theta_2) = \{(0.50, 1, 1, 0.5, 1, 1.50, 1.50, 1)\}$, have been given in Tables 5–8, respectively.

The results using simulated samples have been presented in Tables 1–10. The said results advocate that the proposed estimates are consistent in nature because the estimated values of the parameters converged to the true parametric values with increase in the sample size. The performance of the BEs has been better as compared to ML counterparts. The convergence under BEs was better as compared to MLEs. In additions, the amounts of PRs associated with each estimate become smaller as sample size increases. The performance of loss functions and different approximation methods has been compared on the basis of amounts of PRs associated with each estimate.

It is interesting to note that for smaller choice of true parametric values, the SELF performs better than PLF. Conversely, the performance of PLF seems better for estimation. There is a sense of competition among the performance of different approximation methods; however, the TKA provides the minimum amounts of the PRs among all the approximation methods with few exceptions. The amounts of the PRs are directly proportional with the magnitude of the true parametric values which is in accordance with the theory. In general, the parameter (θ_1) has been estimated more efficiently than the parameter (s).

5. Real-Life Example

In this section, the data regarding the failure times (in minutes) for a specific type of electrical insulation has been used for the illustration of the results obtained in the previous section. The data has been used by Fernandez [24] and is reported by Lawless [25]. The twelve items have been

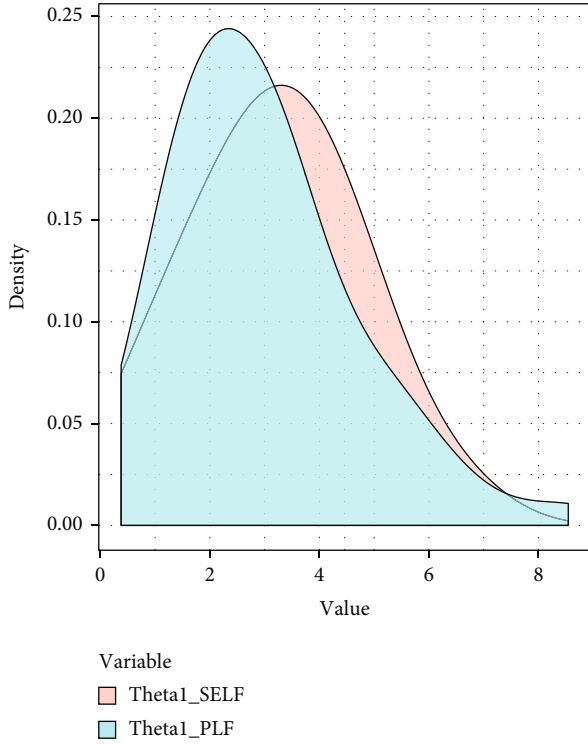


FIGURE 1: The marginal density for θ_1 under QM for dataset.

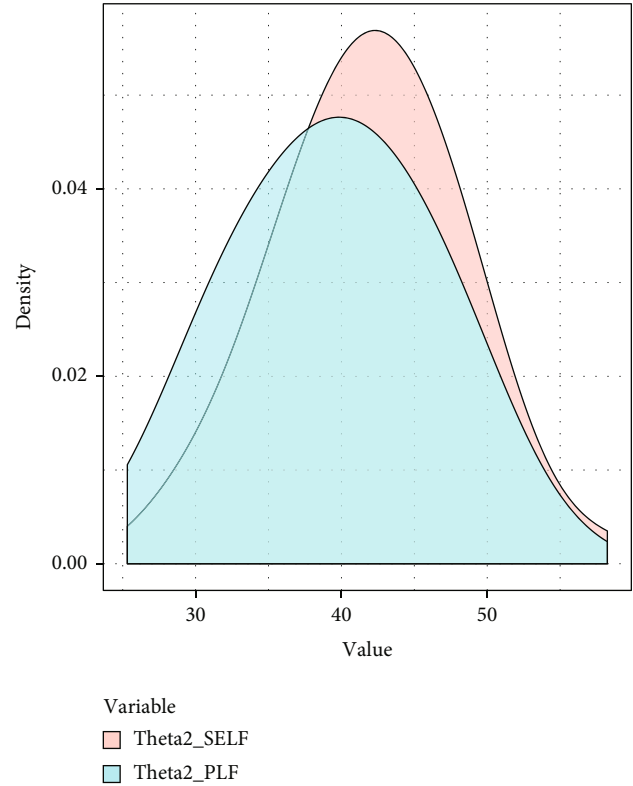


FIGURE 2: The marginal density for θ_2 under QM for dataset-1.

considered for the experiment out of which the experimenter was unable to observe the smallest two failure times and experiment was terminated after the ninth failure. The data is as follows: -, -, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, -, -, and -. Therefore, we have $n = 12$, $r = 3$, $s = 9$, and $m = 7$. The descriptive statistics for the data are mean = 54.94, variance = 689.50, standard error = 9.92, skewness = 0.39, and kurtosis = -1.17. The data has been censored in such a way that same number of observations has been censored from left and right. We have named this dataset as dataset-1. The other dataset used in the study has been named as dataset-2. The dataset-2 represents the failure times of the air condition system of an airplane, reported by Linhart and Zucchini [26]. The dataset-2 consists of observations 1, 3, 5, 7, 11(3), 12, 14(3), 16(2), 20, 21, 23, 42, 47, 52, 62, 71(2), 87, 90, 95, 120(2), 225, 246, and 261. The number in the parenthesis represents the repeated observations. The chi-square and the Kolmogorov-Smirnov tests have been used to confirm that the data can be modeled using TLD. The BEs and corresponding PRs, based on these data, have been presented in Tables 11 and 12, respectively. For convenience, the numerical values for the hyperparameters have been assumed to be $a = b = c = d = 1$ for the estimation.

Tables 11 and 12 contain the BEs and associated PRs using different approximation methods. The amounts for the PRs under PLF are on average smaller to those under SELF. As far as the approximation is concerned, the performance of the TKA seems superior to its counterparts, as the least magnitudes of the PRs are associated with these estimates. Hence, the results obtained by analyzing the real data are in agreement to those obtained from simulated samples. The marginal density plots for model parameters under different approximation methods have been presented in

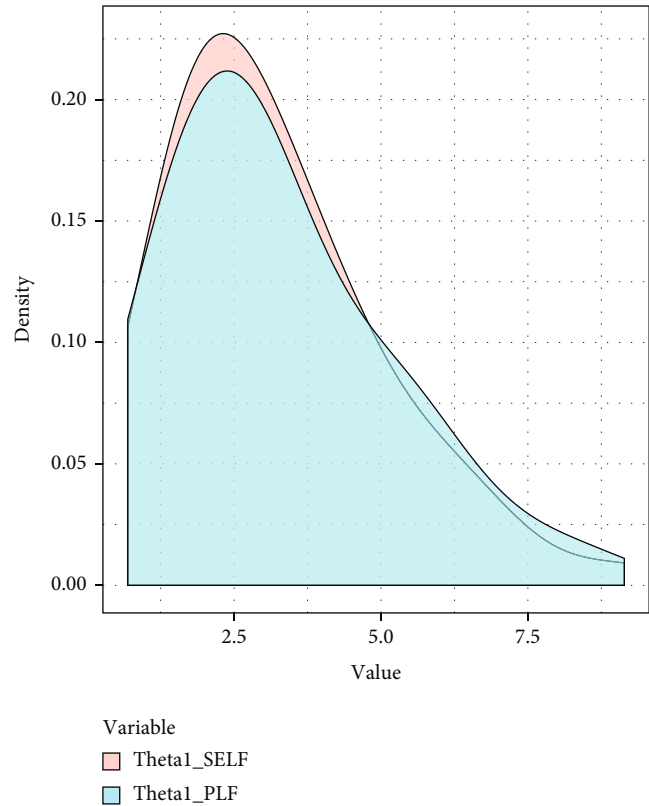


FIGURE 3: The marginal density for θ_1 under LA for dataset-1.

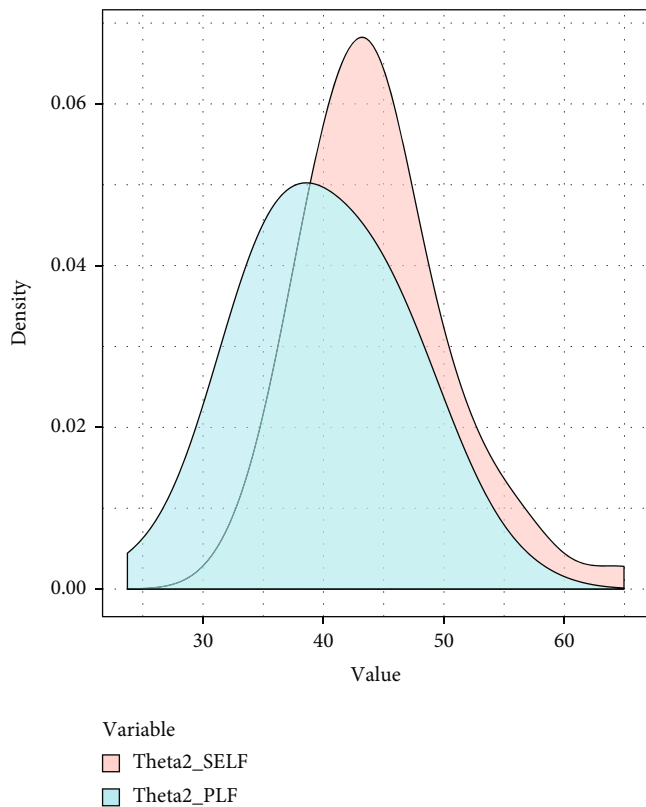


FIGURE 4: The marginal density for θ_2 under LA for dataset-1.

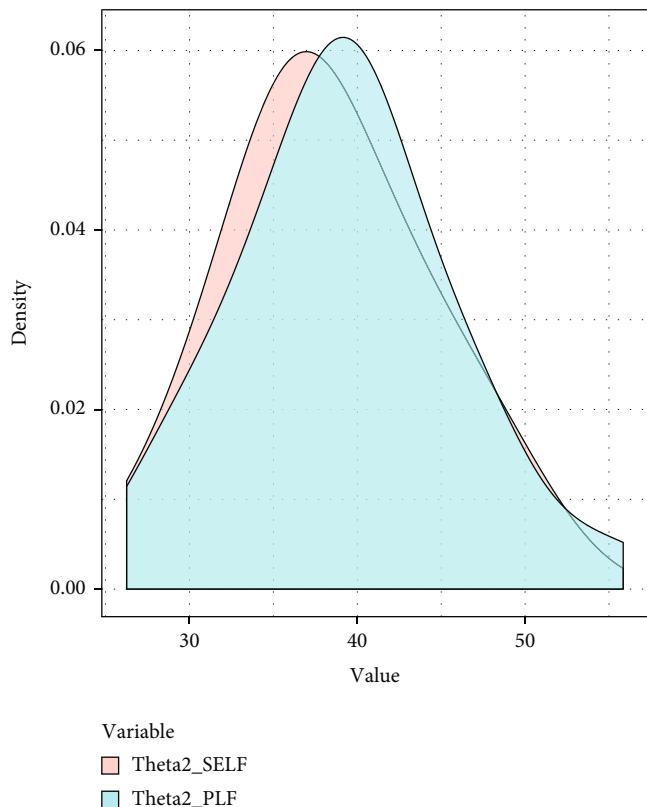


FIGURE 6: The marginal density for θ_2 under TKA for dataset-1.

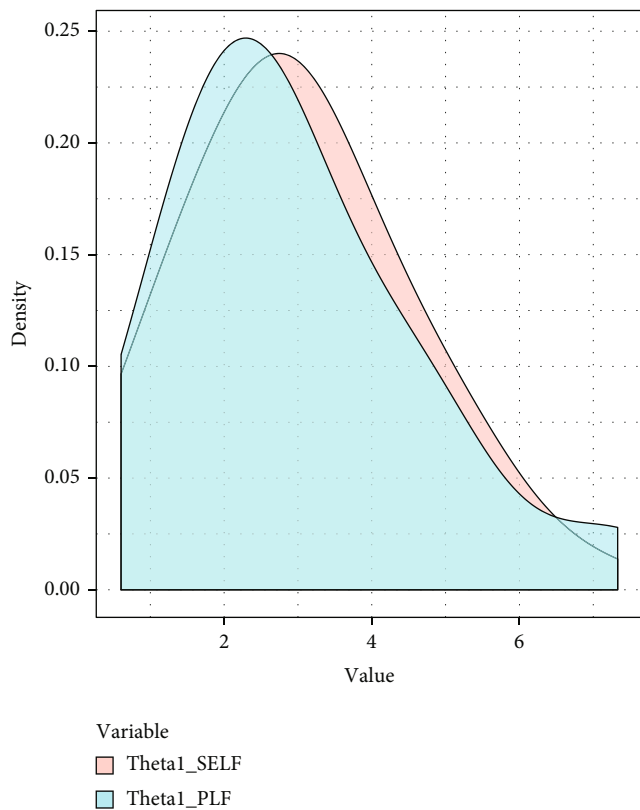


FIGURE 5: The marginal density for θ_1 under TKA for dataset-1.

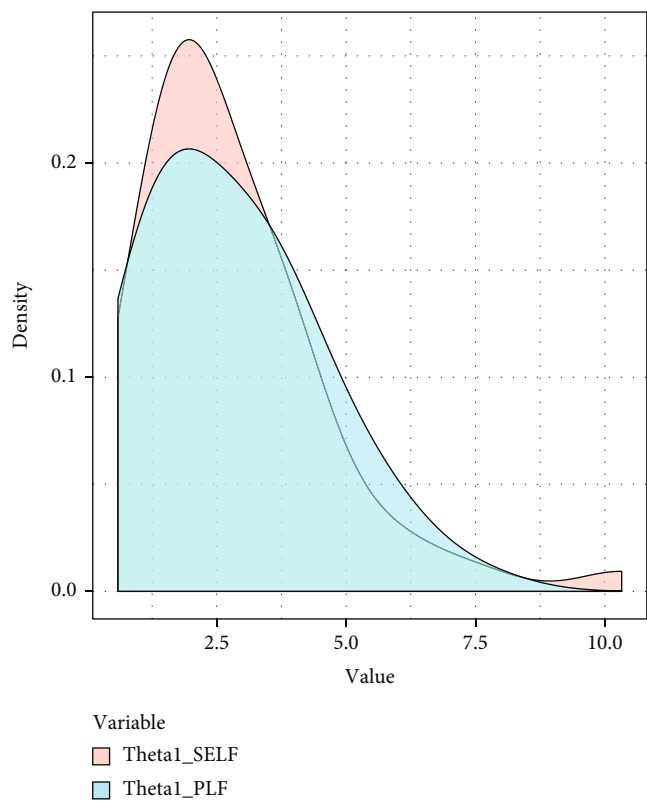


FIGURE 7: The marginal density for θ_1 under GS for dataset-1.

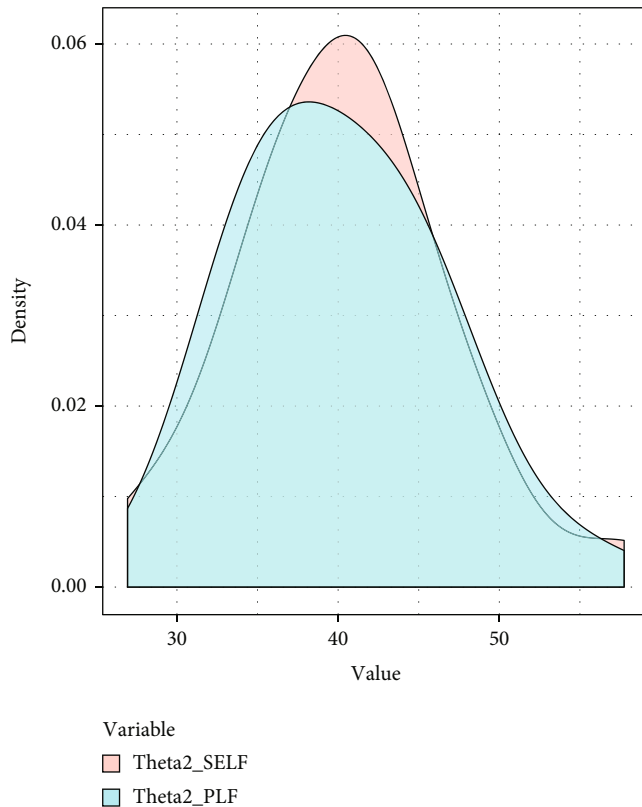


FIGURE 8: The marginal density for θ_2 under GS for dataset-1.

Figures 1–8. For brevity, we have presented the density plots only for dataset-1. The density plots for parameter θ_1 are positively skewed. However, the skewness is not that high for density curves of parameter θ_2 . The density plots under PLF are representing slightly more skewness as compared to those for SELF. From these graphs, it can also be seen that the graphs for marginal densities under SELF and PLF are relatively closer under TKA for both parameters. However, the graphs for marginal densities using other approximation techniques, such as QM, LA, and GS, are quite different under SELF and PLF. This simply suggests that the estimates under TKA are relatively robust with respect to choice of loss functions.

6. Conclusion

This paper considers the posterior analysis for the parameters of the TLD assuming doubly censoring. As the expressions for the BEs cannot be obtained explicitly, we considered different approximation methods to obtain the numerical solutions. The performance of the BEs is compared based on amounts of PRs. The analysis of simulated and real datasets suggests that there is a sense of competition among different approximation methods; however, the performance of the TKA is the best among the approximate methods used in the study. The study can be extended for application of TLD in modeling doubly censored heterogeneous data.

Data Availability

The data used in the paper are available in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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