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Research Article

New Fractional Estimates of Simpson-Mercer Type for Twice Differentiable Mappings Pertaining to Mittag-Leffler Kernel

Saad Ihsan Butt,[1] **Saima Rashid**,[2] **Iram Javed**,[1] **Khuram Ali Khan**,[3] **and Rostin Matendo Mabela**[4]

[1]*COMSATS University Islamabad, Lahore Campus, Pakistan*
[2]*Department of Mathematics, Government College University, Faisalabad 38000, Pakistan*
[3]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[4]*Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Democratic Republic of the Congo*

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

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The main motivation of this study is to introduce a novel auxiliary result of Simpson's formula by employing the Mercer scheme for twice differentiable functions involving the Atangana-Baleanu (AB) fractional integral operator concerned with the Mittag-Leffler as a nonsingular or nonlocal kernel. Thus, by employing Mercer's convexity on twice differentiable mappings along with Hölder's and power-mean inequalities, one can develop a variety of new Simpson's error estimates. Lastly, some applications to q-digamma function and modified Bessel functions are presented. Furthermore, the graphical illustrations described the efficiency and applicability of the proposed technique with success. We make links between our findings and a number of well-known discoveries in the literature. It is hoped that the proposed methodology will provide a new venue in the numerical techniques for calculating the quadrature formulae.

1. Introduction

Convex functions have gained a lot of popularity in recent years. Convex functions are frequently used in various areas of current analysis in a variety of mathematical disciplines. They are magical, especially in optimization theory, because they have so many useful properties. Inequality theory and convex functions have a strong relationship. Convex functions can be used to obtain a variety of important and useful inequalities. Due to wide range of implementations, it is among the most advanced branches of mathematical modeling. Convex functions are the topic of research in a number of disciplines due to their applicability in inequality theory and defined as:

$$\phi(\kappa\varkappa + (1-\kappa)\varkappa_1) \leq \kappa\phi(\varkappa) + (1-\kappa)\phi(\varkappa_1), \tag{1}$$

where $\phi : [\zeta_1, \zeta_2] \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ is a convex function which holds for all $\varkappa, \varkappa_1 \in [\zeta_1, \zeta_2]$ and $\kappa \in [0, 1]$.

Additional information of different types of convexity and their contribution to inequalities can be found here, see [1, 2]. The improvement and exploration of the integral inequalities referring to convex functions is primarily motivated based on the research and findings presented in these books. Because of their wide range of implementations such as probability theory, information theory, computational problems, and optimization, the Jensen and related inequalities are essential and well-known inequalities for convex functions. See [3, 4] and references there in.

One of the most significant inequality that we may say is the natural extension of convex function is Jensen-Mercer inequality [5] given as:

$$\phi\left(\zeta_1 + \zeta_2 - \sum_{\mathrm{j}=1}^{\mathfrak{n}} \Theta_{\mathrm{j}}\,\varkappa_{\mathrm{j}}\right) \leq \phi(\zeta_1) + \phi(\zeta_2) - \sum_{\mathrm{j}=1}^{\mathfrak{n}} \Theta_{\mathrm{j}}\,\phi(\varkappa_{\mathrm{j}}), \tag{2}$$

where ϕ on $[\zeta_1, \zeta_2]$ is a convex function $\forall \varkappa_j \in [\zeta_1, \zeta_2]$ and all $\Theta_j \in [0, 1]$ where

$$\sum_{j=1}^{n} \Theta_j = 1. \tag{3}$$

It is the most effective inequality in predicting the estimations of bounds of distance functions in information theory [6].

Many researchers put an effort in this direction to build new results such as fractional variants of Hermite Jensen-Mercer type inequalities along with applications [7], Moradi and Furuichi worked for the improvement and generalization of Jensen-Mercer-type inequalities [8], Kian and Moslehian worked for the improvement of operator Jensen-Mercer inequality [9], Niezgoda worked on generalization of Mercer's result on convex function [10], and Harovath gave some notes on Jensen-Mercer inequality [11]. A lot of work has been done on Jensen-Mercer-type inequality in Yang's Calculus, one can see [12, 13].

An inequality which is notable as Simpson's inequality is as follows.

Theorem 1 (see [14]). *Suppose that $\phi : [\zeta_1, \zeta_2] \longrightarrow \mathbb{R}$ is a four-time continuously differentiable mapping on (ζ_1, ζ_2), and let $\|\phi^{(4)}\|_\infty = \sup_{\varkappa \in (\zeta_1, \zeta_2)} |\phi^{(4)}(\varkappa)| < \infty$, then the following inequality holds:*

$$\left| \frac{1}{3}\left[\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} + 2\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right] - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa \right| \le \frac{1}{2880}\left\|\phi^{(4)}\right\|_\infty (\zeta_2-\zeta_1)^4. \tag{4}$$

The results of Simpson type inequalities for convex mappings have been looked by various writers because convex theory is an excellent technique to deal with a sizable number of issues from various mathematical disciplines. Specifically, differentiable functions are utilized to demonstrate some Simpson's type inequalities for s-convex functions [15], and then, the inequality extended to Riemann-Liouville fractional integrals [16]. A lot of work has been done utilizing this inequality for first derivative, one can see [17, 18].

Sarikaya et al. [19] explored numerous Simpson type inequalities for functions whose second derivatives are convex. The first and second outcomes on fractional Simpson inequality for twice differentiable functions were established in [14, 20]. With the help of these articles, the aim of this paper is to extend the results given in [19] for twice differentiable functions to generalized fractional integrals. Nowadays, twice differentiable functions are topic of interest for most of the researchers. We will use Mercer convexity along with twice differentiability for Simpson type inequalities to improve our outcomes and give new bounds.

Here, a lemma that is given in [14] stated as follows.

Lemma 2. *If there is a mapping $\phi : [\zeta_1, \zeta_2] \longrightarrow \mathbb{R}$ that is absolutely continuous on (ζ_1, ζ_2) considering $\phi'' \in L_1([\zeta_1, \zeta_2])$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[\phi(\zeta_1) + 4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right) + \phi(\zeta_2)\right] \\
&\quad - \frac{2^{\varpi-1}\Gamma(\varpi+1)}{(\zeta_2-\zeta_1)^{\varpi}}\left[J^{\varpi}_{\zeta_2-}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right) + J^{\varpi}_{\zeta_1+}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right] \\
&= \frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\int_0^1\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \\
&\quad \times \left[\phi''\left(\left(\frac{1+\kappa}{2}\zeta_2 + \frac{1-\kappa}{2}\zeta_1\right)\right) + \phi''\left(\left(\frac{1+\kappa}{2}\zeta_1 + \frac{1-\kappa}{2}\zeta_2\right)\right)\right]d\kappa.
\end{aligned} \tag{5}$$

We will extend this result for a new fractional integral operator along with mercer convexity. It is not easy to deal with double derivative; we have worked on double derivative for a new operator along with Mercer convexity. It is completely new idea, and in this way, the generalized outcomes will increase its worth and captures interest of many scholars toward this field.

The generalization of classical calculus, fractional calculus, is widely used in sciences, particularly engineering. The classical calculus offers an excellent method for modelling and explaining numerous essential dynamic processes in most sections of applied sciences. The hypothesis of fractional calculus was developed to merge and generalise n-fold integration and integer-order differentiation. The field of applied sciences includes fractional analysis. Many results based on fractional models have been published in various fields of science [21].

The fractional operators of integral and derivative helps in improving the relationships between mathematics and other specialisations by providing solutions that are more closely related to real-world problems. Fractional integral and derivative operators have evolved over time [22, 23]. Some fractional numerical simulations can be seen in [24, 25]. In their review article "Fractional calculus in the sky," [26] D. Baleanu and R. P. Agrwal, two esteemed professors, provide the most recent compact review of fractional calculus.

Definition 3 (see [27]). Let $\zeta_2 > \zeta_1$, $\varpi \in [0, 1]$ and $\phi \in H^1(\zeta_1, \zeta_2)$, then the new fractional derivative is:

$${}^{ABC}_{\zeta_1}D^{\varpi}_{\kappa}[\phi(\kappa)] = \frac{B(\varpi)}{1-\varpi}\int_{\zeta_1}^{\kappa}\phi'(\varkappa)E_{\varpi}\left[-\varpi\frac{(\kappa-\varkappa)^{\varpi}}{(1-\varpi)}\right]d\varkappa. \tag{6}$$

Definition 4 (see [27]). Let $\phi \in H^1(\zeta_1, \zeta_2)$, $\zeta_1 > \zeta_2$, $\varpi \in [0, 1]$, then we have:

$${}^{ABR}_{\zeta_1}D^{\varpi}_{\kappa}[\phi(\kappa)] = \frac{B(\varpi)}{1-\varpi}\frac{d}{d\kappa}\int_{\zeta_1}^{\kappa}\phi(\varkappa)E_{\varpi}\left[-\varpi\frac{(\kappa-\varkappa)^{\varpi}}{(1-\varpi)}\right]d\varkappa. \tag{7}$$

Definition 5 (see [27]). Let $\phi \in H^1(\zeta_1, \zeta_2)$, then with nonlocal kernel, the fractional integral operator is defined as:

$$ {}_{\zeta_1}^{AB}I_{\kappa}^{\varpi}\{\phi(\kappa)\} = \frac{1-\varpi}{B(\varpi)}\phi(\kappa) + \frac{\varpi}{B(\varpi)\Gamma(\varpi)}\int_{\zeta_1}^{\kappa}\phi(\varkappa_1)(\kappa-\varkappa_1)^{\varpi-1}d\varkappa_1, \tag{8}$$

where $\zeta_2 > \zeta_1, \varpi \in [0, 1]$.

In [28], AB-fractional integral operator's right hand side is:

$$ {}_{\zeta_2}^{AB}I_{\kappa}^{\varpi}\{\phi(\kappa)\} = \frac{1-\varpi}{B(\varpi)}\phi(\kappa) + \frac{\varpi}{B(\varpi)\Gamma(\varpi)}\int_{\kappa}^{\zeta_2}\phi(\varkappa_1)(\varkappa_1-\kappa)^{\varpi-1}d\varkappa_1. \tag{9}$$

Because the normalization function $B(\varpi)$ is positive, any positive function has a positive fractional AB-integral. It is worth noting that the classical integral is obtained when the order is $\varpi \longrightarrow 1$, while the initial function is obtained when the order is $\varpi \longrightarrow 0$. In the theory of integral inequalities involving AB operators, there has been some recent progress. One can see in [29, 30].

The purpose of this analysis is to utilize the AB integral operator to propose novel Mercer type inequalities for convex functions. The utmost objective is to acquire outcomes that create particular Mercer type inequalities by employing AB operators and elaborate the facts more appropriately in terms of the operator's qualities and kernel structure. Inequalities of the classical Mercer type and their different versions are created in the case when we have $\varpi = 1$ in the obtained results. This significant achievement is attributed to the AB fractional integral operator exhibiting the heredity characteristic. The exponential and power law functions are not as good as the generalized Mittag-Leffler function with robust memory entangled in the AB fractional formulation. Furthermore, the Atangana-Baleanu fractional-order derivative is at the same time Liouville-Caputo and Caputo-Fabrizio thus possesses Markovian and non-Markovian properties. Meanwhile, the graphical construction represents the comparison between the error and error estimates clearly with the aid of MATLAB 2021 package. Finally, the obtained outcomes were backed up by diminished outcomes and implementations.

2. Novel Simpson's Atangana-Baleanu Inequalities

Here, we present Mercer type Simpson's inequalities for Atangana-Baleanu integral operator for differentiable functions on (ν_1, ν_2). For this, we give a new Atangana-Baleanu integral operator auxiliary identity that will serve to produce subsequent results for improvements.

Lemma 6. *If there is a mapping $\phi : [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ that is absolutely continuous on (ν_1, ν_2) considering $\phi'' \in L_1([\nu_1, \nu_2])$, where $\zeta_1, \zeta_2 \in [\nu_1, \nu_2]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1) + 4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) + \phi(\nu_1+\nu_2-\zeta_2)\right] \\
&\quad - \frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}_{(\nu_1+\nu_2-\zeta_1)}^{AB}I_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}^{\varpi}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right. \\
&\quad \left. + {}_{(\nu_1+\nu_2-\zeta_2)}^{AB}I_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}^{\varpi}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right] \\
&\quad + \frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&= \frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\int_0^1 \frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \\
&\quad \times \left[\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right. \\
&\quad \left. + \phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right]d\kappa.
\end{aligned} \tag{10}$$

Proof. We note that

$$\begin{aligned}
I_1 &= \frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \\
&\quad \times \left[\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right]d\kappa,
\end{aligned} \tag{11}$$

using integration by parts; we obtain

$$\begin{aligned}
&= \frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \\
&\quad \cdot \left[\frac{\phi'(\nu_1+\nu_2-(((1+\kappa)/2)\zeta_2+((1-\kappa)/2)\zeta_1))}{(\zeta_1-\zeta_2)/2}\right]\Bigg|_0^1 \\
&\quad - \frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left(\frac{2(\varpi+1)}{3}-(\varpi+1)\kappa^{\varpi}\right) \\
&\quad \cdot \left[\frac{\phi'(\nu_1+\nu_2-(((1+\kappa)/2)\zeta_2+((1-\kappa)/2)\zeta_1))}{(\zeta_1-\zeta_2)/2}\right]d\kappa \\
&= \frac{-2(1-2\varpi)}{3(\zeta_1-\zeta_2)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \frac{2}{(\zeta_1-\zeta_2)B(\varpi)\Gamma(\varpi)}\times\int_0^1\left(\frac{2(\varpi+1)}{3}-(\varpi+1)\kappa^{\varpi}\right) \\
&\quad \cdot \left[\phi'\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right]d\kappa \\
&= \frac{2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_2) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \frac{4\varpi(\varpi+1)}{(\zeta_2-\zeta_1)^2B(\varpi)\Gamma(\varpi)}\int_0^1\kappa^{\varpi-1}\phi\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)d\kappa.
\end{aligned} \tag{12}$$

By the change of the variable $\varkappa = \nu_1 + \nu_2 - (((1+\kappa)/2)\zeta_2 + ((1-\kappa)/2)\zeta_1)$ for $\kappa \in [0,1]$, we have

$$
\begin{aligned}
&= \frac{2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_2) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \frac{8\varpi(\varpi+1)2^{\varpi-1}}{B(\varpi)\Gamma(\varpi)(\zeta_2-\zeta_1)^{\varpi+2}}\int_{\nu_1+\nu_2-\zeta_2}^{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi(\varkappa) \\
&\quad \cdot \left(\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)-\varkappa\right)^{\varpi-1}d\varkappa.
\end{aligned}
\tag{13}
$$

Add and subtract $(1-\varpi)/B(\varpi)(8(\varpi+1)2^{\varpi-1})/((\zeta_2-\zeta_1)^{\varpi+2})\phi(\nu_1+\nu_2-((\zeta_1+\zeta_2)/2))$ in Equation (13); we have

$$
\begin{aligned}
&= \frac{2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_2) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \left[\frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right. \\
&\quad + \frac{8\varpi(\varpi+1)2^{\varpi-1}}{B(\varpi)\Gamma(\varpi)(\zeta_2-\zeta_1)^{\varpi+2}}\int_{\nu_1+\nu_2-\zeta_2}^{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi(\varkappa) \\
&\quad \left.\cdot \left(\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)-\mathrm{u}\right)^{\varpi-1}d\varkappa\right] \\
&\quad + \frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&= \frac{2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_2) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) - \frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}} \\
&\quad \cdot \left({}_{(\nu_1+\nu_2-\zeta_2)}^{AB}I^{\varpi}_{(\nu_1+\nu_2-((\zeta_1+\zeta_2)/2))}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right) \\
&\quad + \frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right).
\end{aligned}
\tag{15}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \\
&\quad \times \left[\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right]d\kappa,
\end{aligned}
\tag{16}
$$

using integration by parts; we obtain

$$
\begin{aligned}
&= \frac{-2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_1) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \frac{4\varpi(\varpi+1)}{(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\int_0^1\kappa^{\varpi-1}\phi \\
&\quad \cdot \left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)d\kappa.
\end{aligned}
\tag{17}
$$

By the change of the variable $\varkappa = \nu_1 + \nu_2 - (((1+\kappa)/2)\zeta_1 + ((1-\kappa)/2)\zeta_2)$ for $\kappa \in [0,1]$, we have

$$
\begin{aligned}
&= \frac{-2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_1) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&-\frac{8\varpi(\varpi+1)2^{\varpi-1}}{B(\varpi)\Gamma(\varpi)(\zeta_2-\zeta_1)^{\varpi+2}}\int_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}^{\nu_1+\nu_2-\zeta_1}\phi(\varkappa) \\
&\cdot\left(\varkappa-\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right)^{\varpi-1}d\varkappa.
\end{aligned}
\tag{19}
$$

Add and subtract $(1-\varpi/B(\varpi))(8(\varpi+1)2^{\varpi-1}/(\zeta_2-\zeta_1)^{\varpi+2})\phi(\nu_1+\nu_2-\zeta_1+\zeta_2/2)$ in Equation (19); we have

$$
\begin{aligned}
&= \frac{-2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_1) \\
&\quad + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2 B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad - \left[\frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.
\end{aligned}
$$

$$+\frac{8\varpi(\varpi+1)2^{\varpi-1}}{B(\varpi)\Gamma(\varpi)(\zeta_2-\zeta_1)^{\varpi+2}}\int_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}^{\nu_1+\nu_2-\zeta_1}\phi(\varkappa) \cdot\left(\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)-\varkappa\right)^{\varpi-1}d\varkappa] + \frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right), \tag{20}$$

$$= \frac{-2(1-2\varpi)}{3(\zeta_2-\zeta_1)B(\varpi)\Gamma(\varpi)}\phi'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) + \frac{4(\varpi+1)}{3(\zeta_2-\zeta_1)^2B(\varpi)\Gamma(\varpi)}\phi(\nu_1+\nu_2-\zeta_1) + \frac{8(\varpi+1)}{3(\zeta_2-\zeta_1)^2B(\varpi)\Gamma(\varpi)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) - \frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}} \cdot\left({}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right) + \frac{(1-\varpi)}{B(\varpi)}\frac{8(\varpi+1)2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi+2}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right). \tag{21}$$

Add Equations (15) and (21), also multiplying both sides by $(\zeta_2-\zeta_1)^2/8(\varpi+1)$, then we get

$$\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right] - \frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) + {}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right] + \frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right). \tag{22}$$

This concludes Lemma's proof. □

Remark 7. If we choose $\varpi=1$, $\nu_1=\zeta_1$, and $\nu_2=\zeta_2$ in Lemma 6, then we have

$$\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa = \frac{(\zeta_2-\zeta_1)^2}{48}\int_0^1\left[4\kappa-3\kappa^2-1\right] \cdot\left[\phi''\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)+\phi''\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right]d\kappa, \tag{23}$$

which is a new equality in literature.

Remark 8. If we choose $\nu_1=\zeta_1$ and $\nu_2=\zeta_2$ in Lemma 6, then we have

$$\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}} \cdot\left[{}^{AB}_{\zeta_2}I^{\varpi}_{(\zeta_1+\zeta_2/2)}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+{}^{AB}_{\zeta_1}I^{\varpi}_{\zeta_1+\zeta_2/2}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right] + \frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right) = \frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\int_0^1\frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right) \times\left[\phi''\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)+\phi''\left(\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right]d\kappa, \tag{24}$$

which is a new equality via the Atangana-Baleanu in fractional calculus.

Theorem 9. *Let ϕ be defined as in Lemma 6, and if $|\phi''|$ is convex on $[\nu_1,\nu_2]$, then we have the following inequality:*

$$\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right] - \frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) + {}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right] + \frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right| \le \frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}Y_1(\varpi)\left[2|\phi''(\nu_1)|+2|\phi''(\nu_2)|-\left[|\phi''(\zeta_1)|+|\phi''(\zeta_2)|\right]\right], \tag{25}$$

where $Y_1(\varpi)$ is defined by

$$Y_1(\varpi)=\begin{cases}\dfrac{1}{B(\varpi)\Gamma(\varpi)}\dfrac{(1-\varpi)^2}{3(\varpi+2)}, & \text{if } 0<\varpi\le\dfrac{1}{2},\\[2ex] \dfrac{1}{B(\varpi)\Gamma(\varpi)}\left(2\left(\dfrac{(\kappa_{\varpi})^{\varpi+2}}{(\varpi+2)}-\dfrac{(1-2\varpi)\kappa_{\varpi}+(\varpi+1)(\kappa_{\varpi})^2}{3}\right)+\dfrac{(1-\varpi)^2}{3(\varpi+2)}\right), & \text{if } \varpi>\dfrac{1}{2}.\end{cases} \tag{26}$$

Proof. We will start the proof by looking at modulus in Lemma 6,

$$\begin{aligned}
&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\
&\quad-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.\\
&\quad\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]\\
&\quad\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\
&\le\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|\\
&\quad\times\left[\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right|\right.\\
&\quad\left.+\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right|\right]d\kappa.
\end{aligned}\tag{27}$$

By using the convexity of $|\phi''|$ with Jensen-Mercer Inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\
&\quad-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.\\
&\quad\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]\\
&\quad\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\
&\le\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|\\
&\quad\times\left[|\phi''(\nu_1)|+|\phi''(\nu_2)|-\left(\frac{1+\kappa}{2}|\phi''(\zeta_2)|+\frac{1-\kappa}{2}|\phi''(\zeta_1)\right)\right.\\
&\quad\left.+|\phi''(\nu_1)|+|\phi''(\nu_1)|-\left(\frac{1+\kappa}{2}|\phi''(\zeta_1)|+\frac{1-\kappa}{2}|\phi''(\zeta_2)\right)\right]d\kappa.
\end{aligned}\tag{28}$$

⇒ To evaluate the above integral, assume the mapping $\varsigma:[0,1]\longrightarrow\mathbb{R}$ where

$$\varsigma(\kappa)=\frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\text{ with }\varpi>0.\tag{29}$$

(1) If $0<\varpi\le(1/2)$, then we have

$$\int_0^1|\varsigma(\kappa)|d\kappa=\frac{1}{B(\varpi)\Gamma(\varpi)}\frac{1-\varpi^2}{3(\varpi+2)}.\tag{30}$$

(2) If $\varpi>(1/2)$, then there exists a real number κ_{ϖ} such that $0<\varpi<1$, and we have

$$\int_0^1|\varsigma(\kappa)|d\kappa=\frac{1}{B(\varpi)\Gamma(\varpi)}\left(2\left(\frac{(\kappa_{\varpi})^{\varpi+2}}{\varpi+2}-\frac{(1-2\varpi)\kappa_{\varpi}+(\varpi+1)(\kappa_{\varpi})^2}{3}\right)+\frac{1-\varpi^2}{3(\varpi+2)}\right).\tag{31}$$

Hence, we have

$$\begin{aligned}
&\Rightarrow\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\
&\quad-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.\\
&\quad\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]\\
&\quad\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\
&\le\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}Y_1(\varpi)\left[2|\phi''(\nu_1)|+2|\phi''(\nu_2)|-\left[|\phi''(\zeta_1)|+|\phi''(\zeta_2)|\right]\right].
\end{aligned}\tag{32}$$

This concludes theorem's proof. □

Remark 10. If we choose $\varpi=1$ and $\kappa_{\varpi}=1/3$, in Theorem 9, then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\
&\quad\left.-\frac{1}{\zeta_2-\zeta_1}\int_{\nu_1+\nu_2-\zeta_2}^{\nu_1+\nu_2-\zeta_1}\phi(\varkappa)d\varkappa\right|\\
&\quad\le\frac{(\zeta_2-\zeta_1)^2}{162}\left[2|\phi''(\nu_1)|+2|\phi''(\nu_2)|-\left[|\phi''(\zeta_1)|+|\phi''(\zeta_2)|\right]\right],
\end{aligned}\tag{33}$$

which is Mercer variant of an identity proved by Sarikaya et al. in [19].

Remark 11. If we choose $\varpi=1$, $\nu_1=\zeta_1$, and $\nu_2=\zeta_2$ in Theorem 9, then $\kappa_{\varpi}=1/3$, and we have the inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa\right|\\
&\quad\le\frac{(\zeta_2-\zeta_1)^2}{162}\left[|\phi''(\zeta_1)|+|\phi''(\zeta_2)|\right],
\end{aligned}\tag{34}$$

which is proved by Sarikaya et al. in [19].

Remark 12. If we choose $\nu_1=\zeta_1$ and $\nu_2=\zeta_2$ in Theorem 9, then we have the inequality

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]\right.$$
$$\left.-\frac{2^{\varpi-1}\Gamma(1+\varpi)}{(\zeta_2-\zeta_1)^{\varpi}}\left[J^{\varpi}_{\zeta_2-}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+J^{\varpi}_{\zeta_1+}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right]\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\Omega_1(\varpi)\left[\left|\phi''(\zeta_1)\right|+\left|\phi''(\zeta_2)\right|\right]\Big|, \tag{35}$$

which is proved by Hezenci et al. in [14].

Remark 13. If we choose $\varpi=1$ and $\kappa_{\varpi}=1/3$ in Remark 12 then we have the inequality

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{162}\left[\left|\phi''(\zeta_1)\right|+\left|\phi''(\zeta_2)\right|\right], \tag{36}$$

which is proved by Sarikaya et al. in [19].

Theorem 14. *Let ϕ be defined as in Lemma 6, then for $\mathfrak{q}>1$, there is a mapping $|\phi''|^{\mathfrak{q}}$ that is convex on $[\nu_1,\nu_2]$, then we have identity:*

$$\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.$$
$$-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.$$
$$\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]$$
$$\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}Y_1(\varpi,p)\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\left[2\left|\phi''(\nu_1)\right|^{\mathfrak{q}}+2\left|\phi''(\nu_2)\right|^{\mathfrak{q}}\right.\right.$$
$$\left.\left.-\left(\left|\phi''(\zeta_2)\right|^{\mathfrak{q}}+\left|\phi''(\zeta_1)\right|^{\mathfrak{q}}\right)\right]\right)^{1/\mathfrak{q}}, \tag{37}$$

where $\mathfrak{p},\mathfrak{q}>1$ are conjugate exponents and Y is defined by

$$Y_1(\varpi,\mathfrak{p})=\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right|^{p}d\kappa\right)^{1/\mathfrak{p}}. \tag{38}$$

Proof. Using Hölder's inequality in Lemma 6, we obtain

$$\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.$$
$$-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.$$
$$\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]$$
$$\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\times\left\{\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|^{\mathfrak{p}}d\kappa\right)^{1/\mathfrak{p}}\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\phi''\right.\right.$$
$$\left.\left.\cdot\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right|^{\mathfrak{q}}d\kappa\right)^{1/\mathfrak{q}}$$
$$+\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|^{\mathfrak{p}}d\kappa\right)^{1/\mathfrak{p}}$$
$$\left.\cdot\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right|^{\mathfrak{q}}d\kappa\right)^{1/\mathfrak{q}}\right\}. \tag{39}$$

By using the convexity of $|\phi''|^{\mathfrak{q}}$ with Jensen-Mercer Inequality, we obtain

$$\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.$$
$$-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.$$
$$\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]$$
$$\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|^{\mathfrak{p}}d\kappa\right)^{1/\mathfrak{p}}$$
$$\times\left\{\frac{1}{B(\varpi)\Gamma(\varpi)}\left(\int_0^1\left(\left|\phi''(\nu_1)\right|^{q}+\left|\phi''(\nu_2)\right|^{q}\right.\right.\right.$$
$$\left.\left.\left.-\left(\frac{1+\kappa}{2}\left|\phi''(\zeta_2)\right|^{q}+\frac{1-\kappa}{2}\left|\phi''(\zeta_1)\right|^{q}\right)\right)d\kappa\right)^{1/\mathfrak{q}}$$
$$+\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left(\left|\phi''(\nu_1)\right|^{q}+\left|\phi''(\nu_2)\right|^{q}\right.\right.$$
$$\left.\left.\left.-\left(\frac{1+\kappa}{2}\left|\phi''(\zeta_2)\right|^{q}+\frac{1-\kappa}{2}\left|\phi''(\zeta_1)\right|^{q}\right)\right)d\kappa\right)^{1/\mathfrak{q}}\right\}. \tag{40}$$

$$\Rightarrow\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.$$
$$-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.$$
$$\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]$$
$$\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|$$

$$\leq \frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|^{\mathfrak{p}}d\kappa\right)^{1/\mathfrak{p}} \times\left\{\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\left(|\phi''(\nu_1)|^{\mathfrak{q}}+|\phi''(\nu_2)|^{\mathfrak{q}}-\frac{3|\phi''(\zeta_2)|^{\mathfrak{q}}+|f''(\zeta_1)|^{\mathfrak{q}}}{4}\right)\right)^{1/\mathfrak{q}} +\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\left(|\phi''(\nu_1)|^{\mathfrak{q}}+|\phi''(\nu_2)|^{\mathfrak{q}}-\frac{|\phi''(\zeta_2)|^{\mathfrak{q}}+3|\phi''(\zeta_1)|^{\mathfrak{q}}}{4}\right)\right)^{1/\mathfrak{q}}\right\}. \tag{41}$$

This concludes theorem's proof. □

Corollary 15. *If we choose $\varpi=1$, $\nu_1=\zeta_1$, and $\nu_2=\zeta_2$ in Theorem 14, then $t_{\varpi}=1/3$, and we have the inequality*

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa\right| \leq \frac{(\zeta_2-\zeta_1)^2}{162}Y(1,\mathfrak{p})\left[|\phi''(\zeta_1)|^{\mathfrak{q}}+|\phi''(\zeta_2)|^{\mathfrak{q}}\right]^{1/\mathfrak{q}}, \tag{42}$$

which is given by Hezenci et al. in [14].

Theorem 16. *Let ϕ be defined as in Lemma 6, then for $\mathfrak{q}>1$, there is a mapping $|\phi''|^{\mathfrak{q}}$ that is convex on $[\nu_1,\nu_2]$, then we have identity:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\&-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]\\&\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}Y_1(\varpi)^{1-1/\mathfrak{q}}\left\{\left(Y_1(\varpi)\left[|\phi''(\nu_1)|^{\mathfrak{q}}+|\phi''(\nu_2)|^{\mathfrak{q}}\right]-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_2)|^{\mathfrak{q}}+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_1)|^{\mathfrak{q}}}{2}\right)^{1/\mathfrak{q}}\right.\\&\left.+\left(Y_1(\varpi)\left[|\phi''(\nu_1)|^{\mathfrak{q}}+|\phi''(\nu_2)|^{\mathfrak{q}}\right]-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_1)|^{\mathfrak{q}}+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_2)|^{\mathfrak{q}}}{2}\right)^{1/\mathfrak{q}}\right\},\end{aligned} \tag{43}$$

where $Y_1(\varpi)$ is defined in Theorem 9 and $Y_2(\varpi)$ is defined by

$$Y_2(\varpi)=\begin{cases}\dfrac{1}{B(\varpi)\Gamma(\varpi)}\dfrac{\left(3-\varpi-2\varpi^2\right)}{18(\varpi+3)}, & \text{if } 0<\varpi\leq\dfrac{1}{2},\\ \dfrac{1}{B(\varpi)\Gamma(\varpi)}\left(2\left(\dfrac{(\kappa_{\varpi})^{\varpi+3}}{(\varpi+3)}-\dfrac{3(1-2\varpi)(\kappa_{\varpi})^2+4(\varpi+1)(\kappa_{\varpi})^3}{18}\right)+\dfrac{(1-\varpi)^2}{3(\varpi+2)}\right), & \text{if } \varpi>\dfrac{1}{2}.\end{cases} \tag{44}$$

Proof. By applying the power-mean inequality in Lemma 6, we get

$$\begin{aligned}&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right.\\&-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^{\varpi}}\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right]\\&\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^{\varpi}}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right|\\&\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\times\left[\frac{1}{B(\varpi)\Gamma(\varpi)}\left(\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|d\kappa\right)^{1-1/\mathfrak{q}}\right.\\&\times\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|\times\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right|^{\mathfrak{q}}d\kappa\right)^{1/\mathfrak{q}}\\&+\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|d\kappa\right)^{1-1/\mathfrak{q}}\\&\times\left(\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\right|\left.\times\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right|^{\mathfrak{q}}d\kappa\right)^{1/\mathfrak{q}}\right].\end{aligned} \tag{45}$$

⇒ To evaluate the above integral, assume the mapping $\varsigma:[0,1]\longrightarrow\mathbb{R}$, where

$$\varsigma(\kappa)=\frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^{\varpi}\right)\kappa\right]\right)\ \text{with}\ \varpi>0. \tag{46}$$

(1) Let us consider $0<\varpi\leq(1/2)$, then we have

$$\int_0^1 |\varsigma(\kappa)|\kappa d\kappa = \frac{1}{B(\varpi)\Gamma(\varpi)}\left(\frac{3-\varpi-2\varpi^2}{18(\varpi+3)}\right). \tag{47}$$

(2) If $\varpi > (1/2)$, then there exists a real number κ_ϖ such that $0 < \varpi < 1$, and we have

$$\int_0^1 |\varsigma(\kappa)|\kappa d\kappa = \frac{1}{B(\varpi)\Gamma(\varpi)}\left(2\left(\frac{(\kappa_\varpi)^{\varpi+3}}{\varpi+3} - \frac{3(1-2\varpi)(\kappa_\varpi)^2 + 4(\varpi+1)(\kappa_\varpi)^3}{18}\right) + \frac{3+\varpi-2\varpi^2}{18(\varpi+3)}\right). \tag{48}$$

Since $|\phi''|^q$ is convex and taking into account Jensen-Mercer Inequality, we obtain

$$\begin{aligned}
&\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^\varpi\right)\kappa\right]\right)\right| \\
&\quad\cdot\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_2+\frac{1-\kappa}{2}\zeta_1\right)\right)\right|^q d\kappa \\
&\le \frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^\varpi\right)\kappa\right]\right)\right| \\
&\quad\times\left[|\phi''(\nu_1)|^q+|\phi''(\nu_2)|^q\right. \\
&\quad\left.-\left(\frac{1+\kappa}{2}|\phi''(\zeta_2)|^q+\frac{1-\kappa}{2}|\phi''(\zeta_1)|^q\right)\right]d\kappa \\
&= Y_1(\varpi)\left[|\phi''(\nu_1)|^q|+\phi''(\nu_2)|^q\right] \\
&\quad-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_2)|^q+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_1)|^q}{2}.
\end{aligned} \tag{49}$$

And similarly,

$$\begin{aligned}
&\frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^\varpi\right)\kappa\right]\right)\right| \\
&\quad\cdot\left|\phi''\left(\nu_1+\nu_2-\left(\frac{1+\kappa}{2}\zeta_1+\frac{1-\kappa}{2}\zeta_2\right)\right)\right|^q d\kappa \\
&\le \frac{1}{B(\varpi)\Gamma(\varpi)}\int_0^1\left|\left(\frac{1}{3}\left[1-2\varpi(1-\kappa)+\left(2-3\kappa^\varpi\right)\kappa\right]\right)\right| \\
&\quad\times\left[|\phi''(\nu_1)|^q+|\phi''(\nu_2)|^q\right. \\
&\quad\left.-\left(\frac{1+\kappa}{2}|\phi''(\zeta_1)|^q+\frac{1-\kappa}{2}|\phi''(\zeta_2)|^q\right)\right]d\kappa \\
&= Y_1(\varpi)\left[|\phi''(\nu_1)|^q+|\phi''(\nu_2)|^q\right] \\
&\quad-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_1)|^q+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_2)|^q}{2}.
\end{aligned} \tag{50}$$

Finally, we obtain

$$\begin{aligned}
&\Rightarrow\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right.\right. \\
&\quad\left.+\phi(\nu_1+\nu_2-\zeta_2)\right]-\frac{2^{\varpi-1}}{(\zeta_2-\zeta_1)^\varpi} \\
&\quad\cdot\left[{}^{AB}_{(\nu_1+\nu_2-\zeta_1)}I^{\varpi}_{(\nu_1+\nu_2-\zeta_1+\zeta_2/2)}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right. \\
&\quad\left.+{}^{AB}_{(\nu_1+\nu_2-\zeta_2)}I^{\varpi}_{\nu_1+\nu_2-\zeta_1+\zeta_2/2}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right] \\
&\quad\left.+\frac{2(1-\varpi)2^{\varpi-1}}{B(\varpi)(\zeta_2-\zeta_1)^\varpi}\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)\right| \\
&\le\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}Y_1(\varpi)^{1-1/q}\left\{\left(Y_1(\varpi)\left[|\phi''(\nu_1)|^q+|\phi''(\nu_2)|^q\right]\right.\right. \\
&\quad\left.-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_2)|^q+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_1)|^q}{2}\right)^{1/q} \\
&\quad+\left(Y_1(\varpi)\left[|\phi''(\nu_1)|^q+|\phi''(\nu_2)|^q\right]\right. \\
&\quad\left.\left.-\frac{(Y_1(\varpi)+Y_2(\varpi))|\phi''(\zeta_1)|^q+(Y_1(\varpi)-Y_2(\varpi))|\phi''(\zeta_2)|^q}{2}\right)^{1/q}\right\}.
\end{aligned} \tag{51}$$

This completes the proof. □

Remark 17. If we choose $\varpi = 1$ and $\kappa_\varpi = (1/3)$, in Theorem 16, then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[\phi(\nu_1+\nu_2-\zeta_1)+4\phi\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\nu_1+\nu_2-\zeta_2)\right]\right. \\
&\quad\left.-\frac{1}{\zeta_2-\zeta_1}\int_{\nu_1+\nu_2-\zeta_2}^{\nu_1+\nu_2-\zeta_1}\phi(\varkappa)d\varkappa\right| \\
&\le(\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/q}\left[\left(\frac{1}{162}|\phi''(\nu_1)|^q+\frac{1}{162}|\phi''(\nu_2)|^q\right.\right. \\
&\quad\left.-\left(\frac{59}{3^5 2^7}|\phi''(\zeta_1)|^q+\frac{133}{3^5 2^7}|\phi''(\zeta_2)|^q\right)\right)^{1/q} \\
&\quad+\left(\frac{1}{162}|\phi''(\nu_1)|^q+\frac{1}{162}|\phi''(\nu_2)|^q\right. \\
&\quad\left.\left.-\left(\frac{59}{3^5 2^7}|\phi''(\zeta_2)|^q+\frac{133}{3^5 2^7}|\phi''(\zeta_1)|^q\right)\right)^{1/q}\right],
\end{aligned} \tag{52}$$

which is Mercer variant of an identity proved by Sarikaya et al. in [19].

Remark 18. If we take $\varpi = 1$, $\nu_1 = \zeta_1$, and $\nu_2 = \zeta_2$ in Theorem 16, then Theorem 16 reduce to

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa\right|$$
$$\leq(\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/\mathrm{q}}\left[\left(\frac{59}{3^5 2^7}\left|\phi''(\zeta_1)\right|^{\mathrm{q}}+\frac{133}{3^5 2^7}\left|\phi''(\zeta_2)\right|^{\mathrm{q}}\right)^{1/\mathrm{q}}+\left(\frac{59}{3^5 2^7}\left|\phi''(\zeta_2)\right|^{\mathrm{q}}+\frac{133}{3^5 2^7}\left|\phi''(\zeta_1)\right|^{\mathrm{q}}\right)^{1/\mathrm{q}}\right], \tag{53}$$

which is proved by Sarikaya et al. in [19].

Remark 19. If we take $\nu_1=\zeta_1$ and $\nu_2=\zeta_2$ in Theorem 16, then Theorem 16 reduce to

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{2^{\varpi-1}\Gamma(1+\varpi)}{(\zeta_2-\zeta_1)^{\varpi}}\cdot\left[J_{\zeta_2-}^{\varpi}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+J_{\zeta_1+}^{\varpi}\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right]\right|$$
$$\leq\frac{(\zeta_2-\zeta_1)^2}{8(\varpi+1)}\Omega_1(\varpi)^{1-1/q}\cdot\left\{\left(\frac{(\Omega_1(\varpi)+\Omega_2(\varpi))\left|\phi''(\zeta_2)\right|^{\mathrm{q}}+(\Omega_1(\varpi)-\Omega_2(\varpi))\left|\phi''(\zeta_1)\right|^{\mathrm{q}}}{2}\right)^{1/q}+\left(\frac{(\Omega_1(\varpi)+\Omega_2(\varpi))\left|\phi''(\zeta_1)\right|^{\mathrm{q}}+(\Omega_1(\varpi)-\Omega_2(\varpi))\left|\phi''(\zeta_2)\right|^{\mathrm{q}}}{2}\right)^{1/q}\right\}, \tag{54}$$

which is proved by Hezenci et al. in [14].

Remark 20. If we take $\varpi=1$ in Remark 19, then we have

$$\left|\frac{1}{6}\left[\phi(\zeta_1)+4\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\phi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varkappa)d\varkappa\right|$$
$$\leq(\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/\mathrm{q}}\left[\left(\frac{59}{3^5 2^7}\left|\phi''(\zeta_1)\right|^{\mathrm{q}}+\frac{133}{3^5 2^7}\left|\phi''(\zeta_2)\right|^{\mathrm{q}}\right)^{1/\mathrm{q}}+\left(\frac{59}{3^5 2^7}\left|\phi''(\zeta_2)\right|^{\mathrm{q}}+\frac{133}{3^5 2^7}\left|\phi''(\zeta_1)\right|^{\mathrm{q}}\right)^{1/\mathrm{q}}\right], \tag{55}$$

which is proved by Sarikaya et al. in [19].

3. Applications

3.1. Q-Digamma Function. The $\varphi_{\mathbf{q}}$-digamma function, which is described as the logarithmic derivative of the q-gamma function, is an essential function related to the $\mathbf{q}$-gamma function. A few papers had also additionally been utilized that explore the monotonicity and complete monotonicity characteristics for functions linked with the $\mathbf{q}$-gamma and $\mathbf{q}$-digamma functions, which tends to result in remarkable inequalities. One can see in [31, 32].

Assume the $\mathbf{q}$-analogue of the digamma function φ for $0<\mathbf{q}<1$ is the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ and is (see [33, 34]) given as:

$$\varphi_{\mathbf{q}}=-\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{q^{k+\varkappa}}{1-q^{k+\varkappa}}=-\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{q^{k\varkappa}}{1-q^{k\varkappa}}. \tag{56}$$

For $\mathbf{q}\geq 1$ and $\varkappa>0$,$\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ can be given as:

$$\begin{aligned}\varphi_{\mathbf{q}}&=-\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\varkappa-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\varkappa)}}{1-\mathbf{q}^{-(k+\varkappa)}}\right]\\&=-\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\varkappa-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\varkappa}}{1-\mathbf{q}^{-k\varkappa}}\right].\end{aligned} \tag{57}$$

Proposition 21. *Assume that $\zeta_1,\zeta_2,\nu_1,\nu_2\in\mathbb{R}$ such that $0<\nu_1<\nu_2$,$q\geq 1$,$0<q<1$, and $q^{-1}=1-p^{-1}$. Then, the following inequality is valid:*

$$\left|\frac{1}{6}\left\{\varphi_q'(\nu_1+\nu_2-\zeta_1)+4\varphi_q'\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right)+\varphi_q'(\nu_1+\nu_2-\zeta_2)\right\}-\frac{\varphi_q(\nu_1+\nu_2-\zeta_1)-\varphi_q(\nu_1+\nu_2-\zeta_2)}{\zeta_2-\zeta_1}\right|$$
$$\begin{aligned}\leq(\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/q}&\left[\left\{\frac{1}{162}\left|\varphi_q^{(3)}(\nu_1)\right|^q+\frac{1}{162}\left|\varphi_q^{(3)}(\nu_2)\right|^q-\left(\frac{59}{3^5 2^7}\left|\varphi_q^{(3)}(\zeta_1)\right|^q+\frac{133}{3^5 2^7}\left|\varphi_q^{(3)}(\zeta_2)\right|^q\right)\right\}^{1/q}\right.\\&\left.+\left\{\frac{1}{162}\left|\varphi_q^{(3)}(\nu_1)\right|^q+\frac{1}{162}\left|\varphi_q^{(3)}(\nu_2)\right|^q-\left(\frac{59}{3^5 2^7}\left|\varphi_q^{(3)}(\zeta_2)\right|^q+\frac{133}{3^5 2^7}\left|\varphi_q^{(3)}(\zeta_1)\right|^q\right)\right\}^{1/q}\right].\end{aligned} \tag{58}$$

Proof. The assertion can be obtained immediately by using Remark 17 with the $\phi(\varkappa)\longrightarrow\varphi_q'(\varkappa)$ for all $\mathbf{q}>0$, and consequently, $\phi''(\varkappa):=\varphi_{\mathbf{q}}^{(3)}(\varkappa)$ is convex on the same interval $(0,\infty)$. □

Proposition 22. *Assume that ζ_1,ζ_2 are the real numbers such that $0<\zeta_1<\zeta_2$,$q\geq 1$,$0<\mathbf{q}<1$, and $q^{-1}=1-p^{-1}$. Then, the following inequality is valid:*

$$\left|\frac{1}{6}\left\{\varphi_q'(\zeta_1)+4\varphi_q'\left(\frac{\zeta_1+\zeta_2}{2}\right)+\varphi_q'(\zeta_2)\right\}-\frac{\varphi_q(\zeta_2)-\varphi_q(\zeta_1)}{\zeta_2-\zeta_1}\right|$$
$$\leq(\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/q}\left[\left\{\frac{59}{3^5 2^7}\left|\varphi_{\mathbf{q}}^{(3)}(\zeta_1)\right|^q+\frac{133}{3^5 2^7}\left|\varphi_{\mathbf{q}}^{(3)}(\zeta_2)\right|^q\right\}^{1/q}+\left\{\frac{59}{3^5 2^7}\left|\varphi_{\mathbf{q}}^{(3)}(\zeta_2)\right|^q+\frac{133}{3^5 2^7}\left|\varphi_{\mathbf{q}}^{(3)}(\zeta_1)\right|^q\right\}^{1/q}\right]. \tag{59}$$

Proof. The assertion can be obtained immediately by using Remark 18 with the $\phi(\varkappa)\longrightarrow\varphi_{\mathbf{q}}'(\varkappa)$ for all $\mathbf{q}>0$, and consequently, $\phi''(\varkappa):=\varphi_{\mathbf{q}}^{(3)}(\varkappa)$ is convex on the same interval $(0,\infty)$. □

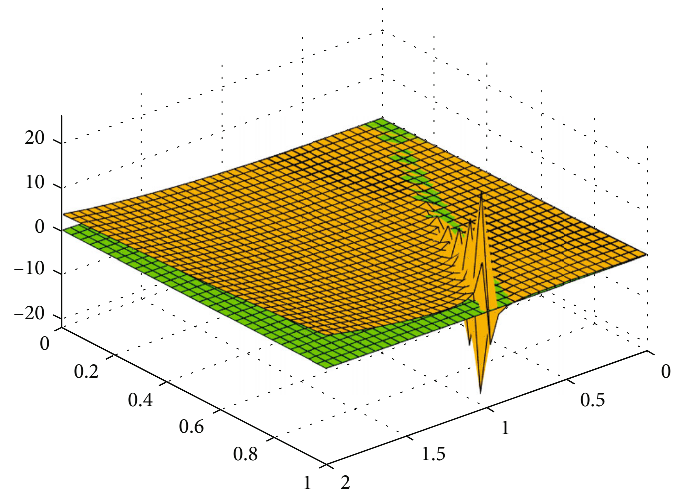

Figure 1: Three-dimensional illustration of the error and error bounds for (43).

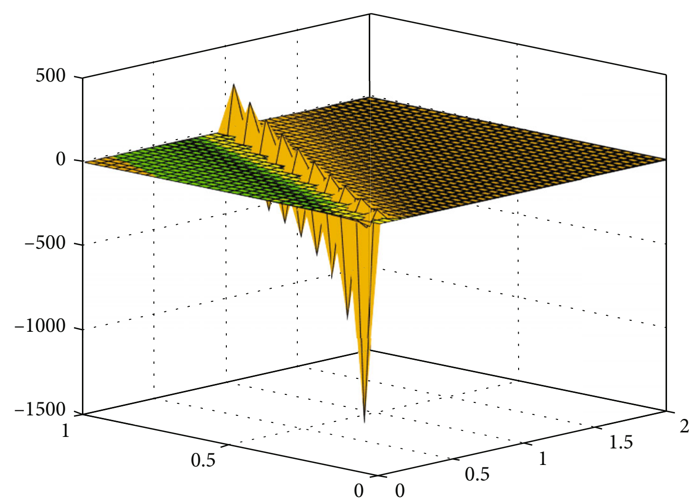

Figure 2: Three-dimensional illustration of the error and error bounds for (53).

3.2. Modified Bessel Function. Bessel functions were named after Friedrich Wilhelm Bessel (1784-1846); however, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessels functions in 1732. Numerous results about Bessel functions have been established by utilizing its generating function (see [35]).

We know the first type of modified Bessel function $\mathfrak{B}\tau_1$, which has the series interpretation (see [33], p.77).

$$\mathfrak{B}_{\tau_1}(\varkappa) = \Sigma_{\mathfrak{n}\geq 0} \frac{(\varkappa/2)^{\tau_1+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\tau_1+\mathfrak{n}+1)}. \tag{60}$$

where $\varkappa \in \mathfrak{R}$ and $\tau_1 > -1$, while the second type modified Bessel function ϕ_{τ_1} (see [33], p.78) is usually defined as

$$\phi_{\tau_1}(\varkappa) = \frac{\pi}{2}\frac{\mathfrak{B}_{-\tau_1}(\varkappa) - \mathfrak{B}_{\tau_1}(\varkappa)}{\sin \tau_1 \pi}. \tag{61}$$

Consider the function $\Psi_{\tau_1}(\varkappa)$: $\mathfrak{R} \longrightarrow [1,\infty)$ defined by

$$\Psi_{\tau_1}(\varkappa) = 2^{\tau_1}\Gamma(\tau_1+1)\varkappa^{-\tau_1}\phi_{\tau_1}(\varkappa), \tag{62}$$

Here, first, second- and third-order derivative formula of $\Psi_{\tau_1}(\varkappa)$ is given as in [33]:

$$\Psi_{\tau_1}'(\varkappa) = \frac{\varkappa}{2(\tau_1+1)}\Psi_{\tau_1+1}(\varkappa), \tag{63}$$

$$\Psi_{\tau_1}''(\varkappa) = \frac{\varkappa^2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\varkappa) + \frac{1}{2(\tau_1+1)}\Psi_{\tau_1+1}(\varkappa), \tag{64}$$

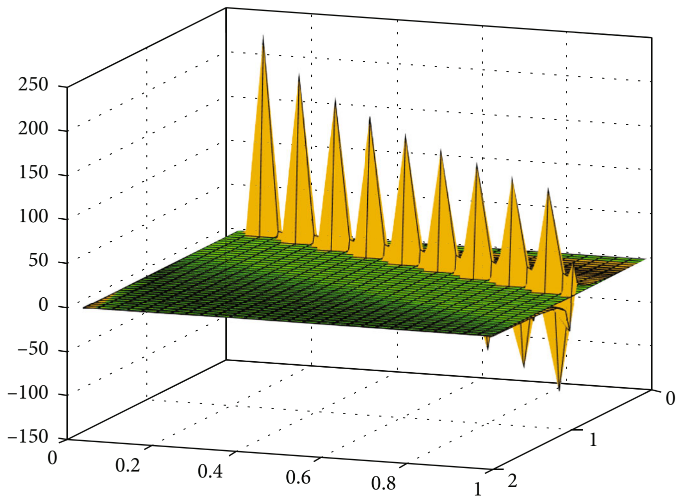

Figure 3: Three-dimensional illustration of the error and error bounds for (55).

$$\Psi_{\tau_1}'''(\varkappa) = \frac{\varkappa^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\varkappa) + \frac{3\varkappa}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\varkappa). \tag{65}$$

Proposition 23. *Suppose that $\zeta_1, \zeta_2, \nu_1, \nu_2 \in \mathbb{R}$ such that $0 < \nu_1 < \nu_2$, and $\tau_1 > -1$. Then, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Bigg\{\frac{(\nu_1+\nu_2-\zeta_1)}{2(\tau_1+1)}\Psi_{\tau_1+1}(\nu_1+\nu_2-\zeta_1) \\
&\quad + 4\frac{(\nu_1+\nu_2-((\zeta_1+\zeta_2)/2))}{2(\tau_1+1)}\Psi_{\tau_1+1}\left(\nu_1+\nu_2-\frac{\zeta_1+\zeta_2}{2}\right) \\
&\quad + \frac{\nu_1+\nu_2-\zeta_2}{2(\tau_1+1)}\Psi_{\tau_1+1}(\nu_1+\nu_2-\zeta_2)\Bigg\} - \frac{\Psi_m(\nu_1+\nu_2-\zeta_1)-\Psi_m(\nu_1+\nu_2-\zeta_2)}{\zeta_2-\zeta_1}\Bigg| \\
&\leq (\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/q} \times \Bigg[\Bigg\{\frac{1}{162}\left(\frac{\nu_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\nu_1)\right. \\
&\quad \left.+ \frac{3\nu_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\nu_1)\right)^q + \frac{1}{162}\left(\frac{\nu_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\nu_2)\right. \\
&\quad \left.+ \frac{3\nu_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\nu_2)\right)^q - \Bigg\{\frac{59}{3^5 2^7}\left(\frac{\zeta_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_1)\right. \\
&\quad \left.+ \frac{3\zeta_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_1)\right)^q + \frac{133}{3^5 2^7}\left(\frac{\zeta_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_2)\right. \\
&\quad \left.+ \frac{3\zeta_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_2)\right)^q\Bigg\}\Bigg\}^{1/q} + \Bigg\{\frac{1}{162}\left(\frac{\nu_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\nu_1)\right. \\
&\quad \left.+ \frac{3\nu_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\nu_1)\right)^q + \frac{1}{162}\left(\frac{\nu_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\nu_2)\right. \\
&\quad \left.+ \frac{3\nu_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\nu_2)\right)^q - \Bigg\{\frac{59}{3^5 2^7}\left(\frac{\zeta_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_2)\right. \\
&\quad \left.+ \frac{3\zeta_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_2)\right)^q + \frac{133}{3^5 2^7}\left(\frac{\zeta_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_1)\right. \\
&\quad \left.+ \frac{3\zeta_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_1)\right)^q\Bigg\}\Bigg\}^{1/q}\Bigg].
\end{aligned} \tag{66}$$

Proof. The required result follows immediately from Remark 17 utilizing $\phi(\varkappa) = \Psi_{\tau_1}'(\varkappa)$, $\varkappa > 0$, and the identities (66) and (63). □

Proposition 24. *Suppose that $\tau_1 > -1$ and $0 < \zeta_1 < \zeta_2$. Then, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Bigg\{\frac{\zeta_1}{2(\tau_1+1)}\Psi_{\tau_1+1}(\zeta_1) + 4\frac{\zeta_1+\zeta_2}{4(\tau_1+1)}\Psi_{\tau_1+1}\left(\frac{\zeta_1+\zeta_2}{2}\right) + \frac{\zeta_2}{2(\tau_1+1)}\Psi_{\tau_1+1}(\zeta_2)\Bigg\} \\
&\quad - \frac{\Psi_m(\zeta_2)-\Psi_m(\zeta_1)}{\zeta_2-\zeta_1}\Bigg| \\
&\leq (\zeta_2-\zeta_1)^2\left(\frac{1}{162}\right)^{1-1/q} \times \Bigg[\Bigg\{\frac{59}{3^5 2^7}\left(\frac{\zeta_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_1)\right. \\
&\quad \left.+ \frac{3\zeta_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_1)\right)^q + \frac{133}{3^5 2^7}\left(\frac{\zeta_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_2)\right. \\
&\quad \left.+ \frac{3\zeta_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_2)\right)^q\Bigg\}^{1/q} \\
&\quad + \Bigg\{\frac{59}{3^5 2^7}\left(\frac{\zeta_2^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_2) + \frac{3\zeta_2}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_2)\right)^q \\
&\quad + \frac{133}{3^5 2^7}\left(\frac{\zeta_1^3}{4(\tau_1+1)(\tau_1+2)(\tau_1+3)}\Psi_{\tau_1+3}(\zeta_1) + \frac{3\zeta_1}{4(\tau_1+1)(\tau_1+2)}\Psi_{\tau_1+2}(\zeta_1)\right)^q\Bigg\}^{1/q}\Bigg].
\end{aligned} \tag{67}$$

Proof. The required result follows immediately from Remark 18 utilizing $\phi(\varkappa) = \Psi_{\tau_1}'(\varkappa), \varkappa > 0$, and the identities (66) and (63). □

4. Conclusion

The study deal with the investigation of Simpson-Mercer type inequalities for twice differentiable functions. Adopting the novel approach, we extended the study of Simpson-Mercer type integral inequalities using Hölder's and power-mean integral inequalities via Atangana-Baleanu (AB) fractional integral operators. It is interesting to extend such findings to the Atangana-Baleanu (AB) operator and also to other convexities. Various representations were used to explain the findings to clarify the important aspects of the fractional inequalities under consideration, such as Figures 1–3, which illustrate the error bounds for the dominant findings. We believe that our newly introduced idea and concept will have very deep research in the captivating field of numerical analysis and inequalities. The strategy's effective and productive execution is investigated and confirmed in order to show that it may be applied to coordinated convex functions that emerge in fractional calculus, especially to fractal and fractional integral operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

S.I. Butt provided the main ideas of the article and is a major contributor in starting the initial draft and conceptualization. S. Rashid dealt with the methodology and investigation. I. Javed contributed to editing of the original draft and handle latex work. K. A. Khan performed the validation, formal analysis, and writing revised version. R. M. Mabela performed review and editing along with the submission of manuscript. All authors read and approved the final manuscript.

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