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Research Article

Boundedness of Multilinear Calderón-Zygmund Operators on Grand Variable Herz Spaces

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In this paper, we prove the boundedness of multilinear Calderón-Zygmund operators on product of grand variable Herz spaces. These results generalize the boundedness of multilinear Calderón-Zygmund operators on product of variable exponent Lebesgue spaces and variable Herz spaces.

1. Introduction

There has been increased interest in the study of multilinear singular integral operators in recent years. The class of multilinear singular integrals with standard Calderón-Zygmund kernels provides the foundation and starting point of research of the theory. Such a class of multilinear Calderón-Zygmund operators was introduced and first studied by Coifman and Meyer [1–3] and later by Grafakos and Torres [4]. For the boundedness and other properties of multilinear fractional integrals, we refer to, e.g., [5–8].

Variable Lebesgue spaces were introduced in [9], but stayed under the radar for a considerable amount of time. Apart from some previous sporadic episodes, the research boom on such spaces can be traced back to the foundational paper [10]. Since then, these spaces have attracted much attention of mathematicians, not only because of their connection with harmonic analysis but also due to their usefulness in application to a wide range of problems, see, e.g., [11]. The standard references to the general theory of variable Lebesgue spaces are [12, 13].

The classical definition of Herz spaces was introduced in [14]. Many studies can be found related to these spaces and its variations, which include variable Herz spaces, continual Herz spaces, and Herz spaces with variable smoothness and integrability. For details, see [15–21] and references therein.

Grand Lebesgue spaces on bounded sets, which proved to be useful in application to partial differential equations, were introduced in [22, 23]. In the last years, various operators of harmonic analysis have been intensively studied on grand spaces, see for instance [20, 24–32]. Grand Lebesgue sequence spaces were introduced recently in [33], where several operators of harmonic analysis were studied, e.g., maximal, convolutions, Hardy, Hilbert, and fractional operators.

In this paper, we prove the boundedness of multilinear Calderón-Zygmund operators on grand variable Herz spaces which were introduced in [34]. The present paper is organized in the following way. Apart from the introduction, in Section 2, we recall some definitions and results related to variable exponent spaces. Section 3 contains some details about multilinear Calderón-Zygmund kernels and the proof of the main result.

Notations.

(i) $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

(ii) $\mathbb{Z}$ is the set of integers

(iii) $\mathbb{Z}_-$ is the set of negative integers

(iv) $n,\bar{m} := \{n, n+1, n+2, \cdots, m-1, m\}$ for $n, m \in \mathbb{Z}$ and $n < m$

(v) $B(x, r)$ is the ball of radius r center at the point x

(vi) $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$ for all $k \in \mathbb{Z}$

(vii) $R_{t,\tau} := B_\tau \setminus B_t = \{x : t<|x|<\tau\}$ is a spherical layer

(viii) $R_k := B_k \setminus B_{k-1}$

(ix) $\chi_k := \chi_{R_k}, \chi_{t,\tau} := \chi_{R_{t,\tau}}$

(x) $f \lesssim g$ means that $f \le Cg$ and $f \simeq g$ means that $f \lesssim g \lesssim f$

(xi) constants (often different constant in the same chain of inequalities) will mainly be denoted by c or C

2. Function Spaces with Variable Exponent

In this section, we recall definitions and results related to variable exponent Lebesgue spaces, variable Herz spaces, and grand variable Herz spaces.

2.1. Lebesgue Space with Variable Exponent. For the current section, we refer to [10–13, 35] unless and until stated otherwise. Let q be a real-valued measurable function on $\mathbb{R}^n$ with values in $[1, \infty)$. For $X \subset \mathbb{R}^n$, we suppose that

$$1 \le q^-(X) \le q(x) \le q^+(X) < \infty, \tag{1}$$

where $q^-(X) := \operatorname{ess\,inf}_{x\in X} q(x)$ and $q^+(X) := \operatorname{ess\,sup}_{x\in X} q(x)$. By $L^{q(\cdot)}(\mathbb{R}^n)$, we denote the space of measurable function f on $\mathbb{R}^n$ such that

$$I_{q(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{q(x)} \mathrm{d}x < \infty. \tag{2}$$

It is a Banach space, see [13, 35], endowed with norm:

$$\|f\|_{q(\cdot)} = \inf\left\{\eta > 0 : I_{q(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\}. \tag{3}$$

By q', we denote the conjugate exponent of q, defined by $q'(x) = q(x)/(q(x)-1)$. In the sequel, we use

log condition:

$$|q(x) - q(y)| \le \frac{A}{-\ln|x-y|}, |x-y| \le \frac{1}{2}, x, y \in \mathbb{R}^n, \tag{4}$$

where $A = A(q) > 0$ does not depend on x, y;

decay condition at 0:

$$|q(x) - q(0)| \le \frac{A}{\ln|x|}, |x| \le \frac{1}{2}, \tag{5}$$

holds for some $q(0) \in (1,\infty)$; and

decay condition at ∞: there exists a number $q(\infty) \in (1,\infty)$, such that

$$|q(x) - q(\infty)| \le \frac{A}{\ln(\mathrm{e} + |x|)}, \tag{6}$$

where $A = A(q)$ does not depend on x.

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)| \mathrm{d}y \ (x \in \mathbb{R}^n). \tag{7}$$

We adopt the following notations:

(i) $L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{f : f \in L^{q(\cdot)}(K)$ for all compact subsets $K \subset \mathbb{R}^n\}$

(ii) $\mathscr{P}(\mathbb{R}^n)$ consists of all measurable functions q satisfying $q^- > 1$ and $q^+ < \infty$

(iii) $\mathscr{P}^{\log}_0(\mathbb{R}^n)$ and $\mathscr{P}^{\log}_\infty(\mathbb{R}^n)$ denote the classes of $q \in \mathscr{P}(\mathbb{R}^n)$ which satisfy (5) and (6), respectively

(iv) $\mathscr{B}(\mathbb{R}^n)$ is the set of all $q \in \mathscr{P}(\mathbb{R}^n)$ for which M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$

For the following lemma, we refer to, e.g., [12].

Lemma 1 (Generalized Hölder's Inequality). *Given $p, q \in \mathscr{P}(\mathbb{R}^n)$, define $r \in \mathscr{P}(\mathbb{R}^n)$ by $1/r(x) = 1/p(x) + 1/q(x)$. Then, there exists a constant c such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$, $fg \in L^{r(\cdot)}(\mathbb{R}^n)$ and*

$$\|fg\|_{r(\cdot)} \le c\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}. \tag{8}$$

The following lemma appears in [21].

Lemma 2. *Let $D > 1$ and $q \in \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$. Then*

$$\frac{1}{c_0} r^{n/q(0)} \le \left\|\chi_{R_{r,Dr}}\right\|_{q(\cdot)} \le c_0 r^{n/q(0)}, \text{for } 0 < r \le 1, \tag{9}$$

$$\frac{1}{c_\infty} r^{n/q\infty} \le \left\|\chi_{R_{r,Dr}}\right\|_{q(\cdot)} \le c_\infty r^{n/q\infty}, \text{for } r \le 1, \tag{10}$$

where $c_0 \ge 1$ and $c_\infty \ge 1$ depend on D, but do not depend on r.

2.2. Herz Spaces with Variable Exponent. The classical Herz spaces were first introduced in [14]. We recall the definition of variable exponent Herz spaces.

Definition 3. Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and $q \in \mathscr{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}} < \infty\right\}, \tag{11}$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}} \left\|2^{k\alpha} f\chi_k\right\|^p_{q(\cdot)}\right)^{1/p}. \tag{12}$$

The nonhomogeneous Herz space $K^{\alpha,p}_{q(\cdot)}$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{K^{\alpha,p}_{q(\cdot)}} < \infty\right\}, \tag{13}$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{N}_0} \left\|2^{k\alpha} f\chi_k\right\|^p_{q(\cdot)}\right)^{1/p} + \left\|f\chi_{B(0,1)}\right\|_{q(\cdot)}. \tag{14}$$

For the boundedness of integral operators on Herz type spaces, we refer to, e.g., [18, 19, 36].

2.3. Grand Lebesgue Sequence Space. In this section, we recall the definition of grand Lebesgue sequence space. For the following definition and statements, see [33]. In what follows, $\mathbb{X}$ stands for one of the sets $\mathbb{Z}^n, \mathbb{Z}, \mathbb{N}$, and $\mathbb{N}_0$.

Definition 4. Let $1 \le p < \infty$ and $\theta > 0$. The grand Lebesgue sequence space $l^{p),\theta}$ is defined by the norm

$$\begin{aligned}\|x\|_{l^{p),\theta}(\mathbb{X})} &:= \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k\in\mathbb{X}} |x_k|^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &= \sup_{\varepsilon>0} \varepsilon^{\theta/p(1+\varepsilon)} \|x\|_{l^{p(1+\varepsilon)}(\mathbb{X})}.\end{aligned} \tag{15}$$

2.4. Grand Variable Herz Space. Following [34], we now introduce the grand variable Herz spaces.

Definition 5. Let $\alpha \in L^\infty(\mathbb{R})$, $1 \le p < \infty$, $q : \mathbb{R}^n \longrightarrow [1,\infty)$, and $\theta > 0$. The homogeneous grand variable Herz space is defined by

$$\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})\colon \|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)} < \infty\right\}, \tag{16}$$

where

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|^{p(1+\varepsilon)}_{q(\cdot)}\right)^{1/[p(1+\varepsilon)]} \\ &= \sup_{\varepsilon>0} \varepsilon^{\theta/p(1+\varepsilon)} \|f\|_{\dot{K}^{\alpha(\cdot),p(1+\varepsilon)}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{17}$$

The following lemma, see [15], is helpful to estimate the norm of characteristics functions.

Lemma 6. *Let $q \in \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$ and let $R = B(0,r) \setminus B(0,(r/2))$. If $|R| \le 2^{-n}$, then*

$$\|\chi_R\|_{q(\cdot)} \simeq |R|^{1/q(x)} \simeq |R|^{1/q_\infty}, \tag{18}$$

with the implicit constants independent of r and $x \in R$.

The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, that $q \in \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$.

3. Boundedness of Multilinear Calderón-Zygmund Operators

Consider the multilinear operator T of the form

$$T(f_1,\cdots,f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m)\mathrm{d}y_1\cdots\mathrm{d}y_m, \tag{19}$$

where $x \notin \bigcap_{j=1}^m \mathrm{spt}(f_j)$ and $f_1,\cdots,f_m \in L^\infty_c(\mathbb{R}^n)$, the space of compactly supported functions. Let $K(x,y_1,\cdots,y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m \in (\mathbb{R}^n)^{m+1}$, which satisfies the size estimate

$$|K(x,y_1,\cdots,y_m)| \le \frac{c}{(|x-y_1| + |x-y_2| + \cdots + |x-y_m|)^{mn}}, \tag{20}$$

for some $c > 0$ and all $(x,y_1,\cdots,y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_j$ for some j. For smoothness, assume that for some $\varepsilon > 0$,

$$\left|K(x,y_1,\cdots,y_m) - K\left(x',y_1,\cdots,y_m\right)\right| \le \frac{c|x-x'|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}}, \tag{21}$$

provided that $|x-x'| \le 1/2 \max\{|x-y_1|,\cdots,|x-y_m|\}$ and

$$\begin{aligned}&\left|K\left(x,y_1,\cdots,y_j,\cdots,y_m\right) - K\left(x,y_1,\cdots,y_j',\cdots,y_m\right)\right| \\ &\quad\le \frac{c\left|y_j - y_j'\right|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}},\end{aligned} \tag{22}$$

whenever $|y_j - y_j'| \le 1/2 \max\{|x-y_1|,\cdots,|x-y_m|\}$ for all $j \in \overline{1,m}$.

Such kernels are called Calderón-Zygmund kernels and the class of all functions satisfying (20), (21), and (22) with parameters m, c, and ε will be denoted by m-CZK(c,ε), compare [4]. We say that T be as in (19) is an m-linear Calderón-Zygmund operator, if

(i) The related kernel belongs to m-CZK(c,ε) class

(ii) T is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_1, q_2, \cdots, q_m < \infty$, and $1/q = 1/q_1 + \cdots + 1/q_m$

Grafakos and Torres [4] proved the boundedness of T from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^{q,\infty}$ for some $1 < q_1, q_2, \cdots, q_m < \infty$, and $1/q = 1/q_1 + \cdots + 1/q_m$, and from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$.

The boundedness of the multilinear Calderón-Zygmund operator T on variable exponent Lebesgue spaces was proved in [37], as stated below.

Lemma 7. *Let $q_i \in \mathscr{B}(\mathbb{R}^n)$, $i \in \overline{1,m}$, $q \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$ with $1/q(x) = 1/q_1(x) + \cdots + 1/q_m(x)$, and $(q/s)' \in \mathscr{B}(\mathbb{R}^n)$ for some $0 < s < q^-$. Then, the m-linear Calderón-Zygmund operator T is bounded on the product of variable exponent Lebesgue spaces. Moreover,*

$$\left\| T\left(\vec{f}\right) \right\|_{q(\cdot)} \lesssim \prod_{i=1}^{m} \|f_i\|_{q_i(\cdot)}, \tag{23}$$

with the constant C independent of $\vec{f} = (f_1, \cdots, f_m)$.

We now state and prove the boundedness of multilinear Calderón-Zygmund operator on grand variable Herz spaces.

Theorem 8. *Let $q_i \in \mathscr{B}(\mathbb{R}^n)$, $i \in \overline{1,m}$, $q, q_i \in \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ such that $q_i(0) \le q_i(\infty)$, $1/q(x) = 1/q_1(x) + \cdots + 1/q_m(x)$, $1 < q_i^- \le q_i^+ < \infty$, and $(q/s)' \in \mathscr{B}(\mathbb{R}^n)$ for some $0 < s < q^-$. Let $\theta > 0$, $1 < p_i < \infty$, and $\alpha_i \in L^\infty(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity for $i \in \overline{1,m}$ with*

$$\frac{-n}{q_i(\infty)} < \alpha_i(0) \le \alpha_i(\infty) < n\left(1 - \frac{1}{q_i(0)}\right). \tag{24}$$

Suppose that $\alpha(x) = \sum_{i=1}^m \alpha_i(x)$ and $1/p = \sum_{i=1}^m 1/p_i$. Then, the m-linear Calderón-Zygmund operator T is bounded on the product of grand variable Herz spaces. Moreover,

$$\left\| T\left(\vec{f}\right) \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} \le \prod_{i=1}^{m} \|f_i\|_{\dot{K}_{q_i(\cdot)}^{\alpha_i(\cdot),p_i),\theta}(\mathbb{R}^n)}, \tag{25}$$

with the constant $C > 0$ independent of $\vec{f} = (f_1, \cdots, f_m)$.

Proof. We restrict ourselves to $m = 2$, the general case following in a similar manner. Defining $f_\varphi(x) := f_1(x)\chi_{R_\varphi}(x)$ and $f_\sigma(x) := f_2(x)\chi_{R_\sigma}(x)$, we decompose the component functions of $\vec{f}$ as

$$f_1(x) = \sum_{\varphi \in \mathbb{Z}} f_\varphi(x),\ f_2(x) = \sum_{\sigma \in \mathbb{Z}} f_\sigma(x). \tag{26}$$

For future usage, we divide $\mathbb{Z}$ into the following sets

$$\begin{aligned} L_k &:= \{n \in \mathbb{Z} \mid n \le k-2\}, \\ M_k &:= \{n \in \mathbb{Z} \mid k-1 \le n \le k+1\}, \\ N_k &:= \{n \in \mathbb{Z} \mid n \le k+2\}, \end{aligned} \tag{27}$$

and, for X and Y arbitrary subsets of $\mathbb{Z}$, we define

$$\nu_k^\alpha(X, Y) := \left\| 2^{k\alpha(\cdot)} \sum_{\varphi \in X} \sum_{\sigma \in Y} T\left(f_\varphi, f_\sigma\right)\chi_k \right\|_{q(\cdot)}, \text{ and } \nu_k(X, Y) := \nu_k^0(X, Y). \tag{28}$$

From Definition 5 and (26), we have

$$\begin{aligned} \|T(f_1, f_2)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}} \left\| 2^{k\alpha(\cdot)} T(f_1, f_2)\chi_k \right\|_{q(\cdot)}^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]} \\ &\le \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}} \left\| \sum_{\varphi \in \mathbb{Z}} \sum_{\sigma \in \mathbb{Z}} 2^{k\alpha(\cdot)} T\left(f_\varphi, f_\sigma\right)\chi_k \right\|_{q(\cdot)}^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]} \\ &\lesssim \sum_{j=1}^{9} I_j, \end{aligned} \tag{29}$$

where

$$I_j := \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}} \lambda_j^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]}, \tag{30}$$

$$\begin{aligned} &\lambda_1 = \nu_k^\alpha(L_k, L_k), \quad \lambda_2 = \nu_k^\alpha(L_k, M_k), \quad \lambda_3 = \nu_k^\alpha(L_k, N_k), \\ &\lambda_4 = \nu_k^\alpha(M_k, L_k), \quad \lambda_5 = \nu_k^\alpha(M_k, M_k), \quad \lambda_6 = \nu_k^\alpha(M_k, N_k), \\ &\lambda_7 = \nu_k^\alpha(N_k, L_k), \quad \lambda_8 = \nu_k^\alpha(N_k, M_k), \quad \lambda_9 = \nu_k^\alpha(N_k, N_k). \end{aligned} \tag{31}$$

It is necessary to estimate I_1, I_2, I_3, I_5, I_6, and I_9, since I_4, I_7, and I_8 can be obtained in a similar manner as I_2, I_3, and I_6, respectively. Estimation for I_1: splitting $\mathbb{Z} = \mathbb{Z}_- \cup \mathbb{N}_0$ and by the asymptotic $2^{k\alpha(x)} \simeq 2^{k\alpha(0)}$ $(x \in R_k$ and $k < 0)$ and $2^{k\alpha(x)} \simeq 2^{k\alpha(\infty)}$ $(x \in R_k$ and $k \ge 0)$, we get

$$\begin{aligned} I_1 &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} 2^{k\alpha(0)p(1+\varepsilon)} \nu_k(L_k, L_k)^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]} \\ &\quad + \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} 2^{k\alpha(\infty)p(1+\varepsilon)} \nu_k(L_k, L_k)^{p(1+\varepsilon)} \right)^{1/p(1+\varepsilon)} \\ &=: I_{11} + I_{12}. \end{aligned} \tag{32}$$

For $\varphi, \sigma \in L_k$, $x \in R_k$, $y_1 \in R_\varphi$, and $y_2 \in R_\sigma$, we have

$$|x - y_1| \ge ||x| - |y_1|| \ge 2^{k-1} - 2^\varphi \ge 2^{k-1} - 2^{k-2} \gtrsim 2^k, \tag{33}$$

$$|x - y_2| \ge ||x| - |y_2|| \ge 2^{k-1} - 2^\sigma \ge 2^{k-1} - 2^{k-2} \gtrsim 2^k, \tag{34}$$

yielding

$$\left|T\left(f_\varphi, f_\sigma\right)(x)\right| \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f_\varphi(y_1)\right||f_\sigma(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} dy_1 dy_2 \lesssim 2^{-2kn}\int_{\mathbb{R}^n}\left|f_\varphi(y_1)\right| dy_1 \int_{\mathbb{R}^n}|f_\sigma(y_2)| dy_2. \tag{35}$$

From estimate (35), Hölder's inequality, $1/q(x) = 1/q_1(x) + 1/q_2(x)$, and Lemma 2, we obtain

$$\begin{aligned}\nu_k(L_k, L_k) &\lesssim \sum_{\varphi\in L_k}\sum_{\sigma\in L_k} 2^{-2kn}\left\|f_\varphi\right\|_{q_1(\cdot)}\left\|\chi_\varphi\right\|_{q_1'(\cdot)}\|f_\sigma\|_{q_2(\cdot)}\|\chi_\sigma\|_{q_2'(\cdot)}\|\chi_k\|_{q(\cdot)} \\ &\lesssim \sum_{\varphi\in L_k}\sum_{\sigma\in L_k} 2^{-2kn}\left\|f_\varphi\right\|_{q_1(\cdot)} 2^{\varphi n/q_1{}'(0)}\|f_\sigma\|_{q_2(\cdot)} 2^{\varphi n/q_2{}'(0)} 2^{kn/q(0)} \\ &\lesssim \left(\sum_{\varphi\in L_k} 2^{(k-\varphi)(n/q_1(0)-n)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right) \\ &\quad\cdot\left(\sum_{\sigma\in L_k} 2^{(k-\sigma)(n/q_2(0)-n)}\|f_\sigma\|_{q_2(\cdot)}\right).\end{aligned} \tag{36}$$

Taking into account the previous estimate for $\nu_k(L_k, L_k)$, the equality $1p = 1p_1 + 1p_2$, and Hölder's inequality, we have

$$\begin{aligned}I_{11} &\lesssim \sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_{k\in\mathbb{Z}_-}\left(\sum_{\varphi\in L_k} 2^{(k-\varphi)(n/q_1(0)-n)+k\alpha_1(0)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right]^{1/[p_1(1+\varepsilon)]} \\ &\quad\times\sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma\in L_k} 2^{(k-\sigma)(n/q_2(0)-n)+k\alpha_2(0)}\|f_\sigma\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right]^{1/[p_2(1+\varepsilon)]} \\ &=: I_{111}I_{112}.\end{aligned} \tag{37}$$

By Hölder's inequality, Fubini's theorem for series, $2^{-p_1(1+\varepsilon)} < 2^{-p_1}$, and defining $b_1 := n/q_1(0) - n + \alpha_1(0) < 0$, we obtain

$$\begin{aligned}I_{111} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}_-}\left(\sum_{\varphi\in L_k} 2^{b_1(k-\varphi)\frac{p_1(1+\varepsilon)}{2}} 2^{\varphi\alpha_1(0)p_1(1+\varepsilon)}\left\|f_\varphi\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\right)\right. \\ &\quad\left.\times\left(\sum_{\varphi\in L_k} 2^{b_1(k-\varphi)\frac{(p_1(1+\varepsilon))'}{2}}\right)^{\frac{p_1(1+\varepsilon)}{(p_1(1+\varepsilon))'}}\right)^{1/[p_1(1+\varepsilon)]} \\ &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{\varphi=-\infty}^{-1} 2^{\varphi\alpha_1(0)p_1(1+\varepsilon)}\left\|f_\varphi\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\sum_{k=\varphi+2}^{-1} 2^{b_1(k-\varphi)\frac{p_1(1+\varepsilon)}{2}}\right)^{1/[p_1(1+\varepsilon)]} \\ &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{\varphi\in L_k} 2^{\varphi\alpha_1(0)p_1(1+\varepsilon)}\left\|f_\varphi\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]} \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}.\end{aligned} \tag{38}$$

The estimate $I_{112} \lesssim \|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}$ is obtained, mutatis mutandis, via the estimation for I_{111}. With this estimates at hand, we obtain

$$I_{11} \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{39}$$

To estimate I_{12}, we split as follow

$$\begin{aligned}I_{12} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{N}_0} 2^{k\alpha(\infty)p(1+\varepsilon)}\left\|\sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} T\left(f_\varphi, f_\sigma\right)\chi_k\right\|_{q(\cdot)}^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &\quad+\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{N}_0} 2^{k\alpha(\infty)p(1+\varepsilon)}\left\|\sum_{\varphi=0}^{k-2}\sum_{\sigma=0}^{k-2} T\left(f_\varphi, f_\sigma\right)\chi_k\right\|_{q(\cdot)}^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &=: I_{121} + I_{122}.\end{aligned} \tag{40}$$

The estimate I_{122} follows in similar manner as in I_{11} with simply replaced $\alpha_i(0)$ by $\alpha_i(\infty)$ and used the fact $q_i(0) \le q_i(\infty)$.

For estimate I_{121}, by Hölder's inequality, Lemma 2, and the inequality $q_i(0) \le q_i(\infty)$, we obtain

$$\begin{aligned}\nu_k(\mathbb{Z}_-, \mathbb{Z}_-) &\lesssim \sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} 2^{-2kn}\left\|f_\varphi\right\|_{q_1(\cdot)}\left\|\chi_\varphi\right\|_{q_1'(\cdot)}\|f_\sigma\|_{q_2(\cdot)} \\ &\quad\cdot\|\chi_\sigma\|_{q_2'(\cdot)}\|\chi_k\|_{q(\cdot)} \lesssim \sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} 2^{-2kn}\left\|f_\varphi\right\|_{q_1(\cdot)} \\ &\quad\cdot 2^{\frac{\varphi n}{q_1{}'(0)}}\|f_\sigma\|_{q_2(\cdot)} 2^{\frac{\sigma n}{q_2{}'(0)}} 2^{\frac{kn}{q(\infty)}} \\ &\lesssim \left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n\right)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right) \\ &\quad\cdot\left(\sum_{\sigma\in\mathbb{Z}_-} 2^{(k-\sigma)\left(\frac{n}{q_2(0)}-n\right)}\|f_\sigma\|_{q_2(\cdot)}\right).\end{aligned} \tag{41}$$

From the estimate of $\nu_k(\mathbb{Z}_-, \mathbb{Z}_-)$, the equality $1/p = 1/p_1 + 1/p_2$, and Hölder's inequality, we have

$$\begin{aligned}I_{121} &\lesssim \sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n\right)+k\alpha_1(\infty)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right]^{1/[p_1(1+\varepsilon)]} \\ &\quad\times\sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_{k\in\mathbb{N}_0}\left(\sum_{\sigma\in\mathbb{Z}_-} 2^{(k-\sigma)\left(\frac{n}{q_2(0)}-n\right)+k\alpha_2(\infty)}\|f_\sigma\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right]^{1/[p_2(1+\varepsilon)]} \\ &=: A_1A_2.\end{aligned} \tag{42}$$

Invoking the Hölder inequality and defining $\xi_1 := n/q_1(0) - n + \alpha_1(\infty) < 0$, we have

$$
\begin{aligned}
A_1 &\lesssim \sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi\in\mathbb{Z}_-}2^{(k-\varphi)\xi_1\frac{p_1(1+\varepsilon)}{2}}2^{\varphi\alpha_1(\infty)p_1(1+\varepsilon)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\right)\right.\\
&\quad\left.\times\left(\sum_{\varphi\in\mathbb{Z}_-}2^{(k-\varphi)\xi_1\frac{(p_1(1+\varepsilon))'}{2}}\right)^{\frac{p_1(1+\varepsilon)}{(p_1(1+\varepsilon))'}}\right]^{1/[p_1(1+\varepsilon)]}\\
&\lesssim\sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\xi_1\frac{p_1(1+\varepsilon)}{2}}\left(\sum_{\varphi\in\mathbb{Z}_-}2^{-\varphi\xi_1\frac{p_1(1+\varepsilon)}{2}}2^{\varphi\alpha_1(\infty)p_1(1+\varepsilon)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\right)\right]^{1/[p_1(1+\varepsilon)]}\\
&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{\varphi\in\mathbb{Z}_-}2^{\varphi\alpha_1(\infty)p_1(1+\varepsilon)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]}\lesssim\|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}.
\end{aligned}
\tag{43}
$$

Similar estimate, with the corresponding changes, is obtained for A_2, from which we obtain $I_{12}\lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}$. Hence

$$
I_1\lesssim\|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}.
\tag{44}
$$

Estimation for I_2: as in the case of I_1, we obtain the following estimate

$$
\begin{aligned}
I_2&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha(0)p(1+\varepsilon)}\nu_k(L_k,M_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}\\
&\quad+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(L_k,M_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}\\
&=:I_{21}+I_{22}.
\end{aligned}
\tag{45}
$$

Notice that, for $x\in R_k$, $y_1\in R_{\varphi}$, $y_2\in R_{\sigma}$, $\varphi\in L_k$, and $\sigma\in M_k$, we have

$$
|x-y_1|+|x-y_2|\geq|x-y_1|\geq||x|-|y_1||\gtrsim 2^k,
\tag{46}
$$

from which, taking Lemma 2 into consideration and elementary computations, we obtain

$$
\begin{aligned}
\nu_k(L_k,M_k)&\leq\sum_{\varphi\in L_k}\sum_{\sigma\in M_k}2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n\right)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\\
&\quad\cdot 2^{(\sigma-k)n}2^{\frac{(k-\sigma)n}{q_2(0)}}\|f_{\sigma}\|_{q_2(\cdot)},\ (k\in\mathbb{Z}_-).
\end{aligned}
\tag{47}
$$

From the estimate for $\nu_k(L_k,M_k)$ and Hölder's inequality, we get

$$
\begin{aligned}
I_{21}&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\varphi\in L_k}2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n\right)+k\alpha_1(0)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]}\\
&\quad\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}}\left(\sum_{\sigma\in M_k}2^{k\alpha_2(0)}2^{(\sigma-k)n}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}\\
&=:I_{211}I_{212}.
\end{aligned}
\tag{48}
$$

Notice that $I_{211}=I_{111}$. For the estimate I_{212}, we reason as follows

$$
\begin{aligned}
I_{212}&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha_2(0)p_2(1+\varepsilon)}\left(\sum_{\sigma\in M_k}2^{(\sigma-k)n}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}\\
&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha_2(0)p_2(1+\varepsilon)}\|f_2\chi_k\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}\\
&\lesssim\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}.
\end{aligned}
\tag{49}
$$

The term I_{22} is estimated by

$$
\begin{aligned}
I_{22}&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(\mathbb{Z}_-,M_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}\\
&\quad+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k\left(0,\bar{k}2,M_k\right)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}\\
&=:I_{221}+I_{222}.
\end{aligned}
\tag{50}
$$

To estimate I_{221}, by Hölder's inequality, Lemma 2, and the inequality $q_1(0)\leq q_1(\infty)$, we have

$$
\begin{aligned}
\nu_k(\mathbb{Z}_-,M_k)&\lesssim\sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in M_k}2^{(k-\varphi)(n/q_1(0)-n)}f_{\varphi q_1(\cdot)}\\
&\quad\cdot 2^{(\sigma-k)n}2^{([(k-\sigma)n]/q_2(\infty))}f_{\sigma q_2(\cdot)}.
\end{aligned}
\tag{51}
$$

Thus using (51) and by Hölder's inequality, we have

$$
\begin{aligned}
I_{221}&\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi\in\mathbb{Z}_-}2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n\right)+k\alpha_1(\infty)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]}\\
&\quad\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\sigma\in M_k}2^{k\alpha_2(\infty)}2^{(\sigma-k)n}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}\\
&=:B_1B_2.
\end{aligned}
\tag{52}
$$

The term B_1 is equal to A_1 and for B_2 we use similar arguments as for I_{212}, replacing $\alpha_2(0)$ with $\alpha_2(\infty)$.

For the term I_{222}, by Hölder's inequality and Lemma 2, we have

$$
\nu_k\left(0,\bar{k}2,M_k\right)\lesssim\sum_{\varphi=0}^{k-2}\sum_{\sigma\in M_k}2^{(k-\varphi)(n/q_1(\infty)-n)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}2^{(\sigma-k)n}\|f_{\sigma}\|_{q_2(\cdot)}.
\tag{53}
$$

Taking into consideration (53) and applying Hölder's inequality, we get

$$I_{222} \lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi=0}^{k-2}2^{(k-\varphi)\left(\frac{n}{q_1(\infty)}-n\right)+k\alpha_1(\infty)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]}$$
$$\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\sigma\in M_k}2^{k\alpha_2(\infty)}2^{(\sigma-k)n}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$=: B_1'B_2'. \tag{54}$$

Noting that $B_2' = B_2$ and estimating B_1' in a similar fashion as I_{212}, we obtain $I_{22} \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}$, from which we get

$$I_2 \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{55}$$

Estimation for I_3: we have

$$I_3 \lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha(0)p(1+\varepsilon)}\nu_k(L_k,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}$$
$$+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(L_k,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]}$$
$$=: I_{31}+I_{32}. \tag{56}$$

For $\varphi\in L_k$, $\sigma\in N_k$, $x\in R_k$, $y_1\in R_{\varphi}$, and $y_2\in R_{\sigma}$, we get

$$|x-y_1| \lesssim |x|-|y_1| \gtrsim 2^k,\ |x-y_2| \geq |y_2|-|x| \gtrsim 2^{\sigma}, \tag{57}$$

from which, taking into account the elementary inequality $(2^k+2^{\sigma})^{2n} \gtrsim 2^{kn+\sigma n}$, it follows

$$\left|T\left(f_{\varphi},f_{\sigma}\right)(x)\right| \lesssim 2^{-kn}\int_{\mathbb{R}^n}\left|f_{\varphi}(y_1)\right|\mathrm{d}y_1 2^{-\sigma n}\int_{\mathbb{R}^n}|f_{\sigma}(y_2)|\mathrm{d}y_2,\ (x\in R_k). \tag{58}$$

By Hölder's inequality, Lemma 2, $q_2(0) \lesssim q_2(\infty)$, and $k\in\mathbb{Z}_-$, we obtain

$$\nu_k(L_k,N_k) \lesssim \sum_{\varphi\in L_k}2^{(k-\varphi)(n/q_1(0)-n)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\sum_{\sigma\in N_k}2^{(k-\sigma)n/q_2(\infty)}\|f_{\sigma}\|_{q_2(\cdot)}. \tag{59}$$

To estimate the term I_{31}, using (59) and the Hölder inequality, we obtain

$$I_{31} \lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\varphi\in L_k}2^{(k-\varphi)(n/q_1(0)-n+\alpha_1(0))}2^{\varphi\alpha_1(0)}\left\|f_{\varphi}\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]}$$
$$\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma\in N_k}2^{(k-\sigma)(n/q_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$=: I_{311}I_{312}. \tag{60}$$

We have $I_{311} \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}$, since $I_{311} = I_{111}$.

As for I_{312}, we split as follow

$$I_{312} \lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma=k+2}^{-1}2^{(k-\sigma)(n/q_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma\in\mathbb{N}_0}2^{(k-\sigma)(n/q_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_{\sigma}\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$=: D_1+D_2. \tag{61}$$

By Hölder's inequality, Fubini's theorem for series, the inequality $n/q_2(\infty)+\alpha_2(0)>0$ and $2^{-p_2(1+\varepsilon)} < 2^{-p_2}$, we obtain

$$D_1 \lesssim \sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma=k+2}^{-1}2^{\sigma\alpha_2(0)p_2(1+\varepsilon)}\|f_{\sigma}\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))p_2(1+\varepsilon)}{2}}\right)\right.$$
$$\left.\times\left(\sum_{\sigma=k+2}^{-1}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))(p_2(1+\varepsilon))'}{2}}\right)^{\frac{p_2(1+\varepsilon)}{(p_2(1+\varepsilon))'}}\right]^{1/[p_2(1+\varepsilon)]}$$
$$\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\sum_{\sigma=k+2}^{-1}2^{\sigma\alpha_2(0)}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))p_2(1+\varepsilon)}{2}}\|f_{\sigma}\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{\sigma=-\infty}^{-1}2^{\sigma\alpha_2(0)p_2(1+\varepsilon)}\|f_{\sigma}\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}\sum_{k=-\infty}^{\sigma-2}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))}{2}}\right)^{1/[p_2(1+\varepsilon)]} \lesssim \|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{62}$$

For D_2, using the same argument as above and the inequality $\alpha_2(0) \leqslant \alpha_2(\infty)$, we obtain

$$D_2 \lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{N}_0}2^{\sigma\alpha_2(0)p_2(1+\varepsilon)}\|f_{\sigma}\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))p_2(1+\varepsilon)}{2}}\right)^{1/[p_2(1+\varepsilon)]}$$
$$\lesssim\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{N}_0}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))p_2(1+\varepsilon)}{2}}\sum_{j=\sigma}^{\infty}2^{j\alpha_2(\infty)p_2(1+\varepsilon)}\left\|f_2\chi_{R_j}\right\|_{q_2(\cdot)}^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]}$$
$$\lesssim\sup_{\varepsilon>0}\left(\sum_{k\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{N}_0}2^{\frac{(k-\sigma)((n/q_2(\infty))+\alpha_2(0))p_2(1+\varepsilon)}{2}}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]} \lesssim \|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{63}$$

So $I_{31} \lesssim \|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1,p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2,p_2),\theta}(\mathbb{R}^n)}$.

The term I_{32} can be estimated as

$$\begin{aligned} I_{32} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(\mathbb{Z}_-,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &\quad+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(0,\bar{k}2,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &=: I_{321}+I_{322}. \end{aligned} \tag{64}$$

For $k\in\mathbb{N}_0$, $\varphi\in\mathbb{Z}_-$, $\sigma\in N_k$, $q_1(0)\lesssim q_1(\infty)$, and Lemma 2, we obtain

$$\begin{aligned} \nu_k(\mathbb{Z}_-,N_k) &\lesssim \sum_{\varphi\in\mathbb{Z}_-}2^{(k-\varphi)(n/q_1(0)-n)}\left\|f_\varphi\right\|_{q_1(\cdot)} \\ &\quad\cdot\sum_{\sigma\in N_k}2^{([(k-\sigma)n]/q_2(\infty))}\|f_\sigma\|_{q_2(\cdot)}. \end{aligned} \tag{65}$$

By (65) and Hölder's inequality, we get

$$\begin{aligned} I_{321} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi\in\mathbb{Z}_-}2^{(k-\varphi)\left(\frac{n}{q_1(0)}-n+\alpha_1(\infty)\right)}2^{\varphi\alpha_1(\infty)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]} \\ &\quad\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\sigma\in N_k}2^{(k-\sigma)\left(\frac{n}{q_2(\infty)}+\alpha_2(\infty)\right)}2^{\sigma\alpha_2(\infty)}\|f_\sigma\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]} \\ &=: D_1'D_2'. \end{aligned} \tag{66}$$

Note that $D_1'=A_1$. The estimate for D_2' can be obtained in a similar way as for I_{312}, by replacing $\alpha_2(0)$ with $\alpha_2(\infty)$.

For estimate I_{322}, by Hölder's inequality and Lemma 2, we have

$$\begin{aligned} \nu_k(0,\bar{k}2,N_k) &\lesssim \sum_{\varphi=0}^{k-2}\sum_{\sigma\in N_k}2^{(k-\varphi)(n/q_1(\infty)-n)}\left\|f_\varphi\right\|_{q_1(\cdot)} \\ &\quad\cdot 2^{(k-\sigma)n/q_2(\infty)}\|f_\sigma\|_{q_2(\cdot)}. \end{aligned} \tag{67}$$

Using (67) and Hölder's inequality, we obtain

$$\begin{aligned} I_{322} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\varphi=0}^{k-2}2^{(k-\varphi)(n/q_1(\infty)-n+\alpha_1(\infty))}2^{\varphi\alpha_1(\infty)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]} \\ &\quad\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}\left(\sum_{\sigma\in N_k}2^{(k-\sigma)(n/q_2(\infty)+\alpha_2(\infty))}2^{\sigma\alpha_2(\infty)}\|f_\sigma\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]} \\ &=: D_1''D_2''. \end{aligned} \tag{68}$$

We have $D_2''=D_2'$ and the estimate for the term D_1'' is similar to B_1'. Taking all the estimates into account yields

$$I_3\lesssim\|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{69}$$

Estimation for I_5: We have

$$\begin{aligned} I_5 &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha(0)p(1+\varepsilon)}\nu_k(M_k,M_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &\quad+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(M_k,M_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &=: I_{51}+I_{52}. \end{aligned} \tag{70}$$

By the $L^{q(\cdot)}$-boundedness of T, see Lemma 7, we obtain that

$$\nu_k(M_k,M_k)\lesssim\sum_{\varphi\in M_k}\sum_{\sigma\in M_k}\left\|f_\varphi\right\|_{q_1(\cdot)}\|f_\sigma\|_{q_2(\cdot)}. \tag{71}$$

By Hölder's inequality, we have

$$\begin{aligned} I_{51} &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\varphi\in M_k}2^{k\alpha_1(0)}\left\|f_\varphi\right\|_{q_1(\cdot)}\right)^{p_1(1+\varepsilon)}\right)^{1/[p_1(1+\varepsilon)]} \\ &\quad\times\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}\left(\sum_{\sigma\in M_k}2^{k\alpha_2(0)}\|f_\sigma\|_{q_2(\cdot)}\right)^{p_2(1+\varepsilon)}\right)^{1/[p_2(1+\varepsilon)]} \\ &\lesssim\|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \end{aligned} \tag{72}$$

Similarly, we can obtain similar estimate for I_{52}, replacing $\alpha_i(0)$ by $\alpha_i(\infty)$ with $k\in\mathbb{N}_0$. Therefore

$$I_5\lesssim\|f_1\|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),p_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),p_2),\theta}(\mathbb{R}^n)}. \tag{73}$$

Estimation for I_6: we have

$$\begin{aligned} I_6 &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}_-}2^{k\alpha(0)p(1+\varepsilon)}\nu_k(M_k,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &\quad+\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{N}_0}2^{k\alpha(\infty)p(1+\varepsilon)}\nu_k(M_k,N_k)^{p(1+\varepsilon)}\right)^{1/[p(1+\varepsilon)]} \\ &=: I_{61}+I_{62}. \end{aligned} \tag{74}$$

For $k\in\mathbb{Z}_-$, $\varphi\in M_k$, $\sigma\in N_k$, and $x\in R_k$, we have

$$|x-y_2|\le|y_2|-|x|2^{\sigma}, \tag{75}$$

yielding

$$\left|T\left(f_\varphi,f_\sigma\right)(x)\right|\le 2^{-kn}\int_{\mathbb{R}^n}\left|f_\varphi(y_1)\right|\mathrm{d}y_1 2^{-\sigma n}\int_{\mathbb{R}^n}|f_\sigma(y_2)|\mathrm{d}y_2. \tag{76}$$

By Hölder's inequality, Lemma 2, $k \in \mathbb{Z}_-$, and inequality $q_2(0) \leqslant q_2(\infty)$, we obtain

$$\nu_k(M_k, N_k) \lesssim \sum_{\varphi \in M_k} \sum_{\sigma \in N_k} 2^{(\varphi - k)n} \left\| f_\varphi \right\|_{q_1(\cdot)} 2^{([(k-\sigma)n]/q_2(\infty))} \| f_\sigma \|_{q_2(\cdot)}. \tag{77}$$

By (77) and invoking Hölder's inequality, we get

$$\begin{aligned} I_{61} &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} \left(\sum_{\varphi \in M_k} 2^{(\varphi - k)n} 2^{k\alpha_1(0)} \left\| f_\varphi \right\|_{q_1(\cdot)} \right)^{p_1(1+\varepsilon)} \right)^{1/[p_1(1+\varepsilon)]} \\ &\quad \times \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} \left(\sum_{\sigma \in N_k} 2^{(k-\sigma)(n/q_2(\infty) + \alpha_2(0))} 2^{\sigma\alpha_2(0)} \| f_\sigma \|_{q_2(\cdot)} \right)^{p_2(1+\varepsilon)} \right)^{1/[p_2(1+\varepsilon)]} \\ &=: I_{611} I_{612}. \end{aligned} \tag{78}$$

Note that the estimate I_{611} is equal to that of I_{51} and I_{612} is obtained by similar argument used in I_{312}.

For $k \in \mathbb{N}_0$, we have

$$\nu_k(M_k, N_k) \lesssim \sum_{\varphi \in M_k} \sum_{\sigma \in N_k} 2^{(\varphi - k)n} \left\| f_\varphi \right\|_{q_1(\cdot)} 2^{([(k-\sigma)n]/q_2(\infty))} \| f_\sigma \|_{q_2(\cdot)}. \tag{79}$$

Using (79) and Hölder's inequality, we get

$$\begin{aligned} I_{62} &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} \left(\sum_{\varphi \in M_k} 2^{(\varphi - k)n} 2^{k\alpha_1(\infty)} \left\| f_\varphi \right\|_{q_1(\cdot)} \right)^{p_1(1+\varepsilon)} \right)^{1/[p_1(1+\varepsilon)]} \\ &\quad \times \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} \left(\sum_{\sigma \in N_k} 2^{(k-\sigma)\left(\frac{n}{q_2(\infty)} + \alpha_2(\infty)\right)} 2^{\sigma\alpha_2(\infty)} \| f_\sigma \|_{q_2(\cdot)} \right)^{p_2(1+\varepsilon)} \right)^{1/[p_2(1+\varepsilon)]} \\ &=: I_{621} I_{622}. \end{aligned} \tag{80}$$

The estimate I_{621} is similar to I_{52} and $I_{622} = D_2''$. Hence

$$I_6 \lesssim \| f_1 \|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot), p_1), \theta}(\mathbb{R}^n)} \| f_2 \|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot), p_2), \theta}(\mathbb{R}^n)}. \tag{81}$$

Estimation for I_9: we estimate I_9 as usual

$$\begin{aligned} I_9 &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} 2^{k\alpha(0)p(1+\varepsilon)} \nu_k(N_k, N_k)^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]} \\ &\quad + \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} 2^{k\alpha(\infty)p(1+\varepsilon)} \nu_k(N_k, N_k)^{p(1+\varepsilon)} \right)^{1/[p(1+\varepsilon)]} \\ &=: I_{91} + I_{92}. \end{aligned} \tag{82}$$

For $\varphi \in N_k$, $y_1 \in R_\varphi$, and $x \in R_k$, we have $|x - y_1| \, 2^\varphi$, whereas for $\sigma \in N_k$, $y_2 \in R_\sigma$, and $x \in R_k$, we have $|x - y_2| \, 2^\sigma$. Under the assumptions for the obtained inequalities, we have

$$\left| T\left(f_\varphi, f_\sigma\right)(x) \right| \lesssim 2^{-\varphi n} \int_{\mathbb{R}^n} \left| f_\varphi(y_1) \right| \mathrm{d}y_1 2^{-\sigma n} \int_{\mathbb{R}^n} |f_\sigma(y_2)| \mathrm{d}y_2. \tag{83}$$

When $k \in \mathbb{Z}_-$, by Hölder's inequality, Lemma 2, and $q_j(0) \leqslant q_j(\infty)$, we obtain

$$\nu_k(N_k, N_k) \lesssim \sum_{\varphi \in N_k} \sum_{\sigma \in N_k} 2^{(k-\varphi)n/q_1(\infty)} 2^{(k-\sigma)n/q_2(\infty)} \left\| f_\varphi \right\|_{q_1(\cdot)} \| f_\sigma \|_{q_2(\cdot)}. \tag{84}$$

Thus

$$\begin{aligned} I_{91} &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} \left(\sum_{\varphi \in N_k} 2^{(k-\varphi)(n/q_1(\infty) + \alpha_1(0))} 2^{\varphi\alpha_1(0)} \left\| f_\varphi \right\|_{q_1(\cdot)} \right)^{p_1(1+\varepsilon)} \right)^{1/[p_1(1+\varepsilon)]} \\ &\quad \times \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{Z}_-} \left(\sum_{\sigma \in N_k} 2^{(k-\sigma)(n/q_1(\infty) + \alpha_1(0))} 2^{\sigma\alpha_2(0)} \| f_\sigma \|_{q_2(\cdot)} \right)^{p_2(1+\varepsilon)} \right)^{1/[p_2(1+\varepsilon)]} \\ &=: I_{911} I_{912}. \end{aligned} \tag{85}$$

The estimates for the terms I_{911} and I_{912} are obtained in the same way as for I_{312}. Therefore

$$I_{91} \leq C \| f_1 \|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot), p_1), \theta}(\mathbb{R}^n)} \| f_2 \|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot), p_2), \theta}(\mathbb{R}^n)}. \tag{86}$$

Finally, for the term I_{92}, we have

$$\begin{aligned} I_{92} &\lesssim \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} \left(\sum_{\varphi \in N_k} 2^{(k-\varphi)(n/q_1(\infty) + \alpha_1(\infty))} 2^{\varphi\alpha_1(\infty)} \left\| f_\varphi \right\|_{q_1(\cdot)} \right)^{p_1(1+\varepsilon)} \right)^{1/[p_1(1+\varepsilon)]} \\ &\quad \times \sup_{\varepsilon > 0} \left(\varepsilon^\theta \sum_{k \in \mathbb{N}_0} \left(\sum_{\sigma \in N_k} 2^{(k-\sigma)(n/q_2(\infty) + \alpha_2(\infty))} 2^{\sigma\alpha_2(\infty)} \| f_\sigma \|_{q_2(\cdot)} \right)^{p_2(1+\varepsilon)} \right)^{1/[p_2(1+\varepsilon)]} \\ &=: I_{921} I_{922}. \end{aligned} \tag{87}$$

The terms I_{921} and I_{922} can be estimated in a similar manner as for that I_{322}. Thus

$$I_9 \lesssim \| f_1 \|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot), p_1), \theta}(\mathbb{R}^n)} \| f_2 \|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot), p_2), \theta}(\mathbb{R}^n)}. \tag{88}$$

Taking into account the estimates (44), (55), (69), (73), (81), and (88), we get

$$\| T(f_1, f_2) \|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p), \theta}(\mathbb{R}^n)} \lesssim \| f_1 \|_{\dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot), p_1), \theta}(\mathbb{R}^n)} \| f_2 \|_{\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot), p_2), \theta}(\mathbb{R}^n)}, \tag{89}$$

which completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Authors' Contributions

All the authors have contributed equally in preparation and finalization of the manuscript.

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