Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4855173, 11 pages
https://doi.org/10.1155/2022/4855173

Research Article

Strong Convergence of a New Hybrid Iterative Scheme for Nonexpensive Mappings and Applications

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Received 27 October 2021; Revised 2 January 2022; Accepted 1 February 2022; Published 19 March 2022

Academic Editor: Santosh Kumar

In the article, we have proposed a new type of hybrid iterative scheme which is a hybrid of Picard and Thakur et al. repetitive schemes. This new hybrid iterative scheme converges faster than all leading schemes like Picard-S* hybrid, Picard-S, Picard-Ishikawa hybrid, Picard-Mann hybrid, Thakur et al. and Abbas and Nazir, S-iterative, Ishikawa and Mann iterative schemes for contraction mapping. By using the Picard-Thakur hybrid iterative scheme, we can find the solution of delay differential equations and also prove some convergence results for nonexpansive mapping in a uniformly convex Banach space.

1. Introduction

In this article, the set of all positive integers is denoted by I^+. Let N denote the nonempty convex subset of a normed space and S be its convex subset, and $\mathcal{V} : S \longrightarrow S$ is called contraction mapping if $\|\mathcal{V}_j - \mathcal{V}_k\| \leq \delta \|j - k\|$ for all $j, k \in S$ and $\delta \in (0, 1)$. If $\delta = 1$, then, the mapping $\mathcal{V}$ is called nonexpansive mapping. An element $j \in S$ is said to be a fixed point of $\mathcal{V}$ if $\mathcal{V} j = j$, and the set of fixed points of $\mathcal{V}$ is denoted by $F(\mathcal{V})$.

In 1890, Picard [1] presented an iterative scheme for approximating the fixed point which is defined by the sequence $\{j_n\}$ as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathcal{V} j_n, \end{cases} \quad n \in I^+. \tag{1}$$

The Krasnoselskii [2] iterative sequence $\{u_n\}$ is defined as

$$\begin{cases} u_1 = u \in S, \\ u_{n+1} = (1-\mu) u_n + \mu \mathcal{V} u_n, \end{cases} \quad n \in I^+, \tag{2}$$

where $\mu \in (0, 1)$.

In 1953, Mann [3] proposed an iterative scheme which is defined as

$$\begin{cases} v_1 = v \in S, \\ v_{n+1} = (1-\theta_n) v_n + \theta_n \mathcal{V} v_n, \end{cases} \quad n \in I^+, \tag{3}$$

where $\{\theta_n\} \in (0, 1)$.

In 1974, Ishikawa [4] gave the concept of the two-step iterative scheme and the sequence $\{w_n\}$ of this iterative is defined as

$$\begin{cases} w_1 = w \in S, \\ w_{n+1} = (1-\theta_n)w_n + \theta_n \mathscr{V} t_n, n \in I^+, \\ t_n = (1-\vartheta_n)w_n + \vartheta_n \mathscr{V} w_n, \end{cases} \quad (4)$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

In 2007, Agarwal et al. [5] introduced a more generalized form of the Ishikawa iterative scheme and they called it the S -iterative scheme and the sequence $\{p_n\}$ of the iterative scheme is defined as

$$\begin{cases} p_1 = p \in S, \\ p_{n+1} = (1-\theta_n)\mathscr{V} p_n + \theta_n \mathscr{V} q_n, n \in I^+, \\ q_n = (1-\vartheta_n)p_n + \vartheta_n \mathscr{V} p_n, \end{cases} \quad (5)$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

In 2016, Sahu et al. [6] and Thakur et al. [7] proposed a new scheme which converges faster than all the existing schemes. The iterative sequence $\{k_n\}$ of this scheme is defined as

$$\begin{cases} k_1 = k \in S, \\ k_{n+1} = (1-\theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n, \\ l_n = (1-\vartheta_n)m_n + \vartheta_n \mathscr{V} m_n, \\ m_n = (1-\sigma_n)k_n + \sigma_n \mathscr{V} k_n, \end{cases} n \in I^+, \quad (6)$$

where $\{\theta_n\}, \{\vartheta_n\}$, and $\{\sigma_n\} \in (0,1)$.

Thakur et al. [7] proposed another iterative scheme which converges faster than all the above schemes and the iterative sequence $\{j_n\}$ of Thakur et al. is defined as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = \mathscr{V}((1-\theta_n)j_n + \theta_n \mathscr{V} l_n), \\ l_n = (1-\vartheta_n)j_n + \vartheta_n \mathscr{V} j_n), \end{cases} n \in I^+, \quad (7)$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

Recently, Lamba and Panwar [8] introduced a new three-step iteration process for Susuzki's nonexpansive mapping and called it the Ap iterative scheme whose rate of convergence is faster than those of the leading schemes. The sequence of the Ap iterative scheme is defined as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = \mathscr{V}((1-\theta_n)\mathscr{V} j_n + \theta_n \mathscr{V} l_n), \\ l_n = \mathscr{V}((1-\vartheta_n)j_n + \vartheta_n \mathscr{V} j_n), \end{cases} n \in I^+, \quad (8)$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

Many physical problems of engineering and applied sciences are mostly constructed in the form of fixed point equations. In the existence of a fixed point equation involving an operator, $\mathscr{V}$ is guaranteed but the exact solution is not possible. We can only approximate the solution which becomes very relevant and this necessitated various iterative schemes [9–14]. Also, the iterative schemes are used for solving different problems like minimization, equilibrium, viscosity approximation, and many more problems in different spaces [15–18].

The Picard iterative scheme is the simplest iteration to estimate the solution of a fixed point equation. Chidume [19] introduced some basic results on this iterative scheme. Chidume generalized and improved the existing results of the fixed point equation in [20]. Okeke and Abbas [21] proved the convergence and almost $\mathscr{V}$-stability of Mann-type and Ishikawa-type random iterative schemes.

In 2013, Khan [22] proposed the Picard-Mann hybrid iterative scheme. The sequence $\{r_n\}$ of this scheme is defined as

$$\begin{cases} r_1 = r \in S, \\ r_{n+1} = \mathscr{V} s_n, \\ s_n = (1-\theta_n)r_n + \theta_n \mathscr{V} r_n, \end{cases} n \in I^+, \quad (9)$$

where $\{\theta_n\} \in (0,1)$.

In 2017, Okeke and Abbas [23] proposed the Picard-Krasnoselskii hybrid iterative scheme and the sequence $\{r_n\}$ of this iterative scheme is defined as

$$\begin{cases} r_1 = r \in S, \\ r_{n+1} = \mathscr{V} s_n, \\ s_n = (1-\nu)r_n + \nu \mathscr{V} r_n, \end{cases} n \in I^+, \quad (10)$$

where $\nu \in (0,1)$.

In 2019, Okeke [24] proposed the Picard-Ishikawa hybrid iterative scheme and the sequence $\{f_n\}$ of this iteration defined as

$$\begin{cases} f_1 = f \in S, \\ f_{n+1} = \mathscr{V} g_n, \\ g_n = (1-\theta_n) f_n + \theta_n \mathscr{V} h_n, \\ h_n = (1-\vartheta_n) f_n + \vartheta_n \mathscr{V} f_n, \end{cases} n \in I^+, \tag{11}$$

where $\{\theta_n\}$ and $\{\vartheta_n\} \in (0,1)$.

Recently, Srivastava [25] introduced a new type of hybrid iterative scheme from Picard and S-iteration (Picars-S hybrid iterative scheme). The sequence $\{a_n\}$ of the scheme is defined as

$$\begin{cases} a_1 = a \in S, \\ a_{n+1} = \mathscr{V} b_n, \\ b_n = (1-\theta_n) \mathscr{V} a_n + \theta_n \mathscr{V} c_n, \\ c_n = (1-\vartheta_n) a_n + \vartheta_n \mathscr{V} a_n, \end{cases} n \in I^+, \tag{12}$$

where $\{\theta_n\}$ and $\{\vartheta_n\} \in (0,1)$.

Also Lamba and Panwar [26] introduced another hybrid scheme from Picard and S^*-iteration (Picard-S^* hybrid iterative scheme) and the sequence $\{a_n\}$ of the scheme is defined as

$$\begin{cases} a_1 = a \in S, \\ a_{n+1} = \mathscr{V} b_n, \\ b_n = (1-\theta_n) \mathscr{V} a_n + \theta_n \mathscr{V} c_n, n \in I^+, \\ c_n = (1-\vartheta_n) \mathscr{V} a_n + \vartheta_n \mathscr{V} d_n, \\ d_n = (1-\sigma_n) a_n + \sigma_n \mathscr{V} a_n, \end{cases} \tag{13}$$

where $\{\theta_n\}$, $\{\vartheta_n\}$, and $\{\sigma_n\} \in (0,1)$.

With the motivation towards the usage of hybridization of iterative schemes, we proposed another type of hybrid scheme which is the Picard-Thakur hybrid iterative scheme, defined by the sequence $\{j_n\}$ as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = (1-\theta_n) \mathscr{V} m_n + \theta_n \mathscr{V} l_n, n \in I^+, \\ l_n = (1-\vartheta_n) m_n + \vartheta_n \mathscr{V} m_n, \\ m_n = (1-\sigma_n) j_n + \sigma_n \mathscr{V} j_n, \end{cases} \tag{14}$$

where $\{\theta_n\}$, $\{\vartheta_n\}$ and $\{\sigma_n\} \in (0,1)$.

Rhoades [27] commented on the convergence of two iterative schemes that converges to a certain fixed point is as follows:

Let $\{a_n\}$ and $\{b_n\}$ be the two fixed point iterative schemes that converge to a certain fixed point j^* of a given operator $\mathscr{V}$. The sequence $\{a_n\}$ is better than $\{b_n\}$ if

$$\|a_n - j^*\| \le \|b_n - j^*\| \quad \forall n \in I^+. \tag{15}$$

2. Preliminaries

Berinde and Takens [10] gave the following definitions.

Definition 1 (see [10]). Let $\{t_n\}$ and $\{w_n\}$ be the two sequences of the real number converging to t and w, respectively. Suppose that

$$\lim_{n\longrightarrow\infty} \frac{|t_n - t|}{|w_n - w|} = k. \tag{16}$$

(i) If $k=0$, then, $\{t_n\} \longrightarrow t$ faster than $\{w_n\} \longrightarrow w$

(ii) If $0 < k < \infty$, then, the rate of convergence of both sequences are the same

Definition 2 (see [10]). Let $\{t_n\}$ and $\{w_n\}$ be the two sequences of a fixed point iterative scheme, both converges to a fixed point ξ for a given operator $\mathscr{V}$ and $\{p_n\}$,$\{q_n\}$ are two sequences of positive numbers. Suppose that the error estimates,

$$\begin{aligned} \|t_n - \xi\| &\le p_n \quad \forall n \in I^+, \\ \|w_n - \xi\| &\le q_n \quad \forall n \in I^+, \end{aligned} \tag{17}$$

are available and $\{p_n\}$,$\{q_n\}$ converge to zero. If $\{p_n\}$ converges faster than $\{q_n\}$, then, $\{t_n\}$ converges faster than $\{w_n\} \longrightarrow \xi$. Most of the literature on the iterative schemes deals with the convergence rate and some analyzes its stability [28–34]. For proving the results, we need the following lemma.

Lemma 3 (see [35]). *Let $\{r_n\} \in \mathbb{R}^+$ be a sequence with $r_{n+1} \le (1-\tau_n) r_n$. If $\{\tau_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} = \infty$, then, $\lim_{n\longrightarrow\infty} r_n = 0$.*

Definition 4 (see [36]). Let S be a subset of Banach space B which is nonempty closed and convex. A mapping $\mathscr{V} : S \longrightarrow S$ is demiclosed w.r.t. $b \in B$, if for each sequence $\{j_n\}$ in S and $a \in S$, $\{j_n\}$ converges weakly at a and $\{\mathscr{V} j_n\}$ converges strongly at $b \Rightarrow \mathscr{V} a = b$.

Definition 5 (see [37]). A Banach space B is said to satisfy Opial's condition if for any sequence $\{j_n\} \in B$,$\{j_n\} \rightharpoonup a$, implies that

$$\liminf_{n\longrightarrow\infty} \|j_n - a\| \le \liminf_{n\longrightarrow\infty} \|j_n - b\|, \tag{18}$$

for all $b \in B$ with $a \neq b$.

Lemma 6 (see [38]). *Let B be a uniformly convex Banach space and $0 < x \le \rho_n \le y < 1 \forall n \in \mathbb{I}^+$. Let $\{j_n\}$, $\{k_n\}$ be the two sequences such that $\limsup_{n\longrightarrow\infty}\|j_n\| \le l$, $\limsup_{n\longrightarrow\infty}\|k_n\| \le l$, and $\limsup_{n\longrightarrow\infty}\|(1-\sigma_n)j_n + \sigma_n k_n\| = l$ hold for some $l \ge 0$, then $\lim_{n\longrightarrow\infty}\|j_n - k_n\| = 0$.*

Lemma 7 (see [36]). *Let $\mathscr{V} : S \longrightarrow S$ be a nonexpansive mapping with Opial's property. If $\{j_n\} \rightharpoonup a$ and $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V} j_n\| = 0$, then, $\mathscr{V} a = a$, i.e., $I - \mathscr{V}$ is demiclosed at zero, where I is the identity mapping on B.*

Proposition 8 (see [39]). *Let S be a subset of Banach space B and $\mathscr{V} : S \longrightarrow S$ a nonexpansive mapping. Then, for all $p, q \in S$*

$$\|p - \mathscr{V} q\| \le 3\|p - \mathscr{V} p\| + \|p - q\|. \tag{19}$$

Senter and Dotson [40] introduced the concept of condition (I) which is defined as

Definition 9. Let $\mathscr{V}$ be a self-mapping on S which is said to satisfy condition (I), if there is an increasing function $Z : [0,\infty) \longrightarrow [0,\infty)$ with $Z(0) = 0$ and $Z(t) > 0$, for all $t > 0$ such that

$$d(j, \mathscr{V}(j)) \ge Z(d(j, F(\mathscr{V}))), \quad \forall j \in S, \tag{20}$$

where $d(j, F(\mathscr{V})) = \inf\{d(j, j^*) : j^* \in F(\mathscr{V})\}$.

In this article, we proposed a new hybrid iterative scheme which converges faster than Mann [3], Ishikawa [4], S-iteration [5], Abbas et al. [9], Thakur et al. [7], Picard-Mann hybrid [22], Picard-Krasnoselskii [23], Picard-Ishikawa [24], and Picard-S hybrid iterative schemes [25]. Recently, Srivastava [25] already proved that the Picard-S hybrid iterative scheme converges faster than all of the above iterative schemes. Therefore, we show that our hybrid iterative scheme converges faster than all the leading schemes. We find the solution of delay differential equations using our proposed hybrid iterative scheme while in last section, we prove some results of this scheme for nonexpansive mapping in the uniformly convex Banach space.

3. Convergence Analysis

This section deals with the rate of convergence of the Picard-Thakur hybrid iterative scheme (14) with Picard-S (12), Picard-Ishikawa (11), Picard-Mann (9), and Thakur et al. (6).

Proposition 10. *Assume that S be a nonempty closed convex subset of a normed space N and let $\mathscr{V} : S \longrightarrow S$ be a contraction mapping. Suppose that the iterative schemes (12), (11), (10), (9), and (6) converge to the same fixed point j^* of $\mathscr{V}$ where $\{\theta_n\}$, $\{\vartheta_n\}$, and $\{\sigma_n\}$ are sequences in $(0, 1)$ such that $0 < \mu \le \{\theta_n\}, \{\vartheta_n\}, \{\sigma_n\} < 1, \forall n \in I^+$, and for some μ and $\delta \in (0, 1)$. Then, the Picard-Thakur hybrid iterative scheme (14) converges faster than all the other schemes.*

Proof. Let j^* be a fixed point of an operator $\mathscr{V}$. Using the definition of contractive mapping and the Thakur et al. iterative scheme (6), we have

$$\begin{aligned}
\|k_{n+1} - j^*\| &= \|(1-\theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n - j^*\| \\
&\le (1-\theta_n)\|\mathscr{V} m_n - j^*\| + \theta_n\|\mathscr{V} l_n - j^*\| \\
&\le (1-\theta_n)\delta\|m_n - j^*\| + \theta_n\delta\|l_n - j^*\| \\
&\le (1-\theta_n)\delta(1-(1-\delta)\sigma_n)\|k_n - j^*\| \\
&\quad + \delta\theta_n(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n)\{1-\theta_n + \vartheta_n(1-(1-\delta)\sigma_n)\}]\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n)(1-(1-(1-\delta)\theta_n\sigma_n)]\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n - (1-\delta)\theta_n\vartheta_n) \\
&\quad + (1-\delta)^2\theta_n\vartheta_n\sigma_n]\|k_n - j^*\| \le \delta[(1-(1-\delta)\sigma_n \\
&\quad - (1-\delta)\theta_n\vartheta_n) + (1-\delta)\theta_n\vartheta_n\sigma_n]\|k_n - j^*\| \\
&\le \delta(1-(1-\delta)\sigma_n)\|k_n - j^*\|.
\end{aligned} \tag{21}$$

Let

$$a_n = \delta^n(1-(1-\delta)\sigma)^n\|k_1 - j^*\|. \tag{22}$$

Now, for (14),

$$\begin{aligned}
\|m_n - j^*\| &= \|(1-\sigma_n j_n + \mathscr{V} j_n - j^*\| \le (1-\sigma_n)\|j_n - j^*\| \\
&\quad + \sigma_n\delta\|j_n - j^*\| \le (1-(1-\delta)\sigma_n)\|j_n - j^*\|,
\end{aligned}$$

$$\begin{aligned}
\|l_n - j^*\| &= \|(1-\vartheta_n)m_n + \vartheta_n\mathscr{V} m_n - j^*\| \le (1-\vartheta_n)\|m_n - j^*\| \\
&\quad + \vartheta_n\delta\|m_n - j^*\| \le (1-(1-\delta)\vartheta_n)\|m_n - j^*\| \\
&\le (1-(1-\delta)\vartheta_n(1-(1-\delta)\sigma_n)\|j_n - j^*\|,
\end{aligned}$$

$$\begin{aligned}
\|k_n - j^*\| &= \|(1-\theta_n)\mathscr{V} m_n + \theta_n\mathscr{V} l_n - j^*\| \\
&\le \delta(1-\theta_n)\|m_n - j^*\| + \delta\theta_n\|l_n - j^*\| \\
&= \delta((1-\theta_n)(1-(1-\delta)\sigma_n)\|j_n - j^*\| \\
&\quad + \theta_n(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)\|j_n - j^*\|) \\
&= \delta(1-(1-\delta)\sigma_n)[1-\theta_n + \theta_n - (1-\delta)\theta_n\vartheta_n] \\
&\quad \cdot \|j_n - j^*\| = \delta(1-(1-\delta)\sigma_n - (1-(1-\delta)\sigma_n) \\
&\quad \cdot ((1-\delta)\theta_n\vartheta_n))\|j_n - j^*\| = \delta(1-(1-\delta)\sigma_n \\
&\quad - (1-\delta)\theta_n\vartheta_n + (1-\delta)^2\theta_n\vartheta_n\sigma_n)\|j_n - j^*\| \\
&\le \delta(1-(1-\delta)\sigma_n - (1-\delta)\theta_n\vartheta_n + (1-\delta)\theta_n\vartheta_n\sigma_n) \\
&\quad \cdot \|j_n - j^*\| \le \delta(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n \\
&\quad - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\|.
\end{aligned} \tag{23}$$

Also,

$$\begin{aligned}
\|j_{n+1} - j^*\| &= \|\mathscr{V} k_n - j^*\| \le \delta\|k_n - j^*\| \\
&\le \delta(\delta(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\| \\
&\le \delta^2(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\|.
\end{aligned} \tag{24}$$

Let

$$b_n = \delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n\|j_1-j^*\|. \tag{25}$$

Then,

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{\delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n\|j_1-j^*\|}{\delta^n(1-(1-\delta)\sigma)^n\|k_1-j^*\|} \\ &= \frac{\delta^{n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n\|j_1-j^*\|}{(1-(1-\delta)\sigma)^n\|k_1-j^*\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{26}$$

Thus, $\{j_n\}$ converges faster than $\{k_n\}$, i.e., the Picard-Thakur iterative scheme converges faster than the Thakur iterative scheme. Similarly, the inequality proved in Proposition 3.1 of the Picard-S hybrid iterative scheme [25] is as follows:

$$c_n = \delta^{2n}(1-(1-\delta)\theta\vartheta)^n\|a_1-j^*\|. \tag{27}$$

Then,

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{\delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n\|j_1-j^*\|}{\delta^{2n}(1-(1-\delta)\theta\vartheta)^n\|a_1-j^*\|} \\ &= \frac{(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n\|j_1-j^*\|}{(1-(1-\delta)\theta\vartheta)^n\|a_1-j^*\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{28}$$

Thus, $\{j_n\}$ converges faster than $\{a_n\}$., i.e., the Picard-Thakur iterative scheme converges faster than the Picard-S iterative scheme. Similarly, we can show that Picard-Thakur hybrid iterative scheme (14) converges faster than (11), (10), and (9). □

Next, we gave an example to show that the Picard-Thakur hybrid iterative scheme (14) converges faster than the Picard-S hybrid, Picard-Ishikawa hybrid, Picard-Mann hybrid, and Thakur iterative schemes.

Example 11. Let $\mathscr{V} : S \longrightarrow S$ where $S = [0, 2] \subset N = \mathbb{R}$ be an operator defined by

$$\mathscr{V}(j) = \begin{cases} 1, & \text{if } j \in [0,1], \\ \sqrt{\dfrac{4-j^2}{3}}, & \text{if } j \in [1,2]. \end{cases} \tag{29}$$

Choose $\theta_n = (n+2)/(n+6)$, $\vartheta_n = (n^2+1)/(n^2+n+1)$, $\sigma_n = \sqrt{(n+1)/(2n+7)}$, for each $n \in I^+$ with an initial value $j_1 = 0.6.\mathscr{V}$ is nonexpansive mapping. All the iterative schemes converge to the fixed point $j^* = 1$. Clearly, in the Table 1 and Figure 1, we can see that the Picard-Thakur hybrid iterative scheme (14) converges faster than the schemes discussed above.

4. Application: Delay Differential Equations

In this section, we can find the solution of the delay differential equation by using our proposed iterative scheme.

Let the space of all continuous real-valued functions be denoted by $C([u, v])$ on closed interval $[u, v]$ endowed with the Chebyshev norm $\|j-m\|_\infty$ and defined as $\|j-m\|_\infty = \sup_{r\in[u,v]}|j(r)-m(r)|$, and it is clear that in [41] that $(C([u, v], \|.\|_\infty))$ is a Banach space. Now, consider the following delay differential equation

$$j'(r) = \psi(r, j(r), j(r-\gamma)), \quad r \in [r_0, v], \tag{30}$$

with initial condition

$$j(r) = \zeta(r), \quad r \in [r_0-\gamma, r_0]. \tag{31}$$

By the solution of the above delay differential equation, we mean a function $j \in C([r_0-\gamma, v], \mathbb{R}) \cap C^1([r_0, v], \mathbb{R}))$ satisfying (30) and (31).

Assume that the following conditions are satisfied.

(1) $r_0, v \in \mathbb{R}), \gamma > 0$

(2) $\psi \in C([r_0, v] \times \mathbb{R}^2, \mathbb{R})$

(3) $\zeta \in C([r_0-\gamma, v], \mathbb{R})$

(4) There exists $L_\psi > 0$ such that

$$|\psi(r, s_1, s_2) - \psi(r, t_1, t_2)| \le L_\psi \Sigma_{i=1}^2 |s_i - t_i|, \quad \forall s_i, t_i \in \mathbb{R}, r \in [r_0, v] \tag{32}$$

(5) $2L_\psi(v-r_0) < 1$

Now, we construct (30) and (31) by the integral equation as

$$j(r) = \begin{cases} \zeta(r), & r \in [r_0-\gamma, v], \\ \zeta(r_0) + \displaystyle\int_{r_0}^{r} \psi(t, j(t), j(t-\gamma))dt, & r \in [r_0, v]. \end{cases} \tag{33}$$

The following result is the generalization of the result of Coman et al. [42].

Theorem 12. *Let the conditions $*_1$) to $*_5$) be satisfied. Then, (30) and (31) have unique solution $j^* \in C([r_0-\gamma], \mathbb{R}) \cap C^1([r_0 v], \mathbb{R})$ and*

$$j^* = \lim_{n\longrightarrow\infty} \mathscr{V}^n(j), \quad \textit{for any } j \in C([r_0-\gamma, v], \mathbb{R}). \tag{34}$$

Now, by using the Picard-Thakur hybrid iterative scheme (14), we prove the following result.

Theorem 13. *Let the conditions $*_1) - *_5)$ be satisfied. Then, (30) and (31) have a unique solution $j^* \in C([r_0 - \gamma], \mathbb{R}) \cap C^1([r_0, \nu], \mathbb{R})$ and the Picard-Thakurb hybrid iterative scheme (14) converges to j^*.*

Proof. Let $\{j_n\}$ be a sequence generated by the Picard-Thakur hybrid iterative scheme (14) for an operator $\mathscr{V}$ defined by

$$\mathscr{V} j(r) = \begin{cases} \zeta(r), & r \in [r_0 - \gamma, \nu], \\ \zeta(r_0) + \int_{r_0}^{r} \psi(p, j(p), j(p - \gamma))dp, & r \in [r_0, \nu]. \end{cases} \tag{35}$$

Let j^* be a fixed point of $\mathscr{V}$. Now, we prove that $j_n \longrightarrow j^*$ as $n \longrightarrow \infty$. It is easy to see that $j_n \longrightarrow j^*$ as $n \longrightarrow \infty$ for each $r \in [r_0 - \gamma, r_0]$.

Now, for each $r \in [r_0, \nu]$, we have

$$\begin{aligned} \|j_{n+1} - j^*\|_\infty &\le \|\mathscr{V} k_n - j^*\|_\infty \le \sup_{r_0 \in [r_0, \nu]} |\mathscr{V} k_n - \mathscr{V} j^*| \le \sup_{r_0 \in [r_0, \nu]} \Bigg|\zeta(r_0) \\ &\quad + \int_{r_0}^{r} \psi(p, k_n(p), k_n(p - \gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\right)\Bigg| \\ &\le \sup_{r_0 \in [r_0, \nu]} \int_{r_0}^{r} |\psi(p, k_n(p), k_n(p - \gamma)) - \psi(p, j^*(p), j^*(p - \gamma))|dp \\ &\le \sup_{r_0 \in [r_0, \nu]} \int_{r_0}^{r} L_\psi(|k_n(p) - j^*(p)| + |k_n(p - \gamma) - j^*(p - \gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi \sup_{r_0 \in [r_0, \nu]} (|k_n(p) - j^*(p)| + |k_n(p - \gamma) - j^*(p - \gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\|k_n - j^*\|_\infty + \|k_n - j^*\|_\infty\right)dp \le 2L_\psi(\nu - r_0)\|k_n - j^*\|_\infty. \end{aligned} \tag{36}$$

Now,

$$\begin{aligned} \|k_n - j^*\|_\infty &= \|(1 - \theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n - j^*\|_\infty \\ &\le (1 - \theta_n)\|\mathscr{V} m_n - j^*\|_\infty + \theta_n\|\mathscr{V} l_n - j^*\|_\infty, \end{aligned} \tag{37}$$

As

$$\begin{aligned} \|\mathscr{V} l_n - j^*\|_\infty &= \|\mathscr{V} l_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, \nu]} \\ &\quad \cdot \Bigg|\zeta(r_0) + \int_{r_0^r} \psi(p, l_n(p), l_n(p - \gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\right)\Bigg| \le \sup_{r \in [r_0 - \gamma, \nu]} \\ &\quad \cdot \Bigg|\int_{r_0}^{r} \psi(p, l_n(p), l_n(p - \gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\Bigg| \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} |\psi(p, l_n(p), l_n(p - \gamma) - \psi(p, j^*(p), j^*(p - \gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} |\psi(p, l_n(p), l_n(p - \gamma) - \psi(p, j^*(p), j^*(p - \gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} L_\psi(|l_n(p) - j^*(p)| + |l_n(p - \gamma) - j^*(p - \gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\sup_{r \in [r_0 - \gamma, \nu]} |l_n(p) - j^*(p)| + \sup_{r \in [r_0 - \gamma, \nu]} |l_n(p - \gamma) - j^*(p - \gamma)|\right)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\|l_n - j^*\|_\infty + \|l_n - j^*\|_\infty\right)dp \\ &\le 2L_\psi(r - r_0)\|l_n - j^*\|_\infty \le 2L_\psi(\nu - r_0)\|l_n - j^*\|_\infty, \end{aligned} \tag{38}$$

$$\begin{aligned} \|l_n - j^*\|_\infty &= \|(1 - \vartheta_n)m_n + \vartheta_n \mathscr{V} m_n - j^*\|_\infty \\ &\le (1 - \vartheta_n)\|m_n - j^*\|_\infty + \vartheta_n\|\mathscr{V} m_n - j^*\|_\infty. \end{aligned} \tag{39}$$

For

$$\begin{aligned} \|\mathscr{V} m_n - j^*\|_\infty &= \|\mathscr{V} m_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, \nu]} \\ &\quad \cdot \Bigg|\zeta(r_0) + \int_{r_0^r} \psi(p, m_n(p), m_n(p - \gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\right)\Bigg| \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \Bigg|\int_{r_0}^{r} \psi(p, m_n(p), m_n(p - \gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\Bigg| \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} |\psi(p, m_n(p), m_n(p - \gamma) - \psi(p, j^*(p), j^*(p - \gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} |\psi(p, m_n(p), m_n(p - \gamma) - \psi(p, j^*(p), j^*(p - \gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, \nu]} \int_{r_0}^{r} L_\psi(|m_n(p) - j^*(p)| + |m_n(p - \gamma) - j^*(p - \gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\sup_{r \in [r_0 - \gamma, \nu]} |m_n(p) - j^*(p)| + \sup_{r \in [r_0 - \gamma, \nu]} |m_n(p - \gamma) - j^*(p - \gamma)|\right)dp \\ &\le \int_{r_0}^{r} L_\psi(\|m_n - j^*\|m_n - j^*\|\|_\infty + |m_n - j^*\|m_n - j^*\|\|_\infty)\,dp \\ &\le 2L_\psi(r - r_0)\|m_n - j^*\|_\infty \le 2L_\psi(\nu - r_0)\|m_n - j^*\|_\infty, \end{aligned} \tag{40}$$

$$\begin{aligned} \|m_n - j^*\|_\infty &= \|(1 - \sigma_n)j_n + \sigma_n \mathscr{V} j_n - j^*\|_\infty \\ &\le (1 - \sigma_n)\|j_n - j^*\|_\infty + \sigma_n\|\mathscr{V} j_n - j^*\|_\infty, \end{aligned} \tag{41}$$

as

$$\begin{aligned} \|\mathscr{V} j_n - j^*\| &= \|\mathscr{V} j_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, \nu]} \\ &\quad \cdot \Bigg|\zeta(r_0) + \int_{r_0^r} \psi(p, j_n(p), j_n(p - \gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\right)\Bigg| \le \sup_{r \in [r_0 - \gamma, \nu]} \\ &\quad \cdot \Bigg|\int_{r_0}^{r} \psi(p, j_n(p), j_n(p - \gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p - \gamma))dp\Bigg| \end{aligned}$$

Table 1: Convergence behavior of Thakur et al. (7), Ap (8), Picard-S (12), Picard-S^* (13), and Picard-Thakur hybrid Iterative schemes (14).

Steps	Picard-Ishikawa hybrid	Thakur et al.	Ap iterative scheme	Picard-S hybrid	Picard-S^* hybrid	Picard-Thakur hybrid
1	0.6000000000	0.6000000000	0.6000000000	0.6000000000	0.6000000000	0.6000000000
2	1.0172938494	1.0023262974	1.0033992688	1.0028485141	0.9942090597	0.9992233640
3	0.9991010616	0.9999896158	0.9999617294	0.9999670218	0.9998760856	0.9999988412
4	1.0000463544	1.0000000464	1.0000004299	1.0000003808	0.9999973348	0.9999999983
5	0.9999976087	0.9999999997	0.9999999952	0.9999999956	0.9999999427	0.9999999999
6	1.0000001234	1.0000000000	1.0000000001	1.0000000001	0.9999999988	0.9999999999
7	0.9999999936	0.9999999999	0.9999999999	0.9999999999	0.9999999999	1.0000000000
8	1.0000000003	1.0000000000	1.0000000000	1.0000000000	0.9999999999	1.0000000000
9	0.9999999999	1.0000000000	1.0000000000	1.0000000000	0.9999999999	1.0000000000
10	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

$$
\begin{aligned}
&\leq \sup_{r\in[r_0-\gamma,\nu]}\int_{r_0}^{r}|\psi(p,j_n(p),j_n(p-\gamma)-\psi(p,j^*(p),j^*(p-\gamma))|dp\\
&\leq \sup_{r\in[r_0-\gamma,\nu]}\int_{r_0}^{r}|\psi(p,j_n(p),j_n(p-\gamma)-\psi(p,j^*(p),j^*(p-\gamma))|dp\\
&\leq \sup_{r\in[r_0-\gamma,\nu]}\int_{r_0}^{r}L_\psi(|j_n(p)-j^*(p)|+|j_n(p-\gamma)-j^*(p-\gamma)|)dp\\
&\leq \int_{r_0}^{r}L_\psi\left(\sup_{r\in[r_0-\gamma,\nu]}|j_n(p)-j^*(p)|+\sup_{r\in[r_0-\gamma,\nu]}|j_n(p-\gamma)-j^*(p-\gamma)|\right)dp\\
&\leq \int_{r_0}^{r}L_\psi(\|j_n-j^*\|_\infty+\|j_n-j^*\|_\infty)dp\leq 2L_\psi(r-r_0)\|j_n-j^*\|_\infty\\
&\leq 2L_\psi(\nu-r_0)\|j_n-j^*\|_\infty.
\end{aligned} \tag{42}
$$

Putting (42) in (41), we get

$$
\begin{aligned}
\|m_n-j^*\|_\infty &\leq (1-\sigma_n)\|j_n-j^*\|_\infty+\sigma_n 2L_\psi(\nu-r_0)\|j_n-j^*\|_\infty\\
&\leq [1-(1-2L_\psi(\nu-r_0)\sigma_n]\,\|j_n-j^*\|_\infty.
\end{aligned} \tag{43}
$$

Putting (43) in (40), we get

$$
\|\mathscr{V}m_n-j^*\|_\infty\leq 2L_\psi(\nu-r_0)[1-((11-2L_\psi(\nu-r_0)\sigma_n]\,\|j_n-j^*\|_\infty. \tag{44}
$$

Putting (44) and (43) in (39), we get

$$
\begin{aligned}
\|l_n-j^*\|_\infty &\leq (1-\vartheta_n)[1-((11-2L_\psi(\nu-r_0)\sigma_n]\,\|j_n-j^*\|_\infty\\
&\quad+\vartheta_n 2L_\psi(\nu-r_0)[1-(1-2L_\psi(\nu-r_0)\sigma_n]\,\|j_n-j^*\|_\infty\\
&\leq (1-(1-2L_\psi(\nu-r_0)\sigma_n)(1-(1-(1-2L_\psi(\nu-r_0)\vartheta_n)\,\|j_n-j^*\|_\infty\\
&\leq [1-(1-2L_\psi(\nu-r_0)\sigma_n)-(1-(1-2L_\psi(\nu-r_0)\sigma_n)(1-2L_\psi(\nu-r_0)\vartheta_n)]\\
&\quad\cdot\|j_n-j^*\|_\infty\leq[1-(1-2L_\psi(\nu-r_0))\sigma_n-(1-2L_\psi(\nu-r_0))\vartheta_n\\
&\quad+[(1-2L_\psi(\nu-r_0)]^2\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty\leq[1-(1-2L_\psi(\nu-r_0))\sigma_n\\
&\quad-(1-2L_\psi(\nu-r_0))\vartheta_n+(1-2L_\psi(\nu-r_0))\vartheta_n\sigma_n]\,\|j_n-j^*\|_\infty\\
&\leq[1-(1-2L_\psi(\nu-r_0))(\sigma_n-\vartheta_n+\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty.
\end{aligned} \tag{45}
$$

Putting (45) in (38), we get

$$
\|\mathscr{V}l_n-j^*\|_\infty\leq 2L_\psi(\nu-r_0)[1-(1-2L_\psi(\nu-r_0))(\sigma_n-\vartheta_n+\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty. \tag{46}
$$

Putting (46) and (40) in (37), we get

$$
\begin{aligned}
\|k_n-j^*\|_\infty &\leq (1-\theta_n)2L_\psi(\nu-r_0)\|m_n-j^*\|_\infty+\theta_n 2L_\psi(\nu-r_0)\\
&\quad\cdot\|l_n-j^*\|_\infty\leq 2L_\psi(\nu-r_0)[(1-\theta_n)\|m_n-j^*\|_\infty+\theta_n\|l_n-j^*\|_\infty]\\
&\leq 2L_\psi(\nu-r_0)[(1-\theta_n)[1-(1-2L_\psi(\nu-r_0)\sigma_n]\,\|j_n-j^*\|_\infty\\
&\quad+\theta_n[1-(1-2L_\psi(\nu-r_0))(\sigma_n-\vartheta_n+\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty]\\
&\leq 2L_\psi(\nu-r_0)[1-\theta_n)(1-(1-2L_\psi(\nu-r_0)\sigma_n)\\
&\quad+\theta_n[1-(1-2L_\psi(\nu-r_0))(\sigma_n-\vartheta_n+\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty\\
&\leq 2L_\psi(\nu-r_0)[1-\theta_n-(1-2L_\psi(\nu-r_0)\sigma_n+(1-2L_\psi(\nu-r_0)\theta_n\sigma_n\\
&\quad+\theta_n-(1-2L_\psi(\nu-r_0)\theta_n(\sigma_n-\vartheta_n+\vartheta_n\sigma_n)]\,\|j_n-j^*\|_\infty\\
&\leq 2L_\psi(\nu-r_0)[1-(1-2L_\psi(\nu-r_0)(\sigma_n-\theta_n\sigma_n\\
&\quad+\theta_n(\sigma_n+\vartheta_n-\vartheta_n\sigma_n)]\|j_n-j^*\|_\infty\leq 2L_\psi(\nu-r_0)\\
&\quad\cdot[1-(1-2L_\psi(\nu-r_0)(\sigma_n-\theta_n\sigma_n+\theta_n\sigma_n+\theta_n\vartheta_n-\theta_n\vartheta_n\sigma_n)]\,\|j_n\\
&\quad-j^*\|_\infty\leq 2L_\psi(\nu-r_0)[1-(1-2L_\psi(\nu-r_0)(\sigma_n+\theta_n\vartheta_n-\theta_n\vartheta_n\sigma_n)]\\
&\quad\cdot\|j_n-j^*\|_\infty.
\end{aligned} \tag{47}
$$

Let $\sigma_n+\theta_n\vartheta_n-\theta_n\vartheta_n\sigma_n=\rho_n$, and by using condition $*_5$), we have

$$
\|k_n-j^*\|_\infty\leq[1-(1-2L_\psi(\nu-r_0)\rho_n]\,\|j_n-j^*\|_\infty. \tag{48}
$$

Putting (48) in (36), we have

$$
\|j_{n+1}-j^*\|_\infty\leq 2L_\psi(\nu-r_0)[1-(1-2L_\psi(\nu-r_0)\rho_n]\,\|j_n-j^*\|_\infty. \tag{49}
$$

Again, using condition $*_5$), we get

$$
\|j_{n+1}-j^*\|_\infty\leq[1-(1-2L_\psi(\nu-r_0)\rho_n]\,\|j_n-j^*\|_\infty. \tag{50}
$$

Let $(1-2L_\psi(\nu-r_0)\rho_n=\tau_n<1$ and $\|j_n-j^*\|_\infty=r_n$. So, the conditions of Lemma 3 are satisfied. Hence, $\lim_{n\longrightarrow\infty}\|j_n-j^*\|=0$. □

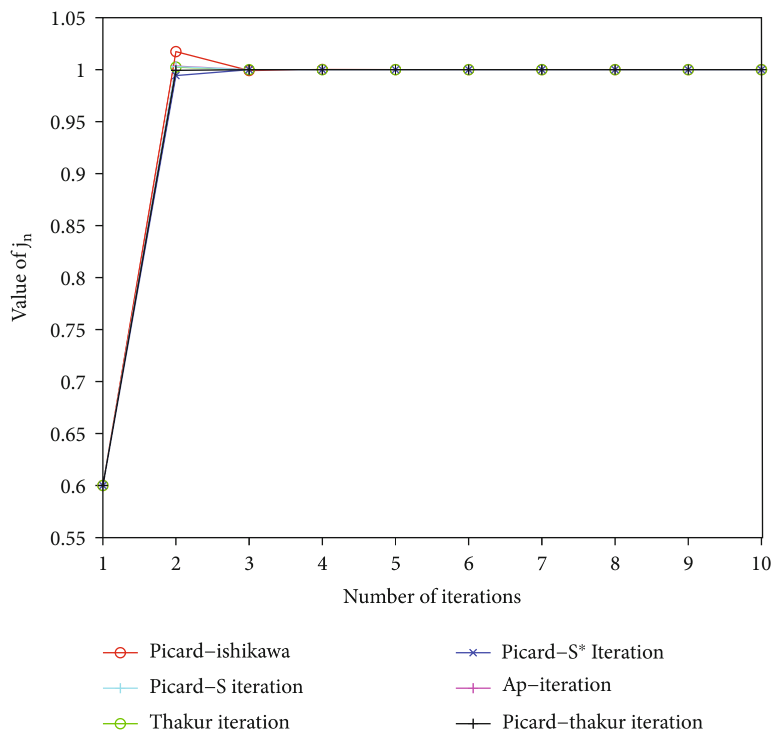

FIGURE 1: Convergence behavior of Thakur et al. (7), Ap (8), Picard-S (12), Picard-S^* (13), and Picard-Thakur hybrid iterative schemes (14).

5. Convergence Results for Nonexpansive Mapping

Lemma 14. *Let S be a nonempty closed and convex subset of uniformly convex Banach space B and $\mathscr{V}: S \longrightarrow S$ be a nonexpansive mapping. If $\{j_n\}$ be a sequence generated by Picard-Thakur hybrid iterative scheme (14) and $F(\mathscr{V}) \neq \varnothing$, then, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists.*

Proof. Let $j^* \in F(\mathscr{V})$, and $\mathscr{V}$ is nonexpansive then

$$\begin{aligned}\|m_n - j^*\| &= \|(1-\sigma_n)j_n + \sigma_n\mathscr{V}j_n - j^*\| \leq (1-\sigma_n)\|j_n \\ &- j^*\| + \sigma_n\|\mathscr{V}j_n - j^*\| \leq (1-\sigma_n)\|j_n - j^*\| + \sigma_n\|j_n \\ &- j^*\| \leq \|j_n - j^*.\end{aligned} \tag{51}$$

Also,

$$\begin{aligned}\|l_n - j^*\| &= \|(1-\vartheta_n)m_n + \vartheta_n\mathscr{V}m_n - j^*\| \leq (1-\vartheta_n)\|m_n \\ &- j^*\| + \vartheta_n\|\mathscr{V}m_n - j^*\| \leq (1-\vartheta_n)\|m_n - j^*\| + \vartheta_n\|m_n \\ &- j^*\| \leq \|m_n - j^*.\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}\|k_n - j^*\| &= \|(1-\theta_n)\mathscr{V}m_n + \theta_n\mathscr{V}l_n - j^*\| \leq (1-\theta_n)\|\mathscr{V}m_n \\ &- j^*\| + \theta_n\|\mathscr{V}l_n - j^*\| \leq (1-\theta_n)\|m_n - j^*\| + \theta_n\|l_n - j^*\| \\ &\leq (1-\theta_n)\|m_n - j^*\| + \theta_n\|m_n - j^*\| \leq \|m_n - j^*\| \leq \|j_n - j^*\|.\end{aligned} \tag{53}$$

Now,

$$\|j_{n+1} - j^*\| = \|\mathscr{V}k_n - j^*\| \leq \|k_n - j^*\| \leq \|j_n - j^*\|. \tag{54}$$

This shows that $\{\|j_n - j^*\|\}$ is a decreasing sequence and bounded below $\forall j^* \in F(\mathscr{V})$. Hence, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. □

Lemma 15. *Let S and $\mathscr{V}: S \longrightarrow S$ be as in Lemma 14. Let $\{j_n\}$ be a sequence defined by Picard-Thakur hybrid iterative scheme (14) with $F(\mathscr{V}) \neq \varnothing$. Then, $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0$.*

Proof. As from the above Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists for each $j^* \in F(\mathscr{V})$. Suppose that for some $l \geq 0$, we have

$$\lim_{n\longrightarrow\infty}\|j_n - j^*\| = l. \tag{55}$$

As from (53), (52), and (51), we have

$$\|m_n - j^*\| \leq \|j_n - j^*\|, \tag{56}$$

$$\|l_n - j^*\| \leq \|j_n - j^*\|, \tag{57}$$

$$\|k_n - j^*\| \leq \|j_n - j^*\|. \tag{58}$$

Taking lim sup as $n \longrightarrow \infty$ of (58), (57), and (56), we get

$$\limsup_{n\longrightarrow\infty}\|m_n - j^*\| \leq l, \tag{59}$$

$$\limsup_{n\longrightarrow\infty}\|l_n - j^*\| \leq l, \tag{60}$$

$$\limsup_{n\longrightarrow\infty}\|k_n - j^*\| \le l. \quad (61)$$

Since $\mathscr{V}$ is nonexpansive, we have

$$\limsup_{n\longrightarrow\infty}\|\mathscr{V}j_n - j^*\| \le l, \quad (62)$$

$$l = \liminf_{n\longrightarrow\infty}\|j_{n+1} - j^*\| = \liminf_{n\longrightarrow\infty}\|\mathscr{V}k_n - j^*\| \le \liminf_{n\longrightarrow\infty}\|k_n - j^*\|, \quad (63)$$

From (63) and (61), we get

$$\lim_{n\longrightarrow\infty}\|k_n - j^*\| = l. \quad (64)$$

Now, from (53), we have

$$\|k_n - j^*\| \le \|m_n - j^*\|. \quad (65)$$

Taking liminf as $n \longrightarrow \infty$, we have

$$\liminf_{n\longrightarrow\infty}\|k_n - j^*\| \le \liminf_{n\longrightarrow\infty}\|m_n - j^*\|, \quad (66)$$

$$l \le \liminf_{n\longrightarrow\infty}\|m_n - j^*\|. \quad (67)$$

So, from (67) and (59), we have

$$l = \lim_{n\longrightarrow\infty}\|m_n - j^*\| = \lim_{n\longrightarrow\infty}\|(1-\sigma_n)j_n + \sigma_n\mathscr{V}j_n - j^*\| = \lim_{n\longrightarrow\infty}\|(1-\sigma_n)(j_n - j^*) + \sigma_n(\mathscr{V}j_n - \mathscr{V}j^*)\|. \quad (68)$$

From (68), (62), and (55) and applying Lemma 6, we get

$$\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0. \quad (69)$$

□

Theorem 16. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Let B be the uniformly convex Banach space which satisfies Opial's condition; then, $\{j_n\}$ converges weakly to a fixed point of $\mathscr{V}$.*

Proof. Let $j^* \in F(\mathscr{V})$; then, by Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. Now, we show that $\{j_n\}$ has a unique weak subsequential limit in $F(\mathscr{V})$.

Let $\{a_n\}$ and $\{b_n\}$ be two subsequences of $\{j_n\}$ and a, b be the weak limits of the subsequences of $\{j_n\}$, respectively. From Lemma 15, $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}(j_n)\| = 0$ and $I - \mathscr{V}$ is demiclosed at zero. By Lemma 7.

Therefore, we get $\mathscr{V}a = a$. For $b \in F(\mathscr{V})$, we follow the same manner.

From Lemma 14, we know that $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. For uniqueness, supposing that $a \ne b$, then, by using Opial's condition,

$$\lim_{n\longrightarrow\infty}\|j_n - a\| = \lim_{n\longrightarrow\infty}\|a_n - a\| < \lim_{n\longrightarrow\infty}\|a_n - b\| = \lim_{n\longrightarrow\infty}\|j_n - b\| = \lim_{n\longrightarrow\infty}\|b_n - b\| < \lim_{n\longrightarrow\infty}\|b_n - a\| = \lim_{n\longrightarrow\infty}\|j_n - a\|. \quad (70)$$

This is a contradiction, so $a = b$. Hence, $\{j_n\}$ converges weakly to $F(\mathscr{V})$. □

Theorem 17. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Then, $\{j_n\}$ converges to a point of $F(\mathscr{V})$ if and only if $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ or $\limsup_{n\longrightarrow\infty}(j_n, F(\mathscr{V})) = 0$, where $d(a_n, F(\mathscr{V})) = \inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\}$.*

Proof. If the sequence $\{j_n\} \longrightarrow j^* \in F(\mathscr{V})$, then, it is oblivious that $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ or $\limsup_{n\longrightarrow\infty}(j_n, F(\mathscr{V})) = 0$.

Conversely, assume that $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$. From Lemma 14,

$\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists, $\forall j^* \in F(\mathscr{V})$. Therefore, by assumption,

$$\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0. \quad (71)$$

Now, to show, the sequence $\{j_n\}$ is cauchy in S. As $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$, for given $\lambda > 0$, there exists $m_0 \in I^+$ such that $\forall n \ge m_0$,

$$d(j_n, F(\mathscr{V})) < \frac{\lambda}{2} \Rightarrow \inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\} < \frac{\lambda}{2}. \quad (72)$$

Particularly, $\inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\} < \lambda/2$. Therefore, there is $j^* \in F(\mathscr{V})$ such that

$$\|j_{m_0} - j^*\| < \frac{\lambda}{2}. \quad (73)$$

Now, for $m, n \ge m_0$,

$$\|j_{n+m} - j_n\| \le \|j_{m+n} - j^*\| + \|j_n - j^*\| \le \|j_{m_0} - j^*\| + \|j_{m_0} - j^*\| = 2\|j_{m_0} - j^*\| < \lambda. \quad (74)$$

This shows that the sequence $\{j_n\}$ is cauchy in S. As $S \subset B$, so, p is a point in S such that $\lim_{n\longrightarrow\infty}j_n = p$. Now, $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ gives that $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0 \Rightarrow p \in F(\mathscr{V})$. □

Theorem 18. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Then, $\{j_n\}$ converges strongly to $F(\mathscr{V}) \ne \varnothing$.*

Proof. By Lemma 15, we have

$$\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0. \quad (75)$$

Since, S is compact, then, let $\{j_{n_k}\}$ be a subsequence of $\{j_n\}$ which converges strongly to j^*, for some $j^* \in S$. By Proposition 8, we have

$$\|j_{n_k} - \mathscr{V} j^*\| \le 3\|j_{n_k} - \mathscr{V} j_{n_k}\| + \|j_{n_k} - j^*\| \quad \forall k \ge 1. \tag{76}$$

Letting $k \longrightarrow \infty$, we get

$$j_{n_k} \longrightarrow \mathscr{V} j^* \Rightarrow \mathscr{V} j^* = j^*, \quad \text{i.e.,} j^* \in F(\mathscr{V}). \tag{77}$$

Also, by Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. Thus, $\{j_n\}$ converges strongly to j^*. □

Now, by using condition (I), we prove the strong convergence result.

Theorem 19. *Let S, $\mathscr{V}$ be as in Lemma 14. Let B be a uniformly convex Banach space which is satisfying condition (I). Then, the sequence $\{j_n\}$ defined by the Picard-Thakur hybrid iterative scheme (14) converges strongly to $F(\mathscr{V}) \neq \varnothing$.*

Proof. As by Lemma 15, we have

$$\lim_{n\longrightarrow\infty} \|j_n - \mathscr{V} j_n\| = 0. \tag{78}$$

By condition (I) and (78), we get

$$\begin{aligned} 0 &\le \lim_{n\longrightarrow\infty} Z(d(j_n, F(\mathscr{V}))) \\ &\le \lim_{n\longrightarrow\infty} \|j_n - \mathscr{V} j_n\| \Rightarrow \lim_{n\longrightarrow\infty} Z(d(j_n, F(\mathscr{V}))) = 0. \end{aligned} \tag{79}$$

Since $Z : [0,\infty) \longrightarrow [0,\infty)$ is an increasing function satisfying $Z(0) = 0, Z(t) > 0 \forall t > 0$.

Hence, we have

$$\lim_{n\longrightarrow\infty} d(j_n, F(\mathscr{V})) = 0. \tag{80}$$

Since all the conditions of Theorem 17 are satisfied, therefore, we can say that $\{j_n\}$ converges strongly to $F(\mathscr{V})$. □

6. Conclusion

In this paper, we present a new hybrid scheme of Picard and Thakur et al. We discuss the convergence of this scheme to the iterative scheme of Mann, Ishikawa, Picard-Mann, Picard-Ishikawa, Picard-S, and Thakur et al. We showed the convergence of Picard-Thakur hybrid iterative with other iterative schemes on graphs and gave application to delay differential equations. We also generalize and extend various results for nonexpansive mapping in a uniformly convex Banach space including [7, 24, 25, 43].

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

Jie Jia analyzed the results and used a software to compare the results, Khurram Shabbir proposed the problem and supervised this work, Khushdil Ahmad wrote the first version of this paper, Nehad Ali Shah verified the results and wrote the final version of this paper, and Thongchai Botmart prepared the example sketch and the plots and arranged the funding for this paper. Jie Jia and Nehad Ali Shah are the first co-authors and contributed equally in this work.

Acknowledgments

This research work is supported by the Higher Education Commission (Islamabad) through the National Research Program for Universities, Grant no. 7359/Punjab/NRPU/R$D/HEC/2017.

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