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Research Article

A Comparative Analysis of Fractional Space-Time Advection-Dispersion Equation via Semi-Analytical Methods

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The approximate solutions of the time fractional advection-dispersion equation are presented in this article. The nonlocal nature of solute movement and the nonuniformity of fluid flow velocity in the advection-dispersion process lead to the formation of a heterogeneous system, which can be modeled using a fractional advection-dispersion equation, which generalizes the classical advection-dispersion equation and replaces the time derivative with the fractional Caputo derivative. Researchers use a variety of numerical techniques to study such fractional models, but the nonlocality of the derivative having fractional order leads to high computation complexity and complex calculations, so the task is to find an efficient technique that requires less computation and provides greater accuracy when numerically solving such models. A innovative techniques, homotopy perturbation method and new iteration method, are used in connection with the Elzaki transform to solve the "fractional advection-dispersion equation" which provides the solution in the convergent series form. When the homotopy perturbation method is used with the Elzaki transform, fast convergent series solutions can be obtained with less computation. By solving some cases of time-fractional advection-dispersion equation with varied initial conditions with the help of new iterative transform method and homotopy perturbation transform method demonstrates the usefulness of the proposed methods.

1. Introduction

For the past 300 years, fractional calculus has been used to generalize the integration and differentiation of integer order to arbitrary order. Due to its nonlocal nature, fractional differential equations are well adapted to explain diverse phenomena in engineering and science, and the researchers' growing interest in this field has led to solving real-world problems in type of fractional differential equations. In addition, fractional derivatives can be used for description in a variety of phenomena that have memory and hereditary properties by mathematical way [1–5]. Fractional order differential equations have been shown to be a valuable tool for revealing hidden characteristics in a variety of real-world processes, including physical sciences, signal processing, electromagnetics, earthquakes, traffic flow, and the study of viscoelastic material properties and many more processes [6–11]. The historical and nonlocal distributed effects are considered via fractional differential coefficients; an outstanding literature on this topic may be found in numerous monographs [12–15]. For this reason, many authors are attracted to knowing the properties of fractional differential equations and vast applications in modeling and engineering fields [16–19].

The ADE is used in the study of solute transport or Brownian motion of particles in a fluid that occurs when advection and particle dispersion occur at the same time [20, 21]. The fractional advection-dispersion equation better represents the phenomenon of anomalous particle diffusion in the transport process; in anomalous diffusion, solute transport is faster or faster than the time's inferred square root given by Baeumer et al. [22]. The equation is used to investigate groundwater pollution, smoke or dust pollution of the atmosphere, and the spread of chemical solutes and pollutant discharges [23]. As a result, FADE has caught the interest of numerous researchers. As a result, the researchers are interested in solving the FADE to determine the solute concentration at a specific time and location [24, 25]. Jaiswal et al. [26] discovered an analytical solution for one-dimensional ADE. Huang et al. [27] developed finite element solutions to the one-dimensional fractional flux ADE. El-Sayed et al. [28] investigated the intermediate fractional ADE. Momani and Odibat [23] used the ADM and variational iteration approach to solve the space-time fractional ADE. In this aspect, Yildirim and Kocak [29] use the homotopy perturbation methodology in Caputo sense to solve the space-time fractional ADE, whereas Hikal and Abu Ibrahim [30] use the Adomian decomposition method. Using the generalized finite rate chemistry model, Alliche and Chikh [31] investigated the nonpremixed chaotic fire of the hydrogen-air downward injector system. Liu et al. [32] investigated various advection-dispersion models using numerical methods. For solar cosmic-ray transport, Rocca et al. [33] established the fractional diffusion-advection equation general solution. Ramani et al. [34] proposed the fractional reduced differential transform method for revisiting the time-fractional Rosenau-Hyman problem's analytical-approximate formulation.

We apply both the novel iterative method presented by Gejji and Jafari [35] and the homotopy perturbation transform method proposed by Madani et al. [36] and Khan and Wu in the current paper [37]. The first technique has been shown to be effective in solving a wide range of nonlinear equations, including algebraic equations, integral equations, ordinary and partial differential equations of integer and fractional order, and systems of equations. The new iterative method is straightforward to explain and use with computer packages, and it produces superior results than the previous Adomain decomposition [38], homotopy perturbation [39], and variational iteration methods [40]. The second technique combines the Elzaki transformation, the homotopy perturbation method, and He's polynomials in a simple manner. The suggested algorithm generates a solution in a rapid convergent series, which could lead to a closed solution. This method has the advantage of being able to combine two powerful methods for finding exact solutions to linear and nonlinear partial differential equations.

2. Basic Definitions

2.1. Definition. The fractional operator D^σ having order σ in Abel-Riemann manner is calculated as [41–43]

$$D^\sigma \nu(\varphi) = \begin{cases} \dfrac{d^j}{d\varphi^j}\nu(\varphi), & \sigma = j, \\ \dfrac{1}{\Gamma(j-\sigma)}\dfrac{d}{d\varphi^j}\displaystyle\int_0^\varphi \dfrac{\nu(\varphi)}{(\varphi-\mu)^{\sigma-j+1}}d\mu, & j-1<\sigma<j, \end{cases} \quad (1)$$

where $j \in Z^+$, $\sigma \in R^+$, and

$$D^{-\sigma}\nu(\varphi) = \frac{1}{\Gamma(\sigma)}\int_0^\varphi (\varphi-\mu)^{\sigma-1}\nu(\mu)d\mu,\ 0<\sigma\leq 1. \quad (2)$$

2.2. Definition. The Abel-Riemann integration operator J^μ having fractional order is given as [35–37]

$$J^\sigma \nu(\varphi) = \frac{1}{\Gamma(\sigma)}\int_0^\varphi (\varphi-\mu)^{\sigma-1}\nu(\varphi)d\varphi, \quad \varphi>0, \quad \sigma>0. \quad (3)$$

With basic properties:

$$\begin{aligned} J^\sigma \varphi^j &= \frac{\Gamma(j+1)}{\Gamma(j+\sigma+1)}\varphi^{j+\mu}, \\ D^\sigma \varphi^j &= \frac{\Gamma(j+1)}{\Gamma(j-\sigma+1)}\varphi^{j-\mu}. \end{aligned} \quad (4)$$

2.3. Definition. The fractional Caputo operator D^σ having order σ is calculated as [41–43]

$${}^C D^\sigma \nu(\varphi) = \begin{cases} \dfrac{1}{\Gamma(j-\sigma)}\displaystyle\int_0^\varphi \dfrac{\nu^j(\mu)}{(\varphi-\mu)^{\sigma-j+1}}d\mu, & j-1<\sigma<j, \\ \dfrac{d^j}{d\varphi^j}\nu(\varphi), & j=\sigma. \end{cases} \quad (5)$$

with the following properties:

$$\begin{aligned} J_\varphi^\sigma D_\varphi^\sigma g(\varphi) &= g(\varphi) - \sum_{k=0}^m g^k(0^+)\frac{\varphi^k}{k!}, \quad \text{for } \varphi>0, \quad \text{and } j-1<\sigma\leq j, j\in N, \\ D_\varphi^\sigma J_\varphi^\sigma g(\varphi) &= g(\varphi). \end{aligned} \quad (6)$$

2.4. Definition. The Elzaki transform of Caputo operator is calculated as [41, 42]

$$E\left[D_\varphi^\sigma g(\varphi)\right] = s^{-\sigma}E[g(\varphi)] - \sum_{k=0}^{j-1} s^{2-\sigma+k}g^{(k)}(0), \quad \text{where} \quad j-1<\sigma<j. \quad (7)$$

3. Idea of New Iterative Transform Method (NITM)

Let us consider the partial differential equation having fractional order in the form of

$$D_\rho^\sigma \zeta(\mu,\rho) + N\zeta(\mu,\rho) + M\zeta(\mu,\rho) = h(\mu,\rho), \quad n\in N, \quad n-1<\sigma\leq n, \quad (8)$$

having initial condition

$$\zeta^k(\mu, 0) = g_k(\mu), \quad k = 0, 1, 2, \cdots, n-1, \tag{9}$$

where N and M are linear and nonlinear components.

By taking the Elzaki transform of Equation (8), we have

$$E\left[D_\rho^\sigma \zeta(\mu, \rho)\right] + E[N\zeta(\mu, \rho) + M\zeta(\mu, \rho)] = E[h(\mu, \rho)]. \tag{10}$$

By using Elzaki differentiation property

$$\begin{aligned} E[\zeta(\mu, \rho)] = &\sum_{k=0}^{m} s^{2-\sigma+k} u^{(k)}(\mu, 0) + s^\sigma E[h(\mu, \rho)] \\ &- s^\sigma E[N\zeta(\mu, \rho) + M\zeta(\mu, \rho)]. \end{aligned} \tag{11}$$

On taking Elzaki inverse transform of Equation (11),

$$\begin{aligned} \zeta(\mu, \rho) = &E^{-1}\left[\left\{\sum_{k=0}^{m} s^{2-\sigma+k} u^k(\mu, 0) + s^\sigma E[h(\mu, \rho)]\right\}\right] \\ &- E^{-1}[s^\sigma E[N\zeta(\mu, \rho) + M\zeta(\mu, \rho)]]. \end{aligned} \tag{12}$$

Now by using iterative technique, we get

$$\zeta(\mu, \rho) = \sum_{m=0}^{\infty} \zeta_m(\mu, \rho), \tag{13}$$

$$N\left(\sum_{m=0}^{\infty} \zeta_m(\mu, \rho)\right) = \sum_{m=0}^{\infty} N[\zeta_m(\mu, \rho)]. \tag{14}$$

The nonlinear term N is recognized as

$$\begin{aligned} N\left(\sum_{m=0}^{\infty} \zeta_m(\mu, \rho)\right) = &\zeta_0(\mu, \rho) + N\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right) \\ &- M\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right). \end{aligned} \tag{15}$$

By substituting Equations (13), (14), and (15) in Equation (12), we get

$$\begin{aligned} \sum_{m=0}^{\infty} \zeta_m(\mu, \rho) = &E^{-1}\left[s^\sigma\left(\sum_{k=0}^{m} s^{2-\mu+k} u^k(\mu, 0) + E[h(\mu, \rho)]\right)\right] \\ &- E^{-1}\left[s^\sigma E\left[N\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right) - M\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right)\right]\right]. \end{aligned} \tag{16}$$

Thus, the iterative formula is given as

$$\begin{aligned} \zeta_0(\mu, \rho) &= E^{-1}\left[s^\sigma\left(\sum_{k=0}^{m} s^{2-\mu+k} u^k(\mu, 0) + s^\sigma E(g(\mu, \rho))\right)\right], \\ \zeta_1(\mu, \rho) &= -E^{-1}[s^\sigma E[N[\zeta_0(\mu, \rho)] + M[\zeta_0(\mu, \rho)]], \\ \zeta_{m+1}(\mu, \rho) &= -E^{-1}\left[s^\sigma E\left[-N\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right) - M\left(\sum_{k=0}^{m} \zeta_k(\mu, \rho)\right)\right]\right], \end{aligned} \tag{17}$$

Lastly, Equations (8) and (9) give series form result for m-term as

$$\zeta(\mu, \rho) \cong \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \cdots + \zeta_m(\mu, \rho), \quad m = 1, 2, \cdots. \tag{18}$$

4. Idea of Homotopy Perturbation Transform Method (HPTM)

Let us consider the fractional partial differential equation having general form.

$$\begin{aligned} D_\rho^\sigma \zeta(\mu, \rho) + M\zeta(\mu, \rho) + N\zeta(\mu, \rho) = h(\mu, \rho), \quad &\rho > 0, \quad 0 < \sigma \le 1, \\ \zeta(\mu, 0) = g(\mu), \quad &\nu \in \mathfrak{R}. \end{aligned} \tag{19}$$

By taking the Elzaki transform of Equation (19)

$$\begin{aligned} E\left[D_\rho^\sigma \zeta(\mu, \rho) + M\zeta(\mu, \rho) + N\zeta(\mu, \rho)\right] &= E[h(\mu, \rho)], \rho > 0, 0 < \sigma \le 1, \\ \zeta(\mu, \rho) = s^2 g(\mu) + s^\sigma E[h(\mu, \rho)] &- s^\sigma E[M\zeta(\mu, \rho) + N\zeta(\mu, \rho)]. \end{aligned} \tag{20}$$

On taking Elzaki inverse transform, we have

$$\zeta(\mu, \rho) = F(x, \rho) - E^{-1}[s^\sigma E\{M\zeta(\mu, \rho) + N\zeta(\mu, \rho)\}], \tag{21}$$

where

$$\begin{aligned} (\mu, \rho) &= E^{-1}\left[s^2 g(\mu) + s^\sigma E[h(\mu, \rho)]\right] \\ &= g(\nu) + E^{-1}[s^\sigma E[h(\mu, \rho)]]. \end{aligned} \tag{22}$$

The perturbation technique of parameter p is given as

$$\zeta(\mu, \rho) = \sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho), \tag{23}$$

where perturbation parameter is denoted by p and $p \in [0, 1]$.

The nonlinear terms can be calculated as

$$N\zeta(\mu, \rho) = \sum_{k=0}^{\infty} p^k H_k(\zeta_k), \tag{24}$$

where H_n represents He's polynomials in terms of $\zeta_0, \zeta_1, \zeta_2, \cdots, \zeta_n$, and can be expressed as

$$H_n(\zeta_0, \zeta_1, \cdots, \zeta_n) = \frac{1}{\sigma(n+1)} D_p^k \left[N\left(\sum_{k=0}^{\infty} p^k \zeta_k \right) \right]_{p=0}, \quad (25)$$

where $D_p^k = \partial^k / \partial p^k$.

Substituting Equations (24) and (25) in Equation (21), we have

$$\begin{aligned} \sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho) &= F(\mu, \rho) - p \\ &\times \left[E^{-1} \left\{ s^\sigma E \left\{ M \sum_{k=0}^{\infty} p^k \zeta_k(\mu, \rho) + \sum_{k=0}^{\infty} p^k H_k(\zeta_k) \right\} \right\} \right]. \end{aligned} \quad (26)$$

On comparison of both sides coefficient of p, we get

$$\begin{aligned} p^0 &: \zeta_0(\mu, \rho) = F(\mu, \rho), \\ p^1 &: \zeta_1(\mu, \rho) = E^{-1}[s^\sigma E(M\zeta_0(\mu, \rho) + H_0(\zeta))], \\ p^2 &: \zeta_2(\mu, \rho) = E^{-1}[s^\sigma E(M\zeta_1(\mu, \rho) + H_1(\zeta))], \\ &\vdots \\ p^k &: \zeta_k(\mu, \rho) = E^{-1}[s^\sigma E(M\zeta_{k-1}(\mu, \rho) + H_{k-1}(\zeta))], k > 0, k \in N. \end{aligned} \quad (27)$$

The $\zeta_k(\mu, \rho)$ term can be calculated easily resulting convergent series. By taking $p \longrightarrow 1$,

$$\zeta(\mu, \rho) = \lim_{M \longrightarrow \infty} \sum_{k=1}^{M} \zeta_k(\mu, \rho). \quad (28)$$

4.1. Example. Consider the time-fractional ADE

$$D_\rho^\sigma \zeta(\mu, \rho) = \ell D_\mu^2 \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho), \quad (29)$$

with initial condition

$$\zeta(\mu, 0) = e^{-\mu}, \quad (30)$$

where ℓ is the ratio of constant diffusivity and the drift velocity. The exact solution is

$$\zeta(\mu, \phi, \rho) = e^{(1+\ell)\rho - \mu}. \quad (31)$$

By taking the Elzaki transform of Eq. (29), we get

$$E[\nu(\mu, \phi, \rho)] = s^2(e^{-\mu}) + s^\sigma E\left[\ell D_\mu^2 \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho)\right]. \quad (32)$$

On taking Elzaki inverse transform, we have

$$\nu(\mu, \phi, \rho) = e^{-\mu} + E^{-1}\left(s^\sigma E\left[\ell D_\mu^2 \zeta(\mu, \rho) - D_\mu \zeta(\mu, \rho)\right]\right). \quad (33)$$

Thus by using NITM, we have

$$\begin{aligned} \zeta_0(\mu, \rho) &= e^{-\mu}, \\ \zeta_1(\mu, \rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_0(\mu, \rho) - D_\mu \zeta_0(\mu, \rho)\right\}\right] \\ &= e^{-\mu} \frac{(\ell+1)\rho^\sigma}{\Gamma(\sigma+1)}, \\ \zeta_2(\mu, \rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_1(\mu, \rho) - D_\mu \zeta_1(\mu, \rho)\right\}\right] \\ &= e^{-\mu} \frac{(\ell+1)^2(\rho^\sigma)^2}{\Gamma(2\sigma+1)}, \\ \zeta_3(\mu, \rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_2(\mu, \rho) - D_\mu \zeta_2(\mu, \rho)\right\}\right] \\ &= e^{-\mu} \frac{(\ell+1)^3(\rho^\sigma)^3}{\Gamma(3\sigma+1)}, \\ &\vdots \\ \zeta_n(\mu, \rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_n(\mu, \rho) - D_\mu \zeta_n(\mu, \rho)\right\}\right] \\ &= e^{-\mu} \frac{(\ell+1)^n(\rho^\sigma)^n}{\Gamma(n\sigma+1)}, n \geq 0. \end{aligned} \quad (34)$$

The series form solution is given as

$$\zeta(\mu, \rho) = \zeta_0(\mu, \rho) + \zeta_1(\mu, \rho) + \zeta_2(\mu, \rho) + \zeta_3(\mu, \rho) + \cdots \zeta_n(\mu, \rho). \quad (35)$$

Thus, we have

$$\begin{aligned} \zeta(\mu, \rho) = e^{-\mu} \Bigg\{ 1 &+ \frac{(\ell+1)\rho^\sigma}{\Gamma(\sigma+1)} + \frac{(\ell+1)^2 \rho^{2\sigma}}{\Gamma(2\sigma+1)} \\ &+ \frac{(\ell+1)^3 \rho^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots + \frac{(\ell+1)^n (\rho^\sigma)^n}{\Gamma(n\sigma+1)} \Bigg\}. \end{aligned} \quad (36)$$

Now by using the HPTM, we have

$$\sum_{n=0}^{\infty} p^n w_n(\mu, \rho) = (e^{-\mu}) + p\left\{E^{-1}\left(s^\sigma E\left[\sum_{n=0}^{\infty} p^n H_n(w)\right]\right)\right\}. \quad (37)$$

By Comparing coefficient of p on both sides, we get:

$$\begin{aligned}
p^0 &: w_0(\mu,\rho) = e^{-\mu},\\
p^1 &: w_1(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_0(w))\}\right] = e^{-\mu}\frac{(\ell+1)\rho^\sigma}{\Gamma(\sigma+1)},\\
p^2 &: w_2(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_1(w))\}\right] = e^{-\mu}\frac{(\ell+1)^2(\rho^\sigma)^2}{\Gamma(2\sigma+1)},\\
p^3 &: w_3(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_2(w))\}\right] = e^{-\mu}\frac{(\ell+1)^3(\rho^\sigma)^3}{\Gamma(3\sigma+1)},\\
&\vdots\\
p^n &: w_n(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_{n-1}(w))\}\right] = e^{-\mu}\frac{(\ell+1)^n(\rho^\sigma)^n}{\Gamma(n\sigma+1)}.
\end{aligned} \tag{38}$$

The solution in series form by means of HPM is given as

$$\zeta(\mu,\rho) = \sum_{n=0}^{\infty} p^n w_n(\mu,\rho). \tag{39}$$

Thus, we have

$$\zeta(\mu,\rho) = e^{-\mu}\left\{1 + \frac{(\ell+1)\rho^\sigma}{\Gamma(\sigma+1)} + \frac{(\ell+1)^2\rho^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{(\ell+1)^3\rho^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots + \frac{(\ell+1)^n(\rho^\sigma)^n}{\Gamma(n\sigma+1)}\right\}. \tag{40}$$

4.2. Example. Consider the time-fractional ADE

$$D_\rho^\sigma\zeta(\mu,\rho) = \ell D_\mu^2\zeta(\mu,\rho) - D_\mu\zeta(\mu,\rho), \tag{41}$$

with initial conditions

$$\zeta(\mu,0) = \mu^3 - \mu^2. \tag{42}$$

By taking the Elzaki transform of Equation (29), we get

$$E[\nu(\mu,\phi,\rho)] = s^2\left(\mu^3-\mu^2\right) + s^\sigma E\left[\ell D_\mu^2\zeta(\mu,\rho) - D_\mu\zeta(\mu,\rho)\right]. \tag{43}$$

On taking Elzaki inverse transform, we have

$$\nu(\mu,\phi,\rho) = \left(\mu^3-\mu^2\right) + E^{-1}\left(s^\sigma E\left[\ell D_\mu^2\zeta(\mu,\rho) - D_\mu\zeta(\mu,\rho)\right]\right). \tag{44}$$

Thus by using NITM, we have

$$\begin{aligned}
\zeta_0(\mu,\rho) &= \mu^3-\mu^2,\\
\zeta_1(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2\zeta_0(\mu,\rho) - D_\mu\zeta_0(\mu,\rho)\right\}\right]\\
&= \left\{-3\mu^2 + 2\mu(1+3\ell) - 2\ell\right\}\frac{\rho^\sigma}{\Gamma(\sigma+1)},\\
\zeta_2(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2\zeta_1(\mu,\rho) - D_\mu\zeta_1(\mu,\rho)\right\}\right]\\
&= \{6\mu - 2 - 12\ell\}\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)},\\
\zeta_3(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2\zeta_2(\mu,\rho) - D_\mu\zeta_2(\mu,\rho)\right\}\right]\\
&= -6\frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}.\\
&\vdots
\end{aligned} \tag{45}$$

The series form solution is given as

$$\zeta(\mu,\rho) = \zeta_0(\mu,\rho) + \zeta_1(\mu,\rho) + \zeta_2(\mu,\rho) + \zeta_3(\mu,\rho) + \cdots \zeta_n(\mu,\rho). \tag{46}$$

Thus, we have

$$\zeta(\mu,\rho) = \left(\mu^3-\mu^2\right) + \left\{-3\mu^2 + 2\mu(1+3\ell) - 2\ell\right\}\frac{\rho^\sigma}{\Gamma(\sigma+1)} + \{6\mu - 2 - 12\ell\}\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)} - 6\frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)} + \cdots. \tag{47}$$

Now by applying the HPTM, we have

$$\sum_{n=0}^{\infty} p^n w_n(\mu,\rho) = \left(e^{-\mu}\right) + p\left\{E^{-1}\left(s^\sigma E\left[\sum_{n=0}^{\infty} p^n H_n(w)\right]\right)\right\}. \tag{48}$$

By comparing coefficient of p on both sides, we get

$$\begin{aligned}
p^0 &: w_0(\mu,\rho) = \mu^3 - \mu^2,\\
p^1 &: w_1(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_0(w))\}\right]\\
&\qquad = \left\{-3\mu^2 + 2\mu(1+3\ell) - 2\ell\right\}\frac{\rho^\sigma}{\Gamma(\sigma+1)},\\
p^2 &: w_2(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_1(w))\}\right]\\
&\qquad = \{6\mu - 2 - 12\ell\}\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)},\\
p^3 &: w_3(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_2(w))\}\right] = -6\frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}.\\
&\vdots
\end{aligned} \tag{49}$$

The solution in series form by means of HPM is given as

$$\zeta(\mu,\rho) = \sum_{n=0}^{\infty} p^n w_n(\mu,\rho). \tag{50}$$

Thus, we have

$$\begin{aligned}
\zeta(\mu,\rho) &= \left(\mu^3 - \mu^2\right) + \left\{-3\mu^2 + 2\mu(1+3\ell) - 2\ell\right\}\frac{\rho^\sigma}{\Gamma(\sigma+1)}\\
&\quad + \{6\mu - 2 - 12\ell\}\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)} - 6\frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)} + \cdots.
\end{aligned} \tag{51}$$

4.3. Example. Consider the time-fractional ADE

$$D_\rho^\sigma \zeta(\mu,\rho) = \ell D_\mu^2 \zeta(\mu,\rho) - D_\mu \zeta(\mu,\rho), \tag{52}$$

with initial conditions

$$\zeta(\mu,0) = \cos(\mu). \tag{53}$$

By taking the Elzaki transform of Equation (29), we get

$$E[\nu(\mu,\phi,\rho)] = s^2(\cos(\mu)) + s^\sigma E\left[\ell D_\mu^2 \zeta(\mu,\rho) - D_\mu \zeta(\mu,\rho)\right]. \tag{54}$$

On taking Elzaki inverse transform, we have

$$\nu(\mu,\phi,\rho) = \cos(\mu) + E^{-1}\left(s^\sigma E\left[\ell D_\mu^2 \zeta(\mu,\rho) - D_\mu \zeta(\mu,\rho)\right]\right). \tag{55}$$

Thus by using NITM, we have

$$\begin{aligned}
\zeta_0(\mu,\rho) &= \cos(\mu),\\
\zeta_1(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_0(\mu,\rho) - D_\mu \zeta_0(\mu,\rho)\right\}\right]\\
&= (\sin(\mu) - \ell\cos(\mu))\frac{\rho^\sigma}{\Gamma(\sigma+1)},\\
\zeta_2(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_1(\mu,\rho) - D_\mu \zeta_1(\mu,\rho)\right\}\right]\\
&= \left(-\cos(\mu) - 2\ell\sin(\mu) + \ell^2\cos(\mu)\right)\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)},\\
\zeta_3(\mu,\rho) &= E^{-1}\left[s^\sigma E\left\{\ell D_\mu^2 \zeta_2(\mu,\rho) - D_\mu \zeta_2(\mu,\rho)\right\}\right]\\
&= \left(-\sin(\mu) + 3\ell\cos(\mu) + 3\ell^2\sin(\mu) - \ell^3\cos(\mu)\right)\\
&\quad \cdot \frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}.\\
&\vdots
\end{aligned} \tag{56}$$

The series form solution is given as

$$\zeta(\mu,\rho) = \zeta_0(\mu,\rho) + \zeta_1(\mu,\rho) + \zeta_2(\mu,\rho) + \zeta_3(\mu,\rho) + \cdots \zeta_n(\mu,\rho). \tag{57}$$

Thus, we have

$$\begin{aligned}
\zeta(\mu,\rho) &= \cos(\mu) + (\sin(\mu) - \ell\cos(\mu))\frac{\rho^\sigma}{\Gamma(\sigma+1)}\\
&\quad + \left(-\cos(\mu) - 2\ell\sin(\mu) + \ell^2\cos(\mu)\right)\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)}\\
&\quad + \left(-\sin(\mu) + 3\ell\cos(\mu) + 3\ell^2\sin(\mu) - \ell^3\cos(\mu)\right)\\
&\quad \cdot \frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}.
\end{aligned} \tag{58}$$

Now by applying the HPTM, we have

$$\sum_{n=0}^{\infty} p^n w_n(\mu,\rho) = \left(e^{-\mu}\right) + p\left\{E^{-1}\left(s^\sigma E\left[\sum_{n=0}^{\infty} p^n H_n(w)\right]\right)\right\}. \tag{59}$$

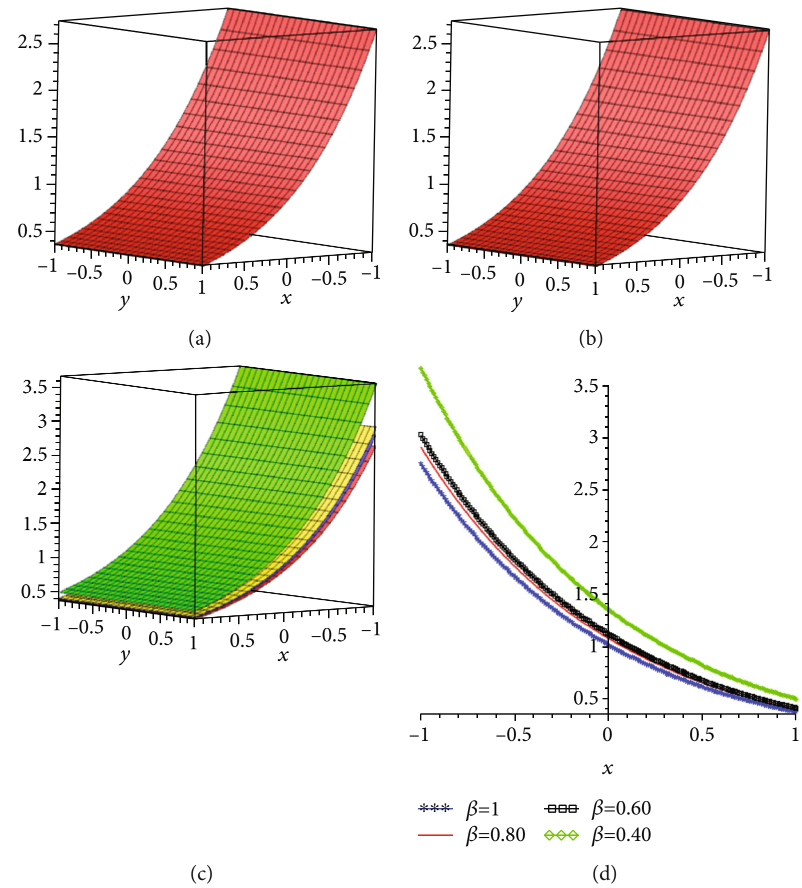

FIGURE 1: Nature of the exact and proposed technique results of example 1.

By Comparing coefficient of p on both sides, we get:

$$\begin{aligned}
p^0 &: w_0(\mu,\rho) = \cos(\mu), \\
p^1 &: w_1(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_0(w))\}\right] \\
&= (\sin(\mu) - \ell\cos(\mu))\frac{\rho^\sigma}{\Gamma(\sigma+1)}, \\
p^2 &: w_2(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_1(w))\}\right] \\
&= \left(-\cos(\mu) - 2\ell\sin(\mu) + \ell^2\cos(\mu)\right) \cdot \frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)}, \\
p^3 &: w_3(\mu,\rho) = \left[E^{-1}\{s^\sigma E(H_2(w))\}\right] \\
&= \left(-\sin(\mu) + 3\ell\cos(\mu) + 3\ell^2\sin(\mu) - \ell^3\cos(\mu)\right)\frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}. \\
&\vdots
\end{aligned} \tag{60}$$

The solution in series form by means of HPM is given as

$$\zeta(\mu,\rho) = \sum_{n=0}^{\infty} p^n w_n(\mu,\rho). \tag{61}$$

Thus, we have

$$\begin{aligned}
\zeta(\mu,\rho) &= \cos(\mu) + (\sin(\mu) - \ell\cos(\mu))\frac{\rho^\sigma}{\Gamma(\sigma+1)} \\
&+ \left(-\cos(\mu) - 2\ell\sin(\mu) + \ell^2\cos(\mu)\right)\frac{(\rho^\sigma)^2}{\Gamma(2\sigma+1)} \\
&+ \left(-\sin(\mu) + 3\ell\cos(\mu) + 3\ell^2\sin(\mu) - \ell^3\cos(\mu)\right) \cdot \frac{(\rho^\sigma)^3}{\Gamma(3\sigma+1)}.
\end{aligned} \tag{62}$$

5. Results and Discussion

We implemented NITM and HPTM for finding the approximate solutions of time-fractional ADE. The analytical solution and exact solution are shown in Figures 1(a) and 1(b) at $\sigma = 1$, whereas Figures 1(c) and 1(d) show the absolute error and the solution at various fractional order. Figures 2 and 3 show the behavior of the proposed method solution at various fractional orders. Table 1 shows the comparison of the exact and suggested methods solution in addition with the absolute error at various fractional order. Finally, the figures and table show that the suggested techniques have higher degree of accuracy and rapid convergence towards the exact results.

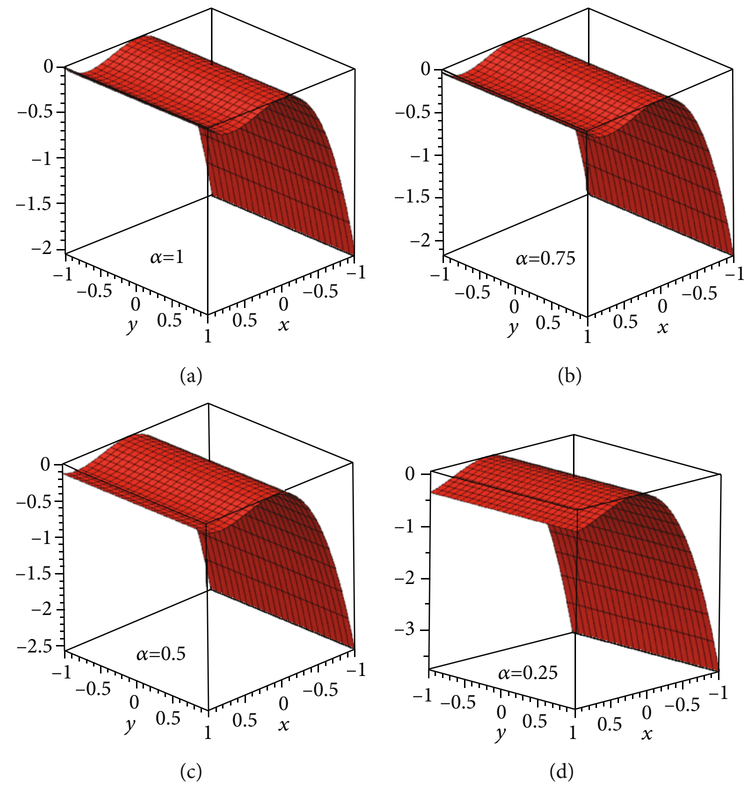

Figure 2: Nature of the proposed method solutions at different fractional orders for problem 2.

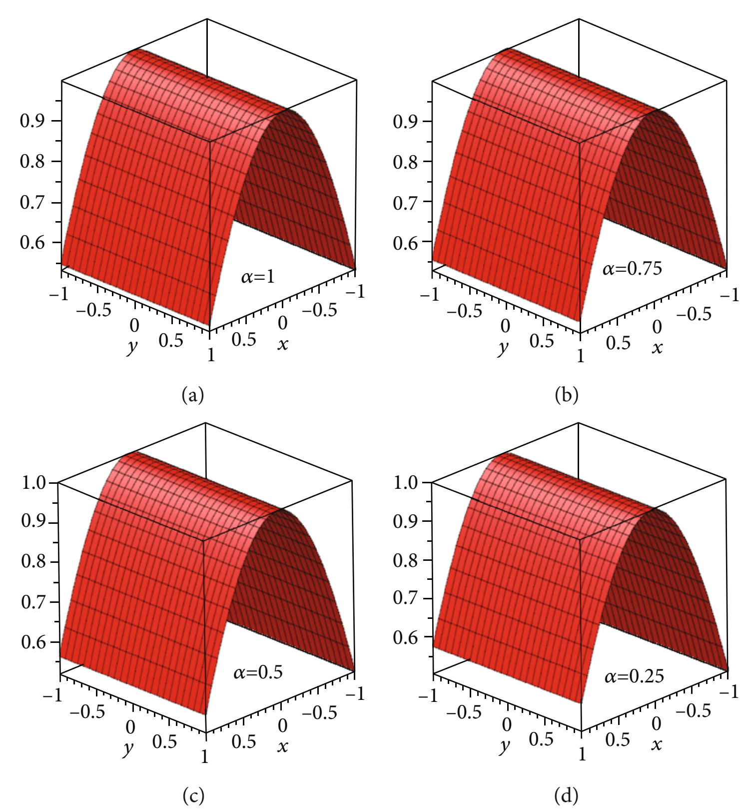

Figure 3: Nature of the proposed method solutions at different fractional orders for problem 3.

Table 1: Analysis of the approximate solution by NITM and HPTM for problem 1.

η	ξ	$\lvert Exact - NITM\rvert$ $\sigma = 0.5$	$\lvert Exact - NITM\rvert$ $\sigma = 1$	$\lvert Exact - HPTM\rvert$ $\sigma = 0.7$	$\lvert Exact - HPTM\rvert$ $\sigma = 1$
0.1	0.5	$7.65212627 \times 10^{-02}$	$3.0000000 \times 10^{-10}$	$2.34826106 \times 10^{-02}$	$3.0000000 \times 10^{-10}$
	1	$4.64124920 \times 10^{-02}$	$2.0000000 \times 10^{-10}$	$1.42429233 \times 10^{-02}$	$2.0000000 \times 10^{-10}$
	1.5	$2.81505993 \times 10^{-02}$	$1.0000000 \times 10^{-10}$	$8.63876960 \times 10^{-03}$	$1.0000000 \times 10^{-10}$
	2	$1.70742016 \times 10^{-02}$	$1.0000000 \times 10^{-10}$	$5.23967870 \times 10^{-03}$	$1.0000000 \times 10^{-10}$
	2.5	$1.03560267 \times 10^{-02}$	$4.0000000 \times 10^{-11}$	$3.17802575 \times 10^{-03}$	$4.0000000 \times 10^{-11}$
	3	$6.28124775 \times 10^{-03}$	$2.0000000 \times 10^{-11}$	$1.92757006 \times 10^{-03}$	$2.0000000 \times 10^{-11}$
	3.5	$3.80976934 \times 10^{-03}$	$2.0000000 \times 10^{-11}$	$1.16913033 \times 10^{-03}$	$2.0000000 \times 10^{-11}$
	4	$2.31074191 \times 10^{-03}$	$1.0000000 \times 10^{-11}$	$7.09113390 \times 10^{-04}$	$1.0000000 \times 10^{-11}$
	4.5	$1.40153581 \times 10^{-03}$	$1.0000000 \times 10^{-11}$	$4.30099010 \times 10^{-04}$	$1.0000000 \times 10^{-11}$
	5	$8.50074443 \times 10^{-04}$	$3.0000000 \times 10^{-12}$	$2.60868239 \times 10^{-04}$	$3.0000000 \times 10^{-12}$
0.2	0.5	$1.09371292 \times 10^{-01}$	$5.8000000 \times 10^{-09}$	$3.65660312 \times 10^{-02}$	$5.8000000 \times 10^{-09}$
	1	$6.63370423 \times 10^{-02}$	$3.5000000 \times 10^{-09}$	$2.21784191 \times 10^{-02}$	$3.5000000 \times 10^{-09}$
	1.5	$4.02354500 \times 10^{-02}$	$2.1000000 \times 10^{-09}$	$1.34518911 \times 10^{-02}$	$2.1000000 \times 10^{-09}$
	2	$2.44040340 \times 10^{-02}$	$1.3000000 \times 10^{-09}$	$8.15898440 \times 10^{-03}$	$1.3000000 \times 10^{-09}$
	2.5	$1.48017948 \times 10^{-02}$	$7.8000000 \times 10^{-10}$	$4.94867420 \times 10^{-03}$	$7.8000000 \times 10^{-10}$
	3	$8.97774240 \times 10^{-03}$	$4.7000000 \times 10^{-10}$	$3.00152262 \times 10^{-03}$	$4.7000000 \times 10^{-10}$
	3.5	$5.44527602 \times 10^{-03}$	$2.9000000 \times 10^{-10}$	$1.82051550 \times 10^{-03}$	$2.9000000 \times 10^{-10}$
0.3	4	$3.30272686 \times 10^{-03}$	$1.7000000 \times 10^{-10}$	$1.10419847 \times 10^{-03}$	$1.7000000 \times 10^{-10}$
	4.5	$2.00320511 \times 10^{-03}$	$1.0000000 \times 10^{-10}$	$6.69730230 \times 10^{-04}$	$1.0000000 \times 10^{-10}$
	5	$1.21500531 \times 10^{-03}$	$6.4000000 \times 10^{-11}$	$4.06211915 \times 10^{-04}$	$6.4000000 \times 10^{-11}$
	0.5	$1.35218250 \times 10^{-01}$	$2.9900000 \times 10^{-08}$	$4.73151582 \times 10^{-02}$	$2.9900000 \times 10^{-08}$
	1	$8.20140147 \times 10^{-02}$	$1.8100000 \times 10^{-08}$	$2.86980942 \times 10^{-02}$	$1.8100000 \times 10^{-08}$
	1.5	$4.97440144 \times 10^{-02}$	$1.1000000 \times 10^{-08}$	$1.74062739 \times 10^{-02}$	$1.1000000 \times 10^{-08}$
	2	$3.01712698 \times 10^{-02}$	$6.7000000 \times 10^{-09}$	$1.05574388 \times 10^{-02}$	$6.7000000 \times 10^{-09}$
	2.5	$1.82998002 \times 10^{-02}$	$4.0400000 \times 10^{-09}$	$6.40341033 \times 10^{-03}$	$4.0400000 \times 10^{-09}$
	3	$1.10993899 \times 10^{-02}$	$2.4500000 \times 10^{-09}$	$3.88386470 \times 10^{-03}$	$2.4500000 \times 10^{-09}$
	3.5	$6.73212028 \times 10^{-03}$	$1.4900000 \times 10^{-09}$	$2.35568301 \times 10^{-03}$	$1.4900000 \times 10^{-09}$
	4	$4.08323736 \times 10^{-03}$	$9.0000000 \times 10^{-10}$	$1.42879398 \times 10^{-03}$	$9.0000000 \times 10^{-10}$
	4.5	$2.47660865 \times 10^{-03}$	$5.4000000 \times 10^{-10}$	$8.66607360 \times 10^{-04}$	$5.4000000 \times 10^{-10}$
	5	$1.50213907 \times 10^{-03}$	$3.3200000 \times 10^{-10}$	$5.25623929 \times 10^{-04}$	$3.32000000 \times 10^{-10}$

6. Conclusion

The solutions for time-fractional ADE are successfully obtained using NITM and HPTM in this paper. The study reveals that the derivative having fractional order, as well as the location and time factors, has an impact on solute concentration. For varying values of the fractional parameter σ, solutions are plotted with spatial and time coordinates for three cases. We compare actual and analytical results with the use of graphs and tables, which are in strong agreement with one another, to demonstrate the effectiveness of the proposed methods. Also, the results achieved by implementing the suggested approaches are compared at various fractional orders, confirming that the result comes closer to the exact solution as the value moves from fractional to integer order. The methods should be extended to solve space-time fractional ADE in two or three dimensions. As a result, the NITM and HPTM are effective methods in finding exact and approximate solutions for nonlinear differential equations arising in science and engineering.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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