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Research Article

A New Measure of Quantum Starlike Functions Connected with Julia Functions

Samir B. Hadid,[1,2] **Rabha W. Ibrahim**,[3] **and Shaher Momani**[1,2,4]

[1] *Department of Mathematics and Sciences, College of Humanities and Sciences, Ajman University, Ajman, UAE*
[2] *Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE*
[3] *IEEE: 94086547, Perrysburg, Ohio, USA*
[4] *Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan*

Correspondence should be addressed to Rabha W. Ibrahim; rabhaibrahim@yahoo.com

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In a complex domain, the investigation of the quantum differential subordinations for starlike functions is newly considered by few research studies. In this note, we arrange a set of necessary conditions utilizing the concept of the quantum differential subordinations for starlike functions related to the set of parametric Julia functions. Our method is based on the usage of quantum Jack lemma, where this lemma is generalized recently by the quantum derivative (Jackson calculus). We illustrate a starlike formula dominated by different types of Julia functions. The sufficient conditions are computed in the quantum and the Julia fractional parameters. We indicate a relationship between these two parameters.

1. Introduction

The notion of differential subordination and superordination (DSS) shows a dynamic model in the investigation of geometric possessions of holomorphic functions in the open unit disk. Lindelof first presented it, while Littlewood [1] did the extraordinary exertion in this area of study. Numerous investigators added information in the application of DSS. Antiquity and the improvement of mechanisms in the area connected with DSS are concisely designated and incorporated in the hardcover by Miller and Mocanu [2]. The main growth in the area of derivative of DSS began by Miller et al. [3]. Generally, the concept is defined for univalent function ω by

$$\varphi \prec \psi \Leftrightarrow \varphi(0) = \psi(0), \tag{1}$$

and $\varphi(\omega) \subset \psi(\omega)$. In general, if there is a function with the properties $\omega(0) = 0, |\omega(\xi)| < |\xi|$, satisfying $\varphi(\xi) = \psi(\omega(\xi))$, then

$$\varphi(\xi) \prec \psi(\xi), \tag{2}$$

where $\xi \in \sqcup := \{\xi \in \mathbb{C}: |\xi| < 1\}$.

Ismail et al. [4] presented a class of complex functions for each fractional number q, $0 < q < 1$ as the class of analytic functions φ on the open unit disk ($\sqcup$), $\varphi(0) = 0, \varphi\prime(0) = 1$, and $|\varphi(q\xi)| \le |\varphi(\xi)|$ on $\sqcup$. This class is investigated, as well as the links between it and other analytic function classes. Agrawal and Sahoo [5] extended this notion by suggesting the q-starlike functions family in a logical order. Srivastava et al. [6] explored the link between the Janowski functions and several known types of q-starlike functions. The Janowski functions are a novel subclass of q-starlike functions that they introduced and presented. Recent investigations can be located in works by Mahmood et al. [7] and Ul-Haq et al. [8].

Parametric Julia functions are usually utilized to determine the upper bound solutions of different types of differential equations of a complex variable [4–11]. In the recent study, we shall extend this concept applying the quantum calculus (Jackson calculus) and employ it to define special classes of analytic function types normalized analytically in the open unit disk $(\varphi(0) = 0, \varphi\prime(0) = 1)$ and dominated by different kinds of the parametric Julia functions. Our method is based on the quantum Jack lemma.

2. Quantum Starlike Formula

The effort of Ma and Minda [12] in this area of studies is not minor as they considered the normalized analytic function $p(0) = 1$ and the condition of a positive real part $\Re(p\prime(\xi)) > 0$. They have formulated the famous subclasses for starlike and convex functions, as follows, respectively:

$$\mathcal{S}^*(p) = \left\{\varphi \in \Delta: \frac{\xi\varphi\prime(\xi)}{\varphi(\xi)} \prec p(\xi), \xi \in \sqcup\right\},$$
$$\mathcal{C}(p) = \left\{\varphi \in \Delta: \frac{\xi\varphi\prime\prime(\xi)}{\varphi\prime(\xi)} + 1 \prec p(\xi), \xi \in \sqcup\right\}, \quad (3)$$

where Δ indicates the class of normalized function $\varphi(0) = 0 = \varphi\prime(0) - 1$.

Quantum calculus (QC) is the novel part of mathematical analysis and its applications and is correspondingly significant for its appearances, both in physics and in mathematics as well. Jackson [13,14] formulated the functions of q-differentiation and q-integration and decorated their meanings for the first stage. Later, Ismail et al. [4] contributed the indication of q-calculus in geometric function theory.

Nowadays, different classes of Ma and Minda are suggested and developed, using QC by researchers. For instant, Seoudy and Aouf [15] introduced subclass of quantum starlike functions involving q-derivative. Recently, Zainab et al. [16] presented a sufficient condition for q-starlikeness using a special curve. In addition, different differential and integral operators are generalized utilizing QC [17–20].

Definition 1. Jackson derivative is indicated in the following difference operator:

$$\left(\partial_q h\right)(\xi) = \frac{h(\xi) - h(q\xi)}{\xi(1-q)}, \quad q \in (0,1), \quad (4)$$

such that

$$\partial_q(\xi^\nu) = \left(\frac{1-q^\nu}{1-q}\right)\xi^{\nu-1}. \quad (5)$$

Moreover, Maclaurin's series representation takes the sum

$$\left(\partial_q h\right)(\xi) = \sum_{\ell=0}^{\infty} h_\ell [\ell]_q \xi^{\ell-1}, \quad (6)$$

where

$$[\ell]_q := \frac{1-q^\ell}{1-q}. \quad (7)$$

Note that

$$\lim_{q\longrightarrow 1^-}\left(\partial_q h\right)(\xi) = h\prime(\xi). \quad (8)$$

The multiplication rule takes the following formula:

$$\partial_q(f(\xi)g(\xi)) = g(\xi)\partial_q f(\xi) + f(q\xi)\partial_q g(\xi)$$
$$= g(q\xi)\partial_q f(\xi) + f(\xi)\partial_q g(\xi). \quad (9)$$

We proceed to define our q-starlike class using the q-parametric Julia functions and connecting with the subclass of normalized functions in $\sqcup$ (Figure 1):

$$J_1^{(\beta)}(\xi) = 1 + \xi - \beta\xi^3 \quad (10)$$

$$J_2^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right)^2, \quad (11)$$

$$J_3^{(\beta)}(\xi) = 1 + \xi - \beta\xi^2, \quad (\beta \in \mathbb{C}, \xi \in \sqcup) \quad (12)$$

Definition 2. For a normalized function $\varphi(\xi) \in \Delta$ of the formula

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad \xi \in \sqcup, \quad (13)$$

the q-starlike is defined by the subordination formula:

$$\frac{\xi\left(\partial_q\varphi\right)(\xi)}{\varphi(\xi)} \prec J_i^{(\beta)}(\xi) \quad (i = 1,2,3, q \in (0,1), \beta \in \mathbb{C}). \quad (14)$$

We denote the subclass of these functions by $\Delta_q^{(\beta)}$, where

$$\left(\partial_q\varphi\right) = 1 + \sum_{n=2}^{\infty} \varphi_n\left(\frac{1-q^n}{1-q}\right)\xi^n. \quad (15)$$

Moreover, a function $\varphi \in \Delta$ is called q-bounded turning if it satisfies the inequality

$$\partial_q\varphi(\xi) \prec J_i^{(\beta)}(\xi). \quad (16)$$

We denote this class by $\mathbb{B}_q^{(\beta)}$.

We aim to find the range of β in terms of q satisfying the inequality (14). For this purpose, we need the following result.

Lemma 1 (see [21]). *Let ϖ be analytic in $\sqcup$, such that $\varpi(0) = 0$. Then, the upper value of ϖ on the circle $|\xi| = 1$ at the point $\xi_0 = re^{i\theta}, \theta \in [-\pi,\pi], q \in (0,1)$, is*

$$\xi_0\left(\partial_q\varpi(\xi_0)\right) = \mu\varpi(\xi_0), \quad \mu \ge 1. \quad (17)$$

3. Results

In this section, we shall illustrate the sufficient conditions on functions $\varphi \in \Delta$ to be in $\Delta_q^{(\beta)}$.

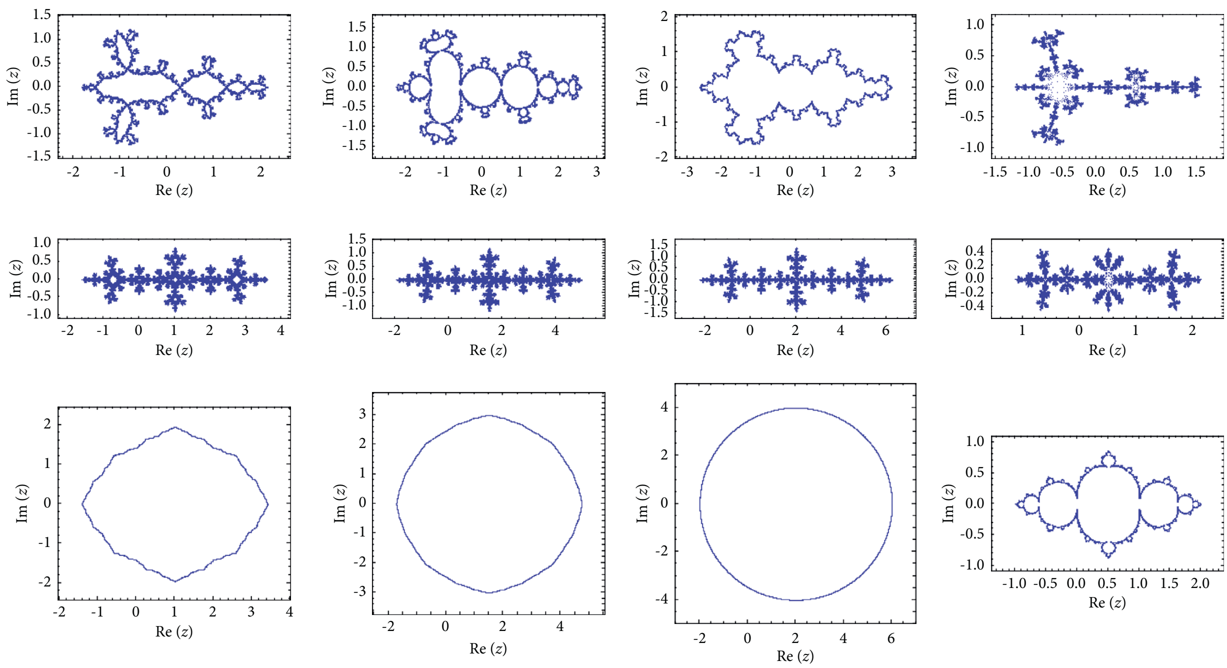

FIGURE 1: Plot of $J_i^{(\beta)}, i = 1, 2, 3$ for $\beta = 1/2, 1/3, 1/4$ and $\beta = 1$, respectively. The plot is connected when $\beta \in (0, 1]$; otherwise, it is disconnected.

Theorem 1. *Let the function $\rho \in \sqcup$, such that $\rho(0) = 1$ and*

$$1 + \xi\left(\partial_q \rho(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \sqcup. \tag{18}$$

If one of the cases

$$\beta \neq \frac{q+1}{q^2 - q + 1}, \beta < \frac{q+1+\sqrt{2}}{q^2 - q + 1}, \beta > \frac{q+1-\sqrt{2}}{q^2 - q + 1}, \quad q \in (0, 1), \tag{19}$$

holds, then

$$\rho(\xi) \prec J_1^{(\beta)}(\xi) = 1 + \xi - \beta\xi^3. \tag{20}$$

Proof. Define a function ϱ as follows:

$$\varrho(\xi) \coloneqq 1 + \xi\left(\partial_q \rho(\xi)\right). \tag{21}$$

By the assumption (18) and the definition of the subordination, we have

$$1 + \xi\left(\partial_q \rho(\xi)\right) = \sqrt{1 + \varpi(\xi)}, \quad \varpi(0) = 0, |\varpi(\xi)| \le |\xi| < 1, \tag{22}$$

which leads to

$$\varpi(\xi) = \varrho^2(\xi) - 1. \tag{23}$$

We aim to show that $|\varrho^2(\xi) - 1| < 1$ for some values $\xi_0 \in \sqcup$, such that

$$\rho(\xi) = 1 + \varpi(\xi) - \beta\varpi^3(\xi). \tag{24}$$

Assume not; then, the above conclusion implies that

$$\varrho(\xi) = 1 + \xi\left(\partial_q\left[1 + \varpi(\xi) - \beta\varpi^3(\xi)\right]\right). \tag{25}$$

By using the rules of Jackson derivative, we obtain

$$\begin{aligned} \partial_q \varpi^2(\xi) &= \partial_q \varpi(\xi)\left[\varpi(\xi) + \varpi(q\xi)\right], \\ \partial_q \varpi^3(\xi) &= \partial_q \varpi(\xi)\left[\varpi^2(\xi) + \varpi(\xi)\varpi(q\xi) + \varpi^2(q\xi)\right]. \end{aligned} \tag{26}$$

Consequently, we get

$$\varrho(\xi) = 1 + W_q(\xi)\left(\xi\partial_q \varpi(\xi)\right), \tag{27}$$

where

$$W_q(\xi) \coloneqq \varpi(\xi) + \varpi(q\xi) - \beta\left(\varpi^2(\xi) + \varpi(\xi)\varpi(q\xi) + \varpi^2(q\xi)\right). \tag{28}$$

But

$$\varpi(q\xi) = \varpi(\xi) - (1-q)\xi\left(\partial_q \varpi(\xi)\right). \tag{29}$$

Hence, this yields

$$W_q(\xi) = \varpi(\xi)\left[2 - 3\beta\varpi(\xi)\right] + \xi\partial_q\varpi(\xi)\left[-(1-q) + (3-q)\varpi(\xi) - \beta(1-q)^2\xi\partial_q\varpi(\xi)\right]. \tag{30}$$

Suppose that there exists a point $\xi_0 \in \sqcup$, such that

$$\max_{|\xi|\le|\xi_0|} |\varpi(\xi)| = |\varpi(\xi_0)| = 1,$$
$$\xi_0\left(\partial_q \varpi(\xi_0)\right) = \mu \varpi(\xi_0), \quad \mu \ge 1. \tag{31}$$

By Jack Lemma 1 and by letting $\varpi(\xi_0) = e^{i\theta}$, we have

$$\begin{aligned}
\left|W_q(\xi_0)\right| &= \left|\varpi(\xi)[2 - 3\beta\varpi(\xi)] + \xi\partial_q\varpi(\xi)\left[-(1-q) + \beta(3-q)\varpi(\xi) - \beta(1-q)^2\xi\partial_q\varpi(\xi)\right]\right|_{\xi=\xi_0} \\
&= \left|e^{i\theta}\left[2 - 3\beta e^{i\theta}\right] + \mu e^{i\theta}\left[-(1-q) + \beta(3-q)e^{i\theta} - \beta(1-q)^2\mu e^{i\theta}\right]\right| \\
&\ge \Re\left(e^{i\theta}\left[2 - 3\beta e^{i\theta}\right] + \mu e^{i\theta}\left[-(1-q) + \beta(3-q)e^{i\theta} - \beta(1-q)^2\mu e^{i\theta}\right]\right) \\
&= \cos(\theta)[2 - 3\beta\cos(\theta)] + \mu\cos(\theta)\left[-(1-q) + \beta(3-q)\cos(\theta) - \beta(1-q)^2\mu\cos(\theta)\right] \\
&= \beta\left[-3 + \mu(3-q) - \mu(1-q)^2\right]\cos^2(\theta) + [2 - \mu(1-q)]\cos(\theta) \\
&= \beta\left[\mu\left(3 - q - (1-q)^2\right) - 3\right]\cos^2(\theta) + [2 - \mu(1-q)]\cos(\theta).
\end{aligned} \tag{32}$$

Accordingly, we conclude that

$$\begin{aligned}
\left|\varrho(\xi)^2 - 1\right|_{\xi=\xi_0} &= \left|\left(1 + W_q(\xi)\left(\xi\partial_q\varpi(\xi)\right)\right)^2 - 1\right|_{\xi=\xi_0} \\
&\ge \left|\left(W_q(\xi_0)\left(\xi_0\partial_q\varpi(\xi_0)\right)\right)^2 - 1\right| \\
&= \left|\left(\mu\beta\left[\mu\left(3 - q - (1-q)^2\right) - 3\right]\cos^3(\theta) + \mu[2 - \mu(1-q)]\cos^2(\theta)\right)^2 - 1\right| \\
&\ge \left|\left(\beta\left[\left(3 - q - (1-q)^2\right) - 3\right] + [1+q]\right)^2 - 1\right| \ge 1,
\end{aligned} \tag{33}$$

provided one of the following cases holds

$$\Upsilon \le -\sqrt{2}, \Upsilon = 0, \Upsilon \ge \sqrt{2}, \tag{34}$$

where

$$\Upsilon := \beta\left[\left(3 - q - (1-q)^2\right) - 3\right] + [1+q]. \tag{35}$$

Hence, we obtain one of the following arguments:

$$\beta = \frac{q+1}{q^2 - q + 1}, \beta \ge \frac{q + 1 + \sqrt{2}}{q^2 - q + 1}, \beta \le \frac{q + 1 - \sqrt{2}}{q^2 - q + 1}, \tag{36}$$

which are all contradict (19), that is

$$\rho(\xi) \prec J_1^{(\beta)}(\xi) = 1 + \xi - \beta\xi^3. \tag{37}$$

□

As a special case, we have the following result.

Corollary 1. *Let $\varphi \in \Delta$ be satisfied the subordination:*

$$1 + \xi\left(\partial_q\left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right)\right) \prec \sqrt{1+\xi}, \quad \xi \in \sqcup. \tag{38}$$

If one of the cases in (21) is occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$\rho(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \tag{39}$$

Obviously, $\rho(0) = 1$. Thus, in virtue of Theorem 1, we have $\varphi \in \Delta_q^{(\beta)}$. □

Similarly, by assuming $\rho(\xi) = \partial_q\varphi(\xi), \varphi \in \Delta$, we have the following result.

Corollary 2. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\partial_q\varphi(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \in \sqcup. \tag{40}$$

If one of the cases in (21) is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

Theorem 2. *Let the function $h \in \sqcup$, such that $h(0) = 1$ and*

$$1 + \xi\left(\partial_q h(\xi)\right) \prec \sqrt{1+\xi}, \quad \xi \in \sqcup, \tag{41}$$

if one of the cases

$$\beta \ne \frac{1}{q+1}, \beta \ne 2, \tag{42}$$

where for $0.418341 < q < 1$,

$$\frac{\left(0.5\left(2q+\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3)\right)}{q+1}\geq\beta \geq\frac{\left(0.5\left(2q-\sqrt{(2q+3)^2-5.62159(q+1)}\right)+3\right)}{q+1}, \tag{43}$$

and for $0<q<0.418341$,

$$\frac{\left(0.5\left(2q+\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3\right)}{q+1}\geq\beta \geq\frac{\left(0.5\left(2q-\sqrt{(2q+3)^2-10.3784(q+1)}\right)+3\right)}{q+1}, \tag{44}$$

hold; then,

$$h(\xi)\prec J_2^{(\beta)}(\xi)=\left(1+\xi-\beta\xi^2\right)^2. \tag{45}$$

Proof. Define a function p as follows:

$$p(\xi) := 1+\xi\left(\partial_q h(\xi)\right). \tag{46}$$

By the assumption (41) and the meninges of the subordination, we have

$$1+\xi\left(\partial_q h(\xi)\right)=\sqrt{1+w(\xi)},\quad w(0)=0, |w(\xi)|\leq|\xi|<1, \tag{47}$$

which yields

$$w(\xi)=p^2(\xi)-1. \tag{48}$$

We have to prove that

$$|w(\xi)|=\left|p^2(\xi)-1\right|<1, \tag{49}$$

for some values $\xi_0\in \sqcup$, such that

$$h(\xi)=\left[1+w(\xi)-\beta w^2(\xi)\right]^2. \tag{50}$$

Assume not; then, the above conclusion imposes

$$p(\xi)=1+\xi\left(\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]^2\right). \tag{51}$$

By using the rules of Jackson derivative and the facts

$$\begin{aligned} w(q\xi)&=w(\xi)-(1-q)\xi\partial_q w(\xi),\\ \partial_q w^2(\xi)&=\partial_q w(\xi)\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right], \end{aligned} \tag{52}$$

we obtain

$$\begin{aligned} &\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]^2=\left[1+w(\xi)-\beta w^2(\xi)\right]\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]\\ &\quad+\left[1+w(q\xi)-\beta w^2(q\xi)\right]\partial_q\left[1+w(\xi)-\beta w^2(\xi)\right]=2\left[1+w(\xi)-\beta w^2(\xi)\right]\\ &\left(\partial_q w(\xi)-\beta\partial_q w(\xi)\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right)=2\left[1+w(\xi)-\beta w^2(\xi)\right]\\ &\partial_q w(\xi)\left(1-\beta\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right). \end{aligned} \tag{53}$$

Consequently, we get

$$p(\xi)=1+2\xi\left[1+w(\xi)-\beta w^2(\xi)\right]\partial_q w(\xi)\left(1-\beta\left[2w(\xi)-(1-q)\xi\partial_q w(\xi)\right]\right). \tag{54}$$

Suppose that there exists a point $\xi_0 \in \sqcup$, such that

$$\max_{|\xi|\le|\xi_0|} |w(\xi)| = |w(\xi_0)| = 1,$$
$$\xi_0(\partial_q w(\xi_0)) = \mu w(\xi_0), \quad \mu \ge 1. \tag{55}$$

We aim to show that

$$|w(\xi)| = |p^2(\xi) - 1| < 1. \tag{56}$$

Our method is based on Jack Lemma 1. Assume not. Then, by consuming $w(\xi_0) = e^{i\theta}$, we get

$$\begin{aligned}
|p^2(\xi_0) - 1| &= \left|\left(1 + 2\xi\left[1 + w(\xi) - \beta w^2(\xi)\right]\partial_q w(\xi)\left(1 - \beta\left[2w(\xi) - (1-q)\xi\partial_q w(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\
&\ge \left|4\left[1 + e^{i\theta} - \beta e^{2i\theta}\right]^2\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right| \\
&\ge \Re\left(4\left[1 + e^{i\theta} - \beta e^{2i\theta}\right]^2\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right) \\
&= 4\left[1 + \cos(\theta) - \beta\cos^2(\theta)\right]^2(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1 \ge 1.
\end{aligned} \tag{57}$$

Then, the solution when $\cos(\theta) = 1$ of

$$\left|4\left[1 + \cos(\theta) - \beta\cos^2(\theta)\right]^2(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1\right| \ge 1 \tag{58}$$

brings one of the following cases:

$$\curlywedge = 0, \curlywedge \ge \sqrt{2}, \curlywedge \le -\sqrt{2}, \tag{59}$$

where

$$\curlywedge := 4(2-\beta)^2(1 - \beta(1+q))^2. \tag{60}$$

Hence, we obtain one of the following arguments:

$$\beta = \frac{1}{q+1}, \beta = 2, \tag{61}$$

and for $0.418341 < q < 1$,

$$\begin{aligned}
&\frac{\left(\left(0.52q + \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1} \le \beta \\
&\le \frac{\left(0.5\left(2q - \sqrt{(2q+3)^2 - 5.62159(q+1)}\right) + 3\right)}{q+1}.
\end{aligned} \tag{62}$$

Moreover, for $0 < q < 0.418341$, we have

$$\begin{aligned}
&\frac{\left(0.5\left(2q + \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1} \le \beta \\
&\le \frac{\left(0.5\left(2q - \sqrt{(2q+3)^2 - 10.3784(q+1)}\right) + 3\right)}{q+1}.
\end{aligned} \tag{63}$$

All the above inequalities contradict the assumptions of the theorem, which lead to

$$h(\xi) \prec J_2^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right)^2. \tag{64}$$

□

Corollary 3. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\frac{\xi(\partial_q\varphi(\xi))}{\varphi(\xi)}\right) \prec \sqrt{1+\xi}. \tag{65}$$

If one of the assumptions of Theorem 2 is occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$p(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \tag{66}$$

Obviously, $p(0) = 1$. Thus, according to Theorem 2, we get $\varphi \in \Delta_q^{(\beta)}$. □

In the same manner of the above result, we obtain the next one when $p(\xi) = \partial_q\varphi(\xi), \varphi \in \Delta$.

Corollary 4. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi(\partial_q\varphi(\xi)) \prec \sqrt{1+\xi}. \tag{67}$$

If one of the assumptions of Theorem 2 is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

Theorem 3. *Let the function $g \in \sqcup$, such that $g(0) = 1$ and*

$$1 + \xi(\partial_q g(\xi)) \prec \sqrt{1+\xi}, \quad \xi \in \sqcup. \tag{68}$$

If one of the cases

$$\beta \neq \frac{1}{q+1};$$
$$\beta \geq -\frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} - 1}{(q+1)}, \quad 0<q<1, \tag{69}$$
$$\beta \leq \frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} + 1}{(q+1)}, \quad 0<q<1,$$

holds, then

$$g(\xi) \prec J_3^{(\beta)}(\xi) = \left(1+\xi-\beta\xi^2\right). \tag{70}$$

Proof. Define a function σ as follows:

$$\sigma(\xi) := 1+\xi\left(\partial_q g(\xi)\right). \tag{71}$$

By the assumption (68) and the meninges of the subordination, we have

$$1+\xi\left(\partial_q g(\xi)\right) = \sqrt{1+u(\xi)}, \quad u(0)=0, |u(\xi)| \leq |\xi| < 1, \tag{72}$$

which yields

$$u(\xi) = \sigma^2(\xi) - 1. \tag{73}$$

We have to prove that

$$|u(\xi)| = \left|\sigma^2(\xi) - 1\right| < 1, \tag{74}$$

for some values $\xi_0 \in \sqcup$, such that

$$g(\xi) = \left[1+u(\xi)-\beta u^2(\xi)\right]. \tag{75}$$

Assume not; then, the above conclusion imposes

$$\sigma(\xi) = 1+\xi\left(\partial_q\left[1+u(\xi)-\beta u^2(\xi)\right]\right). \tag{76}$$

By employing the rules of Jackson derivative and the facts

$$\begin{aligned} u(q\xi) &= u(\xi) - (1-q)\xi\partial_q u(\xi), \\ \partial_q u^2(\xi) &= \partial_q u(\xi)\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right], \end{aligned} \tag{77}$$

we obtain

$$\begin{aligned} \partial_q\left[1+u(\xi)-\beta u^2(\xi)\right] &= \left(\partial_q u(\xi) - \beta\partial_q u(\xi)\left[2u(\xi)-(1-q)\xi\partial_q u(\xi)\right]\right) \\ &= \partial_q u(\xi)\left(1-\beta\left[2u(\xi)-(1-q)\xi\partial_q u(\xi)\right]\right). \end{aligned} \tag{78}$$

Following the above structure, we get

$$\sigma(\xi) = 1+\xi\partial_q u(\xi)\left(1-\beta\left[2u(\xi)-(1-q)\xi\partial_q u(\xi)\right]\right). \tag{79}$$

Suppose that there exists a point $\xi_0 \in \sqcup$, such that

$$\max_{|\xi|\leq|\xi_0|} |u(\xi)| = \left|u(\xi_0)\right| = 1, \tag{80}$$

$$\xi_0\left(\partial_q u(\xi_0)\right) = \mu u(\xi_0), \quad \mu \geq 1. \tag{81}$$

We aim to show that

$$|u(\xi)| = \left|\sigma^2(\xi) - 1\right| < 1. \tag{82}$$

Our method is based on Jack Lemma 1. Assume not. Then, by consuming $u(\xi_0) = e^{i\theta}$, we get

$$\begin{aligned} \left|\sigma^2(\xi_0) - 1\right| &= \left|\left(1 + \xi\partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\ &\geq \left|\left(\xi\partial_q u(\xi)\left(1 - \beta\left[2u(\xi) - (1-q)\xi\partial_q u(\xi)\right]\right)\right)^2 - 1\right|_{\xi=\xi_0} \\ &= \left|\left(\xi_0\partial_q u(\xi_0)\left(1 - \beta\left[2u(\xi_0) - (1-q)\xi_0\partial_q u(\xi_0)\right]\right)\right)^2 - 1\right| \\ &= \left|\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right| \geq \Re\left(\left(\mu e^{i\theta}\left(1 - \beta\left[2e^{i\theta} - (1-q)\mu e^{i\theta}\right]\right)\right)^2 - 1\right) \\ &= (\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1 \geq 1. \end{aligned} \tag{83}$$

Thus, for $\cos(\theta) = 1, \mu = 1$, the solution of

$$\left|(\mu\cos(\theta)(1 - \beta[2\cos(\theta) - (1-q)\mu\cos(\theta)]))^2 - 1\right| \geq 1, \tag{84}$$

provided one of the following cases:

$$\Upsilon = 0, \Upsilon \geq \sqrt{2}, \Upsilon \leq -\sqrt{2}, \tag{85}$$

where

$$\Upsilon \coloneqq (1 - \beta(1+q))^2. \tag{86}$$

Hence, we obtain one of the following arguments:

$$\begin{aligned} \beta &= \frac{1}{q+1}; \\ \beta &\leq -\frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} - 1}{(q+1)}, \quad 0 < q < 1, \\ \beta &\geq \frac{1.18921\sqrt{\left(1/(q+1)^2\right)}q + 1.18921\sqrt{\left(1/(q+1)^2\right)} + 1}{(q+1)}, \quad 0 < q < 1. \end{aligned} \tag{87}$$

All the above inequalities contradict the assumptions of the theorem, which mean that

$$g(\xi) \prec J_3^{(\beta)}(\xi) = \left(1 + \xi - \beta\xi^2\right). \tag{88}$$

□

Corollary 5. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\frac{\xi\left(\partial_q\varphi(\xi)\right)}{\varphi(\xi)}\right) \prec \sqrt{1+\xi}. \tag{89}$$

If one of the assumptions of Theorem 3 occurred, then $\varphi \in \Delta_q^{(\beta)}$.

Proof. Assume

$$\sigma(\xi) = \left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right). \tag{90}$$

Obviously, $\sigma(0) = 1$. Thus, according to Theorem 3, we obtain $\varphi \in \Delta_q^{(\beta)}$. □

Similarly, for $\sigma(\xi) = \partial_q\varphi(\xi)$, we have the following consequence.

Corollary 6. *Let $\varphi \in \Delta$ be the satisfied subordination:*

$$1 + \xi\left(\partial_q\varphi(\xi)\right) \prec \sqrt{1+\xi}. \tag{91}$$

If one of the assumptions of Theorem 3 is occurred, then $\varphi \in \mathbb{B}_q^{(\beta)}$.

4. Conclusion

From above, we investigate the sufficient conditions to obtain the q-subordination of the q-starlike class

$$\left(\frac{\xi\partial_q\varphi(\xi)}{\varphi(\xi)}\right) \prec \mathbb{J}(\xi), \quad \xi \in \sqcup, \tag{92}$$

where $\mathbb{J}(\xi) = J_i^{(\beta)}$, $i = 1, 2, 3$. Differential inequalities are illustrated, involving the q-differential subordination. Nice geometric presentation is included describing the connected

Julia functions of different orders. Our class is called 2D parametric subclass of analytic function, and β is given in terms of q. Note that the case of 1D parametric subclass is given by

$$\mathbb{J}(\xi) = \frac{1+\xi}{1-q\xi}. \tag{93}$$

It is studied in [21], while null parametric subclass is formulated by

$$\mathbb{J}(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2, \tag{94}$$

and it is investigated in [16].

For future works, one can suggest any types of parametric analytic functions (geometric functions) in the open unit disk. The above q-differential subordination formula can be suggested to study the solution of many classes of generalized differential equations such as the class of Briot-Bouquet differential equation (2).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article and read and agreed to the published version of the manuscript.

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