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Research Article

Boundedness for the Modified Fractional Integral Operator from Mixed Morrey Spaces to the Bounded Mean Oscillation Space and Lipschitz Spaces

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In this paper, we establish the boundedness of the modified fractional integral operator from mixed Morrey spaces to the bounded mean oscillation space and Lipschitz spaces, respectively.

1. Introduction

Let $0<\alpha<n$, and the fractional integral operator I_α and the modified fractional integral operator $\widehat{I}_\alpha$ are defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y,$$
$$\widehat{I}_\alpha f(x)=\int_{\mathbb{R}^n}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{\chi_{\{|y|\geq 1\}}(y)}{|y|^{n-\alpha}}\right)f(y)\mathrm{d}y, \tag{1}$$

where f is a locally integrable function on $\mathbb{R}^n$.

The well-known Hardy–Littlewood–Sobolev inequality yields the boundedness of I_α from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1<p<q<\infty$ and $1/p-1/q=\alpha/n$. For the endpoint situation, we know I_α is bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\alpha)}(\mathbb{R}^n)$ (see [1]). Moreover, Peetre [2] proved that $\widehat{I}_\alpha$ is bounded from $L^{n/\alpha}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. Here, $\mathrm{BMO}(\mathbb{R}^n)$ is the bounded mean oscillation space, which consists of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathrm{BMO}}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\left|f(y)-f_{B(x,r)}\right|\mathrm{d}y<\infty, \tag{2}$$

where $B(x,r)$ is the ball centered at x with radius r, $|B(x,r)|$ denotes its Lebesgue measure, and $f_{B(x,r)}=1/|B(x,r)|\int_{B(x,r)}f(y)\mathrm{d}y$. If one regards two functions whose difference is a constant as one, the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathrm{BMO}}$.

Other than Lebesgue spaces, Morrey spaces are also important function spaces to study the boundedness of integral operators in harmonic analysis. The classical Morrey space was introduced by Morrey [3] to study the regularity of elliptic partial differential equations. Now, we recall the definition of the Morrey space $M_q^p(\mathbb{R}^n)$.

For $1\leq q\leq p<\infty$, the classical Morrey space $M_q^p(\mathbb{R}^n)$ consists of all functions $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ with

$$\|f\|_{M_q^p}=\sup_{x\in\mathbb{R}^n,r>0}|B(x,r)|^{(1/p)-(1/q)}\|f\|_{L^q(B(x,r))}<\infty. \tag{3}$$

One can see that Morrey spaces are natural generalizations of Lebesgue spaces. The mapping properties of $\widehat{I}_\alpha$ on Morrey spaces were first studied by Peetre [2] and further generalized by Adams [4]. We refer readers to [5–19] and the references therein for more studies about boundedness of the fractional integral operator on Morrey-type and anisotropic spaces. Recently, the mapping properties of $\widehat{I}_\alpha$ from

Morrey spaces to $\mathrm{BMO}(\mathbb{R}^n)$ and Lipschitz spaces were also obtained in [20–22]. Here, we review the definition of Lipschitz spaces briefly. Let $0<\beta<1$; we say a locally integrable function f belongs to the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|^{1+\beta/n}} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \mathrm{d}y < \infty. \tag{4}$$

If one regards two functions whose difference is a constant as one, the space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathrm{Lip}_\beta}$. As we know, a locally integrable function f which belongs to $\mathrm{Lip}_\beta(\mathbb{R}^n)$ also means that there exists a constant C such that $|f(x) - f(y)| \le C|x-y|^\beta$ for all $x, y \in \mathbb{R}^n$. If C_0 is the smallest constant satisfying the inequality, then $\|f\|_{\mathrm{Lip}_\beta} \sim C_0$.

With the development of the theory of function spaces, Morrey spaces have been extended to different settings. One of the extensions is the mixed Morrey space, which was recently defined by Nogayama et al. [23–25] to uniform mixed Lebesgue spaces and Morrey spaces. Note that there exists another mixed Morrey space by using the iteration of Morrey norm introduced by Ragusa and Scapellato [26]. We refer the readers to [21, 27, 28] for the boundedness of various operators on these mixed Morrey spaces of iteration type.

To give the definition of mixed Morrey spaces $M^p_{\vec{q}}(\mathbb{R}^n)$, we first recall the definition of mixed Lebesgue spaces introduced in [29]. Let $\vec{p} = (p_1, \ldots, p_n) \in (0,\infty]^n$. Then, the mixed Lebesgue norm $\|\cdot\|_{\vec{p}}$ is defined by

$$\|f\|_{\vec{p}} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} \mathrm{d}x_1 \right)^{p_2/p_1} \mathrm{d}x_2 \right)^{p_3/p_2} \cdots \mathrm{d}x_n \right)^{1/p_n}, \tag{5}$$

where f is a measurable function. If $p_j = \infty$ for some $j = 1, \ldots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ to be the set of all measurable functions f with $\|f\|_{\vec{p}} < \infty$.

Now, we can give the definition of mixed Morrey spaces introduced by Nogayama [24].

Definition 1. Let $1 \le \vec{q} < \infty, 1 \le p < \infty$, and $n/p \le \sum_{i=1}^n 1/q_i$. A measurable function f belongs to the mixed Morrey space $M^p_{\vec{q}}(\mathbb{R}^n)$ if and only if

$$\|f\|_{M^p_{\vec{q}}} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{(1/p)-(1/n)\left(\sum_{i=1}^n (1/q_i)\right)} \|f\chi_{B(x,r)}\|_{\vec{q}} < \infty. \tag{6}$$

In [24], the author proved the boundedness of I_α from one mixed Morrey space to another, which inspires us to consider the mapping of $\widetilde{I}_\alpha$ at or beyond the endpoint situation, i.e., the boundedness of $\widetilde{I}_\alpha$ from $M^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ or $\mathrm{Lip}_\beta(\mathbb{R}^n)$.

Throughout the paper, we use the following notations.

Let $L_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. We use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

The letter $\vec{p}$ denotes n-tuples of the numbers in $[0,\infty]$, i.e., $\vec{p} = (p_1, \ldots, p_n)$, where $n \in \mathbb{N}$ and $n \ge 2$. By definition, the inequality, for example, $0 < \vec{p} < \infty$, means $0 < p_i < \infty$ for all i. For $1 \le \vec{p} \le \infty$, we denote $\vec{p}' = (p_1', \ldots, p_n')$, where p_i' satisfies $(1/p_i) + (1/p_i') = 1$.

By $A \lesssim B$, we mean that $A \le CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Main Results

We first recall the boundedness of I_α on mixed Lebesgue spaces, which will be used to prove our main results. Here and in the following, we denote $\vec{q_1} = (q_{11}, q_{12}, \ldots, q_{1n})$ and $\vec{q_2} = (q_{21}, q_{22}, \ldots, q_{2n})$.

Lemma 1. *Let $0 < \alpha < n$ and $1 < \vec{q_1} < \vec{q_2} < \infty$. Then,*

$$\|I_\alpha f\|_{L^{\vec{q_2}}} \lesssim \|f\|_{L^{\vec{q_1}}} \tag{7}$$

if and only if

$$\alpha = \sum_{i=1}^n \frac{1}{q_{1i}} - \sum_{i=1}^n \frac{1}{q_{2i}}. \tag{8}$$

The proof of Lemma 1 can be found in Lemma 3.1 of [30].

In [24], Nogayama established the following result on the boundedness of I_α from one mixed Morrey space to another.

Lemma 2. *Let $0 < \alpha < n$, $1 < p_1, p_2 < \infty$, and $1 < \vec{q_1}, \vec{q_2} < \infty$. Assume that $n/p_1 \le \sum_{i=1}^n 1/q_{1i}$ and $n/p_2 \le \sum_{i=1}^n 1/q_{2i}$. Also, assume that $1/p_2 = 1/p_1 - \alpha/n$ and $\vec{q_1}/p_1 = \vec{q_2}/p_2$. Then, for $f \in M^{p_1}_{\vec{q_1}}(\mathbb{R}^n)$,*

$$\|I_\alpha f\|_{M^{p_2}_{\vec{q_2}}} \lesssim \|f\|_{M^{p_1}_{\vec{q_1}}}. \tag{9}$$

For more studies on the boundedness of operators on mixed Morrey spaces, we refer the readers to [31, 32]. One can see that p_1 in Lemma 2 satisfies $1 < p_1 < n/\alpha$. It is natural to ask what happens if $p_1 = n/\alpha$ or $p_1 > n/\alpha$. In this section, we give an affirmative answer. More precisely, we will establish the boundedness of modified fractional integral operator $\widehat{I}_\alpha$ from $M^{p_1}_{\vec{q_1}}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ or $\mathrm{Lip}_\beta(\mathbb{R}^n)$.

When $p_1 = n/\alpha$ in Lemma 2, we have the following theorem.

Theorem 1. *Let $0<\alpha<n$, $1<\vec{q_1}<\infty$, and $\alpha<\sum_{i=1}^n 1/q_{1i}$. Then, for all $f \in M^{n/\alpha}_{\vec{q_1}}(\mathbb{R}^n)$, we have*

$$\left\|\widehat{I}_\alpha f\right\|_{\mathrm{BMO}} \lesssim \|f\|_{M^{n/\alpha}_{\vec{q_1}}}. \tag{10}$$

Proof. By the definition of $\mathrm{BMO}(\mathbb{R}^n)$, we only need to show for any $B = B(x_0, r)$ and $f \in M^{n/\alpha}_{\vec{q_1}}(\mathbb{R}^n)$, there holds

$$\frac{1}{|B|}\int_B \left|\widehat{I}_\alpha f(x) - \left(\widehat{I}_\alpha f\right)_B\right| dx \lesssim \|f\|_{M^{n/\alpha}_{\vec{q_1}}}. \tag{11}$$

Write $2^jB = B(x_0, 2^jr)$, and let

$$c_1 = \int_{\mathbb{R}^n\backslash 2B} \frac{f(y)}{|x_0 - y|^{n-\alpha}} dy. \tag{12}$$

Then, we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left|\widehat{I}_\alpha f(x) - \left(\widehat{I}_\alpha f\right)_B\right| dx \\
&\le \frac{2}{|B|}\int_B \left|\int_{2B} \frac{f(y)}{|x-y|^{n-\alpha}} dy\right| dx \\
&\quad + \frac{2}{|B|}\int_B \left|\int_{\mathbb{R}^n\backslash 2B} \frac{f(y)}{|x-y|^{n-\alpha}} dy - c_1\right| dx
\end{aligned} \tag{13}$$

$:= \mathrm{I} + \mathrm{II}.$

Define $\vec{q_2}$ such that $1<\vec{q_1}<\vec{q_2}<\infty$ and $\alpha = \sum_{i=1}^n 1/q_{1i} - \sum_{i=1}^n 1/q_{2i}$. For the term I, by using Lemma 1 and Hölder's inequality on mixed Lebesgue spaces, we obtain

$$\begin{aligned}
I &\lesssim \frac{1}{|B|}\int_B \left|I_\alpha(f\chi_{2B})(x)\right| dx \lesssim \frac{1}{|B|}\left\|I_\alpha(f\chi_{2B})\right\|_{\vec{q_2}} \|\chi_B\|_{\vec{q_2}'} \\
&\lesssim \frac{1}{|B|}\|f\chi_{2B}\|_{\vec{q_1}} \|\chi_B\|_{\vec{q_2}'} \lesssim \frac{r^{\sum_{i=1}^n (1/q_{1i}) - \alpha}}{|B|} \|f\|_{M^{n/\alpha}_{\vec{q_1}}} \|\chi_B\|_{\vec{q_2}'} \\
&\lesssim r^{\sum_{i=1}^n (1/q_{1i}) - \sum_{i=1}^n (1/q_{2i}) - \alpha} \|f\|_{M^{n/\alpha}_{\vec{q_1}}} \lesssim \|f\|_{M^{n/\alpha}_{\vec{q_1}}}.
\end{aligned} \tag{14}$$

Now, we turn to estimate II. By a direct computation, we have

$$\begin{aligned}
\mathrm{II} &= \frac{1}{|B|}\int_B \left|\int_{\mathbb{R}^n\backslash 2B} (y) dy\right| dx \\
&\lesssim \frac{r}{|B|}\int_B \left|\int_{\mathbb{R}^n\backslash 2B} \frac{1}{|x_0 - y|^{n-\alpha+1}} f(y) dy\right| dx \\
&\lesssim \sum_{j=1}^\infty \frac{r}{(2^jr)^{n-\alpha+1}} \int_{2^{j+1}B\backslash 2^jB} |f(y)| dy \\
&\lesssim \sum_{j=1}^\infty \frac{r}{(2^jr)^{n-\alpha+1}} \left\|f\chi_{2^{j+1}B}\right\|_{\vec{q_1}} \left\|\chi_{2^{j+1}B}\right\|_{\vec{q_1}'} \\
&\lesssim \sum_{j=1}^\infty \frac{r}{(2^jr)^{\sum_{i=1}^n (1/q_{1i}) - \alpha + 1}} \left\|f\chi_{2^{j+1}B}\right\|_{\vec{q_1}} \\
&= \sum_{j=1}^\infty \frac{r}{2^jr} \frac{1}{(2^jr)^{\sum_{i=1}^n (1/q_{1i}) - \alpha}} \left\|f\chi_{2^{j+1}B}\right\|_{\vec{q_1}} \\
&\lesssim \sum_{j=1}^\infty \frac{1}{2^j} \|f\|_{M^{n/\alpha}_{\vec{q_1}}} = \|f\|_{M^{n/\alpha}_{\vec{q_1}}}.
\end{aligned} \tag{15}$$

From the estimates of I and II, we get (11), which finishes the proof.

When $p_1 > n/\alpha$ in Lemma 2, we have the following mapping property of $\widehat{I}_\alpha$ from $M^{p_1}_{\vec{q_1}}(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha - n/p_1}(\mathbb{R}^n)$. □

Theorem 2. *Let $0<\alpha<n$, $0<\alpha - n/p_1<1$, $1<\vec{q_1}<\infty$, and $\alpha<\sum_{i=1}^n 1/q_{1i}$. Then, for $f \in M^{p_1}_{\vec{q_1}}(\mathbb{R}^n)$, we have*

$$\left\|\widehat{I}_\alpha f\right\|_{\mathrm{Lip}_{\alpha - n/p_1}} \lesssim \|f\|_{M^{p_1}_{\vec{q_1}}}. \tag{16}$$

Proof. The proof is similar to that of Theorem 1.

By the definition of Lipschitz spaces, we only need to show for any $B = B(x_0, r)$ and $f \in M^{p_1}_{\vec{q_1}}(\mathbb{R}^n)$, there holds

$$\frac{1}{|B|^{1+\alpha/n - 1/p_1}}\int_B \left|\widehat{I}_\alpha f(x) - \left(\widehat{I}_\alpha f\right)_B\right| dx \lesssim \|f\|_{M^{p_1}_{\vec{q_1}}}. \tag{17}$$

We also write $2^jB = B(x_0, 2^jr)$ and let

$$c_1 = \int_{\mathbb{R}^n\backslash 2B} \frac{f(y)}{|x_0 - y|^{n-\alpha}} dy. \tag{18}$$

Similar to (13), we have

$$\begin{aligned}&\frac{1}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|\widehat{I}_\alpha f(x)-\left(\widehat{I}_\alpha f\right)_B\right|\mathrm{d}x\\&\leq\frac{2}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|\int_{2B}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y\right|\mathrm{d}x\\&\quad+\frac{2}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|\int_{\mathbb{R}^n\backslash 2B}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y-c_1\right|\mathrm{d}x\\&=I_1+I_2.\end{aligned}\tag{19}$$

Choose $\overrightarrow{q_2}$ such that $1<\overrightarrow{q_1}<\overrightarrow{q_2}<\infty$ and $\alpha=\sum_{i=1}^n 1/q_{1i}-\sum_{i=1}^n 1/q_{2i}$. For the term I_1, by using Lemma 1 and Hölder's inequality on mixed Lebesgue spaces, we get

$$\begin{aligned}I_1&=\frac{1}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|I_\alpha\left(f\chi_{2B}\right)(x)\right|\mathrm{d}x\lesssim\frac{1}{|B|^{1+\alpha/n-1/p_1}}\left\|I_\alpha\left(f\chi_{2B}\right)\right\|_{\overrightarrow{q_2}}\left\|\chi_B\right\|_{\overrightarrow{q_2}'}\\&\lesssim\frac{1}{|B|^{1+\alpha/n-1/p_1}}\left\|f\chi_{2B}\right\|_{\overrightarrow{q_1}}\left\|\chi_B\right\|_{\overrightarrow{q_2}'}\lesssim\frac{r^{\sum_{i=1}^n(1/q_{1i})-(n/p_1)}}{r^{\sum_{i=1}^n(1/q_{2i})+\alpha-(n/p_1)}}\|f\|_{M^{p_1}_{\overrightarrow{q_1}}}\\&=\|f\|_{M^{p_1}_{\overrightarrow{q_1}}}\end{aligned}\tag{20}$$

since $\alpha=\sum_{i=1}^n 1/q_{1i}-\sum_{i=1}^n 1/q_{2i}$.

For I_2, by a similar method as in the proof of Theorem 1, we have

$$\begin{aligned}I_2&=\frac{1}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|\int_{\mathbb{R}^n\backslash 2B}f(y)\mathrm{d}y\right|\mathrm{d}x\\&\lesssim\frac{r}{|B|^{1+\alpha/n-1/p_1}}\int_B\left|\int_{\mathbb{R}^n\backslash 2B}\frac{1}{|x_0-y|^{n-\alpha+1}}f(y)\mathrm{d}y\right|\mathrm{d}x\\&\lesssim\sum_{j=1}^\infty\frac{r^{1-\alpha+n/p_1}}{\left(2^jr\right)^{n-\alpha+1}}\int_{2^{j+1}B\backslash 2^jB}|f(y)|\mathrm{d}y\\&\lesssim\sum_{j=1}^\infty\frac{r^{1-\alpha+n/p_1}}{\left(2^jr\right)^{n-\alpha+1}}\left\|f\chi_{2^{j+1}B}\right\|_{\overrightarrow{q_1}}\left\|\chi_{2^{j+1}B}\right\|_{\overrightarrow{q_1}'}\\&\lesssim\sum_{j=1}^\infty\frac{r^{1-\alpha+n/p_1}\left(2^jr\right)^{\sum_{i=1}^n(1/q_{1i})-(n/p_1)}}{\left(2^jr\right)^{\sum_{i=1}^n(1/q_{1i})-\alpha+1}}\left\|f\chi_{2^{j+1}B}\right\|_{\overrightarrow{q_1}}\\&=\sum_{j=1}^\infty\frac{r^{1-\alpha+n/p_1}}{\left(2^jr\right)^{(1-\alpha+n/p_1)}}\frac{1}{\left(2^jr\right)^{\sum_{i=1}^n(1/q_{1i})-(n/p_1)}}\left\|f\chi_{2^{j+1}B}\right\|_{\overrightarrow{q_1}}\\&\lesssim\sum_{j=1}^\infty\frac{1}{2^{j(1-\alpha+n/p_1)}}\|f\|_{M^{p_1}_{\overrightarrow{q_1}}}\lesssim\|f\|_{M^{p_1}_{\overrightarrow{q_1}}}\end{aligned}\tag{21}$$

since $1-\alpha+n/p_1>0$.

From the estimates of I_1 and I_2, we get (17). The proof is complete.

It is worth mentioning that our results in Theorems 1 and 2 extend the corresponding results of classical Morrey spaces to mixed Morrey spaces. □

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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