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Research Article

On Orthogonal Coupled Fixed Point Results with an Application

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In this manuscript, owing to the concept of orthogonal coupled contraction mappings type I and II, we prove coupled fixed point theorem in orthogonal metric spaces. In order to strengthen our main results, a suitable example is presented. Moreover, the results we obtained supplement and improve previous research findings. A fruitful application is also supplied to endorse our outcomes.

1. Introduction

One of the simple and widely applicable result in nonlinear analysis is Banach contraction principle and it is prolonged in 360° in the last century. One ordinary approach to bolster the Banach contraction principle is to restore the metric by weird generalized metric spaces. In 2017, Eshaghi Gordji et al. [1] established the idea of orthogonality and offered a framework to enlarge the results. In the same year, Eshaghi Gordji and Habibi [2] extended this work and proved some fixed point theorem in generalized O-metric spaces. Later, in 2019, Gordji and Habibi [3] demonstrated fixed point theorems in ε-connected O-metric spaces. By applying altering distance functions, Gungor and Turkoglu [4], in 2019, established fixed point results on O-metric spaces. Using orthogonal F-contraction mappings, Sawangsup et al. [5] proved some fixed point theorem on O-complete metric spaces. In 2021, Beg et al. [6] proved fixed point theorems on O-complete b-metric spaces. In 2021, Arul et al. [7] proved fixed point theorems on O-metric spaces. The concept of O-triangular α-admissibility introduced by Arul et al. [8] proved fixed point theorems on O-metric spaces in 2022.

In 1987, on the other extreme, Gue and Lakshmikantham [9] prompted the conception of a coupled fixed point in partially ordered metric spaces. Following that, Bhaskar and Lakshmikantham [10], in 2006, demonstrated presence of coupled fixed point theorems by utilizing the mixed monotone property. The same coupled fixed point theorems on complete cone metric spaces was exposed by Sabetghadam et al. [11], in 2009. Afterwards, Guneseelan et al. [12] analyzed these results on complex partial b-metric space. Motivated by the above work, here, we prove coupled fixed point theorems on O-complete metric spaces.

2. Preliminaries

In 2021, Gunaseelan et al. [13] proved the following theorem.

Theorem 1. *Let $(\coprod, \varphi)$ be a complete complex partial metric space and the mapping $\Phi : \coprod \times \coprod \longrightarrow \coprod$ such that*

$$\varphi(\Phi(\vartheta, \aleph), \Phi(\sigma, \mu)) \leq \zeta_1 \varphi(\Phi(\vartheta, \aleph), \sigma) + \zeta_2 \varphi(\Phi(\sigma, \mu), \vartheta), \tag{1}$$

where ζ_1 and ζ_2 are nonnegative constants with $\zeta_1 + \zeta_2 < 1$. Then, there exists a unique coupled fixed point of Φ.

Now, let us recall some basic concepts, which will be used in the sequel.

Definition 2 (see [1]). Let $\coprod \neq \varnothing$ and $\perp \subseteq \coprod \times \coprod$ be a binary relation such that or

$$\begin{gathered} \exists \vartheta_0 \in \coprod : (\forall \vartheta \in \coprod, \vartheta \perp \vartheta_0) \\ (\forall \vartheta \in \coprod, \vartheta_0 \perp \vartheta), \end{gathered} \tag{2}$$

then, it is called an orthogonal set (briefly O-set). We denote this O-set by $(\coprod, \perp)$.

Definition 3 (see [1]). Let $(\coprod, \perp)$ be an O-set. A sequence $\{\vartheta_\iota\}$ is called an orthogonal sequence (briefly, O-sequence) if or

$$\begin{gathered} (\forall \iota \in \mathbb{N}, \vartheta_\iota \perp \vartheta_{\iota+1}), \\ (\forall \iota \in \mathbb{N}, \vartheta_{\iota+1} \perp \vartheta_\iota). \end{gathered} \tag{3}$$

Definition 4 (see [1]). A triplet $(\coprod, \perp, \varphi)$ is called an orthogonal metric space (briefly O-metric space) if $(\coprod, \perp)$ is an O-set and $(\coprod, \varphi)$ is a metric space.

Definition 5 (see [1]). Let $(\coprod, \perp)$ be an O-set. A mapping $\Phi : \coprod \times \coprod \longrightarrow \coprod$ is said to be $\perp$-preserving (briefly OP) if $\Phi(\vartheta, \aleph) \perp \Phi(\sigma, \mu)$ whenever $\vartheta \perp \sigma$ and $\aleph \perp \mu$. Also, $\Phi : \coprod \times \coprod \longrightarrow \coprod$ is said to be weakly $\perp$-preserving (briefly WOP) if $\Phi(\vartheta, \aleph) \perp \Phi(\sigma, \mu)$ or $\Phi(\sigma, \mu) \perp \Phi(\vartheta, \aleph)$ whenever $\vartheta \perp \sigma$ and $\aleph \perp \mu$.

Definition 6 (see). Let $\coprod$ be a nonempty set and $\Phi : \coprod \times \coprod \longrightarrow \coprod$ be a mapping. A point $(\aleph, \chi) \in \coprod \times \coprod$ is said to be a coupled fixed point of Φ if $\Phi(\aleph, \chi) = \aleph$ and $\Phi(\chi, \aleph) = \chi$.

3. Main Result

This section presents the new results motivated by Theorem 1 and an O-set, we introduce new O-coupled contraction mappings of type I and II.

Definition 7. Let $(\coprod, \perp, \varphi)$ be an O-metric space. A function $\Phi : \coprod \times \coprod \longrightarrow \coprod$ is called an O-coupled contraction mapping of type I (briefly OCCMT-I) on $(\coprod, \perp, \varphi)$ if for all $\vartheta, \aleph, \sigma, \mu \in \coprod$ with $\vartheta \perp \sigma$ and $\aleph \perp \mu$, $\varphi(\Phi(\vartheta, \aleph), \Phi(\sigma, \mu)) > 0$,

$$\varphi(\Phi(\vartheta, \aleph), \Phi(\sigma, \mu)) \leq \zeta_1 \varphi(\vartheta, \sigma) + \zeta_2 \varphi(\aleph, \mu), \tag{4}$$

where ζ_1 and ζ_2 are nonnegative constants with $\zeta_1 + \zeta_2 < 1$.

Definition 8. Let $(\coprod, \perp, \varphi)$ be an O-metric space. A function $\Phi : \coprod \times \coprod \longrightarrow \coprod$ is called an O-coupled contraction mapping of type II (briefly OCCMT-II) on $(\coprod, \perp, \varphi)$ if for all $\vartheta, \aleph, \sigma, \mu \in \coprod$ with $\vartheta \perp \sigma$ and $\aleph \perp \mu$, $\varphi(\Phi(\vartheta, \aleph), \Phi(\sigma, \mu)) > 0$,

$$\varphi(\Phi(\vartheta, \aleph), \Phi(\sigma, \mu)) \leq \zeta_1 \varphi(\Phi(\vartheta, \aleph), \sigma) + \zeta_2 \varphi(\Phi(\sigma, \mu), \vartheta), \tag{5}$$

where ζ_1 and ζ_2 are nonnegative constants with $\zeta_1 + \zeta_2 < 1$.

Theorem 9. *Let $(\coprod, \perp, \varphi)$ be an O-complete metric space with an orthogonal element $(\vartheta_0, \mathfrak{z}_0)$ and $\Phi : \coprod \times \coprod \longrightarrow \coprod$ be a mapping such that*

(i) Φ is OP

(ii) Φ is OCCMT-I

Then, there exists a unique coupled fixed point of Φ.

Proof. By the definition of an O-set, we can find $\vartheta_0 \in \coprod$ satisfying or

$$\begin{gathered} (\forall \vartheta \in \coprod, \vartheta \perp \vartheta_0) \\ (\forall \vartheta \in \coprod, \vartheta_0 \perp \vartheta), \end{gathered} \tag{6}$$

and we can find $\varpi_0 \in \coprod$ satisfying or

$$\begin{gathered} (\forall \vartheta \in \coprod, \vartheta \perp \varpi_0) \\ (\forall \vartheta \in \coprod, \varpi_0 \perp \vartheta). \end{gathered} \tag{7}$$

□

It follows that $\vartheta_0 \perp \Phi(\vartheta_0, \varpi_0)$ or $\Phi(\vartheta_0, \varpi_0) \perp \vartheta_0$ and $\varpi_0 \perp \Phi(\varpi_0, \vartheta_0)$ or $\Phi(\varpi_0, \vartheta_0) \perp \varpi_0$. Let

$$\begin{gathered} \vartheta_1 \coloneqq \Phi(\vartheta_0, \varpi_0), \vartheta_2 \coloneqq \Phi(\vartheta_1, \varpi_1) = \Phi^2(\vartheta_0, \varpi_0), \cdots, \\ \vartheta_{\iota+1} \coloneqq \Phi(\vartheta_\iota, \varpi_\iota) = \Phi^{\iota+1}(\vartheta_0, \varpi_0), \\ \varpi_1 \coloneqq \Phi(\varpi_0, \vartheta_0), \varpi_2 \coloneqq \Phi(\varpi_1, \vartheta_1) = \Phi^2(\varpi_0, \vartheta_0), \cdots, \\ \varpi_{\iota+1} \coloneqq \Phi(\varpi_\iota, \vartheta_\iota) = \Phi^{\iota+1}(\varpi_0, \vartheta_0). \end{gathered} \tag{8}$$

If $\vartheta_\iota = \vartheta_{\iota+1}$, $\varpi_\iota = \mathfrak{z}_{\iota+1}$ for any $\iota \in \mathbb{N} \cup \{0\}$, then $(\vartheta_\iota, \varpi_\iota)$ is a coupled fixed point of Φ. Suppose that $\vartheta_\iota \neq \vartheta_{\iota+1}$ or $\varpi_\iota \neq \varpi_{\iota+1}$, $\forall \iota \in \mathbb{N} \cup \{0\}$. Then, or

$$\begin{gathered} \varphi(\Phi(\vartheta_\iota, \varpi_\iota), \Phi(\vartheta_{\iota+1}, \varpi_{\iota+1})) > 0, \\ \varphi(\Phi(\varpi_\iota, \vartheta_\iota), \Phi(\mathfrak{z}_{\iota+1}, \vartheta_{\iota+1})) > 0 \end{gathered} \tag{9}$$

for all $\iota \in \mathbb{N} \cup \{0\}$. Since Φ is OP, we have

$$\begin{gathered} \vartheta_\iota \perp \vartheta_{\iota+1} \text{ or } \vartheta_{\iota+1} \perp \vartheta_\iota, \\ \varpi_\iota \perp \varpi_{\iota+1} \quad \text{or} \quad \varpi_{\iota+1} \perp \varpi_\iota, \end{gathered} \tag{10}$$

$\forall \iota \in \mathbb{N} \cup \{0\}$. Therefore, $\{\vartheta_\iota\}$ and $\{\varpi_\iota\}$ are O-sequences.

Since Φ is OCCMT-I,

$$\begin{aligned}\varphi(\vartheta_\iota, \vartheta_{\iota+1}) &\le \zeta_1 \varphi(\vartheta_{\iota-1}, \vartheta_\iota) + \zeta_2 \varphi(\varpi_{\iota-1}, \varpi_\iota),\\ \varphi(\varpi_\iota, \varpi_{\iota+1}) &\le \zeta_1 \varphi(\mathfrak{z}_{\iota-1}, \varpi_\iota) + \zeta_2 \varphi(\vartheta_{\iota-1}, \vartheta_\iota).\end{aligned} \tag{11}$$

Set

$$h_\iota \coloneqq \varphi(\vartheta_\iota, \vartheta_{\iota+1}) + \varphi(\mathfrak{z}_\iota, \varpi_{\iota+1}). \tag{12}$$

Then, we have

$$h_\iota \le (\zeta_1+\zeta_2)\varphi(\vartheta_{\iota-1}, \vartheta_\iota) + (\zeta_1+\zeta_2)\varphi(\varpi_{\iota-1}, \varpi_\iota) = (\zeta_1+\zeta_2)h_{\iota-1}, \quad \forall \iota \in \mathbb{N}. \tag{13}$$

Since $\mathfrak{j} \coloneqq \zeta_2/(1-\zeta_2) < 1$, we get

$$h_\iota \le \mathfrak{j} h_{\iota-1} \le \cdots \le \mathfrak{j}^\iota h_0. \tag{14}$$

If $h_0 = 0$, then

$$\varphi(\vartheta_0, \vartheta_1) + \varphi(\varpi_0, \varpi_1) = 0. \tag{15}$$

Hence, $\vartheta_0 = \vartheta_1 = \Phi(\vartheta_0, \varpi_0)$ and $\varpi_0 = \varpi_1 = \Phi(\varpi_0, \vartheta_0)$, which implies that (ϑ_0, ϖ_0) is a coupled fixed point of Φ. Let $h_0 > 0$. For each $\iota \ge \theta$,

$$\varphi(\vartheta_\iota, \vartheta_\theta) \le \varphi(\vartheta_\iota, \vartheta_{\iota-1}) + \varphi(\vartheta_{\iota-1}, \vartheta_{\iota-2}) + \cdots \varphi(\vartheta_{\theta+1}, \vartheta_\theta). \tag{16}$$

Similarly, we can derive that

$$\varphi(\varpi_\iota, \varpi_\theta) \le \varphi(\varpi_\iota, \varpi_{\iota-1}) + \varphi(\varpi_{\iota-1}, \varpi_{\iota-2}) + \cdots \varphi(\varpi_{\theta+1}, \varpi_\theta). \tag{17}$$

Thus,

$$\begin{aligned}\varphi(\vartheta_\iota, \vartheta_\theta) + \varphi(\varpi_\iota, \varpi_\theta) &\le h_{\iota-1} + h_{\iota-2} + \cdots + h_\theta\\ &\le \left(\mathfrak{j}^{\iota-1} + \mathfrak{j}^{\iota-2} + \cdots + \mathfrak{j}^\theta\right) h_0\\ &\le \frac{\mathfrak{j}^\theta}{1-\mathfrak{j}} h_0 \longrightarrow 0, \ \iota \longrightarrow \infty,\end{aligned} \tag{18}$$

which implies that $\{\vartheta_\iota\}$ and $\{\varpi_\iota\}$ are Cauchy *O*-sequences. By the definition of an *O*-complete, we can find $\vartheta, \varpi \in \coprod$ satisfying $\lim_{\iota \longrightarrow \infty} \vartheta_\iota = \vartheta$ and $\lim_{\iota \longrightarrow \infty} \varpi_\iota = \varpi$. Now,

$$\begin{aligned}\varphi(\Phi(\vartheta, \varpi), \vartheta) \le \varphi(\Phi(\vartheta, \varpi), \vartheta_{\iota+1}) + \varphi(\vartheta_{\iota+1}, \vartheta), \varphi(\Phi(\vartheta, \varpi), \vartheta)\\ - \varphi(\vartheta_{\iota+1}, \vartheta) \le \varphi(\Phi(\vartheta, \varpi), \vartheta_{\iota+1}).\end{aligned} \tag{19}$$

By choice of ϑ and ϖ, we have

$$\begin{aligned}\vartheta \perp \vartheta_\iota \quad &\text{or} \quad \vartheta_\iota \perp \vartheta,\\ \varpi \perp \varpi_\iota \quad &\text{or} \quad \varpi_\iota \perp \varpi.\end{aligned} \tag{20}$$

Since Φ is OCCMT-I, we get

$$\varphi(\Phi(\vartheta, \varpi), \Phi(\vartheta_\iota, \varpi_\iota)) \le \zeta_1 \varphi(\vartheta, \vartheta_\iota) + \zeta_2 \varphi(\varpi, \varpi_\iota). \tag{21}$$

From (19) and (21), we obtain

$$\varphi(\Phi(\vartheta, \varpi), \vartheta) - \varphi(\vartheta_{\iota+1}, \vartheta) \le \zeta_1 \varphi(\vartheta, \vartheta_\iota) + \zeta_2 \varphi(\varpi, \varpi_\iota). \tag{22}$$

As $\iota \longrightarrow \infty$, we get

$$\lim_{\iota \longrightarrow \infty} \varphi(\Phi(\vartheta, \varpi), \vartheta) = 0. \tag{23}$$

Therefore, $\Phi(\vartheta, \varpi) = \vartheta$.

Similarly, we can prove that $\Phi(\varpi, \vartheta) = \varpi$. Assume that (ϑ^*, ϖ^*) is another coupled fixed point of Φ satisfying $(\vartheta, \varpi) \ne (\vartheta^*, \varpi^*)$. Then, $\varphi(\Phi(\vartheta, \varpi), \Phi(\vartheta^*, \varpi^*)) = \varphi(\vartheta, \vartheta^*) > 0$ and $\varphi(\Phi(\varpi, \vartheta), \Phi(\varpi^*, \vartheta^*)) = \varphi(\mathfrak{z}, \varpi^*) > 0$. Since Φ is OP, we get

$$\begin{aligned}\vartheta \perp \vartheta^* \quad &\text{or} \quad \vartheta^* \perp \vartheta,\\ \varpi \perp \varpi^* \quad &\text{or} \quad \varpi^* \perp \varpi.\end{aligned} \tag{24}$$

Since Φ is an OCCMT-I, we get

$$\begin{aligned}\varphi(\vartheta, \vartheta^*) = \varphi(\Phi(\vartheta, \varpi), \Phi(\vartheta^*, \varpi^*)) \le \zeta_1 \varphi(\vartheta, \vartheta^*) + \zeta_2 \varphi(\varpi, \varpi^*),\\ \varphi(\varpi, \varpi^*) = \varphi(\Phi(\varpi, \vartheta), \Phi(\varpi^*, \vartheta^*)) \le \zeta_1 \varphi(\varpi, \varpi^*) + \zeta_2 \varphi(\vartheta, \vartheta^*).\end{aligned} \tag{25}$$

Thus, we have

$$\varphi(\vartheta, \vartheta^*) + \varphi(\varpi, \varpi^*) \le (\zeta_1+\zeta_2)(\varphi(\vartheta, \vartheta^*) + \varphi(\varpi, \varpi^*)). \tag{26}$$

Since $\zeta_1 + \zeta_2 < 1$, we obtain

$$\varphi(\vartheta, \vartheta^*) + \varphi(\varpi, \varpi^*) = 0. \tag{27}$$

Therefore, $\vartheta = \vartheta^*$ and $\varpi = \varpi^*$, which is a absurdity. So, Φ has a unique coupled fixed point.

Example 1. Let $\coprod = \mathbb{R}$ and $\varphi(\aleph, \chi) = |\aleph - \mathfrak{v}|$ for all $\aleph, \chi \in \coprod$. Define a relation $\perp$ on $\coprod$ by

$$\aleph \perp \chi \quad \text{iff} \quad \aleph, \chi \ge 0. \tag{28}$$

Then, $(\coprod, \perp, \varphi)$ is an *O*-complete metric space. Define a mapping $\Phi : \coprod \times \coprod \longrightarrow \coprod$ by $\Phi(\aleph, \chi) = \aleph + 3\chi/5$. Let $\aleph \perp \vartheta$ and $\chi \perp \sigma$. Then, $\aleph, \vartheta \ge 0$ and $\chi, \sigma \ge 0$. Now,

$$\begin{aligned}\Phi(\aleph, \chi) &= \frac{\aleph + 3\chi}{5} \ge 0,\\ \Phi(\vartheta, \sigma) &= \frac{\vartheta + 3\sigma}{5} \ge 0.\end{aligned} \tag{29}$$

It follows that

$$\Phi(\aleph, \chi) \perp \Phi(\vartheta, \sigma). \tag{30}$$

Therefore, Φ is OP. Then, for all $\aleph, \chi, \vartheta, \sigma \in \vartheta$, $\Phi(\aleph, \chi) \ne \Phi(\vartheta, \sigma)$, we get

$$\varphi(\Phi(\aleph,\chi),\Phi(\vartheta,\sigma)) = \left|\frac{\aleph+3\chi}{5} - \frac{\vartheta+3\sigma}{5}\right| \le \left|\frac{\aleph-\vartheta}{5}\right| + \left|\frac{3(\chi-\sigma)}{5}\right| = \frac{|\aleph-\vartheta|}{5} + \frac{3|\chi-\sigma|}{5} = \frac{1}{5}\varphi(\aleph,\vartheta) + \frac{3}{5}\varphi(\chi,\sigma). \quad (31)$$

Therefore, all the hypotheses of Theorem 9 are fulfilled with $\zeta_1(=1/5)+\zeta_2(=3/5)<1$. Hence, Φ has a unique coupled fixed point $(0,0)\in\mathbb{R}\times\mathbb{R}$.

Theorem 10. *Let $(\coprod,\perp,\varphi)$ be an O-complete metric space with an orthogonal element $(\vartheta_0,\mathfrak{z}_0)$ and $\Phi:\coprod\times\coprod\longrightarrow\coprod$ be a mapping such that*

(i) Φ is OP

(ii) Φ is OCCMT-II

Then, there exists a unique coupled fixed point of Φ.

Proof. By the definition of an O-set, we can find $\vartheta_0\in\coprod$ satisfying

$$(\forall\vartheta\in\coprod,\vartheta\perp\vartheta_0) \quad \text{or} \quad (\forall\vartheta\in\coprod,\vartheta_0\perp\vartheta), \quad (32)$$

and we can find $\varpi_0\in\coprod$ satisfying

$$(\forall\vartheta\in\coprod,\vartheta\perp\varpi_0) \quad \text{or} \quad (\forall\vartheta\in\coprod,\varpi_0\perp\vartheta). \quad (33)$$

□

It follows that $\vartheta_0\perp\Phi(\vartheta_0,\varpi_0)$ or $\Phi(\vartheta_0,\varpi_0)\perp\vartheta_0$ and $\varpi_0\perp\Phi(\varpi_0,\vartheta_0)$ or $\Phi(\varpi_0,\vartheta_0)\perp\varpi_0$. Let

$$\begin{aligned}\vartheta_1 &\coloneqq \Phi(\vartheta_0,\varpi_0), \vartheta_2 \coloneqq \Phi(\vartheta_1,\varpi_1) = \Phi^2(\vartheta_0,\varpi_0), \cdots, \vartheta_{\iota+1} \coloneqq \Phi(\vartheta_\iota,\varpi_\iota) = \Phi^{\iota+1}(\vartheta_0,\varpi_0),\\ \varpi_1 &\coloneqq \Phi(\varpi_0,\vartheta_0), \varpi_2 \coloneqq \Phi(\varpi_1,\vartheta_1) = \Phi^2(\varpi_0,\vartheta_0), \cdots, \varpi_{\iota+1} \coloneqq \Phi(\varpi_\iota,\vartheta_\iota) = \Phi^{\iota+1}(\varpi_0,\vartheta_0).\end{aligned} \quad (34)$$

If $\vartheta_\iota=\vartheta_{\iota+1}$, $\varpi_\iota=\mathfrak{z}_{\iota+1}$ for any $\iota\in\mathbb{N}\cup\{0\}$, then, $(\vartheta_\iota,\varpi_\iota)$ is a coupled fixed point of Φ. Suppose that $\vartheta_\iota\neq\vartheta_{\iota+1}$ or $\varpi_\iota\neq\varpi_{\iota+1}$, $\forall\iota\in\mathbb{N}\cup\{0\}$. Then

$$\varphi(\Phi(\vartheta_\iota,\varpi_\iota),\Phi(\vartheta_{\iota+1},\varpi_{\iota+1}))>0 \text{ or } \varphi(\Phi(\varpi_\iota,\vartheta_\iota),\Phi(\mathfrak{z}_{\iota+1},\vartheta_{\iota+1}))>0, \quad (35)$$

for all $\iota\in\mathbb{N}\cup\{0\}$. Since Φ is OP, we have

$$\begin{aligned}&\vartheta_\iota\perp\vartheta_{\iota+1} \quad \text{or} \quad \vartheta_{\iota+1}\perp\vartheta_\iota,\\ &\varpi_\iota\perp\varpi_{\iota+1} \quad \text{or} \quad \varpi_{\iota+1}\perp\varpi_\iota,\end{aligned} \quad (36)$$

$\forall\iota\in\mathbb{N}\cup\{0\}$. Therefore $\{\vartheta_\iota\}$ and $\{\varpi_\iota\}$ are O-sequences. Since Φ is OCCMT-II, we derive that

$$\begin{aligned}\varphi(\vartheta_\iota,\vartheta_{\iota+1}) &= \varphi(\Phi(\vartheta_{\iota-1},\varpi_{\iota-1}),\Phi(\vartheta_\iota,\varpi_\iota)) \le \zeta_1\varphi(\vartheta_\iota,\vartheta_\iota)\\ &\quad + \zeta_2\varphi(\vartheta_{\iota+1},\vartheta_{\iota-1}) \le \zeta_2(\varphi(\vartheta_{\iota+1},\vartheta_\iota)+\varphi(\vartheta_\iota,\vartheta_{\iota-1}))\\ &= \frac{\zeta_2}{1-\zeta_2}\varphi(\vartheta_\iota,\vartheta_{\iota-1}).\end{aligned} \quad (37)$$

Similarly, we can derive that

$$\varphi(\varpi_\iota,\varpi_{\iota+1}) \le \frac{\zeta_2}{1-\zeta_2}\varphi(\varpi_\iota,\varpi_{\iota-1}), \quad (38)$$

From (37) and (38), we derive

$$\varphi(\vartheta_\iota,\vartheta_{\iota+1})+\varphi(\varpi_\iota,\varpi_{\iota+1}) \le \frac{\zeta_2}{1-\zeta_2}(\varphi(\vartheta_\iota,\vartheta_{\iota-1})+\varphi(\varpi_\iota,\varpi_{\iota-1})). \quad (39)$$

Set

$$h_\iota \coloneqq \varphi(\vartheta_\iota,\vartheta_{\iota+1})+\varphi(\mathfrak{z}_\iota,\varpi_{\iota+1}). \quad (40)$$

Then, we have

$$h_\iota \le \frac{\zeta_2}{1-\zeta_2}(\varphi(\vartheta_{\iota-1},\vartheta_\iota)+\varphi(\varpi_{\iota-1},\varpi_\iota)) = \frac{\zeta_2}{1-\zeta_2}h_{\iota-1}, \quad \forall\iota\in\mathbb{N}. \quad (41)$$

Since $\zeta_1+\zeta_2<1$, then $\jmath=\zeta_2/(1-\zeta_2)<1$, we get

$$h_\iota < \jmath h_{\iota-1} \le \cdots \le \jmath^\iota h_0. \quad (42)$$

If $h_0=0$, then

$$\varphi(\vartheta_0,\vartheta_1)+\varphi(\varpi_0,\varpi_1)=0. \quad (43)$$

Hence, $\vartheta_0=\vartheta_1=\Phi(\vartheta_0,\varpi_0)$ and $\varpi_0=\varpi_1=\Phi(\varpi_0,\vartheta_0)$, which implies that (ϑ_0,ϖ_0) is a coupled fixed point of Φ. Let $h_0>0$. For each $\iota\ge\theta$,

$$\varphi(\vartheta_\iota,\vartheta_\theta) \le \varphi(\vartheta_\iota,\vartheta_{\iota-1})+\varphi(\vartheta_{\iota-1},\vartheta_{\iota-2})+\cdots\varphi(\vartheta_{\theta+1},\vartheta_\theta). \quad (44)$$

Similarly, we can derive that

$$\varphi(\varpi_\iota,\varpi_\theta) \le \varphi(\varpi_\iota,\varpi_{\iota-1})+\varphi(\varpi_{\iota-1},\varpi_{\iota-2})+\cdots\varphi(\varpi_{\theta+1},\varpi_\theta). \quad (45)$$

Thus,

$$\begin{aligned}\varphi(\vartheta_\iota,\vartheta_\theta)+\varphi(\varpi_\iota,\varpi_\theta) &\le h_{\iota-1}+h_{\iota-2}+\cdots+h_\theta\\ &\le \left(\jmath^{\iota-1}+\jmath^{\iota-2}+\cdots+\jmath^\theta\right)h_0 \le \frac{\jmath^\theta}{1-\jmath}h_0 \longrightarrow 0,\end{aligned} \quad (46)$$

which implies that $\{\vartheta_\iota\}$ and $\{\varpi_\iota\}$ are Cauchy O-sequences. By the definition of an O-complete, we can find $\vartheta,\varpi\in\coprod$ satisfying $\lim_{\iota\longrightarrow\infty}\vartheta_\iota=\vartheta$ and $\lim_{\iota\longrightarrow\infty}\varpi_\iota=$. Now,

$$\varphi(\Phi(\vartheta,\varpi),\vartheta) \le \varphi(\Phi(\vartheta,\varpi),\vartheta_{\iota+1}) + \varphi(\vartheta_{\iota+1},\vartheta), \varphi(\Phi(\vartheta,\varpi),\vartheta) - \varphi(\vartheta_{\iota+1},\vartheta) \le \varphi(\Phi(\vartheta,\varpi),\vartheta_{\iota+1}). \tag{47}$$

By choice of ϑ and ϖ, we have

$$\begin{aligned} &\vartheta \perp \vartheta_\iota \quad \text{or} \quad \vartheta_\iota \perp \vartheta, \\ &\varpi \perp \varpi_\iota \quad \text{or} \quad \varpi_\iota \perp \varpi. \end{aligned} \tag{48}$$

Since Φ is OCCMT-II, we get

$$\varphi(\Phi(\vartheta,\varpi),\Phi(\vartheta_\iota,\varpi_\iota)) \le \zeta_1\varphi(\Phi(\vartheta,\varpi),\vartheta_\iota) + \zeta_2\varphi(\Phi(\vartheta_\iota,\varpi_\iota),\vartheta). \tag{49}$$

From (47) and (49), we obtain

$$\varphi(\Phi(\vartheta,\varpi),\vartheta) - \varphi(\vartheta_{\iota+1},\vartheta) \le \zeta_1\varphi(\Phi(\vartheta,\varpi),\vartheta_\iota) + \zeta_2\varphi(\vartheta_{\iota+1},\vartheta). \tag{50}$$

As $\iota \longrightarrow \infty$, we get

$$\lim_{\iota\longrightarrow\infty}\varphi(\Phi(\vartheta,\varpi),\vartheta) = 0. \tag{51}$$

Therefore, $\Phi(\vartheta,\varpi) = \vartheta$.

Similarly, we can prove that $\Phi(\varpi,\vartheta) = \varpi$. Assume that (ϑ^*,ϖ^*) is another coupled fixed point of Φ satisfying $(\vartheta,\varpi) \ne (\vartheta^*,\varpi^*)$. Then, $\varphi(\Phi(\vartheta,\varpi),\Phi(\vartheta^*,\varpi^*)) = \varphi(\vartheta,\vartheta^*) > 0$ and $\varphi(\Phi(\varpi,\vartheta),\Phi(\varpi^*,\vartheta^*)) = \varphi(\mathfrak{z},\varpi^*) > 0$. Since Φ is OP, we get

$$\begin{aligned} &\vartheta \perp \vartheta^* \quad \text{or} \quad \vartheta^* \perp \vartheta, \\ &\varpi \perp \varpi^* \quad \text{or} \quad \varpi^* \perp \varpi. \end{aligned} \tag{52}$$

Since Φ is an OCCMT-II, we get

$$\varphi(\vartheta,\vartheta^*) = \varphi(\Phi(\vartheta,\varpi),\Phi(\vartheta^*,\varpi^*)) \le \zeta_1\varphi(\Phi(\vartheta,\varpi),\vartheta^*) + \zeta_2\varphi(\Phi(\vartheta^*,\varpi^*),\vartheta) \tag{53}$$

and $\varphi(\varpi,\varpi^*) = \varphi(\Phi(\varpi,\vartheta),\Phi(\varpi^*,\vartheta^*)) \le \zeta_1\varphi(\Phi(\varpi,\vartheta),\varpi^*) + \zeta_2\varphi(\Phi(\varpi^*,\vartheta^*),\varpi)$.

Thus, we have

$$\varphi(\vartheta,\vartheta^*) + \varphi(\varpi,\varpi^*) \le (\zeta_1+\zeta_2)(\varphi(\vartheta,\vartheta^*) + \varphi(\varpi,\varpi^*)). \tag{54}$$

Since $a + b < 1$, we obtain

$$\varphi(\vartheta,\vartheta^*) + \varphi(\varpi,\varpi^*) = 0. \tag{55}$$

Therefore, $\vartheta = \vartheta^*$ and $\varpi = \varpi^*$, which is a absurdity. So, Φ has a unique coupled fixed point.

Example 2. Let $\coprod = [0,1]$ and $\varphi(\aleph,\chi) = |\aleph - \mathfrak{v}|$ for all $\aleph,\chi \in \coprod$. Define a relation $\perp$ on $\coprod$ by

$$\aleph \perp \chi \text{ iff } \aleph,\chi \ge 0. \tag{56}$$

Then, $(\coprod,\perp,\varphi)$ is an O-complete metric space. Define a mapping $\Phi : \coprod \times \coprod \longrightarrow \coprod$ by $\Phi(\aleph,\chi) = \aleph(1-\chi)$. Let $\aleph \perp \vartheta$ and $\chi \perp \sigma$. Then $\aleph,\vartheta \ge 0$ and $\chi,\sigma \ge 0$. Now

$$\begin{aligned} \Phi(\aleph,\chi) &= \aleph(1-\chi) \ge 0, \\ \Phi(\vartheta,\sigma) &= \vartheta(1-\sigma) \ge 0. \end{aligned} \tag{57}$$

It follows that

$$\Phi(\aleph,\chi) \perp \Phi(\vartheta,\sigma). \tag{58}$$

Therefore, Φ is OP. Then, for all $\aleph,\chi,\vartheta,\sigma \in \vartheta$, $\Phi(\aleph,\chi) \ne \Phi(\vartheta,\sigma)$, we get

$$\begin{aligned} \varphi(\Phi(\aleph,\chi),\Phi(\vartheta,\sigma)) &= |\aleph(1-\chi) - \vartheta(1-\sigma)| \le \frac{3}{4}|\aleph(1-\chi) - \vartheta| \\ &\quad + \frac{1}{5}|\vartheta(1-\sigma) - \aleph| = \frac{3}{4}\varphi(\Phi(\aleph,\chi),\vartheta) \\ &\quad + \frac{1}{5}\varphi(\Phi(\vartheta,\sigma),\aleph). \end{aligned} \tag{59}$$

Therefore, all the hypotheses of Theorem 10 are fulfilled with $\zeta_1(=3/4) + \zeta_2(=1/5) < 1$. Hence, Φ has a unique coupled fixed point $(0,0) \in \mathbb{R} \times \mathbb{R}$.

4. Supportive Application

Let $\mathscr{H} = [0,\mathscr{H}]$ and $\coprod = C(\mathscr{H},\mathbb{R}) = \{\mathfrak{G} : \mathscr{H} \longrightarrow \mathbb{R} \mid \mathfrak{G} \text{ is a continuous function}\}$. Consider the integral equations:

$$\begin{cases} \rho(\chi) = \eth(\chi) + \int_0^{\coprod} \delta(\mathfrak{v},\alpha)\Omega(\alpha,\rho(\alpha),\delta(\alpha))d\alpha, & \chi \in [0,\mathscr{H}], \\ \delta(\chi) = \eth(\chi) + \int_0^{\coprod} \delta(\chi,\alpha)\Omega(\alpha,\delta(\alpha),\rho(\alpha))d\alpha, & \chi \in [0,\mathscr{H}], \end{cases} \tag{60}$$

where

(a) $\eth : \coprod \longrightarrow \mathbb{R}$ and $\Omega : \coprod \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous,

(b) $\delta : \coprod \times \coprod$ is continuous and measurable at $\alpha \in \coprod$, $\forall \chi \in \coprod$;

(c) $\delta(\chi,\alpha) \ge 0, \forall \chi,\alpha \in \coprod$ and $\int_0^{\mathscr{H}} \delta(\chi,\alpha)d\alpha \le 1$, $\forall \chi \in \coprod$.

Theorem 11. *Assume that the conditions (a)–(c) hold. Suppose that*

$$|\Omega(\chi,\rho(\chi),\delta(\chi)) - \Omega(\mathfrak{v},\theta(\chi),\mathfrak{q}(\chi))| \le \frac{|\rho(\chi) - \theta(\chi)|}{2} + \frac{|\delta(\chi) - \mathfrak{q}(\chi)|}{3}, \tag{61}$$

for each $\chi \in \coprod$ and $\forall \rho,\delta \in C(\mathscr{H},\mathbb{R})$. Then, equation (60) has a unique solution in $\coprod$.

Proof. Let $\coprod = \{\kappa \in C(\mathscr{H}, \mathbb{R}) : \kappa(\beta) > 0 \quad \forall \quad \beta \in \mathscr{H}\}$. Define a relation $\perp$ on $\coprod$ by

$$\rho \perp \delta \; iff \; \rho(\beta)\delta(\beta) \geq \rho(\beta) \text{ or } \rho(\beta)\delta(\beta) \geq \delta(\beta) \quad \forall \quad \beta \in [0, \mathscr{H}]. \tag{62}$$

Define a mapping $\varphi : \coprod \times \coprod \longrightarrow [0, \infty)$ by

$$\varphi(\rho, \delta) = \sup_{\beta \in I} |\rho(\chi) - \delta(\chi)|, \tag{63}$$

for all $\rho, \delta \in \coprod$. Thus, $(\coprod, \varphi)$ is a O-complete metric space. Define $\Phi : \coprod \times \coprod \longrightarrow \coprod$ by

$$\Phi(\rho, \delta)(\chi) = \eth(\chi) + \int_0^{\coprod} \delta(\chi, \alpha)\Omega(\alpha, \rho(\alpha), \delta(\alpha))d\alpha, \quad \chi \in [0, \mathscr{H}]. \tag{64}$$

□

For each $\rho, \delta \in \coprod$ with $\rho \perp \delta$ and $\chi \in [0, \mathscr{H}]$, we have

$$\Phi(\rho, \delta)(\chi) = \eth(\chi) + \int_0^{\coprod} \delta(\chi, \alpha)\Omega(\alpha, \rho(\alpha), \delta(\alpha))d\alpha \geq 1. \tag{65}$$

It follows that $[\Phi(\rho, \delta)(\chi)][\Phi(\delta, \rho)(\chi)] \geq \Phi(\rho, \delta)(\chi)$ and so $\Phi(\rho, \delta)(\chi) \perp \Phi(\delta, \rho)(\chi)$. Then, Φ is OP. Let $\rho, \delta, \theta, \mathfrak{q} \in \vartheta$ with $\rho \perp \delta$ and $\theta \perp \mathfrak{q}$. Suppose that $\Phi(\rho, \delta)(\chi) \neq \Phi(\theta, \mathfrak{q})(\chi)$. For every $\chi \in [0, \mathscr{H}]$, we have

$$\begin{aligned} |\Phi(\rho, \delta)(\chi) - \Phi(\theta, \mathfrak{q})(\chi)| &= \int_0^{\mathscr{H}} |\delta(\chi, \alpha)(\Omega(\alpha, \rho(\alpha), \delta(\alpha)) - \Omega(\alpha, \theta(\alpha), \mathfrak{q}(\alpha)))| d\alpha \\ &\leq \int_0^{\mathscr{H}} \delta(\chi, \alpha)|\Omega(\alpha, \rho(\alpha), \delta(\alpha)) - \Omega(\alpha, \theta(\alpha), \mathfrak{q}(\alpha))| d\alpha \\ &\leq \int_0^{\mathscr{H}} \delta(\chi, \alpha)\left(\frac{|\rho(\chi) - \theta(\chi)|}{2} + \frac{|\delta(\chi) - \mathfrak{q}(\chi)|}{3}\right) d\alpha \\ &\leq \left(\frac{|\rho(\chi) - \theta(\chi)|}{2} + \frac{|\delta(\chi) - \mathfrak{q}(\chi)|}{3}\right)\int_0^{\mathscr{H}} \delta(\chi, \alpha) d\alpha \\ &\leq \frac{|\rho(\chi) - \theta(\chi)|}{2} + \frac{|\delta(\chi) - \mathfrak{q}(\chi)|}{3}, \end{aligned} \tag{66}$$

which implies that

$$\varphi(\Phi(\rho, \delta), \Phi(\theta, \mathfrak{q})) \leq a\left(= \frac{1}{2}\right)\varphi(\rho, \theta) + b\left(= \frac{1}{3}\right)\varphi(\delta, \mathfrak{q}). \tag{67}$$

Therefore, all the hypotheses of Theorem 9 are fulfilled and hence equation (60) has a unique solution.

5. Open problem

In this article, we proved coupled fixed point theorems for orthogonal coupled contraction mappings of type I and II in O-complete metric spaces. An illustrative example is presented to strengthen our obtained main results. Agarwal and Karapinar [14] proved coupled fixed point theorems in G-metric spaces in 2013. Here, the intriguing open problem is to investigate the coupled fixed point theorems on orthogonal G-metric spaces instead of coupled fixed point theorems on O-metric space. In 2014, Roldán et al. [15] introduced multidimensional fixed point theorems. It is an intriguing open problem to investigate the orthogonal multidimensional fixed point theorems instead of orthogonal coupled fixed point theorem. In 2016, Roldán, Sintunavarat [17] proved common coupled fixed point theorems in fuzzy metric spaces using the CLRg property. It is an intriguing open problem to investigate the orthogonal common coupled fixed point theorems instead of orthogonal coupled fixed point theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

References

[1] M. Eshaghi Gordji, M. Ramezani, M. De la Sen, and Y. Cho, "On orthogonal sets and Banach fixed point theorem," *Fixed Point Theory*, vol. 18, pp. 569–578, 2017.

[2] M. Eshaghi Gordji and H. Habibi, "Fixed point theory in generalized orthogonal metric space," *Journal of Linear and Topological Algebra*, vol. 6, pp. 251–260, 2017.

[3] M. E. Gordji and H. Habibi, "Fixed point theory in ε -connected orthogonal metric space," *Sahand Communications in Mathematical Analysis*, vol. 16, pp. 35–46, 2019.

[4] N. B. Gungor and D. Turkoglu, "Fixed point theorems on orthogonal metric spaces via altering distance functions," in *AIP Conference Proceedings 2183*, p. 040011, 2019.

[5] K. Sawangsup, W. Sintunavarat, and Y. Cho, "Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 10, 2020.

[6] I. Beg, M. Gunaseelan, and G. Arul Joseph, "Fixed point of orthogonal F-Suzuki contraction mapping on O-complete b-metric spaces with applications," *Journal of Function Spaces*, vol. 2021, Article ID 6692112, 12 pages, 2021.

[7] G. Arul Joseph, M. Gunaseelan, V. Parvaneh, and H. Aydi, "Solving a nonlinear Fredholm integral equation via an orthogonal metric," *Advances in Mathematical Physics*, vol. 2021, Article ID 1202527, 8 pages, 2021.

[8] G. Arul Joseph, M. Gunaseelan, R. L. Jung, and P. Choonkil, "Solving a nonlinear integral equation via orthogonal metric space," *AIMS Mathematics*, vol. 7, no. 1, pp. 1198–1210, 2022.

[9] D. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis:*

Theory, Methods and Applications, vol. 11, no. 5, pp. 623–632, 1987.

[10] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[11] F. Sabetghadam, H. P. Masiha, and A. H. Sanatpour, "Some coupled fixed point theorems in cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 125426, 9 pages, 2009.

[12] M. Gunaseelan, L. N. Mishra, V. N. Mishra, I. A. Baloch, and M. De la Sen, "Application to coupled fixed-point theorems on complex partial b-metric space," *Journal of mathematics*, vol. 2020, Article ID 8881859, 2020.

[13] M. Gunaseelan, L. N. Mishra, and V. N. Mishra, "Generalized coupled fixed point results on complex partial metric space using contractive condition," *Journal of Nonlinear Modeling and Analysis*, vol. 3, pp. 69–80, 2021.

[14] R. P. Agarwal and E. Karapınar, "Remarks on some coupled fixed point theorems in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 2, 2013.

[15] A. Roldán, J. Martinez-Moreno, C. Roldán, and E. Karapinar, "Some remarks on multidimensional fixed point theorems," *Fixed Point Theory*, vol. 15, no. 2, pp. 545–558, 2014.