

Retraction

Retracted: Multicriteria Decision-Making Methods Using Bipolar Neutrosophic Hamacher Geometric Aggregation Operators

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] M. Jamil, F. Afzal, D. Afzal, D. K. Thapa, and A. Maqbool, "Multicriteria Decision-Making Methods Using Bipolar Neutrosophic Hamacher Geometric Aggregation Operators," *Journal of Function Spaces*, vol. 2022, Article ID 5052867, 13 pages, 2022.

Research Article

Multicriteria Decision-Making Methods Using Bipolar Neutrosophic Hamacher Geometric Aggregation Operators

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The study presents a novel conception of aggregation operators (AOs) based on bipolar neutrosophic sets by using Hamacher operations and their application in modeling real-life multicriteria decision-making problems. The neutrosophic set represents incomplete, inconsistent, and indeterminate information effectively. For better understanding in this paper, we have explained all essential definitions and their respective derived neutrosophic sets (NSs) and generalization bipolar neutrosophic sets (BNSs). The primary focus of our work is Hamacher aggregation operators like BN Hamacher weighted geometric (BNHWG), BN Hamacher ordered weighted geometric (BNHOWG), and BN Hamacher hybrid geometric (BNHHG) and their required properties. The proposed scheme provides decision-makers with a comprehensive view of the complexities and vagueness in multicriteria decision-making. As compared to existing methods, these techniques provide comprehensive, increasingly exact, and precise results. Finally, we applied different types of newly introduced AOs and numerical representation on a practical example to demonstrate the effectiveness of the proposed method. Our proposed model and its application have shown improved utility and applicability in the complex decision-making process.

1. Introduction

In the current modern age of community decision-making, data is frequently inadequate, imprecise, and incompatible. Zadeh anticipated the theory of a fuzzy set [1], which deals with vagueness and has applications in a diversity of fields. It does, however, have a flaw in that it can only express a membership value and cannot state any information about nonmembership. To overcome this, Atanassov [2] set up the fundamental concept of intuitionistic fuzzy set (IFS) and its theory to sum up the initiative idea of fuzzy sets. Each and every element of IFS is represented by a pair of membership value (truth-membership) $\mathfrak{F}(\chi)$ as well as nonmembership value (falsity-membership) $f(\chi)$ and satisfies the

conditions $\mathfrak{F}(\chi), f(\chi) \in [0, 1]$ along with $0 \leq \mathfrak{F}(\chi) + f(\chi) \leq 1$. IFSs can only handle incomplete data; they cannot handle indeterminate or unreliable data sets. Smarandache [3] developed the novel neutrosophic set (NS), which added extra indeterminacy membership value $I(\chi)$ with IFS. NS is capable of dealing with knowledge that is incomplete, indeterminate, and contradictory very effectively. When $\mathfrak{F}(\chi) + I(\chi) + f(\chi) < 1$, it represents the information indeterminate. When $\mathfrak{F}(\chi) + I(\chi) + f(\chi) > 1$, it shows that this represents the inconsistent information under a neutrosophic environment.

A single-valued neutrosophic set (SVNS) deals with real-life problems as developed by Wang et al. [4], along with conditions $\mathfrak{F}(\chi), I(\chi), f(\chi) \in [0, 1]$ as well as $0 \leq \mathfrak{F}(\chi) + I(\chi) + f(\chi) \leq 3$.

$\chi) + f(\chi) \leq 3$. Dubois et al. [5] defined the correlation coefficient as well as suggested a method for comparing SVN. The interval-valued neutrosophic set developed by Wang and others [6] broadens the truth, indeterminacy, and false membership range of the value between 0 and 1.

Hamacher's t -norms/ t -conorms [7] are more flexible than algebraic as well as Einstein t -norms/ t -conorms. Many academics have developed the Hamacher operations to address issues involving numerous multicriteria fuzzy decision-making [8–11]. There has not been much research done on Hamacher operations and their applicability to bipolar neutrosophic numbers since the beginning of this field. We developed bipolar neutrosophic Hamacher geometric AOs for multicriteria decision-making by extending Hamacher operations to bipolar neutrosophic sets.

Aggregation operators (AOs) are of great consequence for researchers to attract their attention. Many scientists [12–16] have made a significant contribution toward theory development of IFS since its inception. Based on IFS, Xu and Yager [14] develop the concept of different IF aggregation operators (AOs). They as well used AOs to make decisions related to real life. Einstein aggregation operators (AOs) were developed by Wang and Liu and Chen [17, 18]. Jamil and others [9, 19] develop aggregation operators (AOs) based on bipolar neutrosophic values along with application to group decision-making issues. The bipolar fuzzy set [20–22] has come up at the same time as a different approach in the direction of dealing with ambiguity related to decision-making problems. BFS has both positive and negative membership degrees. The bipolar fuzzy set's membership degree varies from -1 to 1. BFS is very useful in a variety of study domains, including decision-making [6, 18, 23–25]. Wang et al. [10] define bipolar averaging as well as geometric fuzzy aggregation operators (AOs). Deli et al. [26, 27] offered the bipolar neutrosophic set by means of fundamental operations along with the comparison method. Fan and others [28] develop Heronian mean aggregation operators (AOs).

Despite the fact that there is a variety of literature on the topic, the following points about the BNS and Hamacher operations motivated the researcher to conduct a systematic as well as in-depth investigation into the decision analysis. Our most important tools are stated below:

- (1) SVN. This set incorporates the generality of previous sets like classical set, FS, and IFS
- (2) Bipolar fuzzy sets are extremely useful for dealing with unpredictable real-world situations and are useful in dealing with both positive and negative membership values
- (3) The main and foremost intention of the current study include
 - (a) suggesting various bipolar neutrosophic Hamacher AOs and their related properties to our study

- (b) based on BNN, establishing a multicriteria decision-making (MCDM) approach toward real-life problems
- (c) giving a descriptive numerical example of MCDM program

The rest of the research is structured as follows: In Section 2, there are essential definitions as well as their related properties. In Section 3, we introduced BNHWG aggregation operators. In Section 4, these novel AOs are applied to multicriteria decision-making in addition to that of a numerical example. Section 5, at last, proposed a comparative study along with concluding remarks.

2. Preliminaries

We have given a basic definition of the neutrosophic set in the present segment. Different fuzzy sets along with BNS, score, accuracy as well as certainty functions, and Einstein operation are defined.

Definition 1 (see [3]). Consider R to represent a universal set with the neutrosophic set stated below:

$$N = \{(\chi, \mathfrak{F}(\chi), I(\chi), f(\chi)) | \chi \in R\}. \quad (1)$$

The truth-membership is represented by the function $\mathfrak{F} : N \rightarrow Q$, indeterminacy-membership is represented by the function $I : N \rightarrow Q$, and falsity-membership is represented by the function $f : N \rightarrow Q$, where $Q =]0^-, 1^+[$. There is no limitation on the summation of $\mathfrak{F}(\chi)$, $I(\chi)$, and $f(\chi)$, $0^- \leq \mathfrak{F}(\chi) + I(\chi) + f(\chi) \leq 3^+$.

Since applying NS to real-life science as well as business fields is difficult, Ye [29] suggested the idea of SVN as stated.

Definition 2 (see [4]). Consider P to represent a fixed set; the single-valued neutrosophic set (SVNS) of A is stated as

$$A_{NS} = \{(\chi, \mathfrak{F}(\chi), I(\chi), f(\chi)) | \chi \in P\}. \quad (2)$$

The truth-membership is $\mathfrak{F} : A_{NS} \rightarrow L$, indeterminacy-membership is $I : A_{NS} \rightarrow L$, and falsity-membership is $f : A_{NS} \rightarrow L$, where $L = [0, 1]$. The condition of SVNS is $0 \leq \mathfrak{F}(\chi) + I(\chi) + f(\chi) \leq 3$.

Example 1. Let $P = \{\chi_1, \chi_2, \chi_3\}$, then

$$A = \{(\chi_1, 0.1, 0.3, 0.8), (\chi_2, 0.3, 0.6, 0.5), (\chi_3, 0.4, 0.1, 0.7)\} \quad (3)$$

is a SVNS subset of universal set P .

Fundamental operations between two single-valued neutrosophic sets are

$$A_{NS} = \{(\chi, \mathfrak{S}_{A_{NS}}(\chi), I_{A_{NS}}(\chi), f_{A_{NS}}(\chi)) | \chi \in P\}, B_{NS} = \{(\chi, \mathfrak{S}_{B_{NS}}(\chi), I_{B_{NS}}(\chi), f_{B_{NS}}(\chi)) | \chi \in P\} \quad (4)$$

given as the following:

(i) The subset $A_{NS} \subseteq B_{NS}$ represented as

$$\mathfrak{S}_{A_{NS}}(\chi) \leq \mathfrak{S}_{B_{NS}}(\chi), I_{A_{NS}}(\chi) \geq I_{B_{NS}}(\chi), f_{A_{NS}}(\chi) \geq f_{B_{NS}}(\chi) \quad (5)$$

(ii) $A_{NS} = B_{NS}$ represented as

$$\mathfrak{S}_{A_{NS}}(\chi) = \mathfrak{S}_{B_{NS}}(\chi), I_{A_{NS}}(\chi) = I_{B_{NS}}(\chi), f_{A_{NS}}(\chi) = f_{B_{NS}}(\chi) \quad (6)$$

(iii) The complement A'_{NS} is

$$A'_{NS} = \{(\chi, f_{A_{NS}}(\chi), 1 - I_{A_{NS}}(\chi), \mathfrak{S}_{A_{NS}}(\chi)) | \chi \in P\} \quad (7)$$

(iv) $A_{NS} \cap B_{NS}$ represented as

$$A_{NS} \cap B_{NS} = \{(\chi, \min \{\mathfrak{S}_{A_{NS}}(\chi), \mathfrak{S}_{B_{NS}}(\chi)\}, \max \{I_{A_{NS}}(\chi), I_{B_{NS}}(\chi)\}, \max \{f_{A_{NS}}(\chi), f_{B_{NS}}(\chi)\}) | \chi \in P\} \quad (8)$$

(v) The union is defined by

$$A_{NS} \cup B_{NS} = \{(\chi, \max \{\mathfrak{S}_{A_{NS}}(\chi), \mathfrak{S}_{B_{NS}}(\chi)\}, \min \{I_{A_{NS}}(\chi), I_{B_{NS}}(\chi)\}, \min \{f_{A_{NS}}(\chi), f_{B_{NS}}(\chi)\}) | \chi \in P\} \quad (9)$$

Definition 3 (see [5]). Consider two single-valued neutrosophic numbers (SVNNs) $u_1 = (\mathfrak{S}_1, I_1, f_1)$ and $u_2 = (\mathfrak{S}_2, I_2, f_2)$. Thus, the different basic operations between two

SVNNs are defined below:

$$\begin{aligned} u_1 + u_2 &= (\mathfrak{S}_1 + \mathfrak{S}_2 - \mathfrak{S}_1 \mathfrak{S}_2, I_1 I_2, f_1 f_2), \\ u_1 \cdot u_2 &= (\mathfrak{S}_1 \cdot \mathfrak{S}_2, I_1 + I_2 - I_1 I_2, f_1 + f_2 - f_1 f_2), \\ \lambda(u_1) &= (1 - (1 - \mathfrak{S}_1)^\lambda, (I_1)^\lambda, (f_1)^\lambda), \\ (u_1)^\lambda &= ((\mathfrak{S}_1)^\lambda, 1 - (1 - I_1)^\lambda, 1 - (1 - f_1)^\lambda), \end{aligned} \quad (10)$$

where $\lambda > 0$.

Definition 4 (see [5]). Let $u_1 = (\mathfrak{S}_1, I_1, f_1)$ represent a SVNN. The score function for SVNN is $s(u_1)$ stated as

$$s(u_1) = \frac{(\mathfrak{S}_1 + 1 - I_1 + 1 - f_1)}{3}. \quad (11)$$

Definition 5 (see [5]). The accuracy function for a SVNN $u_1 = (\mathfrak{S}_1, I_1, f_1)$ is denoted by $a(u_1)$ defined below:

$$a(u_1) = (\mathfrak{S}_1 - f_1). \quad (12)$$

Definition 6 (see [5]). The certainty function for a SVNN $u_1 = (\mathfrak{S}_1, I_1, f_1)$ is denoted by $c(u_1)$ defined below:

$$c(u_1) = \mathfrak{S}_1. \quad (13)$$

Definition 7 (see [5]). Consider two SVNNs $u_1 = (\mathfrak{S}_1, I_1, f_1)$ and $u_2 = (\mathfrak{S}_2, I_2, f_2)$. Then, the relationship between two SVNNs is stated as below:

- (i) If $s(u_1) > s(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$
- (ii) If $s(u_1) = s(u_2)$ and $a(u_1) > a(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$
- (iii) If $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$, and $c(u_1) > c(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$
- (iv) If $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$, and $c(u_1) = c(u_2)$, then u_1 is equal to u_2 , denoted by $u_1 = u_2$

Definition 8 (see [21]). Consider P as a universal set, and then, the bipolar fuzzy set will be stated as below:

$$F = \{(\chi, \mathfrak{S}_F^+(\chi), f_F^-(\chi)) | \chi \in P\}, \quad (14)$$

where $\mathfrak{S}_F^+(\chi): F \rightarrow L$ is the positive membership degree and $f_F^-(\chi): F \rightarrow K$ is the negative membership degree; here, $L = [1, 0]$, $K = [-1, 0]$.

Definition 9 (see [27]). Consider A as a bipolar neutrosophic set (BNS) within universal set P stated below:

$$A = \{(\chi, \mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi), \mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)) | \chi \in P\}. \quad (15)$$

Let $\mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi) = BN^+$ and $\mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi) = BN^-$, where $\mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi)$ represent the positive membership function for the truth value, indeterminate value, and false value of an element $\chi \in P$ and $\mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)$ represent the negative membership function for the truth value, indeterminate value, and false value for an element $\chi \in P$. Then, $BN^+ : A \rightarrow L$ and $BN^- : A \rightarrow K$, where $L = [1, 0]$, $K = [-1, 0]$. There is a condition that $0 \leq \mathfrak{S}^+(\chi) + I^+(\chi) + f^+(\chi) + \mathfrak{S}^-(\chi) + I^-(\chi) + f^-(\chi) \leq 6$.

Example 2. Let $P = \{\chi_1, \chi_2, \chi_3\}$, then

$$A = \{(\chi_1, 0.4, 0.3, 0.5, -0.6, -0.7, -0.8), (\chi_2, 0.6, 0.8, 0.2, -0.5, -0.8, -0.4), (\chi_3, 0.7, 0.5, 0.9, -0.6, -0.5, -0.4)\} \quad (16)$$

is a BNS subset of universal set P .

Fundamental operations [27] for BNSs are shown below.

Consider $A_1 = \{(\chi, \mathfrak{S}_1^+(\chi), I_1^+(\chi), f_1^+(\chi), \mathfrak{S}_1^-(\chi), I_1^-(\chi), f_1^-(\chi)) | \chi \in P\}$ and $A_2 = \{(\chi, \mathfrak{S}_2^+(\chi), I_2^+(\chi), f_2^+(\chi), \mathfrak{S}_2^-(\chi), I_2^-(\chi), f_2^-(\chi)) | \chi \in P\}$ as two BNSs.

(i) Then $A_1 \subseteq A_2$ if and only if

$$\begin{aligned} \mathfrak{S}_1^+(\chi) &\leq \mathfrak{S}_2^+(\chi), I_1^+(\chi) \leq I_2^+(\chi), f_1^+(\chi) \geq f_2^+(\chi), \\ \mathfrak{S}_1^-(\chi) &\geq \mathfrak{S}_2^-(\chi), I_1^-(\chi) \geq I_2^-(\chi), f_1^-(\chi) \leq f_2^-(\chi) \end{aligned} \quad (17)$$

(ii) $A_1 = A_2$ if and only if

$$\begin{aligned} \mathfrak{S}_1^+(\chi) &= \mathfrak{S}_2^+(\chi), I_1^+(\chi) = I_2^+(\chi), f_1^+(\chi) = f_2^+(\chi), \\ \mathfrak{S}_1^-(\chi) &= \mathfrak{S}_2^-(\chi), I_1^-(\chi) = I_2^-(\chi), f_1^-(\chi) = f_2^-(\chi) \end{aligned} \quad (18)$$

(iii) The union is defined as below:

$$(A_1 \cup A_2) = \left\{ \left(\max(\mathfrak{S}_1^+(\chi), \mathfrak{S}_2^+(\chi)), \frac{I_1^+(\chi) + I_2^+(\chi)}{2}, \min(f_1^+(\chi), f_2^+(\chi)), \min(\mathfrak{S}_1^-(\chi), \mathfrak{S}_2^-(\chi)), \frac{I_1^-(\chi) + I_2^-(\chi)}{2}, \max(f_1^-(\chi), f_2^-(\chi)) \right) \right\} \quad (19)$$

(iv) The intersection is defined as

$$(A_1 \cap A_2) = \left\{ \left(\min(\mathfrak{S}_1^+(\chi), \mathfrak{S}_2^+(\chi)), \frac{I_1^+(\chi) + I_2^+(\chi)}{2}, \max(f_1^+(\chi), f_2^+(\chi)), \max(\mathfrak{S}_1^-(\chi), \mathfrak{S}_2^-(\chi)), \frac{I_1^-(\chi) + I_2^-(\chi)}{2}, \min(f_1^-(\chi), f_2^-(\chi)) \right) \right\} \quad (20)$$

(v) Let $A = \{(\chi, \mathfrak{S}^+(\chi), I^+(\chi), f^+(\chi), \mathfrak{S}^-(\chi), I^-(\chi), f^-(\chi)) | \chi \in P\}$ and be a BNS. Then, the complement A' is defined as

$$\begin{aligned} \mathfrak{S}_{A'}^+(\chi) &= \{1^+\} - \mathfrak{S}_A^+(\chi), I_{A'}^+(\chi) = \{1^+\} - I_A^+(\chi), f_{A'}^+(\chi) = \{1^+\} - f_A^+(\chi), \\ \mathfrak{S}_{A'}^-(\chi) &= \{1^-\} - \mathfrak{S}_A^-(\chi), I_{A'}^-(\chi) = \{1^-\} - I_A^-(\chi), f_{A'}^-(\chi) = \{1^-\} - f_A^-(\chi). \end{aligned} \quad (21)$$

Definition 10 (see [27]). Let us consider two bipolar neutrosophic numbers (BNNs) $u_1 = (\mathfrak{S}_1^+, I_1^+, f_1^+, \mathfrak{S}_1^-, I_1^-, f_1^-)$ and $u_2 = (\mathfrak{S}_2^+, I_2^+, f_2^+, \mathfrak{S}_2^-, I_2^-, f_2^-)$. Then, the basic operations between two BNNs are stated below:

$$\begin{aligned} u_1 + u_2 &= (\mathfrak{S}_1^+ + \mathfrak{S}_2^+ - \mathfrak{S}_1^+ \mathfrak{S}_2^+, I_1^+ I_2^+, f_1^+ f_2^+, -\mathfrak{S}_1^- \mathfrak{S}_2^-, -(I_1^- I_2^- - I_1^- I_2^-), -(f_1^- f_2^- - f_1^- f_2^-)), \\ u_1 \cdot u_2 &= (\mathfrak{S}_1^+ \mathfrak{S}_2^+, I_1^+ + I_2^+ - I_1^+ I_2^+, f_1^+ + f_2^+ - f_1^+ f_2^+, -(\mathfrak{S}_1^- \mathfrak{S}_2^- - \mathfrak{S}_1^- \mathfrak{S}_2^-), -I_1^- I_2^-, -f_1^- f_2^-), \\ \lambda(u_1) &= (1 - (1 - \mathfrak{S}_1^+)^{\lambda}, (I_1^+)^{\lambda}, (f_1^+)^{\lambda}, -(\mathfrak{S}_1^-)^{\lambda}, -(I_1^-)^{\lambda}, -(1 - (1 - (f_1^-)^{\lambda}))), \\ (u_1)^{\lambda} &= ((\mathfrak{S}_1^+)^{\lambda}, 1 - (1 - I_1^+)^{\lambda}, 1 - (1 - f_1^+)^{\lambda}, -((1 - (1 - \mathfrak{S}_1^-)^{\lambda})), -(I_1^-)^{\lambda}, -(f_1^-)^{\lambda}), \end{aligned} \quad (22)$$

where $\lambda > 0$.

Definition 11 (see [27]). The score function for a BNN $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$ is denoted by $S(u)$ defined as below:

$$S(u) = \frac{1}{6} (\mathfrak{S}^+ + 1 - I^+ + 1 - f^+ + 1 + \mathfrak{S}^- - I^- - f^-). \quad (23)$$

Definition 12 (see [27]). The accuracy function for a BNN $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$ is denoted by $a(u)$ defined as below:

$$a(u) = \mathfrak{S}^+ - f^+ + \mathfrak{S}^- - f^-. \quad (24)$$

Definition 13 (see [27]). Let $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$ represent a BNN, then certainty function for BNN u is stated as

$$c(u) = \mathfrak{S}^+ - f^-. \quad (25)$$

Definition 14 (see [27]). Suppose $u_1 = (\mathfrak{S}_1^+, I_1^+, f_1^+, \mathfrak{S}_1^-, I_1^-, f_1^-)$ and $u_2 = (\mathfrak{S}_2^+, I_2^+, f_2^+, \mathfrak{S}_2^-, I_2^-, f_2^-)$ are bipolar neutrosophic numbers (BNNs), so the comparison method among BNNs is stated below:

- (i) If $S(u_1) > S(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$
- (ii) If $S(u_1) = S(u_2)$ and $a(u_1) > a(u_2)$, then u_1 is superior to u_2 , denoted by $u_1 > u_2$

(iii) If condition $S(u_1) = S(u_2), a(u_1) = a(u_2)$, and $c(u_1) > c(u_2)$, then u_1 is greater than u_2 , denoted by $u_1 > u_2$

(iv) If condition $S(u_1) = S(u_2), a(u_1) = a(u_2)$, and $c(u_1) = c(u_2)$, then u_1 is equal to u_2 denoted by $u_1 = u_2$

Hamacher [7] proposed a more generalized t -norm and t -conorm. Hamacher product \otimes represent a t -norm as well as Hamacher sum \oplus represent a t -conorm, where

$$T(P, \beta) = P \otimes \beta = \frac{P\beta}{\gamma + (1 - \gamma)(P + \beta - P\beta)}, \quad \gamma > 0,$$

$$T''(P, \beta) = P \oplus \beta = \frac{P + \beta - P\beta - (1 - \gamma)P\beta}{1 - (1 - \gamma)P\beta}, \quad \gamma > 0. \tag{26}$$

When $\gamma = 1$, the Hamacher t -norm as well as t -conorm will be reduced to algebraic t -norm as well as t -conorm, respectively,

$$T(P, \beta) = P \otimes \beta = P\beta, \tag{27}$$

$$T''(P, \beta) = P \oplus \beta = P + \beta - P\beta.$$

When $\gamma = 2$, the Hamacher t -norm as well as t -conorm will be equivalent to the Einstein t -norm as well as t -conorm, respectively [21]:

$$T(P, \beta) = P \otimes \beta = \frac{P\beta}{1 + (1 - P)(1 - \beta)}, \tag{28}$$

$$T''(P, \beta) = P \oplus \beta = \frac{P + \beta}{1 + P\beta}.$$

Now, we are introducing the Hamacher operation of bipolar neutrosophic set; at the same time, the concept of Hamacher sum, Hamacher product, Hamacher scalar multiple, and Hamacher exponential operation for BNN is stated below.

Definition 15. Let $u = (\mathfrak{S}^+, I^+, f^+, \mathfrak{S}^-, I^-, f^-)$, $u_1 = (\mathfrak{S}_1^+, I_1^+, f_1^+, \mathfrak{S}_1^-, I_1^-, f_1^-)$, and $u_2 = (\mathfrak{S}_2^+, I_2^+, f_2^+, \mathfrak{S}_2^-, I_2^-, f_2^-)$ be three bipolar neutrosophic values and $\lambda > 0$ represent any of the real values; then, basic Hamacher operations are $\gamma > 0$:

$$u_1 \oplus u_2 = \left(\frac{\mathfrak{S}_1^+ + \mathfrak{S}_2^+ - \mathfrak{S}_1^+ \mathfrak{S}_2^+ - (1 - \gamma) \mathfrak{S}_1^+ \mathfrak{S}_2^+}{1 - (1 - \gamma) \mathfrak{S}_1^+ \mathfrak{S}_2^+}, \frac{I_1^+ I_2^+}{\gamma + (1 - \gamma)(I_1^+ + I_2^+ - I_1^+ I_2^+)}, \frac{f_1^+ f_2^+}{\gamma + (1 - \gamma)(f_1^+ + f_2^+ - f_1^+ f_2^+)}, \frac{-\mathfrak{S}_1^- \mathfrak{S}_2^-}{\gamma + (1 - \gamma)(\mathfrak{S}_1^- + \mathfrak{S}_2^- - \mathfrak{S}_1^- \mathfrak{S}_2^-)}, \frac{-(-I_1^- - I_2^- - I_1^- I_2^- - (1 - \gamma) I_1^- I_2^-)}{1 - (1 - \gamma) I_1^- I_2^-}, \frac{-(-f_1^- - f_2^- - f_1^- f_2^- - (1 - \gamma) f_1^- f_2^-)}{1 - (1 - \gamma) f_1^- f_2^-} \right),$$

$$u_1 \otimes u_2 = \left(\frac{\mathfrak{S}_1^+ \mathfrak{S}_2^+}{\gamma + (1 - \gamma)(\mathfrak{S}_1^+ + \mathfrak{S}_2^+ - \mathfrak{S}_1^+ \mathfrak{S}_2^+)}, \frac{I_1^+ + I_2^+ - I_1^+ I_2^+ - (1 - \gamma) I_1^+ I_2^+}{1 - (1 - \gamma) I_1^+ I_2^+}, \frac{f_1^+ + f_2^+ - f_1^+ f_2^+ - (1 - \gamma) f_1^+ f_2^+}{1 - (1 - \gamma) f_1^+ f_2^+}, \frac{-(-\mathfrak{S}_1^- - \mathfrak{S}_2^- - \mathfrak{S}_1^- \mathfrak{S}_2^- - (1 - \gamma) \mathfrak{S}_1^- \mathfrak{S}_2^-)}{1 - (1 - \gamma) \mathfrak{S}_1^- \mathfrak{S}_2^-}, \frac{-I_1^- I_2^-}{\gamma + (1 - \gamma)(I_1^- + I_2^- - I_1^- I_2^-)}, \frac{-f_1^- f_2^-}{\gamma + (1 - \gamma)(f_1^- + f_2^- - f_1^- f_2^-)} \right),$$

$$\lambda(u) = \left(\frac{(1 + (\gamma - 1)\mathfrak{S}^+)^\lambda - (1 - \mathfrak{S}^+)^\lambda}{(1 + (\gamma - 1)\mathfrak{S}^+)^\lambda + (\gamma - 1)(1 - \mathfrak{S}^+)^\lambda}, \frac{\gamma(I^+)^\lambda}{(1 + (\gamma - 1)(1 - I^+)^\lambda + (\gamma - 1)(I^+)^\lambda)}, \frac{\gamma(f^+)^\lambda}{(1 + (\gamma - 1)(1 - f^+)^\lambda + (\gamma - 1)(f^+)^\lambda)}, \frac{-\gamma|\mathfrak{S}^-|^\lambda}{(1 + (\gamma - 1)|\mathfrak{S}^-|^\lambda + (\gamma - 1)|\mathfrak{S}^-|^\lambda)}, \frac{-\gamma|I^-|^\lambda}{(1 + (\gamma - 1)(1 + |I^-|)^\lambda + (\gamma - 1)|I^-|^\lambda)}, \frac{-((1 + (\gamma - 1)|f^-|)^\lambda - (1 + f^-)^\lambda)}{(1 + (\gamma - 1)|f^-|^\lambda + (\gamma - 1)(1 + f^-)^\lambda)} \right),$$

$$(u)^\lambda = \left(\frac{(\mathfrak{S}^+)^\lambda}{(1 + (\gamma - 1)(1 - \mathfrak{S}^+)^\lambda + (\gamma - 1)(\mathfrak{S}^+)^\lambda)}, \frac{(1 + (\gamma - 1)I^+)^\lambda - (1 - I^+)^\lambda}{(1 + (\gamma - 1)I^+)^\lambda + (\gamma - 1)(1 - I^+)^\lambda}, \frac{(1 + (\gamma - 1)f^+)^\lambda - (1 - f^+)^\lambda}{(1 + (\gamma - 1)f^+)^\lambda + (\gamma - 1)(1 - f^+)^\lambda}, \frac{-((1 + (\gamma - 1)|\mathfrak{S}^-|)^\lambda - (1 + \mathfrak{S}^-)^\lambda)}{(1 + (\gamma - 1)|\mathfrak{S}^-|^\lambda + (\gamma - 1)(1 + \mathfrak{S}^-)^\lambda)}, \frac{-\gamma|I^-|^\lambda}{(1 + (\gamma - 1)(1 + |I^-|)^\lambda + (\gamma - 1)|I^-|^\lambda)}, \frac{-\gamma|f^-|^\lambda}{(1 + (\gamma - 1)(1 + |f^-|)^\lambda + (\gamma - 1)|f^-|^\lambda)} \right). \tag{29}$$

3. Bipolar Neutrosophic Hamacher AOs

In this section, we develop a number of basic properties for the bipolar neutrosophic Hamacher weighted geometric aggregation operator, bipolar neutrosophic Hamacher ordered weighted geometric aggregation operator, and bipolar neutrosophic Hamacher hybrid geometric aggregation operator.

3.1. Bipolar Neutrosophic Hamacher Weighted Geometric AO. Let $u_\ell = (\mathfrak{S}_\ell^+, I_\ell^+, f_\ell^+, \mathfrak{S}_\ell^-, I_\ell^-, f_\ell^-)$ be a family of BNNs, where $\ell \in \{1, 2, 3, \dots, n\}$.

Definition 16. The (BNHWG) aggregation operator is defined as follows:

$$BNHWG_\nu(u_1, u_2, u_3, \dots, u_n) = \bigotimes_{\ell=1}^n (\nu_\ell u_\ell) = \nu_1 u_1 \otimes \nu_2 u_2 \otimes \dots \otimes \nu_n u_n, \tag{30}$$

where $\nu = (\nu_1, \nu_2, \nu_3, \dots, \nu_n)^T$ represent the associated weighting vector for u_ℓ , such that $\nu_\ell > 0$ and $\sum_{\ell=1}^n \nu_\ell = 1$.

Theorem 17. The BNHWG operators give in return a bipolar neutrosophic value (BNV) with

$$BNHWG_\nu(u_1, u_2, \dots, u_n) = \left(\frac{\frac{\gamma \mathfrak{B}_1}{\mathfrak{B}_2 + (\gamma - 1)\mathfrak{B}_1}, \frac{\zeta_1 - \zeta_2}{\zeta_1 + (\gamma - 1)\zeta_2}, f_1 - f_2}{\frac{\mathfrak{B}_3 - \mathfrak{B}_4}{f_1 + (\gamma - 1)f_2}, \frac{-\gamma \zeta_3}{\mathfrak{B}_3 + (\gamma - 1)\mathfrak{B}_4}, \frac{-\gamma f_3}{\zeta_3 + (\gamma - 1)\zeta_3}}, -\gamma f_3 \right), \tag{31}$$

where

$$\mathfrak{B}_1 = \prod_{\ell=1}^n (\mathfrak{S}_\ell^+)^\nu, \mathfrak{B}_2 = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 - \mathfrak{S}_\ell^+)^\nu), \mathfrak{B}_3 = \prod_{\ell=1}^n (1 + (\gamma - 1)|\mathfrak{S}_\ell^-|^\nu), \mathfrak{B}_4 = \prod_{\ell=1}^n (1 + \mathfrak{S}_\ell^-)^\nu, \zeta_1 = \prod_{\ell=1}^n (1 + (\gamma - 1)I_\ell^+)^\nu, \zeta_2 = \prod_{\ell=1}^n (1 - I_\ell^+)^\nu, \zeta_3 = \prod_{\ell=1}^n |I_\ell^-|^\nu, \zeta_4 = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 + |I_\ell^-|)^\nu), f_1 = \prod_{\ell=1}^n (1 + (\gamma - 1)f_\ell)^\nu, f_2 = \prod_{\ell=1}^n (1 - f_\ell)^\nu, f_3 = \prod_{\ell=1}^n |f_\ell|^\nu, f_4 = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 + |f_\ell|)^\nu),$$

where $v = (v_1, v_2, v_3, \dots, v_n)^T$ represent the associated weighting vector for u_ℓ , such that $v_\ell > 0$ and $\sum_{\ell=1}^n v_\ell = 11$, $\gamma > 0$.

Proof. Now by mathematical induction.

For $n = 2$,

$$v_1 u_1 = \left(\frac{\gamma(\mathfrak{S}_1^+)^{v_1}}{(1 + (\gamma - 1)(1 - \mathfrak{S}_1^+))^{v_1} + (\gamma - 1)(\mathfrak{S}_1^+)^{v_1}} \frac{(1 + (\gamma - 1)I_1^+)^{v_1} - (1 - I_1^+)^{v_1}}{(1 + (\gamma - 1)I_1^+)^{v_1} + (\gamma - 1)(1 - I_1^+)^{v_1}}, \right. \\ \cdot \frac{(1 + (\gamma - 1)f_1^+)^{v_1} - (1 - f_1^+)^{v_1}}{(1 + (\gamma - 1)f_1^+)^{v_1} + (\gamma - 1)(1 - f_1^+)^{v_1}}, \frac{-((1 + (\gamma - 1)|\mathfrak{S}_1^-|)^{v_1} - (1 + \mathfrak{S}_1^-)^{v_1})}{(1 + (\gamma - 1)|\mathfrak{S}_1^-|)^{v_1} + (\gamma - 1)(1 + \mathfrak{S}_1^-)^{v_1}}, \\ \cdot \frac{-\gamma|I_1^-|^{v_1}}{(1 + (\gamma - 1)(1 + I_1^-))^{v_1} + (\gamma - 1)|I_1^-|^{v_1}}, \frac{-\gamma|f_1^-|^{v_1}}{(1 + (\gamma - 1)(1 + f_1^-))^{v_1} + (\gamma - 1)|f_1^-|^{v_1}} \Big), \\ v_2 u_2 = \left(\frac{\gamma(\mathfrak{S}_2^+)^{v_2}}{(1 + (\gamma - 1)(1 - \mathfrak{S}_2^+))^{v_2} + (\gamma - 1)(\mathfrak{S}_2^+)^{v_2}} \frac{(1 + (\gamma - 1)I_2^+)^{v_2} - (1 - I_2^+)^{v_2}}{(1 + (\gamma - 1)I_2^+)^{v_2} + (\gamma - 1)(1 - I_2^+)^{v_2}}, \right. \\ \cdot \frac{(1 + (\gamma - 1)f_2^+)^{v_2} - (1 - f_2^+)^{v_2}}{(1 + (\gamma - 1)f_2^+)^{v_2} + (\gamma - 1)(1 - f_2^+)^{v_2}}, \frac{-((1 + (\gamma - 1)|\mathfrak{S}_2^-|)^{v_2} - (1 + \mathfrak{S}_2^-)^{v_2})}{(1 + (\gamma - 1)|\mathfrak{S}_2^-|)^{v_2} + (\gamma - 1)(1 + \mathfrak{S}_2^-)^{v_2}}, \\ \cdot \frac{-\gamma|I_2^-|^{v_2}}{(1 + (\gamma - 1)(1 + I_2^-))^{v_2} + (\gamma - 1)|I_2^-|^{v_2}}, \frac{-\gamma|f_2^-|^{v_2}}{(1 + (\gamma - 1)(1 + f_2^-))^{v_2} + (\gamma - 1)|f_2^-|^{v_2}} \Big), \quad (33)$$

and for

$$v_1 u_1 = \left(\frac{\gamma(\mathfrak{S}_1^+)^{v_1}}{(1 + (\gamma - 1)(1 - \mathfrak{S}_1^+))^{v_1} + (\gamma - 1)(\mathfrak{S}_1^+)^{v_1}} \frac{(1 + (\gamma - 1)I_1^+)^{v_1} - (1 - I_1^+)^{v_1}}{(1 + (\gamma - 1)I_1^+)^{v_1} + (\gamma - 1)(1 - I_1^+)^{v_1}}, \right. \\ \cdot \frac{(1 + (\gamma - 1)f_1^+)^{v_1} - (1 - f_1^+)^{v_1}}{(1 + (\gamma - 1)f_1^+)^{v_1} + (\gamma - 1)(1 - f_1^+)^{v_1}}, \frac{-((1 + (\gamma - 1)|\mathfrak{S}_1^-|)^{v_1} - (1 + \mathfrak{S}_1^-)^{v_1})}{(1 + (\gamma - 1)|\mathfrak{S}_1^-|)^{v_1} + (\gamma - 1)(1 + \mathfrak{S}_1^-)^{v_1}}, \\ \cdot \frac{-\gamma|I_1^-|^{v_1}}{(1 + (\gamma - 1)(1 + I_1^-))^{v_1} + (\gamma - 1)|I_1^-|^{v_1}}, \frac{-\gamma|f_1^-|^{v_1}}{(1 + (\gamma - 1)(1 + f_1^-))^{v_1} + (\gamma - 1)|f_1^-|^{v_1}} \Big). \quad (34)$$

So, satisfied for $n = 1$, put $n = r$ for equation (34),

$$\text{BNHWG}_v(u_1, u_2, \dots, u_r) = \left(\frac{\gamma \prod_{\ell=1}^r (\mathfrak{S}_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 - \mathfrak{S}_\ell^+))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (\mathfrak{S}_\ell^+)^{v_\ell}}, \right. \\ \cdot \frac{\prod_{\ell=1}^r (1 + (\gamma - 1)I_\ell^+)^{v_\ell} - \prod_{\ell=1}^r (1 - I_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)I_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 - I_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^r (1 + (\gamma - 1)f_\ell^+)^{v_\ell} - \prod_{\ell=1}^r (1 - f_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)f_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 - f_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^r (1 - (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} - \prod_{\ell=1}^r (1 - \mathfrak{S}_\ell^-)^{v_\ell}}{\prod_{\ell=1}^r (1 - (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 + \mathfrak{S}_\ell^-)^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^r |I_\ell^-|^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 + I_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r |I_\ell^-|^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^r |f_\ell^-|^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 + f_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r |f_\ell^-|^{v_\ell}} \Big). \quad (35)$$

If equation (34) is true $n = r$, then we show that (34) is

true for $n = r + 1$; thus,

$$\text{BNHWG}_v(u_1, u_2, \dots, u_r) = \left(\frac{\gamma \prod_{\ell=1}^r (\mathfrak{S}_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 - \mathfrak{S}_\ell^+))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (\mathfrak{S}_\ell^+)^{v_\ell}}, \right. \\ \cdot \frac{\prod_{\ell=1}^r (1 + (\gamma - 1)I_\ell^+)^{v_\ell} - \prod_{\ell=1}^r (1 - I_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)I_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 - I_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^r (1 + (\gamma - 1)f_\ell^+)^{v_\ell} - \prod_{\ell=1}^r (1 - f_\ell^+)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)f_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 - f_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^r (1 + (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} - \prod_{\ell=1}^r (1 + \mathfrak{S}_\ell^-)^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r (1 + \mathfrak{S}_\ell^-)^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^r |I_\ell^-|^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 + I_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r |I_\ell^-|^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^r |f_\ell^-|^{v_\ell}}{\prod_{\ell=1}^r (1 + (\gamma - 1)(1 + f_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^r |f_\ell^-|^{v_\ell}} \Big) \\ \otimes \left(\frac{\gamma (\mathfrak{S}_{r+1}^+)^{v_{r+1}}}{(1 + (\gamma - 1)(1 - \mathfrak{S}_{r+1}^+))^{v_{r+1}} + (\gamma - 1) \prod_{\ell=1}^r (\mathfrak{S}_{r+1}^+)^{v_{r+1}}}, \right. \\ \cdot \frac{(1 + (\gamma - 1)I_{r+1}^+)^{v_{r+1}} - (1 - I_{r+1}^+)^{v_{r+1}}}{(1 + (\gamma - 1)I_{r+1}^+)^{v_{r+1}} + (\gamma - 1)(1 - I_{r+1}^+)^{v_{r+1}}}, \\ \cdot \frac{(1 + (\gamma - 1)f_{r+1}^+)^{v_{r+1}} - (1 - f_{r+1}^+)^{v_{r+1}}}{(1 + (\gamma - 1)f_{r+1}^+)^{v_{r+1}} + (\gamma - 1)(1 - f_{r+1}^+)^{v_{r+1}}}, \\ \cdot \frac{(1 + (\gamma - 1)|\mathfrak{S}_{r+1}^-|)^{v_{r+1}} - (1 + \mathfrak{S}_{r+1}^-)^{v_{r+1}}}{(1 + (\gamma - 1)|\mathfrak{S}_{r+1}^-|)^{v_{r+1}} + (\gamma - 1)(1 + \mathfrak{S}_{r+1}^-)^{v_{r+1}}}, \\ \cdot \frac{-\gamma |I_{r+1}^-|^{v_{r+1}}}{(1 + (\gamma - 1)(1 + I_{r+1}^-))^{v_{r+1}} + (\gamma - 1) |I_{r+1}^-|^{v_{r+1}}}, \\ \cdot \frac{-\gamma |f_{r+1}^-|^{v_{r+1}}}{(1 + (\gamma - 1)(1 + f_{r+1}^-))^{v_{r+1}} + (\gamma - 1) |f_{r+1}^-|^{v_{r+1}}} \Big) \\ = \left(\frac{\gamma \prod_{\ell=1}^{r+1} (\mathfrak{S}_\ell^+)^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)(1 - \mathfrak{S}_\ell^+))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} (\mathfrak{S}_\ell^+)^{v_\ell}}, \right. \\ \cdot \frac{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)I_\ell^+)^{v_\ell} - \prod_{\ell=1}^{r+1} (1 - I_\ell^+)^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)I_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} (1 - I_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)f_\ell^+)^{v_\ell} - \prod_{\ell=1}^{r+1} (1 - f_\ell^+)^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)f_\ell^+)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} (1 - f_\ell^+)^{v_\ell}}, \\ \cdot \frac{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} - \prod_{\ell=1}^{r+1} (1 + \mathfrak{S}_\ell^-)^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)|\mathfrak{S}_\ell^-|)^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} (1 + \mathfrak{S}_\ell^-)^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^{r+1} |I_\ell^-|^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)(1 + I_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} |I_\ell^-|^{v_\ell}}, \\ \cdot \frac{-\gamma \prod_{\ell=1}^{r+1} |f_\ell^-|^{v_\ell}}{\prod_{\ell=1}^{r+1} (1 + (\gamma - 1)(1 + f_\ell^-))^{v_\ell} + (\gamma - 1) \prod_{\ell=1}^{r+1} |f_\ell^-|^{v_\ell}} \Big). \quad (36)$$

Thus, equation (34) is true for $n = r + 1$. Hence, equation (34) is true for all n . \square

Theorem 18 (idempotency). Let $u_\ell = (\mathfrak{S}_\ell^+, I_\ell^+, f_\ell^+, \mathfrak{S}_\ell^-, I_\ell^-, f_\ell^-)$ ($\ell = 1, 2, \dots, n$) be a set of equal bipolar neutrosophic values, that is, $u_\ell = u$:

$$\text{BNHWG}_v(u_1, u_2, \dots, u_n) = u. \quad (37)$$

Theorem 19 (bounded). Suppose $u^- = \min_\ell u_\ell, u^+ = \max_\ell u_\ell$,

$$u^- \leq \text{BNHWG}_v(u_1, u_2, \dots, u_n) \leq u^+. \quad (38)$$

Theorem 20 (monotonicity). Let $u_\ell = (\mathfrak{S}_\ell^+, I_\ell^+, f_\ell^+, \mathfrak{S}_\ell^-, I_\ell^-, f_\ell^-)$ ($\ell = 1, 2, \dots, n$) and $u'_\ell = (\mathfrak{S}'_\ell^+, I'_\ell^+, f'_\ell^+, \mathfrak{S}'_\ell^-, I'_\ell^-, f'_\ell^-)$ ($\ell = 1, 2, \dots, n$) be two collections of bipolar neutrosophic values. If $u_\ell \leq u'_\ell$ then

$$\text{BNHWG}_v(u_1, u_2, \dots, u_n) \leq \text{BNHWG}_v(u'_1, u'_2, u'_3, \dots, u'_n). \quad (39)$$

Now, we will look at two particular examples of the BNHWG operator.

- (i) If $\gamma = 1$, then the BNHWG operator is equivalent to the basic algebraic operator bipolar neutrosophic weighted geometric (BNWG) operator:

$$\begin{aligned} \text{BNHWG}_v(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_\ell) \\ &= \left(\prod_{\ell=1}^n (\mathfrak{S}_\ell^+)^{v_\ell}, 1 - \prod_{\ell=1}^n (1 - I_\ell^+)^{v_\ell}, 1 - \prod_{\ell=1}^n (1 - f_\ell^+)^{v_\ell}, - \left(1 - \left(\prod_{\ell=1}^n (1 + \mathfrak{S}_\ell^-)^{v_\ell} \right) \right), \right. \\ &\quad \left. - \prod_{\ell=1}^n |I_\ell^-|^{v_\ell}, - \prod_{\ell=1}^n |f_\ell^-|^{v_\ell} \right) \end{aligned} \tag{40}$$

- (ii) If $\gamma = 2$, then the bipolar neutrosophic Einstein weighted geometric (BNEWG) operator is equivalent to the BNHWG operator:

$$\begin{aligned} \text{BNHWG}_v(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_\ell) = \left(\frac{2 \prod_{\ell=1}^n (\mathfrak{S}_\ell^+)^{v_\ell}}{\prod_{\ell=1}^n (2 - \mathfrak{S}_\ell^+)^{v_\ell} + \prod_{\ell=1}^n (\mathfrak{S}_\ell^+)^{v_\ell}}, \right. \\ &\quad \times \frac{\prod_{\ell=1}^n (1 + I_\ell^+)^{v_\ell} - \prod_{\ell=1}^n (1 - I_\ell^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + I_\ell^+)^{v_\ell} + \prod_{\ell=1}^n (1 - I_\ell^+)^{v_\ell}}, \frac{\prod_{\ell=1}^n (1 + f_\ell^+)^{v_\ell} - \prod_{\ell=1}^n (1 - f_\ell^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + f_\ell^+)^{v_\ell} + \prod_{\ell=1}^n (1 - f_\ell^+)^{v_\ell}}, \\ &\quad - \frac{\prod_{\ell=1}^n (1 + |\mathfrak{S}_\ell^-|)^{v_\ell} - \prod_{\ell=1}^n (1 + \mathfrak{S}_\ell^-)^{v_\ell}}{\prod_{\ell=1}^n (1 + |\mathfrak{S}_\ell^-|)^{v_\ell} + \prod_{\ell=1}^n (1 + \mathfrak{S}_\ell^-)^{v_\ell}}, - \frac{2 \prod_{\ell=1}^n |I_\ell^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + I_\ell^-)^{v_\ell} + \prod_{\ell=1}^n |I_\ell^-|^{v_\ell}}, \\ &\quad \left. \times \frac{-2 \prod_{\ell=1}^n |f_\ell^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + f_\ell^-)^{v_\ell} + \prod_{\ell=1}^n |f_\ell^-|^{v_\ell}} \right) \end{aligned} \tag{41}$$

3.2. Bipolar Neutrosophic Hamacher Ordered Weighted Geometric AO

Definition 21. The bipolar neutrosophic Hamacher ordered weighted geometric (BNHOWG) aggregation operator is defined as

$$\begin{aligned} \text{BNHOWG}_v(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_{\sigma(\ell)}) \\ &= v_1 u_{\sigma(1)} \otimes v_2 u_{\sigma(2)} \otimes v_3 u_{\sigma(3)} \otimes \dots \otimes v_n u_{\sigma(n)}, \end{aligned} \tag{42}$$

where $(\sigma(1), \dots, \sigma(n-1), \sigma(n))$ represent permutation of $(1, 2, 3, \dots, n-1, n)$ with $u_{\sigma(\ell-1)} \geq u_{\sigma(\ell)}$ ($\ell = 2, \dots, n-1, n$) and $v = (v_1, v_2, v_3, \dots, v_n)^T$ represent associated weighting vector for u_ℓ , such that $v_\ell > 0$ and $\sum_{\ell=1}^n v_\ell = 1$, $\gamma > 0$.

Theorem 22. The BNHOWG operators give in return a BNV with

$$\text{BNHOWG}_v(u_1, u_2, \dots, u_n) = \left(\frac{\gamma \mathfrak{B}_{\partial 1}^-}{\mathfrak{B}_{\partial 2}^- + (\gamma - 1) \mathfrak{B}_{\partial 1}^-}, \frac{\zeta_{\partial 1}^- - \zeta_{\partial 2}^-}{\zeta_{\partial 1}^- + (\gamma - 1) \zeta_{\partial 2}^-}, f_{\partial 1}^- - f_{\partial 2}^- \right) \left(\frac{\mathfrak{B}_{\partial 3}^- - \mathfrak{B}_{\partial 4}^-}{f_{\partial 1}^- + (\gamma - 1) f_{\partial 2}^-}, \frac{-\gamma \zeta_{\partial 3}^-}{\mathfrak{B}_{\partial 3}^- + (\gamma - 1) \mathfrak{B}_{\partial 4}^-}, \zeta_{\partial 3}^- + (\gamma - 1) \zeta_{\partial 4}^-}, -\gamma f_{\partial 3}^- \right) \tag{43}$$

where

$$\begin{aligned} \mathfrak{B}_{\partial 1}^- &= \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}, \mathfrak{B}_{\partial 2}^- = \prod_{\ell=1}^n (1 + (\gamma - 1) (1 - \mathfrak{S}_{\sigma(\ell)}^+))^{v_\ell}, \mathfrak{B}_{\partial 3}^- \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1) |\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell}, \mathfrak{B}_{\partial 4}^- = \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}, \zeta_{\partial 1}^- \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1) I_{\sigma(\ell)}^+)^{v_\ell}, \zeta_{\partial 2}^- = \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}, \zeta_{\partial 3}^- \\ &= \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}, \zeta_{\partial 4}^- = \prod_{\ell=1}^n (1 + (\gamma - 1) (1 - I_{\sigma(\ell)}^-))^{v_\ell}, f_{\partial 1}^- \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1) f_{\sigma(\ell)}^+)^{v_\ell}, f_{\partial 2}^- = \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}, f_{\partial 3}^- \\ &= \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell}, f_{\partial 4}^- = \prod_{\ell=1}^n (1 + (\gamma - 1) (1 + f_{\sigma(\ell)}^-))^{v_\ell}, \end{aligned} \tag{44}$$

where $(\sigma(1), \dots, \sigma(n-1), \sigma(n))$ represent permutation of $(1, 2, 3, \dots, n-1, n)$ with $u_{\sigma(\ell-1)} \geq u_{\sigma(\ell)}$ ($\ell = 2, \dots, n-1, n$) and $v = (v_1, v_2, v_3, \dots, v_n)^T$ represent associated weighting vector for u_ℓ , such that $v_\ell > 0$ and $\sum_{\ell=1}^n v_\ell = 1$, $\gamma > 0$.

Proof. The theorem is straightforward. \square

Theorem 23 (idempotency). Consider $u_\ell = (\mathfrak{S}_\ell^+, I_\ell^+, f_\ell^+, \mathfrak{S}_\ell^-, I_\ell^-, f_\ell^-)$ ($\ell = 1, 2, \dots, n$) to represent a set of equal BNN that is $u_\ell = u$:

$$\text{BNHOWG}_v(u_1, u_2, \dots, u_n) = u. \tag{45}$$

Theorem 24 (bounded). Suppose $u^- = \min_\ell u_\ell, u^+ = \max_\ell u_\ell$,

$$u^- \leq \text{BNHOWG}_v(u_1, u_2, \dots, u_n) \leq u^+. \tag{46}$$

Theorem 25 (monotonicity). Let $u_\ell = (\mathfrak{S}_\ell^+, I_\ell^+, f_\ell^+, \mathfrak{S}_\ell^-, I_\ell^-, f_\ell^-)$ ($\ell = 1, 2, \dots, n$) and $u'_\ell = (\mathfrak{S}'_{\ell^+}, I'_{\ell^+}, f'_{\ell^+}, \mathfrak{S}'_{\ell^-}, I'_{\ell^-}, f'_{\ell^-})$ ($\ell = 1, 2, \dots, n$) be a set of two bipolar neutrosophic values. If $u_\ell \leq u'_\ell$, then

$$\text{BNHOWG}_v(u_1, u_2, \dots, u_n) \leq \text{BNHOWG}_v(u'_1, u'_2, \dots, u'_n). \tag{47}$$

At the moment, we will look at two particular examples related to BNHOWG AO.

(i) The bipolar neutrosophic ordered weighted geometric (BNOWG) operator is equivalent to the BNHOWG operator if $\gamma = 1$:

$$\begin{aligned} \text{BNOWG}_v(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_\ell) = \left(\prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}, 1 - \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}, 1 \right. \\ &\quad \left. - \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}, - \left(1 - \left(\prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell} \right) \right), - \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}, - \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell} \right) \end{aligned} \tag{48}$$

(ii) The bipolar neutrosophic Einstein ordered weighted geometric (BNEOWG) operator is equivalent to the BNHOWG operator, if $\gamma = 2$:

$$\begin{aligned} \text{BNEOWG}_v(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_\ell) = \left(\frac{2 \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (2 - \mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}}, \right. \\ &\quad \times \frac{\prod_{\ell=1}^n (1 + I_{\sigma(\ell)}^+)^{v_\ell} - \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + I_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}}, \frac{\prod_{\ell=1}^n (1 + f_{\sigma(\ell)}^+)^{v_\ell} - \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + f_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}}, \\ &\quad \frac{\prod_{\ell=1}^n (1 + |\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell} - \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}}{\prod_{\ell=1}^n (1 + |\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell} + \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}}, \frac{-2 \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + I_{\sigma(\ell)}^-)^{v_\ell} + \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}}, \\ &\quad \left. \times \frac{-2 \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + f_{\sigma(\ell)}^-)^{v_\ell} + \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell}} \right) \end{aligned} \tag{49}$$

3.3. Bipolar Neutrosophic Hamacher Hybrid Geometric AO

Definition 26. The bipolar neutrosophic Hamacher hybrid weighted geometric aggregation (BNHHG) operator is defined as

$$\begin{aligned} \text{BNHHG}_{w,v}(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell u_{\sigma(\ell)}) \\ &= v_1 \dot{u}_{\sigma(1)} \otimes v_2 \dot{u}_{\sigma(2)} \otimes v_3 \dot{u}_{\sigma(3)} \otimes \dots \otimes v_n \dot{u}_{\sigma(n)}, \end{aligned} \tag{50}$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ represent associated weighting vector for u_ℓ along with $w_\ell \in [0, 1]$, $\sum_{\ell=1}^n w_\ell = 1$, $\dot{u}_{\sigma(\ell)}$ represent the ℓ th biggest element of the BN arguments $\dot{u}_\ell (\dot{u}_\ell = (nv_\ell)u_\ell)$, $v = (v_1, v_2, v_3, \dots, v_{n-1}, v_n)$ represent the weighting vector of bipolar neutrosophic argument u_ℓ with $w_\ell \in [0, 1]$, $\sum_{\ell=1}^n v_\ell = 1$. Here, n represent the balancing coefficient.

Note that the BNHHG operator reduces to the BNHWG operator if $w = (1/n, 1/n, \dots, 1/n)^T$ as well as the BNHOWG operator if $v = (1/n, 1/n, \dots, 1/n)$.

Theorem 27. The BNHHG operators give in return a BNV with

$$\text{BNHHWG}_{w,v}(u_1, u_2, \dots, u_n) = \left(\frac{\frac{\gamma \dot{b}_{\delta 1}}{\dot{b}_{\delta 2} + (\gamma - 1)\dot{b}_{\delta 1}}, \frac{\dot{\zeta}_{\delta 1} - \dot{\zeta}_{\delta 2}}{\dot{\zeta}_{\delta 1} + (\gamma - 1)\dot{\zeta}_{\delta 2}}, \dot{f}_{\delta 1} - \dot{f}_{\delta 2}}{\frac{\dot{f}_{\delta 1} + (\gamma - 1)\dot{f}_{\delta 2}, \frac{\dot{b}_{\delta 3} - \dot{b}_{\delta 4}}{\dot{b}_{\delta 3} + (\gamma - 1)\dot{b}_{\delta 4}}, \frac{\gamma \dot{\zeta}_{\delta 3}}{\dot{\zeta}_{\delta 4} + (\gamma - 1)\dot{\zeta}_{\delta 3}}, -\dot{f}_{\delta 3}}{\dot{f}_{\delta 1} + (\gamma - 1)\dot{f}_{\delta 2}}}, \right) \tag{51}$$

where

$$\begin{aligned} \dot{b}_{\delta 1} &= \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}, \dot{b}_{\delta 2} = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 - \mathfrak{S}_{\sigma(\ell)}^+))^{v_\ell}, \dot{b}_{\delta 3} \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1)|\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell}, \dot{b}_{\delta 4} = \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}, \dot{\zeta}_{\delta 1} \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1)I_{\sigma(\ell)}^+)^{v_\ell}, \dot{\zeta}_{\delta 2} = \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}, \dot{\zeta}_{\delta 3} \\ &= \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}, \dot{\zeta}_{\delta 4} = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 - I_{\sigma(\ell)}^-))^{v_\ell}, \dot{f}_{\delta 1} \\ &= \prod_{\ell=1}^n (1 + (\gamma - 1)f_{\sigma(\ell)}^+)^{v_\ell}, \dot{f}_{\delta 2} = \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}, \dot{f}_{\delta 3} \\ &= \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell}, \dot{f}_{\delta 4} = \prod_{\ell=1}^n (1 + (\gamma - 1)(1 + f_{\sigma(\ell)}^-))^{v_\ell}, \end{aligned} \tag{52}$$

where $w = (w_1, w_2, w_3, \dots, w_{n-1}, w_n)$ represent the associated weighting vector for u_ℓ along with $w_\ell \in [0, 1]$, $\sum_{\ell=1}^n w_\ell = 1$, $\dot{u}_{\sigma(\ell)}$ represent the ℓ th biggest element of the BN arguments $\dot{u}_\ell (\dot{u}_\ell = (nv_\ell)u_\ell)$, $v = (v_1, v_2, v_3, \dots, v_{n-1}, v_n)$ represent the weighting vector of bipolar neutrosophic argument u_ℓ with $w_\ell \in [0, 1]$, $\sum_{\ell=1}^n v_\ell = 1$. Here, n represent the balancing coefficient, $\gamma > 0$.

Proof. The proof is followed from Theorem 17. \square

Now, we will look at two particular cases of the BNHOWG operator.

(i) The bipolar neutrosophic hybrid geometric (BNHG) operator is equivalent to the BNHHG operator, if $\gamma = 1$:

$$\begin{aligned} \text{BNHHG}_{w,v}(u_1, u_2, \dots, u_n) &= \bigotimes_{\ell=1}^n (v_\ell \dot{u}_\ell) = \left(\prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}, 1 - \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}, 1 \right. \\ &\quad \left. - \prod_{\ell=1}^n (1 - f_{\sigma(\ell)}^+)^{v_\ell}, - \left(1 - \left(\prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell} \right) \right), - \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}, - \prod_{\ell=1}^n |f_{\sigma(\ell)}^-|^{v_\ell} \right) \end{aligned} \tag{53}$$

(ii) The bipolar neutrosophic Einstein hybrid geometric (BNHG) operator is equivalent to the BNHHG operator, if $\gamma = 2$:

TABLE 1: BNN decision matrix D_1 .

	L_1	L_2	L_3	L_4
M_1	(0.6,0.50,0.4,-0.5,-0.4,-0.3)	(0.3,0.6,0.1,-0.4,-0.5,-0.4)	(0.4,0.5,0.30,-0.1,-0.6,-0.7)	(0.3,0.7,0.2,-0.4,-0.5,-0.6)
M_2	(0.7,0.5,0.40,-0.1,-0.9,-0.3)	(0.2,0.4,0.9,-0.4,-0.8,-0.4)	(0.2,0.4,0.7,-0.5,-0.3,-0.9)	(0.2,0.8,0.5,-0.2,-0.3,-0.5)
M_3	(0.6,0.2,0.4,-0.9,-0.2,-0.3)	(0.9,0.4,0.3,-0.6,-0.3,-0.2)	(0.5,0.3,0.7,-0.40,-0.6,-0.9)	(0.4,0.6,0.5,-0.3,-0.9,-0.8)
M_4	(0.1,0.5,0.4,-0.70,-0.6,-0.3)	(0.80,0.5,0.3,-0.7,-0.1,-0.4)	(0.5,0.8,0.7,-0.6,-0.40,-0.6)	(0.4,0.6,0.5,-0.6,-0.4,-0.7)

TABLE 2: BNN decision matrix D_2 .

	L_1	L_2	L_3	L_4
M_1	(0.1,0.6,0.3,-0.7,-0.9,-0.3)	(0.4,0.5,0.4,-0.3,-0.7,-0.9)	(0.5,0.60,0.2,-0.70,-0.2,-0.8)	(0.2,0.5,0.5,-0.8,-0.3,-0.9)
M_2	(0.50, 0.30,0.7,-0.5,-0.2,-0.4)	(0.4,0.8,0.7,-0.3,-0.6,-0.5)	(0.50, 0.30,0.8,-0.3,-0.4,-0.6)	(0.5,0.7,0.4,-0.6,-0.5,-0.4)
M_3	(0.5,0.3,0.60,-0.70,-0.5,-0.6)	(0.3,0.8, 0.10,-0.7,-0.4,-0.7)	(0.4,0.6,0.20,-0.5,-0.7,-0.3)	(0.5,0.7,0.2,-0.4,-0.3,-0.8)
M_4	(0.4,0.6,0.3,-0.8,-0.4,-0.5)	(0.4,0.4,0.5,-0.2,-0.5,-0.6)	(0.6,0.4,0.9,-0.5,-0.40,-0.70)	(0.8,0.2,0.4,-0.7,-0.3,-0.5)

TABLE 3: BNN decision matrix D_3 .

	L_1	L_2	L_3	L_4
M_1	(0.3,0.70,0.5,-0.8,-0.2,-0.4)	(0.5,0.3,0.4,-0.6,-0.1,-0.9)	(0.5,0.4,0.6,-0.7,-0.2,-0.5)	(0.4,0.6,0.5,-0.2,-0.8,-0.5)
M_2	(0.1,0.60,0.2,-0.6,-0.7,-0.4)	(0.3,0.9, 0.60,-0.8,-0.3,-0.5)	(0.8,0.5,0.6,-0.2,-0.7,-0.1)	(0.5,0.8,0.4,-0.5,-0.3,-0.6)
M_3	(0.40, 0.50,0.6,-0.1,-0.5,-0.3)	(0.5,0.3,0.4,-0.6,-0.7,-0.4)	(0.6,0.7,0.3,-0.6,-0.1,-0.7)	(0.3,0.7,0.5,-0.2,-0.7,-0.6)
M_4	(0.70, 0.4,0.8,-0.3,-0.7,-0.6)	(0.7,0.4,0.5,-0.3,-0.5,-0.3)	(0.3,0.7,0.4,-0.8,-0.3,-0.5)	(0.3,0.5,0.9,-0.8,-0.3,-0.5)

TABLE 4: Collective BNN decision matrix R .

(a)

	L_1	L_2
M_1	(0.3062,0.6274,0.4331,-0.7102,-0.3488,-0.3472)	(0.4127,0.4403,0.3155,-0.4896,-0.2545,-0.7288)
M_2	(0.2653,0.5178,0.3816,-0.4515,-0.6161,-0.3676)	(0.2828,0.7943,0.7450,-0.6281,-0.4779,-0.4683)
M_3	(0.4748,0.3777,0.5457,-0.5819,-0.3868,-0.3792)	(0.5552,0.4634,0.3138,-0.6218,-0.4968,-0.3720)
M_4	(0.3738,0.4740,0.6281,-0.5613,-0.6027,-0.5486)	(0.6599,0.4312,0.4441,-0.4264,-0.3216,-0.3792)

(b)

	L_3	L_4
M_1	(0.4683,0.4740,0.4462,-0.5630,-0.2856,-0.6132)	(0.3214,0.6148,0.4181,-0.4204,-0.5867,-0.6015)
M_2	(0.5064,0.4331,0.6786,-0.3168,-0.4968,-0.3084)	(0.3868,0.7827,0.4312,-0.4415,-0.3339,-0.5260)
M_3	(0.5260,0.5818,0.4264,-0.5252,-0.2717,-0.6563)	(0.3640,0.6724,0.4462,-0.2719,-0.6563,-0.6967)
M_4	(0.4064,0.6900,0.6449,-0.6999,-0.3472,-0.5669)	(0.4071,0.4801,0.7555,-0.7309,-0.3276,-0.5559)

TABLE 5: Ranking for all alternatives.

γ	Operators	Ranking
1	BNHWG aggregation operator [27]	$M_3 < M_1 < M_4 < M_2$
2	BNHWG aggregation operator	$M_3 < M_1 < M_2 < M_4$
3	BNHWG aggregation operator	$M_3 < M_1 < M_4 < M_2$

$$\text{BNEHG}_{w,v}(u_1, u_2, \dots, u_n) = \bigotimes_{\ell=1}^n (v_\ell u_\ell) = \left(\frac{2 \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (2 - \mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (\mathfrak{S}_{\sigma(\ell)}^+)^{v_\ell}}, \right. \\ \times \frac{\prod_{\ell=1}^n (1 + I_{\sigma(\ell)}^+)^{v_\ell} - \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + I_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (1 - I_{\sigma(\ell)}^+)^{v_\ell}}, \frac{\prod_{\ell=1}^n (1 + J_{\sigma(\ell)}^+)^{v_\ell} - \prod_{\ell=1}^n (1 - J_{\sigma(\ell)}^+)^{v_\ell}}{\prod_{\ell=1}^n (1 + J_{\sigma(\ell)}^+)^{v_\ell} + \prod_{\ell=1}^n (1 - J_{\sigma(\ell)}^+)^{v_\ell}}, \\ \left. \frac{\prod_{\ell=1}^n (1 + |\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell} - \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}}{\prod_{\ell=1}^n (1 + |\mathfrak{S}_{\sigma(\ell)}^-|)^{v_\ell} + \prod_{\ell=1}^n (1 + \mathfrak{S}_{\sigma(\ell)}^-)^{v_\ell}}, \frac{-2 \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + I_{\sigma(\ell)}^-)^{v_\ell} + \prod_{\ell=1}^n |I_{\sigma(\ell)}^-|^{v_\ell}}, \right. \\ \left. \times \frac{-2 \prod_{\ell=1}^n |J_{\sigma(\ell)}^-|^{v_\ell}}{\prod_{\ell=1}^n (2 + J_{\sigma(\ell)}^-)^{v_\ell} + \prod_{\ell=1}^n |J_{\sigma(\ell)}^-|^{v_\ell}} \right) \quad (54)$$

4. Multicriteria Decision-Making Based on Bipolar Neutrosophic Einstein AOs

This section comprises multicriteria application based on bipolar neutrosophic Hamacher aggregation operators with crisp numbers as attribute weights and BNNs as attribute values.

4.1. Algorithm. Let $M = \{M_1, M_2, \dots, M_m\}$ be the set of m alternatives, $L = \{L_1, L_2, \dots, L_n\}$ be the set or collection of n criterions, $D = \{D_1, D_2, \dots, D_{k-1}, D_k\}$ represent finite k decision-makers. Suppose $v = (v_1, v_2, \dots, v_n)^T$ is the weighted vector for decision-makers $\bar{D}^s (s = 1, 2, 3, \dots, k - 1, k)$, with $v_\ell \in [0, 1]$ and $\sum_{\ell=1}^n v_\ell = 1$. Let $w = (w_1, w_2, \dots, w_{n-1}, w_n)^T$ represent the associated weighting vector of the attribute set $L = \{L_1, L_2, \dots, L_n\}$ such that $w_\ell \in [0, 1]$ and $\sum_{\ell=1}^n w_\ell = 1$. The decision-maker evaluates an alternative criterion; values are defined by BN values. Let $u_{ij}^{(s)} = [(\mathfrak{S}_{ij}^+, I_{ij}^+, f_{ij}^+, \mathfrak{S}_{ij}^-, I_{ij}^-, f_{ij}^-)]_{m \times n}$ represent decision-making matrices supplied by the decision-maker and $u_{ij}^{(s)}$ represent a BNN for L_{ij} criterion-associated alternatives. We have the condition $\mathfrak{S}_{ij}^+, I_{ij}^+, f_{ij}^+, \mathfrak{S}_{ij}^-, I_{ij}^-$ as well as $f_{ij}^- \in [0, 1]$ with condition $0 \leq \mathfrak{S}_{ij}^+, I_{ij}^+, f_{ij}^+, \mathfrak{S}_{ij}^-, I_{ij}^-, f_{ij}^- \leq 6$ where $i = 1, 2, 3, \dots, m$ as well as $j = 1, 2, 3, \dots, n - 1, n$.

Step 1. Construction of matrices (Tables 1–3) for decision $\bar{D}^s = [u_{ij}^{(s)}]_{m \times n} (s = 1, 2, 3, \dots, k - 1, k)$.

Step 2. Computing (Table 4) $\text{BNHWG}_v(r_{i1}, r_{i2}, \dots, r_{in})$ for every $i = 1, 2, \dots, m - 1, m$:

$$r_i = \left(\mathfrak{S}_i^+, I_i^+, f_i^+, \mathfrak{S}_i^-, I_i^-, f_i^- \right) = \text{BNHWG}_v(r_{i1}, r_{i2}, \dots, r_{in}) \\ = \bigotimes_{\ell=1}^n (v_\ell r_{i\ell}) = \left(\frac{\gamma \mathcal{B}_1}{\mathcal{B}_2 + (\gamma - 1) \mathcal{B}_1}, \frac{\zeta_1 - \zeta_2}{\zeta_1 + (\gamma - 1) \zeta_2}, f_1 - f_2, \right. \\ \left. \frac{f_1 + (\gamma - 1) f_3}{\mathcal{B}_3 - \mathcal{B}_4}, \frac{-\gamma \zeta_3}{\zeta_4 + (\gamma - 1) \zeta_3}, -\gamma f_3 \right) \quad (55)$$

Step 3. Compute the score values $S(r_{i\hat{i}}) (i = 1, 2, \dots, m - 1, m)$.

Step 4. Ranking of various software systems for $\text{BNHWG}_v(u_{i\hat{i}1}, u_{i\hat{i}2}, \dots, u_{i\hat{i}n})$ as per their scores.

Step 5. Choose the most excellent possible alternative(s). Now, we give a numerical example as follows.

4.2. Illustrative Example. We consider a Medicine Business Company which needs to employ a professional manager. The organization creates a working group of three decision-makers for this reason, with weighting vectors $v = (0.3, 0.2, 0.5)^T$. There are several considerations to consider when choosing the most knowledgeable manager, but in this case, the committee only considers the four criteria mentioned below, with weighting vector $\omega = (0.2, 0.4, 0.1, 0.3)^T$. Following the first screening exam, four managers $M_i (i = 1, 2, 3, 4)$ will proceed to the next round of the procedure. The committee must make a decision based on the four attributes listed below.

- (1) L_1 : salary
- (2) L_2 : experience
- (3) L_3 : working skill
- (4) L_4 : dealing with public

Step 1. Construct the decision matrices.

Step 2. Compute $\text{BNHWG}_v(u_1, u_2, u_3, \dots, u_n)$ for every $(i = 1, 2, \dots, m - 1, m)$:

$$r_i = \left(\mathfrak{S}_i^+, I_i^+, f_i^+, \mathfrak{S}_i^-, I_i^-, f_i^- \right) = \text{BNHWG}_v(r_{i1}, r_{i2}, \dots, r_{in}) \\ = \bigotimes_{\ell=1}^n (v_\ell r_{i\ell}) = \left(\frac{\gamma \mathcal{B}_1}{\mathcal{B}_2 + (\gamma - 1) \mathcal{B}_1}, \frac{\zeta_1 - \zeta_2}{\zeta_1 + (\gamma - 1) \zeta_2}, f_1 - f_2, \right. \\ \left. \frac{f_1 + (\gamma - 1) f_3}{\mathcal{B}_3 - \mathcal{B}_4}, \frac{-\gamma \zeta_3}{\zeta_4 + (\gamma - 1) \zeta_3}, -\gamma f_3 \right) \quad (56)$$

Step 3. Now, we calculate the score function, for $\gamma = 3$:

$$r_1 = (0.3667, 0.5380, 0.3836, -0.5280, -0.3603, -0.5948), \\ r_2 = (0.3274, 0.7190, 0.5915, -0.5117, -0.4581, -0.4457), \\ r_3 = (0.4748, 0.5286, 0.4137, -0.5088, -0.4915, -0.4883), \\ r_4 = (0.4936, 0.4839, 0.6088, -0.5856, -0.3742, -0.4805), \\ S(r_1) = 0.4787, S(r_2) = 0.4015, S(r_3) = 0.5006, S(r_4) = 0.4450. \quad (57)$$

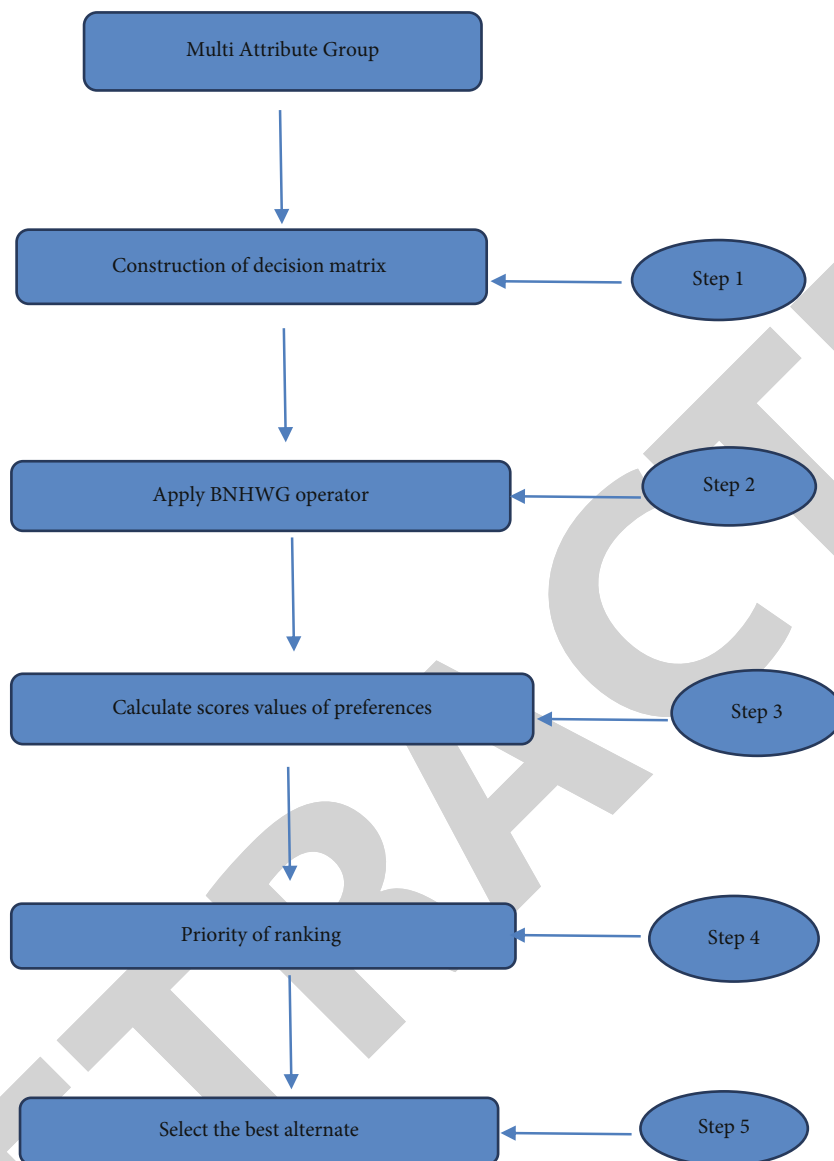


FIGURE 1: Flow chart for MCDM.

Step 4. By calculating the scores for $\gamma = 3$, we have

$$M_3 < M_1 < M_4 < M_2. \quad (58)$$

Step 5. Thus, the best option is M_3 .

5. Comparison

So far, the researchers have used a variety of decision-making techniques. These tools are the following: Chen [18] utilize FSs, and afterward, Atanassov [2] utilize IFs; BFSs is used by Dubois et al. [5], and Zavadskas et al. [23] made use of NSs; Deli et al. [27] made use of bipolar neutrosophic soft sets, while Deli et al. [26] made use of BNSs as well as a variety of other research decisions. In this paper, we use Hamacher operations to apply bipolarity to neutrosophic sets.

The advantage of our proposed methods is that this paper's aggregation operators are more general and versatile. M_3 is, however, the most attractive investment firm.

6. Conclusion

The aim of this paper is to investigate various BNS AOs and apply Hamacher t -norms/ t -conorms to multicriteria community decision-making, with BNS values as the criteria. We proposed bipolar neutrosophic Hamacher aggregation operators motivated by Hamacher operations. To begin with, we discussed BN Hamacher aggregation operators and their required properties. These AOs are BNHWA, BNHOWA, and BNHHWG along with their cases. Finally, we presented a framework for making multicriteria decision-making. An illustrative example applied to our proposed AO is selecting the best manager for Medicine Company. When we take $\gamma = 1$, our proposed AO gives results

similar to that of Deli et al. [27], but if $\gamma = 2$, results vary (Table 5). The findings in the current manuscript demonstrate that our intended approaches are more accurate as well as realistic in practice. We plan to expand the anticipated model extended to other domains and their application in our future research, like pattern recognition and risk analysis.

Data Availability

All data related to the manuscript is contained in it.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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