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*Research Article*

# Littlewood–Paley Characterization for Musielak–Orlicz–Hardy Spaces Associated with Self-Adjoint Operators

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Let $(X, d, \mu)$ be a metric measure space endowed with a metric $d$ and a non-negative Borel doubling measure $\mu$. Let $L$ be a non-negative self-adjoint operator on $L^2(X)$. Assume that the (heat) kernel associated to the semigroup $e^{-tL}$ satisfies a Gaussian upper bound. In this paper, we prove that the Musielak–Orlicz–Hardy space $H_{\varphi,L}(X)$ associated with $L$ in terms of the Lusin-area function and the Musielak–Orlicz–Hardy space $H_{L,G,\varphi}(X)$ associated with $L$ in terms of the Littlewood–Paley function coincide and their norms are equivalent. To do this, we first establish the discrete characterization of these two spaces. It improves the known results in the literature.

## 1. Introduction

The metric measure space $(X, d, \mu)$ is a set $X$ equipped with a metric $d$ and a non-negative Borel doubling measure $\mu$ on $X$. Let $f \in L^2(X)$ and $L$ be a densely defined operator on $L^2(X)$ which satisfies the following two conditions:

(i) **(H1)** $L$ is a non-negative self-adjoint operator on $L^2(X)$.

(ii) **(H2)** The kernel of $e^{-tL}$, denoted by $p_t(x, y)$, is a measurable function on $X \times X$ satisfying the Gaussian estimates, i.e., there exist $C_1, C_2 > 0$ such that

$$\left|p_t(x, y)\right| \leq \frac{C_1}{V(x, \sqrt{t})} e^{-\left(d(x,y)^2/C_2 t\right)}, \quad (1)$$

holds for all $t > 0$ and $x, y \in X$, where $V(x, \sqrt{t}) = \mu(B(x, \sqrt{t}))$.

The Littlewood–Paley function $G_L(f)$ and Lusin-area function $S_L(f)$ associated with the heat semigroup generated by $L$ are given by

$$G_L(f)(x) = \left(\int_0^\infty \left|t^2 L e^{-t^2 L} f(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$
$$S_L(f)(x) = \left(\int_0^\infty \int_{d(x,y)<t} \left|t^2 L e^{-t^2 L} f(x)\right|^2 \frac{d\mu(y)}{\mu(B(x,t))} \frac{dt}{t}\right)^{1/2}. \quad (2)$$

In this paper, we focus on the characterization of the Musielak–Orlicz–Hardy spaces $H_{\varphi,L}$ and $H_{L,G,\varphi}$, where the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function (cf. Definition 6 below).

*Definition 1.* Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function. A function $f \in H^2(X)$ is said to be in $\widetilde{H}_{\varphi,L}(X)$ if $S_L(f) \in L^\varphi(X)$ (cf. Definition 7 below). Moreover, we define

$$\|f\|_{\widetilde{H}_{\varphi,L}(X)} = \left\|S_L(f)\right\|_{L^\varphi} = \inf\left\{\lambda \in (0,\infty): \int_X \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) d\mu(x) \leq 1\right\}. \quad (3)$$

The Musielak–Orlicz–Hardy space $H_{\varphi,L}(X)$ is defined to be the complement space of $\widetilde{H}_{\varphi,L}(X)$.

*Definition 2.* Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function. A function $f \in H^2(X)$ is said to be in $\widetilde{H}_{L,G,\varphi}(X)$ if $G_L(f) \in L^{\varphi}(X)$. Moreover, we define

$$\|f\|_{\widetilde{H}_{L,G,\varphi}(X)} = \left\|G_L(f)\right\|_{L^{\varphi}} = \inf\left\{\lambda \in (0,\infty): \int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda}\right) d\mu(x) \le 1\right\}. \tag{4}$$

The Musielak–Orlicz–Hardy space $H_{L,G,\varphi}(X)$ is defined to be the complement space of $\widetilde{H}_{L,G,\varphi}(X)$.

Recently, the study of Hardy spaces associated with operators has been attracting great interest. It was initiated by Auscher et al. who studied the Hardy space $H^1_L(\mathbb{R}^n)$ with operators $L$ in [1], where the heat kernel of $L$ satisfies the pointwise Poisson upper bounded condition. Later on, Duong and Yan [2, 3] presented the adapted BMO theory on condition that the heat kernel of $L$ satisfies the pointwise Gaussian estimate. In [4], Yan established the theory of Hardy space $H^p_L(\mathbb{R}^n)$ for $0 < p < 1$ associated with the operator $L$ satisfying Davies–Gaffney estimates.

It is a natural question to ask the behavior of weighted Hardy space $H^p_{L,\omega}(\mathbb{R}^n)$ associated with an operator $L$ and an appropriate weight $\omega$. A pioneering investigation work of the weighted Hardy space $H^1_{L,\omega}(\mathbb{R}^n)$ associated with the Schrödinger operator $L$ was the paper by Song and Yan [5]. In 2016, Duong et al. [6] considered the weighted Hardy spaces $H^p_{L,S,\omega}(\mathbb{R}^n)$ and $H^p_{L,G,\omega}(\mathbb{R}^n)$ on homogeneous space $X$ for $0 < p \le 1$ and obtained the equivalence of these two kinds by adding Moser-type conditions, where the operator $L$ has the kernel satisfying Gaussian upper bound. Shortly after that, the equivalence of these two kinds spaces was characterized by Hu [7] without assuming the Moser-type boundedness condition.

In 2014, Ky [8] introduced the Musielak–Orlicz–Hardy space $H_{\varphi}(\mathbb{R}^n)$ by using growth function $\varphi$. Naturally, the Musielak–Orlicz–Hardy space $H_{\varphi,L}$ which is defined by means of the Lusin-area function associated with an operator $L$ was introduced and studied in [9], where $L$ satisfies Davies–Gaffney estimates. Unfortunately, the characterization of $H_{\varphi,L}$ required an extra assumption that $\varphi$ satisfies the uniformly reverse Hölder condition (cf. [9]).

Motivated by the above, we are concerned with the Musielak–Orlicz spaces $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ which we define by means of the Lusin-area function and the Littlewood–Paley function on homogeneous space $X$. Our aim in the present paper is to prove that the two kinds of Musielak–Orlicz spaces are equivalent. Our main result is stated as follows.

**Theorem 1.** *Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function of uniformly lower type $p_1$. Then, the spaces $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ coincide and their norms are equivalent.*

Theorem 1 obtains the behavior of Littlewood–Paley $g$-function $G_L$ on $H_{\varphi,L}$ and partly improves the result in [9]. To make it clear, we first establish the discrete characterization of the Musielak–Orlicz spaces $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ and state these results as follows.

**Theorem 2.** *Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function of uniformly lower type $p_1$. Let $f \in H_{\varphi,L}(X) \cap L^2(X)$. Then, for all $M \in \mathbb{N}$ with $M > (nq(\varphi)/2p_1)$, $f$ has an $\mathbf{AT}_{\mathscr{L},M}$-expansion such that*

$$\|f\|_{H_{\varphi,L}(X)} \cong \left\|W_f\right\|_{L^{\varphi}(X)}. \tag{5}$$

**Theorem 3.** *Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function of uniformly lower type $p_1$. Let $f \in H_{L,G,\varphi}(X) \cap L^2(X)$. Then, for all $M \in \mathbb{N}$ with $M > (nq(\varphi)/2p_1)$, $f$ has an $\mathbf{AT}_{\mathscr{L},M}$-expansion such that*

$$\|f\|_{H_{L,G,\varphi}(X)} \cong \left\|W_f\right\|_{L^{\varphi}(X)}. \tag{6}$$

Theorems 2 and 3 extend the results in [6, 7], respectively. Also, we extend the results in [9] by removing the assumption of uniformly reverse Hölder condition. As a consequence of Theorems 2 and 3, we immediately get Theorem 1.

The paper is organized as follows. Section 2 contains some basic definitions and lemmas concerning metric measure spaces, growth functions, Musielak–Orlicz space, and $\mathbf{AT}_{\mathscr{L},M}$-family. The aim of Section 3 is to prove Theorem 2 and establish the characterization of Musielak–Orlicz–Hardy space $H_{\varphi,L}$. We develop a method to unify the different control terms of inner integral. The aim of Section 4 is to prove Theorem 3 and set up the characterization of Musielak–Orlicz–Hardy space $H_{L,G,\varphi}$. We borrow the ideas from [6,10]. Consequently, we get that the characterization of Musielak–Orlicz–Hardy space by means of $H_{\varphi,L}$ and $H_{L,G,\varphi}$ is equivalent.

Most of the notations we use are standard. $C$ denotes a positive constant that may change from line to line and we use the subscript for the sake of eliminating confusion. We write $A \cong B$ if there exist constants $C_1$ and $C_2$ which are independent of $A$ and $B$ such that $C_1 B \le A \le C_2 B$. For a measurable set $A$, $|A|$ denotes the Lebesgue measure of $A$ and $\chi_A$ is the characteristic function.

## 2. Basic Concepts and Lemmas

*2.1. Metric Measure Spaces.* A metric measure space $(X, d, \mu)$ is a set $X$ equipped with a metric $d$ and a non-negative Borel doubling measure $\mu$ on $X$. Fix $x \in X$ and let $r \in (0,\infty)$, and we denote the open ball centered at $x$ with radius $r$ by

$$B(x,r) = \{y \in X: d(x,y) < r\}, \tag{7}$$

and set $V(x,r) = \mu(B(x,r))$.

*Definition 3.* A space of homogeneous type $(X, d, \mu)$ is a set $X$ with a metric $d$ and a non-negative measure $\mu$ on $X$, so that there exists a constant $C_D \in [1, \infty)$ such that for all $x \in X$ and $r > 0$,

$$V(x, 2r) \le C_D V(x, r) < \infty. \tag{8}$$

Definition 3 was introduced by Coifman and Weiss [11]. The property of $\mu$ in (8) is the doubling condition and it implies the strong $n$ homogeneity property, i.e., for some constant $C > 0$ and homogeneity $n$,

$$V(x, \lambda r) \le C\lambda^n V(x, r) \tag{9}$$

holds uniformly for all $\lambda \in [1, \infty)$, $x \in X$, and $r > 0$.

Let $C_D$ be as in (8) and set $m = \log_2 C_D$, and Grigor'yan et al. have shown that (see [12])

$$V(x, R) \le C_D \left[\frac{R + d(x, y)}{r}\right]^m V(y, r) \tag{10}$$

holds for all $x, y \in X$ and $0 < r \le R < \infty$. It is easy to verify, by doubling condition (8), that for any $N > n$, there exists a constant $C_N$ such that for all $x \in X$ and $t > 0$,

$$\int_X \left(1 + t^{-1} d(x, y)\right)^{-N} \mathrm{d}\mu(y) \le C_N V(x, t). \tag{11}$$

The dyadic cube decomposition on spaces of homogeneous type comes from Christ [13] as follows:

**Lemma 1.** *Let $(X, d, \mu)$ be a space of homogeneous type. Then, there exists a collection of open subsets $\{Q_\alpha^k \subset X\colon k \in \mathbb{Z}, \alpha \in I_k\}$ and constants $\delta \in (0, 1)$ and $0 < C_1, C_2 < \infty$ such that*

*(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$, $\forall k$; for each fixed $k$, if $\alpha \ne \beta$, then $Q_\alpha^k \cap Q_\beta^k = \varnothing$.*

*(ii) For any $\alpha, \beta, k, l$, if $k \le l$, then either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \varnothing$.*

*(iii) For each $(k, \alpha)$ and each $l < k$, there is a unique $\beta \in I_l$ such that $Q_\alpha^k \subset Q_\beta^l$.*

*(iv) Diameter $(Q_\alpha^k) \le C_1 \delta^k$.*

*(v) Each $Q_\alpha^k$ contains some ball $B(z_\alpha^k, C_2 \delta^k)$, where $z_\alpha^k \in X$.*

The sets $Q_\alpha^k$ are analogues of the Euclidean dyadic cubes; it may help to think of $Q_\alpha^k$ as being essentially a cube of ball of diameter roughly $\delta^k$ with center $z_\alpha^k$. We then set $\ell(Q_\alpha^k) = C_1 \delta^k$. It is worthy pointing out that the precise value of $C_1$ is non-essential (cf. Christ [13]). Here and in what follows, we assume $C_1 = \delta^{-1}$.

*2.2. Growth Functions.* We first recall the Orlicz function. A non-decreasing function $\Phi\colon [0, \infty) \longrightarrow [0, \infty)$ is called an Orlicz function if $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\lim_{t \longrightarrow \infty} \Phi(t) = \infty$ (cf. Yang [9]).

The function $\Phi$ is said to be of upper type $p$ (resp., lower type $p$) for some $p \in [0, \infty)$, if for all $t \in [1, \infty)$ (resp., $t \in [0, 1]$) and $s \in [0, \infty)$, there is a constant $C > 0$ such that $\Phi(st) \le Ct^p \Phi(s)$. $\Phi$ is said to be of type $(p_1, p_2)$ if it is of both upper type $p_1$ and lower type $p_2$.

Given a function $\varphi\colon X \times [0, \infty) \longrightarrow [0, \infty)$, for any $x \in X$, $\varphi(x, \cdot)$ is an Orlicz function. If there exists a constant $C > 0$ such that for all $x \in X$, $t \in [1, \infty)$ (resp., $t \in [0, 1]$)) and $s \in [0, \infty)$,

$$\varphi(x, st) \le Ct^p \varphi(x, s), \tag{12}$$

then $\varphi$ is said to be of uniformly upper type $p$ (resp., uniformly lower type $p$). Moreover, $\varphi$ is said to be of positive uniformly upper type (resp., uniformly lower type) if it is of uniformly upper type (resp., uniformly lower type) $p$ for some $p \in (0, \infty)$.

Let $\varphi\colon X \times [0, \infty) \longrightarrow [0, \infty)$. If for all $t \in [0, \infty)$, $x \mapsto \varphi(x, t)$ is measurable and for all bounded subsets $K$ of $X$,

$$\int_K \sup_{t \in (0, \infty)} \left\{\varphi(x, t)\left[\int_K \varphi(y, t) \mathrm{d}\mu(y)\right]^{-1}\right\} \mathrm{d}\mu(x) < \infty. \tag{13}$$

Then, $\varphi(\cdot, t)$ is said to be uniformly locally integrable (cf. [8]).

We next recall the uniformly Muckenhoupt condition in [9, 14].

*Definition 4.* Let $\varphi\colon X \times [0, \infty) \longrightarrow [0, \infty)$ be uniformly locally integrable. The function $\varphi(\cdot, t)$ is said to satisfy the uniformly Muckenhoupt condition for some $q \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_q(X)$, if

$$\mathbb{A}_q(\varphi) = \sup_{t \in (0, \infty)} \sup_{B \subset X} \left\{\frac{1}{\mu(B)} \int_B \varphi(x, t) \mathrm{d}\mu(x)\right\} \left\{\frac{1}{\mu(B)} \int_B [\varphi(y, t)]^{(-q'/q)} \mathrm{d}\mu(x)\right\}^{(q/q')} < \infty, \tag{14}$$

when $q \in (1, \infty)$ and $(1/q) + (1/q') = 1$, or

$$\mathbb{A}_1(\varphi) = \sup_{t\in(0,\infty)} \sup_{B\subset X} \frac{1}{\mu(B)} \int_B \varphi(x,t)\mathrm{d}\mu(x) \left( \operatorname*{ess\,sup}_{y\in B} [\varphi(y,t)]^{-1} \right) < \infty. \quad (15)$$

Here the first supremum is taken over all $t \in (0,\infty)$ and the second one is taken over all balls $B \subset X$.

We define $\mathbb{A}_\infty(X) = \cup_{q\in[1,\infty)} \mathbb{A}_q(X)$ and let

$$q(\varphi) = \inf\{q \in [1,\infty): \quad \varphi \in \mathbb{A}_q(X)\} \quad (16)$$

be the critical indices of $\varphi$. Moreover, we denote

$$\varphi(E,t) = \int_E \varphi(x,t)\mathrm{d}\mu(x), \quad (17)$$

for any measurable subset $E$ of $X$ and $t \in [0,\infty)$. Let $\mathcal{M}$ be the Hardy–Littlewood maximal function on $X$, namely, for all $x \in X$,

$$\mathcal{M}(f)(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\mathrm{d}\mu(y), \quad (18)$$

where the supremum is taken over all balls $B$ containing $x$. The following lemma on the properties of $\mathbb{A}_\infty(X)$ is Lemma 2.8 in [9].

**Lemma 2**

*(i)* $\mathbb{A}_1(X) \subset \mathbb{A}_p(X) \subset \mathbb{A}_q(X)$ *for* $1 \le p \le q < \infty$.

*(ii) If* $\varphi \in \mathbb{A}_p(X)$ *with* $p \in (1,\infty)$, *then there exist some* $q \in (1,p)$ *such that* $\varphi \in \mathbb{A}_q(X)$.

*(iii) If* $\varphi \in \mathbb{A}_p(X)$ *with* $p \in (1,\infty)$, *then there exists a constant* $C > 0$ *such that for all measurable functions* $f$ *on* $X$ *and* $t \in [0,\infty)$,

$$\int_X [\mathcal{M}(f)(x)]^p \varphi(x,t)\mathrm{d}\mu(x) \le C \int_X |f(x)|^p \varphi(x,t)\mathrm{d}\mu(x). \quad (19)$$

*(iv) If* $\varphi \in \mathbb{A}_p(X)$ *with* $p \in (1,\infty)$, *then there exists a constant* $C > 0$ *such that for all balls* $B \subset X$ *and measurable set* $E \subset B$ *and* $t \in [0,\infty)$,

$$\frac{\varphi(B,t)}{\varphi(E,t)} \le C \left[\frac{\mu(B)}{\mu(E)}\right]^p. \quad (20)$$

We now introduce the notion of growth functions (cf. [8, 9]).

*Definition 5.* Let $\varphi$: $X \times [0,\infty) \longrightarrow [0,\infty)$. Then, $\varphi(x,t)$ is a Musielak–Orlicz function, if

(i) $\varphi(x,\cdot)$: $[0,\infty) \longrightarrow [0,\infty)$ is an Orlicz function for all $x \in X$.

(ii) $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

*Definition 6.* Let $\varphi$: $X \times [0,\infty) \longrightarrow [0,\infty)$. Then, $\varphi$ is called a growth function, if the following hold.

(i) $\varphi$ is a Musielak–Orlicz function.

(ii) $\varphi \in \mathbb{A}_\infty(X)$.

(iii) $\varphi$ is of positive uniformly upper type $p_1$ for some $p_1 \in (0,1]$ and of uniformly lower type $p_2$ for some $p_2 \in (0,1]$.

**Lemma 3.** *Let* $\varphi$ *be a growth function and set* $\tilde{\varphi}(x,t) = \int_0^t (\varphi(x,s)/s)ds$ *for all* $(x,t) \in X \times [0,\infty)$. *Then,* $\tilde{\varphi}$ *is a growth function, which is equivalent to* $\varphi$, *and* $\tilde{\varphi}(x,\cdot)$ *is continuous and strictly increasing.*

*2.3. Musielak–Orlicz Spaces.* In this section, we study the Musielak–Orlicz spaces associated with the growth function $\varphi$.

*Definition 7.* The Musielak–Orlicz space $L^\varphi(X)$ denotes the set of all measurable function $f$ on $X$ with $\int_X \varphi(x,|f(x)|)\mathrm{d}\mu(x) < \infty$ and the Luxembourg norm

$$\|f\|_{L^\varphi(X)} = \inf\left\{\lambda \in (0,\infty): \int_X \varphi\left(x, \frac{|f(x)|}{\lambda}\right)\mathrm{d}\mu \le 1\right\}. \quad (21)$$

The space $L^\varphi(\ell^p, X)$ is defined to be the set of all $\{f_j\}_{j\in\mathbb{Z}}$ satisfying $[\sum_j |f_j|^p]^{1/p} \in L^\varphi(X)$ and let

$$\left\|\{f_j\}_j\right\|_{L^\varphi(\ell^p,X)} = \left\|\left[\sum_j |f_j|^p\right]^{1/p}\right\|_{L^\varphi(X)}. \quad (22)$$

We have the following Fefferman–Stein vector-valued inequality of Musielak–Orlicz type (cf. [15]).

**Lemma 4.** *Let* $p \in (1,\infty]$, $\varphi$ *be a Musielak–Orlicz function with uniformly lower type* $p_1$ *and upper type* $p_2$, $q \in (1,\infty)$, *and* $\varphi \in \mathbb{A}_q(X)$. *If* $q(\varphi) < p_1 \le p_2 < \infty$, *then there exists a constant* $C > 0$ *such that, for all* $\{f_j\}_{j\in\mathbb{Z}} \in L^\varphi(\ell^p, X)$,

$$\int_X \varphi\left(x, \left[\sum_j \mathcal{M}(f_j)(x)^p\right]^{1/p}\right)\mathrm{d}\mu(x) \le C \int_X \varphi\left(x, \left[\sum_j |f_j(x)|^p\right]^{1/p}\right)\mathrm{d}\mu(x). \quad (23)$$

**Corollary 1.** *Let $p, \varphi$ be as in Lemma 4. Then, for all $r \in (0, (p_1/q(\varphi)))$ and $\{f_j\}_{j\in\mathbb{Z}} \in L^\varphi(\ell^p, X)$, there exists a constant $C > 0$ such that*

$$\int_X \varphi\left(x, \left[\sum_j \mathcal{M}(f_j)(x)^p\right]^{(1/rp)}\right) d\mu(x) \le C \int_X \varphi\left(x, \left[\sum_j |f_j(x)|^p\right]^{(1/rp)}\right) d\mu(x). \tag{24}$$

*Proof.* Fix $r \in (0, (p_1/q(\varphi)))$ and let $\widetilde{\varphi}(x,t) = \varphi(x, t^{1/r})$. We claim that $\widetilde{\varphi}$ is of uniformly lower type $p_1/r$ and upper type $p_2/r$. In fact, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \widetilde{\varphi}(x, st) &= \varphi\left(x, s^{1/r}t^{1/r}\right) \le C_1 t^{p_1/r}\varphi\left(x, s^{1/r}\right) = C_1 t^{p_1/r}\widetilde{\varphi}(x,s),\\ \widetilde{\varphi}(x, st) &= \varphi\left(x, s^{1/r}t^{1/r}\right) \le C_2 t^{p_2/r}\varphi\left(x, s^{1/r}\right) = C_2 t^{p_2/r}\widetilde{\varphi}(x,s), \end{aligned} \tag{25}$$

for all $t \in [1,\infty)$, $x \in X$ and $s \in [0,\infty)$. Meanwhile, one has

$$q(\widetilde{\varphi}) < \frac{p_1}{r} \le \frac{p_2}{r} < \infty, \tag{26}$$

since $q(\varphi) = q(\widetilde{\varphi})$. Therefore, Lemma 4 yields

$$\begin{aligned} \int_X \varphi\left(x, \left[\sum_j \mathcal{M}(f_j)(x)^p\right]^{1/rp}\right) d\mu(x) &= \int_X \widetilde{\varphi}\left(x, \left[\sum_j \mathcal{M}(f_j)(x)^p\right]^{1/p}\right) d\mu(x)\\ &\le C\int_X \widetilde{\varphi}\left(x, \left[\sum_j |f_j(x)|^p\right]^{1/p}\right) d\mu(x) = C\int_X \varphi\left(x, \left[\sum_j |f_j(x)|^p\right]^{1/rp}\right) d\mu(x). \end{aligned} \tag{27}$$

It finishes the proof of Corollary 1. □

*2.4.* $\mathbf{AT}_{\mathscr{L},M}$*-Family and Decomposition Theorem.* In this section, we assume that the space $X$ satisfies the strong homogeneity property (9) with homogeneous dimension $n$. In view of Lemma 1, the space $X$ possesses a dyadic decomposition analogous to the Euclidean dyadic cubes, i.e., there exists a collection of open subsets $\{Q_\alpha^k \subset X\colon k \in \mathbb{Z}, \alpha \in I_k\}$ such that for every $k \in \mathbb{Z}$,

$$X = \bigcup_{\alpha\in I_k} Q_\alpha^k, \tag{28}$$

where $I_k$ is some index set and $Q_\alpha^k$ has the properties as in Lemma 1. Such open subsets $\{Q_\alpha^k \subset X\colon k \in \mathbb{Z}, \alpha \in I_k\}$ are said to be a family of dyadic cubes of $X$ (cf. [6]).

*Definition 8.* Suppose that the operator $L$ satisfies **(H1)** and **(H2)** and $M \in \mathbb{N}$. Then, a collection of functions $\{a_Q\}_{Q\text{: Dyadic}}$ in $L^2(X)$ is said to be an $\mathbf{AT}_{\mathscr{L},M}$-family associated with an operator $L$, if for every dyadic $Q$, there exists a function $\mathscr{D}(L^{2M})$ such that

(i) $a_Q = L^M(b_Q)(x)$.
(ii) $\operatorname{supp}(L^k(b_Q)) \subset 3Q$, $k = 0, 1, \ldots, 2M$.
(iii) $|(\ell(Q)^2 L)^k(b_Q)\ (x)| \le \ell(Q)^{2M} V(Q)^{-(1/2)}$, $k = 0, 1, \ldots, 2M$.

Here, $\mathscr{D}(T)$ denotes the domain of an unbounded operator $T$ and $T^k$, which is the $k$-fold composition of $T$ with itself, in the sense of unbounded operators.

For a function $f$ in $L^2(X)$, if there exists sequence $s = \{s_Q\}_{Q\text{: dyadic}}$, $0 \le s_Q < \infty$, and an $\mathbf{AT}_{\mathscr{L},M}$-family $\{a_Q\}_{Q\text{: dyadic}}$ in $L^2(X)$ such that

$$f = \sum_{Q\text{:dyadic}} s_Q a_Q, \tag{29}$$

we say that $f$ has an $\mathbf{AT}_{\mathscr{L},M}$-expansion. Then, we denote the function related to the sequence $s = \{s_Q\}_{Q\text{:dyadic}}$ by $W_f(x)$ and

$$W_f(x) = \left(\sum_{Q\text{:dyadic}} \left[V(Q)^{-(1/2)}|s_Q|\chi_Q(x)\right]^2\right)^{(1/2)}. \tag{30}$$

With the notation above, we have the following characterization of $L^2(X)$.

**Proposition 1.** *Suppose that the operator L satisfies* **(H1)** *and* **(H2)***. Let $f \in L^2(X)$. Then, for $M \in \mathbb{N}$, $f$ has an* $\mathbf{AT}_{\mathscr{L},M}$*-expansion*

$$f = \sum_{Q\text{:dyadic}} s_Q a_Q. \tag{31}$$

*Moreover, let $Q_\alpha^k$ and $\delta$ be as in Lemma 1. Then,*

$$s_{Q_\alpha^k}=\left(\int_{\delta^{k+1}}^{\delta^k}\int_{Q_\alpha^k}\left|t^2Le^{-t^2L}f(y)\right|^2 d\mu(y)\frac{dt}{t}\right)^2. \quad (32)$$

*Proof.* The proof of Proposition 1 can be found in [6, Theorem 3.2]. □

## 3. The Proof of Theorem 2

In this section, we establish a characterization of the Musielak–Orlicz–Hardy space $H_{\varphi,L}$, where the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function.

For every $v\in(0,\infty)$ and $x\in X$, let $\Gamma_v(x)=\{(y,t)\in X\times(0,\infty): d(x,y)<vt\}$ be the cone of aperture $v$ and vertex $x\in X$. For any closed subset $F$ of $X$, we denote the union of all cones with vertices in $F$ by

$$\mathscr{R}_v(F)=\bigcup_{x\in F}\Gamma_v(x). \quad (33)$$

When $v=1$, $\Gamma(x)$ and $\mathscr{R}(F)$ stand for $\Gamma_1(x)$ and $\mathscr{R}_1(x)$, respectively. Given an open subset $O$ of $X$, we establish Lemma 5 of $\mathscr{R}(O^{\complement})$ on the geometric properties. We also remark that Aguilera and Segovia [16] obtained the same result in the case of Euclidean space.

**Lemma 5.** *Suppose that $(X,d,\mu)$ is a space of homogeneous type and there exists a constant $C_D>1$ such that (8) holds. Let $O$ be an open subset of $X$ and $F=O^{\complement}$. For $v>1$, we denote $O^*$ by*

$$O^*=\left\{x\in X: \mathscr{M}(\chi_O)(x)>(4v)^{-2\log_2 C_D}\right\}, \quad (34)$$

*and write $F^*=(O^*)^{\complement}$. Then,*

*(i) $\mathscr{R}_v(F^*)\subset\mathscr{R}(F)$.*

*(ii) There exists a constant $C_v$ such that*

$$V(z,t)<C_v\mu(B(z,t)\cap F) \quad (35)$$

*hods for $(z,t)\in\mathscr{R}_v(F^*)$.*

*Proof.* It suffices to show that the lemma holds when $\mathscr{R}_v(F^*)\neq\varnothing$ since it is trivial if $\mathscr{R}_v(F^*)=\varnothing$. We first prove (i) on the condition that $\mathscr{R}_v(F^*)\neq\varnothing$, which implies $O\neq X$.

Let $(z,t)\in\mathscr{R}_v(F^*)$. We thus have $z\in F$ or $z\in O$. It is easy to see that $(z,t)\in\mathscr{R}(F)$ since $d(z,z)=0<t$ in the case $z\in F$ and then $\mathscr{R}_v(F^*)\subset\mathscr{R}(F)$ holds. The proof of (i) is reduced to the verification in the case $z\in O$.

Suppose $z\in O$ and let $\delta=\operatorname{dist}(z,F)$. Then, $0<\delta<\infty$ and $B(z,\delta)\subset O$ since $F$ is closed and non-empty. For every $(z,t)\in\mathscr{R}_v(F^*)$, we have $y\in F^*$ such that $d(z,y)<vt$. Thus, writing $r=\delta+d(z,y)$, we get $B(z,\delta)\subset B(y,r)$ and

$$B(z,\delta)\subset B(z,\delta)\cap O\subset B(y,r)\cap O. \quad (36)$$

Hence,

$$V(z,\delta)\le\mu(B(y,r)\cap O)\le(4v)^{-2\log_2 C_D}V(y,r). \quad (37)$$

By using (10) twice, we have

$$\begin{aligned}V(y,r)&\le C_D\left(r\delta^{-1}\right)^{\log_2 C_D}V(y,\delta)\\&\le C_D^2\left(r\delta^{-1}\right)^{\log_2 C_D}\left(1+\delta^{-1}d(y,z)\right)^{\log_2 C_D}V(z,\delta)\\&=\left(2r\delta^{-1}\right)^{2\log_2 C_D}V(z,\delta),\end{aligned} \quad (38)$$

and then

$$\delta\le\frac{r}{2v}=\frac{\delta+d(z,y)}{2v}<\frac{\delta+vt}{2v}. \quad (39)$$

It follows that $\delta<t$ since $v>1$. Recalling the definition of $\delta$, we get $x\in F$ such that $d(x,z)<t$, which implies $(z,t)\in\mathscr{R}(F)$. It completes the proof of (i).

Next, we prove (ii). Given $(z,t)\in\mathscr{R}_v(F^*)$, we get $y\in F^*$ such that $d(z,y)<vt$. Thus, $B(z,t)\subset B(y,(1+v)t)$ and

$$\begin{aligned}\mu(B(z,t)\cap O)&\le\mu(B(y,(1+v)t)\cap O)\\&\le(4v)^{-2\log_2 C_D}V(y,(1+v)t).\end{aligned} \quad (40)$$

Therefore,

$$\begin{aligned}\mu(B(z,t)\cap O)&\le(4v)^{-2\log_2 C_D}C_D V(y,t)\\&\le C_D^2(4v)^{-2\log_2 C_D}(1+v)^{\log_2 C_D}\\&\quad\left(1+t^{-1}d(y,z)\right)^{\log_2 C_D}V(z,t)\\&<\left(\frac{1+v}{2v}\right)^{2\log_2 C_D}V(z,t).\end{aligned} \quad (41)$$

We obtain

$$\left[1-\left(\frac{1+v}{2v}\right)^{2\log_2 C_D}\right]V(z,t)<\mu(B(z,t)\cap F), \quad (42)$$

since $V(z,t)=\mu(B(z,t)\cap O)+\mu(B(z,t)\cap F)$, and complete the proof of (ii). It finishes the proof of Lemma 5. □

For all $v\in(0,\infty)$, $f\in L^2(X)$ and $x\in X$, the variant Lusin-area function associated with $L$ is given by

$$S_{L,v}(f)(x)=\left(\int_0^\infty\int_{d(x,y)<vt}\left|t^2Le^{-t^2L}(f)(y)\right|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{1/2}. \quad (43)$$

Lemmas 6 and 7 extend the results in [14, 16] for the operator $S_{L,v}$.

**Lemma 6.** *Suppose that the operator $L$ satisfies **(H1)** and **(H2)**. Let $\varphi\in\mathbb{A}_p(X)$ for $1\le p<\infty$ and $O,O^*,F,F^*$ be as in Lemma 5. Then, there exists a finite constant $C$, which is independent of $O$, such that for all $\lambda\in(0,\infty)$ and $f\in L^2(X)$,*

$$\int_{F^*}\left|S_{L,v}(f)(x)\right|^2\varphi(x,\lambda)d\mu(x)\le C\int_F\left|S_L(f)(x)\right|^2\varphi(x,\lambda)d\mu(x), \quad (44)$$

*where $S_L$ is the short hand of $S_{L,1}$.*

*Proof.* Given $x \in F^*$ and $(y,t) \in \Gamma_\nu(x)$, we observe that $d(x,y) < \nu t$ and hence by (10),

$$V(x,t)^{-1} \le C_D\left(1+t^{-1}d(x,y)\right)^{\log_2 C_D} V(y,t)^{-1} < C_D(1+\nu)^{\log_2 C_D} V(y,t)^{-1}. \tag{45}$$

It follows that

$$\begin{aligned}&\int_{F^*} \left|S_{L,\nu}(f)(x)\right|^2 \varphi(x,\lambda)\mathrm{d}\mu(x)\\ &\le C_D(1+\nu)^{\log_2 C_D}\int_{F^*}\left(\int_{\Gamma_\nu(x)}\left|t^2Le^{-t^2L}(f)(y)\right|^2\frac{\mathrm{d}\mu(y)\mathrm{d}t}{V(y,t)t}\right)\varphi(x,\lambda)\mathrm{d}\mu(x)\\ &= C_{\nu,D}\int_{\mathscr{R}_\nu(F^*)}\left|t^2Le^{-t^2L}(f)(y)\right|^2 V(y,t)^{-1}\varphi(B(y,\nu t)\cap F^*,\lambda)\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t}.\end{aligned} \tag{46}$$

Then, applying Lemma 2 to the sets $B(y,t)$ and $B(y,\nu t)$, $B(y,t)\cap F$ and $B(y,t)$, respectively, we get

$$\varphi(B(y,\nu t),\lambda) \le C(2\nu)^{\log_2 C_D}\varphi(B(y,t),\lambda), \tag{47}$$

$$\varphi(B(y,t),\lambda) \le C\left(\frac{V(y,t)}{\mu(B(y,t)\cap F)}\right)^p \varphi(B(y,t)\cap F,\lambda). \tag{48}$$

Therefore, (47), (48), and Lemma 5 yield

$$\varphi(B(y,\nu t),\lambda) \le C\varphi(B(y,t)\cap F,\lambda). \tag{49}$$

Thus, $\int_{F^*}|S_{L,\nu}(f)(x)|^2\varphi(x,\lambda)\mathrm{d}\mu(x)$ is bounded by

$$C\int_{\mathscr{R}_\nu(F^*)}\left|t^2Le^{-t^2L}(f)(y)\right|^2 V(y,t)^{-1}\varphi(B(y,t)\cap F,\lambda)\frac{d\mu(y)\mathrm{d}t}{t}. \tag{50}$$

Finally, in view of $\mathscr{R}_\nu(F^*) \subset \mathscr{R}(F)$ (see Lemma 5), it follows immediately that (50) is bounded by

$$\begin{aligned}&C\int_{\mathscr{R}(F)}\left|t^2Le^{-t^2L}(f)(y)\right|^2 V(y,t)^{-1}\varphi(B(y,t)\cap F,\lambda)\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t}\\ &= C\int_F\left(\int_{\Gamma_\nu(x)}\left|t^2Le^{-t^2L}(f)(y)\right|^2\frac{\mathrm{d}\mu(y)\mathrm{d}t}{V(y,t)t}\right)\varphi(x,\lambda)\mathrm{d}\mu(x)\\ &\le C\int_F\left|S_L(f)(x)\right|^2\varphi(x,\lambda)\mathrm{d}\mu(x),\end{aligned} \tag{51}$$

where we use the fact that

$$V(y,t)^{-1} \le C_D\left(1+t^{-1}d(x,y)\right)^{\log_2 C_D} V(x,t)^{-1} < C_D^2 V(x,t)^{-1}, \tag{52}$$

for $(y,t) \in \Gamma_\nu(x)$ in the last line. It finishes the proof of Lemma 6. □

**Lemma 7.** *Suppose that the operator L satisfies **(H1)** and **(H2)**. Let $\varphi$ be a growth function and $\varphi \in \mathbb{A}_q(X)$ for $1 < q < \infty$. Then, there exists a constant $C_\nu > 0$ such that*

$$\int_X \varphi\left(x, S_{L,\nu}(f)(x)\right)\mathrm{d}\mu(x) \le C_\nu \int_X \varphi(x, S_L(f)(x))\mathrm{d}\mu(x) \tag{53}$$

*holds for all $\nu \in (0,\infty)$ and all measurable functions $f$.*

*Proof.* It suffices to show that Lemma 7 holds in the case $\nu \in (1,\infty)$ since the conclusion is trivial if $\nu \in (0,1]$. Given $\lambda \in (0,\infty)$, we introduce the notations

$$\begin{aligned}O_\lambda &= \{x \in X : S_L(f)(x) > \lambda\},\\ O_\lambda^* &= \left\{x \in X : \mathscr{M}\left(\chi_{O_\lambda}\right)(x) > (4\nu)^{-\log_2 C_D}\right\},\end{aligned} \tag{54}$$

where $\mathscr{M}$ is the Hardy–Littlewood maximal function. Noting that $\varphi \in \mathbb{A}_q(X)$, Lemma 2 yields

$$\begin{aligned}\varphi(O_\lambda^*,\lambda) &= \varphi\left(\left\{x \in X : \mathscr{M}\left(\chi_{O_\lambda}\right)(x) > (4\nu)^{-\log_2 C_D}\right\},\lambda\right)\\ &\le \int_X (4\nu)^{q\log_2 C_D}\left(\mathscr{M}\left(\chi_{O_\lambda}\right)(x)\right)^q\varphi(x,\lambda)\mathrm{d}\mu(x)\\ &\le C\varphi(O_\lambda,\lambda).\end{aligned} \tag{55}$$

Writing $F_\lambda = O_\lambda^{\complement}$, $F_\lambda^* = (O_\lambda^*)^{\complement}$ and applying Lemma 6, we obtain

$$\int_{F_\lambda^*}\left|S_{L,\nu}(f)(x)\right|^2\varphi(x,\lambda)\mathrm{d}\mu(x) \le \int_{F_\lambda}\left|S_L(f)(x)\right|^2\varphi(x,\lambda)\mathrm{d}\mu(x). \tag{56}$$

Thus, by using (55) and (56), we have

$$\begin{aligned}
&\varphi\left(\left\{x \in X\colon S_{L,\nu}(f)(x) > \lambda\right\}, \lambda\right) \\
&\le \varphi\left(O_\lambda^*, \lambda\right) + \varphi\left(\left\{x \in F_\lambda^*\colon S_{L,\nu}(f)(x) > \lambda\right\}, \lambda\right) \\
&\le C\varphi\left(O_\lambda, \lambda\right) + \frac{1}{\lambda^2}\int_{F_\lambda^*} \left|S_{L,\nu}(f)(x)\right|^2 \varphi(x, \lambda)\,\mathrm{d}\mu(x) \\
&\le C\left[\varphi\left(O_\lambda, \lambda\right) + \frac{1}{\lambda^2}\int_{F_\lambda} \left|S_L(f)(x)\right|^2 \varphi(x, \lambda)\,\mathrm{d}\mu(x)\right] \\
&\le C\left[\varphi\left(O_\lambda, \lambda\right) + \frac{1}{\lambda^2}\int_0^t t\varphi\left(\{x \in X\colon S_L(f)(x) > t\}, \lambda\right)\mathrm{d}t\right].
\end{aligned} \tag{57}$$

Therefore, we employ (57) together with the assumption $\nu \in (1, \infty)$, Lemma 3, and the uniformly upper type 1 of $\varphi$ to get

$$\begin{aligned}
\int_X \varphi\left(x, S_{L,\nu}(f)(x)\right)\mathrm{d}\mu(x) &\le C\int_0^\infty \frac{1}{\lambda}\varphi\left(\left\{x \in X\colon S_{L,\nu}(f)(x) > \lambda\right\}, \lambda\right)\mathrm{d}\lambda \\
&\le C\int_0^\infty \frac{1}{\lambda}\varphi\left(O_\lambda, \lambda\right)\mathrm{d}\lambda + C\int_0^\infty \frac{1}{\lambda^3}\int_0^t t\varphi\left(\{x \in X\colon S_L(f)(x) > t\}, \lambda\right)\mathrm{d}t\,\mathrm{d}\lambda \\
&\le C\int_0^\infty \frac{1}{\lambda}\varphi\left(\{x \in X\colon S_L(f)(x) > \lambda\}, \lambda\right)\mathrm{d}\lambda \\
&\quad + C\int_0^\infty \frac{1}{\lambda^3}\int_0^t t\varphi\left(\{x \in X\colon S_L(f)(x) > t\}, \lambda\right)\mathrm{d}t\,\mathrm{d}\lambda \\
&\le C\int_X \varphi\left(x, S_L(f)(x)\right)\mathrm{d}\mu(x) \\
&\quad + C\int_0^\infty \varphi\left(\{x \in X\colon S_L(f)(x) > t\}, t\right)\int_t^\infty \frac{\mathrm{d}\lambda}{\lambda^2}\mathrm{d}t \\
&\le C\int_X \varphi\left(x, S_L(f)(x)\right)\mathrm{d}\mu(x).
\end{aligned} \tag{58}$$

It finishes the proof of Lemma 7. □

Lemma 8 says that the sequence $\left\{s_{Q_\alpha^k}\right\}_{\alpha \in I_k}$ can be majorized by the Hardy–Littlewood maximal operator $\mathscr{M}$ on $(X, d, \mu)$ (cf. [17], pp.147, where we take $r = 1$).

**Lemma 8.** *Suppose $0 < q \le 1$ and $N > (n/q)$. Fix $k \in \mathbb{Z}$ and let $\left\{s_{Q_\alpha^k}\right\}_{\alpha \in I_k}$ be as in Proposition 1. Then, for any subsequence $I_k' \subset I_k$ and for every $x \in X$,*

$$\sum_{\alpha \in I_k'} \frac{\left|s_{Q_\alpha^k}\right|}{\left[1 + \ell\left(Q_\alpha^k\right)^{-1} d\left(x, y_\alpha^k\right)\right]^N} \le C\left[\mathscr{M}\left(\sum_{\alpha \in I_k'} \left|s_{Q_\alpha^k}\right|^q \chi_{Q_\alpha^k}\right)\right]^{1/q}, \tag{59}$$

*where $y_\alpha^k$ is the center of $Q_\alpha^k$ and $C$ depends only on $n$ and $N - (n/q)$.*

*Proof.* of Theorem 2. Fix $f \in H_{\varphi,L}(X) \cap L^2(X)$, and we let $\lambda_0 = \|f\|_{H_{\varphi,L}(X)}$ and $\lambda_1 = \|W_f\|_{L^\varphi(X)}$. It suffices to show that for all $\lambda \in (0, \infty)$, we have

$$\int_X \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right)\mathrm{d}\mu(x) \cong \int_X \varphi\left(x, \frac{\left|W_f(x)\right|}{\lambda}\right)\mathrm{d}\mu(x). \tag{60}$$

In fact, since (60) holds for all $\lambda \in (0, \infty)$, there exists a constant $C_0$ such that

$$\int_X \varphi\left(x, \frac{S_L(f)(x)}{\lambda_1}\right)\mathrm{d}\mu(x) \le C_0\int_X \varphi\left(x, \frac{\left|W_f(x)\right|}{\lambda_1}\right)\mathrm{d}\mu(x) \le C_0. \tag{61}$$

Using (12) and (61), we have

$$\int_X \varphi\left(x, \frac{G_L(f)(x)}{C_1\lambda_1}\right) d\mu(x) \le 1, \tag{62}$$

for some constant $C_1$, which implies $\lambda_0 \le C_1\lambda_1$. Analogously, there exists a constant $C_2$ such that $\lambda_1 \le C_2\lambda_0$ and we get the desired result.

We now turn to prove (60). Given $(x,k) \in X \times \mathbb{Z}$, by Lemma 1, there exists a unique $\alpha \in I_k$ such that $x \in Q_\alpha^k$. We denote such $Q_\alpha^k$ by $Q_x^k$ and write

$$\begin{aligned} W_f(x) &= \left\{\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\left[\mu\left(Q_\alpha^k\right)^{-(1/2)}\left|s_{Q_\alpha^k}\right|\chi_{Q_\alpha^k}(x)\right]^2\right\}^{(1/2)} \\ &= \left\{\sum_{k\in\mathbb{Z}}\mu\left(Q_x^k\right)^{-1}\left|s_{Q_x^k}\right|^2\right\}^{(1/2)} \\ &= \left\{\sum_{k\in\mathbb{Z}}\int_{\delta^{k+1}}^{\delta^k}\mu\left(Q_x^k\right)^{-1}\int_{Q_x^k}\left|t^2Le^{-t^2L}f(y)\right|^2 d\mu(y)\frac{dt}{t}\right\}^{(1/2)}, \end{aligned} \tag{63}$$

where constants $\delta \in (0,1)$ satisfy Lemma 1 and the last line is obtained by using Proposition 1.

Moreover, for any fixed $(x,k) \in X \times \mathbb{Z}$, Lemma 1 also tells us that there are $z_x^k \in Q_x^k$ and constants $C_1 \in (0,1), C_2 = \delta^{-1}$ such that

$$B\left(z_x^k, C_1\delta^k\right) \subset Q_x^k \subset B\left(x, C_2\delta^k\right) \subset B\left(x, C_2\delta^{-1}t\right), \tag{64}$$

for all $t \in (\delta^{k+1}, \delta^k)$. Consequently,

$$\begin{aligned} \mu\left(Q_\alpha^k\right)^{-1} &\le V\left(z_x^k, C_1\delta^k\right)^{-1} \le C\left(1 + \frac{d(x,z_x^k)}{C_1\delta^k}\right)^m V\left(x, C_1\delta^k\right)^{-1} \\ &\le CV\left(x, C_1\delta^k\right)^{-1} \le CV\left(x,\delta^k\right)^{-1} \le CV(x,t)^{-1}, \end{aligned} \tag{65}$$

where we use (10) and the fact that $V(x,\delta^k) \le C_D C_1^{-m} V(x, C_1\delta^k)$ with $C_1 \in (0,1)$. Hence, (63) and (65) yield

$$\begin{aligned} W_f(x) &\le C\left\{\sum_{k\in\mathbb{Z}}\int_{\delta^{k+1}}^{\delta^k} V(x,t)^{-1}\int_{B(x,C_2\delta^{-1}t)}\left|t^2Le^{-t^2L}f(y)\right|^2 d\mu(y)\frac{dt}{t}\right\}^{1/2} \\ &= C\left\{\int_0^\infty\int_{d(x,y)<\delta^{-2}t}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right\}^{1/2} \\ &= CS_{L,\delta^{-2}}(f)(x). \end{aligned} \tag{66}$$

Thus, by using Lemma 7, we deduce that

$$\int_X \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) d\mu(x) \ge C\int_X \varphi\left(x, \frac{\left|W_f(x)\right|}{\lambda}\right) d\mu(x). \tag{67}$$

It remains to establish the reverse inequality of (67). Let $\delta$ be as in Lemma 1. In view of Proposition 1, we write

$$f = \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k} s_{Q_\alpha^k} a_{Q_\alpha^k}, \tag{68}$$

and get

$$\begin{aligned}
&S_L(f)(x)\\
&=\left(\int_0^\infty\int_{d(x,y)<t}\left|t^2Le^{-t^2L}f(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&=\left(\int_0^\infty\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&=\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\le\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k>j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\right)^{1/2}\frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\\
&+\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k\le j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\right)^{1/2}\\
&=I_1+I_2.
\end{aligned}\tag{69}$$

We firstly estimate the inner integral of $I_1$. For any $k>j$ and $\alpha\in I_k$, noting $a_{Q_\alpha^k}=L^Mb_{Q_\alpha^k}$, we have

$$\begin{aligned}
\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|&=\left|t^2L^{M+1}e^{-t^2L}\left(b_{Q_\alpha^k}\right)(y)\right|\\
&=t^{-2M}\left|\left(t^2L\right)^{M+1}e^{-t^2L}\left(b_{Q_\alpha^k}\right)(y)\right|.
\end{aligned}\tag{70}$$

Since $M>(nq(\varphi)/2p_1)$ with $n$ given as in (9), we can choose some $q$ satisfying Corollary 1 such that $2M>(n/q)$. Thus, there is some $N>0$ such that $2M>N>(n/q)$. Then, applying Definition 8, the upper bound of the kernel $(t^2L)^{M+1}e^{-t^2L}$ (cf. [18], Proposition 3.1), and (11), we get

$$\begin{aligned}
&\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|\\
&\le\frac{C_5}{V(y,t)}t^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\int_{3Q_\alpha^k}e^{-\left(d(y,z)^2/C_6t^2\right)}\mathrm{d}\mu(z)\\
&\le Ct^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(\frac{t}{t+d\left(y,z_\alpha^k\right)}\right)^N,
\end{aligned}\tag{71}$$

where $z_\alpha^k$ is the center of $Q_\alpha^k$. Since $d(x,y)<t$, we further have

$$\begin{aligned}
&\left(\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\right)^{1/2}\\
&\le Ct^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(\int_{d(x,y)<t}\left(\frac{t+d(x,y)}{t+d\left(x,z_\alpha^k\right)}\right)^{2N}\frac{\mathrm{d}\mu(y)}{V(x,t)}\right)^{1/2}\\
&\le Ct^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(1+t^{-1}d\left(x,z_\alpha^k\right)\right)^{-N},
\end{aligned}\tag{72}$$

Hence, Lemma 8 yields the inner integral of $I_1$ which is bounded by

$$\begin{aligned}
&\left(\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k>j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\right)^{1/2}\\
&\le C\sum_{k>j}\sum_{\alpha\in I_k}t^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+t^{-1}d\left(x,z_\alpha^k\right)\right]^N}\\
&\le C\sum_{k>j}\delta^{(2M-N)(k-j)}\sum_{\alpha\in I_k}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,z_\alpha^k\right)\right]^N}\\
&\le C\sum_{k>j}\delta^{(2M-N)(k-j)}\left[\mathcal{M}\left(\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^q\mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}\right)(x)\right]^{1/q}.
\end{aligned}\tag{73}$$

Secondly, we estimate the inner integral of $I_2$. For any $k\le j$ and $\alpha\in I_k$, we write

$$\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|=t^2\left|e^{-t^2L}\left(L\left(a_{Q_\alpha^k}\right)\right)(y)\right|.\tag{74}$$

Then, using Definition 8, Gaussian estimate (1), and inequality (11), we obtain

$$\begin{aligned}
&\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|\\
&\le\frac{C_5}{V(y,t)}t^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\int_{3Q_\alpha^k}e^{-\left(d(y,z)^2/C_6t^2\right)}\mathrm{d}\mu(z)\\
&\le Ct^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(1+\ell\left(Q_\alpha^k\right)^{-1}\mathrm{d}\left(y,z_\alpha^k\right)\right)^{-N}.
\end{aligned}\tag{75}$$

Since $d(x,y)<t\le\ell(Q_\alpha^k)$, we further have

$$\begin{aligned}
&\left(\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(y)\right|^2\frac{\mathrm{d}\mu(y)}{V(x,t)}\right)^{1/2}\\
&\le Ct^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(\int_{d(x,y)<t}\left(\frac{\ell(Q_\alpha^k)+d(x,y)}{\ell(Q_\alpha^k)+d(x,z_\alpha^k)}\right)^{2N}\frac{\mathrm{d}\mu(y)}{V(x,t)}\right)^{1/2}\\
&\le Ct^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(1+\ell\left(Q_\alpha^k\right)d\left(x,z_\alpha^k\right)\right)^{-N},
\end{aligned}\tag{76}$$

Hence, Lemma 8 yields the inner integral of $I_2$ which is bounded by

$$\left(\int_{d(x,y)<t}\left|t^2Le^{-t^2L}\left(\sum_{k\le j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(y)\right|^2\frac{d\mu(y)}{V(x,t)}\right)^{1/2}$$
$$\le C\sum_{k\le j}\sum_{\alpha\in I_k}t^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,z_\alpha^k\right)\right]^N}$$
$$\le C\sum_{k\le j}\delta^{2(j-k)}\sum_{\alpha\in I_k}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,z_\alpha^k\right)\right]^N}$$
$$\le C\sum_{k\le j}\delta^{2(j-k)}\left[\mathcal{M}\left(\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^q\mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}\right)(x)\right]^{1/q}. \tag{77}$$

Fix $j\in\mathbb{Z}$, and we let $\beta>0$ and

$$\tau=\begin{cases}1, & k>j;\\ -1, & k\le j.\end{cases} \tag{78}$$

Writing $F_k(x)=\mathcal{M}\left(\sum_{\alpha\in I_k}|s_{Q_\alpha^k}|^q\mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}\right)(x)$, we now turn to estimate

$$J=\left(\sum_k\delta^{\beta\tau(k-j)}F_k(x)^{(1/q)}\right)^2. \tag{79}$$

Since

$$\delta^{\beta\tau(k-j)}=\frac{\beta\delta^\beta}{1-\delta^\beta}\int_{\delta^{\tau(k-j)}}^{\delta^{\tau(k-j)-1}}s^{\beta-1}ds, \tag{80}$$

we have

$$\begin{aligned}J&=C_\beta\left(\sum_k\int_{\delta^{\tau(k-j)}}^{\delta^{\tau(k-j)-1}}F_k(x)^{(1/q)}s^{\beta-1}ds\right)^2\\
&=C_\beta\left(\int_0^1\sum_k\chi_{E_k}(s)F_k(x)^{1/q}s^{\beta-1}ds\right)^2\\
&\le C_\beta\left(\int_0^1 s^{\beta-1}ds\right)\left(\int_0^1\left(\sum_k\chi_{E_k}(s)F_k(x)^{1/q}\right)^2s^{\beta-1}ds\right)\\
&\le C_\beta\int_0^1\left(\sum_k\chi_{E_k}(s)F_k(x)^{1/q}\right)^2s^{\beta-1}ds\\
&=C_\beta\sum_k\int_{\delta^{\tau(k-j)}}^{\delta^{\tau(k-j)-1}}F_k(x)^{2/q}s^{\beta-1}ds\\
&=C_\beta\sum_k\delta^{\beta\tau(k-j)}F_k(x)^{2/q},\end{aligned} \tag{81}$$

where $E_k=[\delta^{\tau(k-j)},\delta^{\tau(k-j)-1}]$. In view of inequalities (73)–(81), taking $\beta=2M-N,\tau=1$, and $\beta=2,\tau=-1$ respectively, we get

$$\begin{aligned}S_L(f)(x)&\le C\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|\sum_{k>j}\delta^{(2M-N)(k-j)}F_k(x)^{1/q}\right|^2\frac{dt}{t}\right.\\
&\quad\left.+\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|\sum_{k\le j}\delta^{2(j-k)}F_k(x)^{1/q}\right|^2\frac{dt}{t}\right)^{1/2}\\
&\le C\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\sum_{k>j}\delta^{(2M-N)(k-j)}F_k(x)^{2/q}\frac{dt}{t}\right.\\
&\quad\left.+\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\sum_{k\le j}\delta^{2(j-k)}F_k(x)^{2/q}\frac{dt}{t}\right)^{1/2}\\
&=C\left(\sum_{k\in\mathbb{Z}}F_k(x)^{2/q}\left(\sum_{k>j}\delta^{(2M-N)(k-j)}+\sum_{k\le j}\delta^{2(j-k)}\right)\right)^{1/2}\\
&\le C\left(\sum_{k\in\mathbb{Z}}F_k(x)^{2/q}\right)^{1/2}.\end{aligned} \tag{82}$$

Therefore, (82) and Corollary 1 yield

$$
\begin{aligned}
\int_X \varphi\left(x, \frac{S_L(f)(x)}{\lambda}\right) d\mu(x) &\leq C \int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}} F_k(x)^{2/q}\right)^{1/2}\right) d\mu(x) \\
&\leq C \int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}}\left(\sum_{\alpha\in I_k} \left|s_{Q_\alpha^k}\right|^q \mu\left(Q_\alpha^k\right)^{-(q/2)} \chi_{Q_\alpha^k}(x)\right)^{2/q}\right)^{1/2}\right) d\mu(x) \\
&= C \int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k} \left|s_{Q_\alpha^k}\right|^2 \mu\left(Q_\alpha^k\right)^{-1} \chi_{Q_\alpha^k}(x)\right)^{1/2}\right) d\mu(x) \\
&= C \int_X \varphi\left(x, \frac{W_f(x)}{\lambda}\right) d\mu(x),
\end{aligned}
\tag{83}
$$

which gives the reverse inequality of (67). It finishes the proof of Theorem 2. □

## 4. The Proof of Theorem 3

In this section, we establish a characterization of the Musielak–Orlicz–Hardy space $H_{L,G,\varphi}$, where the operator $L$ satisfies **(H1)** and **(H2)** and $\varphi$ is a growth function. Our proof will borrow some ideals from Duong et al. [6].

We first recall some basic definitions and facts about Fefferman–Stein type maximal function, referring to [7] for a complete account.

Given $f \in L^2(X)$, $a > 0$, and $(x,t) \in X \times (0,\infty)$, the Fefferman–Stein type maximal function is defined as

$$
\mathscr{M}_{a,L}^*(f)(x,t) = \operatorname*{ess\,sup}_{y\in X} \frac{\left|t^2 L e^{-t^2 L} f(y)\right|}{\left[1 + t^{-1} d(x,y)\right]^a}. \tag{84}
$$

Lemma 9 is useful (cf. [7]).

**Lemma 9.** *Suppose the operator $L$ satisfies **(H1)** and **(H2)**. Let $m$ be as in (10). Then, for any $\beta, r > 0$ and $a > (m/2)$, there exists a constant $C > 0$ such that*

$$
\left|\mathscr{M}_{a,L}^*(f)\left(x, 2^{-l}t\right)\right|^r \leq C \sum_{j=l}^{\infty} 2^{-(j-l)\beta r} \int_X \frac{\left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x)\right|^r}{V\left(z, 2^{-l}\right)\left[1 + 2^l d(x,z)\right]^{ar}} d\mu(z) \tag{85}
$$

*holds for all $f \in L^2(X)$, $l \in \mathbb{Z}$, $x \in X$, and $t \in [1,2)$.*

We also need Lemma 10, and its proof is standard, which we omit here.

**Lemma 10.** *Let $n$ and $m$ be as in (9) and (10), and $N > n + m$. Then, there exists a constant $C > 0$ such that*

$$
\int_X \frac{|f(x)|}{V(x,t)\left[1 + t^{-1} d(x,y)\right]^N} d\mu(x) \leq C \mathscr{M}(f)(y) \tag{86}
$$

*holds for all measurable functions $f$ on $(X, d, \mu)$, $t > 0$, and each $y \in X$.*

*Proof.* of Theorem 3. Fix $f \in H_{L,G,\varphi}(X) \cap L^2(X)$, and we let $\lambda_0 = \|f\|_{H_{L,G,\varphi}(X)}$ and $\lambda_1 = \|W_f\|_{L^\varphi(X)}$. It suffices to show that for all $\lambda \in (0,\infty)$, we have

$$
\int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda}\right) d\mu(x) \cong \int_X \varphi\left(x, \frac{\left|W_f(x)\right|}{\lambda}\right) d\mu(x). \tag{87}
$$

In fact, since (87) holds for all $\lambda \in (0,\infty)$, there exists a constant $C_0$ such that

$$
\int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda_1}\right) d\mu(x) \leq C_0 \int_X \varphi\left(x, \frac{\left|W_f(x)\right|}{\lambda_1}\right) d\mu(x) \leq C_0. \tag{88}
$$

Using (12) and (88), we have

$$
\int_X \varphi\left(x, \frac{G_L(f)(x)}{C_1 \lambda_1}\right) d\mu(x) \leq 1, \tag{89}
$$

for some constant $C_1$, which implies $\lambda_0 \leq C_1 \lambda_1$. Analogously, there exists a constant $C_2$ such that $\lambda_1 \leq C_2 \lambda_0$ and we get the desired result.

We now fix arbitrary $\lambda \in (0,\infty)$ and turn to prove (88). Given $(x,k) \in X \times \mathbb{Z}$, by Lemma 1, there exists a unique $\alpha \in I_k$ such that $x \in Q_\alpha^k$. We denote such $Q_\alpha^k$ by $Q_x^k$ and write

$$\begin{aligned} W_f(x) &= \left\{ \sum_{k\in\mathbb{Z}} \sum_{\alpha\in I_k} \left[ \mu\left(Q_\alpha^k\right)^{-(1/2)} \left|s_{Q_\alpha^k}\right| \chi_{Q_\alpha^k}(x) \right]^2 \right\}^{(1/2)} \\ &= \left\{ \sum_{k\in\mathbb{Z}} \mu\left(Q_x^k\right)^{-1} \left|s_{Q_x^k}\right|^2 \right\}^{(1/2)} \\ &= \left\{ \sum_{k\in\mathbb{Z}} \int_{\delta^{k+1}}^{\delta^k} \mu\left(Q_x^k\right)^{-1} \int_{Q_x^k} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \right\}^{(1/2)}, \end{aligned} \tag{90}$$

where constants $\delta \in (0,1)$ satisfy Lemma 1 and the last line is obtained by using Proposition 1.

Moreover, for any fixed $(x,k) \in X \times \mathbb{Z}$, Lemma 1 also tells us that there are $z_x^k \in Q_x^k$ and constants $C_3 \in (0,1), C_4 > 0$ such that

$$B\left(z_x^k, C_3\delta^k\right) \subset Q_x^k \subset B\left(x, C_4\delta^k\right) \subset B\left(x, C_4\delta^{-1}t\right) = B_x, \tag{91}$$

for all $t \in (\delta^{k+1}, \delta^k)$. Consequently, by inequalities (9) and (10), we have

$$\begin{aligned} &\int_{\delta^{k+1}}^{\delta^k} \mu\left(Q_x^k\right)^{-1} \int_{Q_x^k} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \\ &\le \int_{\delta^{k+1}}^{\delta^k} \mu\left(B\left(z_x^k, C_3\delta^k\right)^{-1} \int_{B(x,C_4\delta^k)} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \\ &\le C \int_{\delta^{k+1}}^{\delta^k} \frac{\mu\left(B\left(x, C_4\delta^k\right)\right)}{\mu\left(B\left(z_x^k, C_3\delta^k\right)\right)} \operatorname*{esssup}_{y\in B(x,C_4\delta^k)} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \frac{\mathrm{d}t}{t} \\ &\le C \int_{\delta^{k+1}}^{\delta^k} \operatorname*{esssup}_{y\in B_x} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \frac{\mathrm{d}t}{t} \\ &\le C \int_{\delta^{k+1}}^{\delta^k} \left[\mathscr{M}_{a,L}^*(f)(x,t)\right]^2 \frac{\mathrm{d}t}{t}, \end{aligned} \tag{92}$$

for some appropriate constant $C$, where $\mathscr{M}_{a,L}^*(f)(x,t)$ is the Fefferman–Stein type maximal function with some large enough constant $a$ to be chosen, and the last line follows from

$$\begin{aligned} \operatorname*{esssup}_{y\in B_x} \left|t^2 L e^{-t^2 L} f(y)\right|^2 &= \operatorname*{esssup}_{y\in B_x} \frac{\left|t^2 L e^{-t^2 L} f(y)\right|^2}{\left[1 + t^{-1} d(x,y)\right]^{2a}} \left[1 + t^{-1} d(x,y)\right]^{2a} \\ &\le \left(1 + C_1\delta^{-1}\right) \left[\mathscr{M}_{a,L}^*(f)(x,t)\right]^2. \end{aligned} \tag{93}$$

Hence, (90) and (92) yield

$$\begin{aligned} W_f(x) &\le C \left\{ \int_0^\infty \left[\mathscr{M}_{a,L}^*(f)(x,t)\right]^2 \frac{\mathrm{d}t}{t} \right\}^{1/2} \\ &\le C \left\{ \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \left[\mathscr{M}_{a,L}^*(f)(x,t)\right]^2 \frac{\mathrm{d}t}{t} \right\}^{1/2} \\ &= C \left\{ \sum_{k\in\mathbb{Z}} \int_1^2 \left[\mathscr{M}_{a,L}^*(f)\left(x, 2^{-k}t\right)\right]^2 \frac{\mathrm{d}t}{t} \right\}^{1/2}. \end{aligned} \tag{94}$$

Thus, by using Lemma 9, we deduce that for any $\beta, r > 0$ and $a > (m/2)$, there exists a constant $C$ such that

$$\left|\mathscr{M}_{a,L}^*(f)\left(x, 2^{-k}t\right)\right|^r \le C \sum_{j=k}^\infty 2^{-(j-k)\beta r} \int_X \frac{\left|\left(2^{-j}t\right)^2 L e^{-(2^{-j}t)^2 L} f(z)\right|^r}{V\left(z, 2^{-k}\right)\left[1 + 2^k d(x,z)\right]^{ar}} \mathrm{d}\mu(z). \tag{95}$$

As in Corollary 1, let $r \in (0,1)$ with $p = (2/r) > 1$. Fix $\beta > 0$ and choose $a > (m/2)$ such that $ar > m + n$. Then, we integrate on both sides of (95) and employ Minkowski inequality and Lemma 10 to get

$$\begin{aligned}
&\left(\int_1^2 \left|\mathcal{M}_{a,L}^{*}(f)\left(x,2^{-k}y\right)\right|^2 \frac{\mathrm{d}t}{t}\right)^{r/2} \\
&\leq C\left(\int_1^2 \left|\sum_{j=k}^{\infty} 2^{-(j-k)\beta r}\int_X \frac{\left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(z)\right|^r}{V\left(z,2^{-k}\right)\left[1+2^k d(x,z)\right]^{ar}}\mathrm{d}\mu(z)\right|^{2/r}\frac{\mathrm{d}t}{t}\right)^{r/2} \\
&= C\sum_{j=k}^{\infty} 2^{-(j-k)\beta r}\left(\int_1^2 \left|\int_X \frac{\left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(z)\right|^r}{V\left(z,2^{-k}\right)\left[1+2^k d(x,z)\right]^{ar}}\mathrm{d}\mu(z)\right|^{2/r}\frac{\mathrm{d}t}{t}\right)^{r/2} \\
&\leq C\sum_{j=k}^{\infty} 2^{-(j-k)\beta r}\int_X \frac{\left[\int_1^2 \left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(z)\right|^2 (\mathrm{d}t/t)\right]^{r/2}}{V\left(z,2^{-k}\right)\left[1+2^k d(x,z)\right]^{ar}}\mathrm{d}\mu(z) \\
&= C\int_X \frac{\sum_{j=k}^{\infty} 2^{-(j-k)\beta r}\left[\int_1^2 \left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(z)\right|^2 (\mathrm{d}t/t)\right]^{r/2}}{V\left(z,2^{-k}\right)\left[1+2^k d(x,z)\right]^{ar}}\mathrm{d}\mu(z) \\
&\leq C\mathcal{M}\left(\sum_{j=k}^{\infty} 2^{-(j-k)\beta r}\left[\int_1^2 \left|\left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(\cdot)\right|^2 \frac{\mathrm{d}t}{t}\right]^{r/2}\right)(x) \\
&= C\mathcal{M}\left(G_k\right)(x).
\end{aligned} \tag{96}$$

Thus, by using Corollary 1, we have

$$\begin{aligned}
&\int_X \varphi\left(x,\frac{\left|W_f(x)\right|}{\lambda}\right)\mathrm{d}\mu(x) \\
&\leq \int_X \varphi\left(x,\frac{C\left(\sum_{k\in\mathbb{Z}}\int_1^2 \left[\mathcal{M}_{a,L}^{*}(f)\left(x,2^{-k}t\right)\right]^2 (\mathrm{d}t/t)\right)^{1/2}}{\lambda}\right)\mathrm{d}\mu(x) \\
&= \int_X \varphi\left(x,\frac{C\left(\sum_{k\in\mathbb{Z}}\left[\left(\int_1^2 \left[\mathcal{M}_{a,L}^{*}(f)\left(x,2^{-k}t\right)\right]^2 (\mathrm{d}t/t)\right)^{r/2}\right]^{2/r}\right)^{1/2}}{\lambda}\right)\mathrm{d}\mu(x) \\
&\leq \int_X \varphi\left(x,\frac{C\left(\sum_{k\in\mathbb{Z}}\left[\mathcal{M}\left(G_k\right)(x)\right]^{2/r}\right)^{1/2}}{\lambda}\right)\mathrm{d}\mu(x) \\
&\leq C\int_X \varphi\left(x,\frac{\left(\sum_{k\in\mathbb{Z}} G_k(x)^{2/r}\right)^{1/2}}{\lambda}\right)\mathrm{d}\mu(x).
\end{aligned} \tag{97}$$

We now turn to estimate $G_k^{2/r}(x)$. Since

$$2^{-(j-k)\beta r} = \frac{\beta r}{1-2^{-\beta r}} \int_{2^{j-k}}^{2^{j-k+1}} s^{-\beta r-1} \mathrm{d}s, \tag{98}$$

for any $k \in \mathbb{Z}$, we have

$$\begin{aligned} G_k(x) &= C \sum_{j=k}^{\infty} \int_{2^{j-k}}^{2^{j-k+1}} \left[ \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right]^{r/2} \frac{\mathrm{d}s}{s^{\beta r+1}} \\ &= C \sum_{j=k}^{\infty} \int_1^{\infty} \left[ \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right]^{r/2} \chi_{E_j}(s) \frac{\mathrm{d}s}{s^{\beta r+1}} \\ &= C \int_1^{\infty} \sum_{j=k}^{\infty} \left[ \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right]^{r/2} \chi_{E_j}(s) \frac{\mathrm{d}s}{s^{\beta r+1}}, \end{aligned} \tag{99}$$

where $E_j = [2^{j-k}, 2^{j-k+1}]$. Using Hölder's inequality, we obtain

$$\begin{aligned} G_k(x)^{2/r} &= \left[ C \int_1^{\infty} \sum_{j=k}^{\infty} \left[ \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right]^{r/2} \chi_{E_j}(s) \frac{\mathrm{d}s}{s^{\beta r+1}} \right]^{2/r} \\ &\leq C \left( \int_1^{\infty} s^{-\beta r-1} \mathrm{d}s \right)^{4/r(2-r)} \\ &\quad \times \int_1^{\infty} \left( \sum_{j=k}^{\infty} \left[ \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right]^{r/2} \chi_{E_j}(s) \right)^{2/r} \frac{\mathrm{d}s}{s^{\beta r+1}} \\ &= C \int_1^{\infty} \left( \sum_{j=k}^{\infty} \left( \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right) \chi_{E_j}(s) \right) \frac{\mathrm{d}s}{s^{\beta r+1}} \\ &= C \sum_{j=k}^{\infty} \left( \int_{2^{j-k}}^{2^{j-k+1}} \frac{\mathrm{d}s}{s^{\beta r+1}} \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right) \\ &= C \sum_{j=k}^{\infty} \left( 2^{-(j-k)\beta r} \int_1^2 \left| \left(2^{-j}t\right)^2 L e^{-\left(2^{-j}t\right)^2 L} f(x) \right|^2 \frac{\mathrm{d}t}{t} \right). \end{aligned} \tag{100}$$

Summarizing all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} G_k(x)^{2/r} &\le C\sum_{k\in\mathbb{Z}}\sum_{j\ge k}\left(2^{-(j-k)\beta r}\int_1^2\left|\left(2^{-j}t\right)^2 Le^{-\left(2^{-j}t\right)^2 L} f(x)\right|^2\frac{\mathrm{d}t}{t}\right)\\ &= C\sum_{j\in\mathbb{Z}}\sum_{k\le j}\left(2^{-(j-k)\beta r}\int_1^2\left|\left(2^{-j}t\right)^2 Le^{-\left(2^{-j}t\right)^2 L} f(x)\right|^2\frac{\mathrm{d}t}{t}\right)\\ &= C\left(1-2^{-\beta r}\right)^{-1}\sum_{j\in\mathbb{Z}}\int_1^2\left|\left(2^{-j}t\right)^2 Le^{-\left(2^{-j}t\right)^2 L} f(x)\right|^2\frac{\mathrm{d}t}{t}\\ &= C\sum_{j\in\mathbb{Z}}\int_{2^{-j}}^{2^{-j+1}}\left|t^2Le^{-t^2L}f(x)\right|^2\frac{\mathrm{d}t}{t}\\ &= C\int_0^\infty\left|t^2Le^{-t^2L}f(x)\right|^2\frac{\mathrm{d}t}{t}\\ &= C\left(G_L(f)(x)\right)^2.\end{aligned} \tag{101}$$

Therefore, (11), (97), and (101) yield

$$\int_X\varphi\left(x,\frac{G_L(f)(x)}{\lambda}\right)\mathrm{d}\mu(x)\ge C\int_X\varphi\left(x,\frac{\left|W_f(x)\right|}{\lambda}\right)\mathrm{d}\mu(x). \tag{102}$$

It reduces to show the reverse inequality of (102). Let $\delta$ be as in Lemma 1. In view of Proposition 1, we write

$$f=\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}, \tag{103}$$

and get

$$\begin{aligned}G_L(f)(x) &= \left(\int_0^\infty\left|t^2Le^{-t^2L}f(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &= \left(\int_0^\infty\left|t^2Le^{-t^2L}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &= \left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|t^2Le^{-t^2L}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &\le \left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|t^2Le^{-t^2L}\left(\sum_{k>j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &\quad+\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|t^2Le^{-t^2L}\left(\sum_{k\le j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &= I_3+I_4.\end{aligned} \tag{104}$$

We now estimate the integrand function of $I_3$. For any $k>j$ and $\alpha\in I_k$, noting $a_{Q_\alpha^k}=L^Mb_{Q_\alpha^k}$, we have

$$\begin{aligned}\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(x)\right| &= \left|t^2L^{M+1}e^{-t^2L}\left(b_{Q_\alpha^k}\right)(x)\right|\\ &= t^{-2M}\left|\left(t^2L\right)^{M+1}e^{-t^2L}\left(b_{Q_\alpha^k}\right)(x)\right|.\end{aligned} \tag{105}$$

Since $M>(nq(\varphi)/2p_1)$ with $n$ given as in (9), we can choose some $q$ satisfying Corollary 1 such that $2M>(n/q)$. Thus, there is some $N>0$ such that $2M>N>(n/q)$. Then, applying Definition 8, the upper bound of the kernel $(t^2L)^{M+1}e^{-t^2L}$ (cf. [18], Proposition 3.1), and (11), we get

$$\begin{aligned}&\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(x)\right|\\ &\le\frac{C_5}{V(x,t)}t^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\int_{3Q_\alpha^k}e^{-\left(d(x,y)^2/C_6t^2\right)}\mathrm{d}\mu(y)\\ &\le Ct^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(\frac{t}{t+d\left(x,y_\alpha^k\right)}\right)^N,\end{aligned} \tag{106}$$

where $y_\alpha^k$ is the center of $Q_\alpha^k$. Hence, by Lemma 8, we have

$$\begin{aligned}&\left|t^2Le^{-t^2L}\left(\sum_{k>j}\sum_{\alpha\in I_k}s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|\\ &\le C\sum_{k>j}\sum_{\alpha\in I_k}t^{-2M}\ell\left(Q_\alpha^k\right)^{2M}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+t^{-1}d\left(x,y_\alpha^k\right)\right]^N}\\ &\le C\sum_{k>j}\delta^{(2M-N)(k-j)}\sum_{\alpha\in I_k}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,y_\alpha^k\right)\right]^N}\\ &\le C\sum_{k>j}\delta^{(2M-N)(k-j)}\left[\mathscr{M}\left(\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^q\mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}\right)(x)\right]^{1/q}.\end{aligned} \tag{107}$$

For the integrand function of $I_4$ in the case $k\le j$ and $\alpha\in I_k$, we write

$$\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(x)\right| = t^2\left|e^{-t^2L}\left(L\left(a_{Q_\alpha^k}\right)\right)(x)\right|. \tag{108}$$

Then, using Definition 8, Gaussian estimate (1), and inequality (11), we obtain

$$\begin{aligned}&\left|t^2Le^{-t^2L}\left(a_{Q_\alpha^k}\right)(x)\right|\\ &\leq \frac{C_5}{V(x,t)}t^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\int_{3Q_\alpha^k} e^{-\left(d(x,y)^2/C_6t^2\right)}\mathrm{d}\mu(y)\\ &\leq Ct^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\left(1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,y_\alpha^k\right)\right)^{-N}.\end{aligned} \tag{109}$$

Consequently,

$$\begin{aligned}&\left|t^2Le^{-t^2L}\left(\sum_{k\leq j}\sum_{\alpha\in I_k} s_{Q_\alpha^k}a_{Q_\alpha^k}\right)(x)\right|\\ &\leq C\sum_{k\leq j}\sum_{\alpha\in I_k} t^2\ell\left(Q_\alpha^k\right)^{-2}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,y_\alpha^k\right)\right]^N}\\ &\leq C\sum_{k\leq j}\delta^{2(j-k)}\sum_{\alpha\in I_k}\mu\left(Q_\alpha^k\right)^{-(1/2)}\frac{\left|s_{Q_\alpha^k}\right|}{\left[1+\ell\left(Q_\alpha^k\right)^{-1}d\left(x,y_\alpha^k\right)\right]^N}\\ &\leq C\sum_{k\leq j}\delta^{2(j-k)}\left[\mathcal{M}\left(\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^q\mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}\right)(x)\right]^{1/q}.\end{aligned} \tag{110}$$

Similar to the discussion in (82) and combining (104)–(110), we get

$$\begin{aligned}G_L(f)(x) &\leq C\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|\sum_{k>j}\delta^{(2M-N)(k-j)}G_k(x)^{1/q}\right|^2\frac{\mathrm{d}t}{t}\right.\\ &\quad\left.+\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\left|\sum_{k\leq j}\delta^{2(j-k)}G_k(x)^{1/q}\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &\leq C\left(\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\sum_{k>j}\delta^{(2M-N)(k-j)}G_k(x)^{2/q}\frac{\mathrm{d}t}{t}\right.\\ &\quad\left.+\sum_{j\in\mathbb{Z}}\int_{\delta^j}^{\delta^{j-1}}\sum_{k\leq j}\delta^{2(j-k)}G_k(x)^{2/q}\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &= C\left(\sum_{k\in\mathbb{Z}}G_k(x)^{2/q}\left(\sum_{k>j}\delta^{(2M-N)(k-j)}+\sum_{k\leq j}\delta^{2(j-k)}\right)\right)^{1/2}\\ &\leq C\left(\sum_{k\in\mathbb{Z}}G_k(x)^{2/q}\right)^{1/2}.\end{aligned} \tag{111}$$

Therefore, (111) and Corollary 1 yield

$$\begin{aligned}\int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda}\right)d\mu(x) &\le C\int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}} G_k(x)^{2/q}\right)^{1/2}\right)d\mu(x)\\ &\le C\int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}}\left(\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^q \mu\left(Q_\alpha^k\right)^{-(q/2)}\chi_{Q_\alpha^k}(x)\right)^{2/q}\right)^{1/2}\right)d\mu(x)\\ &= C\int_X \varphi\left(x, \lambda^{-1}\left(\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\left|s_{Q_\alpha^k}\right|^2 \mu\left(Q_\alpha^k\right)^{-1}\chi_{Q_\alpha^k}(x)\right)^{1/2}\right)d\mu(x)\\ &= C\int_X \varphi\left(x, \frac{W_f(x)}{\lambda}\right)d\mu(x),\end{aligned} \tag{112}$$

which gives the reverse inequality of (102). It finishes the proof of Theorem 2. □

## Data Availability

No data were used to support this study.

## Disclosure

This work is a renewed version of [19].

## Conflicts of Interest

The authors declare that they have no conflicts of interest.


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