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Research Article

Boundedness of One Class of Integral Operators from Second Order Weighted Sobolev Space to Weighted Lebesgue Space

Aigerim Kalybay

KIMEP University, 4 Abay Ave., 050010 Almaty, Kazakhstan

Correspondence should be addressed to Aigerim Kalybay; kalybay@kimep.kz

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In the paper, for a certain class of Hardy operators with kernels, we consider the problem of their boundedness from a second order weighted Sobolev space to a weighted Lebesgue space.

1. Introduction

Let $I=(0,\infty)$ and $1<p,q<\infty$. Let u and v be positive functions locally integrable on the interval I. In addition, suppose that $v^{-p'}\in L_1^{loc}(I)$, where $p'=p/p-1$.

Let $W_{p,v}^2\equiv W_p^2(v,I)$ be a set of functions $f:I\longrightarrow\mathbb{R}$ having generalized derivatives up to the second order on I with the finite norm

$$\|f\|_{W_{p,v}^2}=\|vf''\|_p+|f'(1)|+|f(1)|, \tag{1}$$

where $\|\cdot\|_p$ is the standard norm of the space $L_p(I)$, $1<p<\infty$.

In the paper, we consider the problem of boundedness of the integral operator

$$\mathscr{K}f(x)=\int_0^x K(x,s)f(s)ds, x>0, \tag{2}$$

with a kernel $K(x,s)\geq 0$ from the weighted space $W_{p,v}^2$ to the weighted space $L_{q,u}\equiv L_q(u,I)$ with the norm $\|f\|_{L_{q,u}}=\|uf\|_q$. This problem is equivalent to the validity of the following inequality

$$\|u\mathscr{K}f\|_q\leq C\|f\|_{W_{p,v}^2}, f\in W_{p,v}^2. \tag{3}$$

Let $C_0^\infty(I)$ be the set of compactly supported functions infinitely time continuously differentiable on I. Due to the assumptions on v, we have that $C_0^\infty(I)\subset W_{p,v}^2$. Denote by $\dot{W}_{p,v}^2\equiv\dot{W}_p^2(v,I)$ the closure of the set $C_0^\infty(I)$ with respect to norm defined by (1). Depending on the behaviour of the function v at zero and infinity, the set $C_0^\infty(I)$ can be dense or not dense in the space $W_{p,v}^2$, i.e., $\dot{W}_{p,v}^2=W_{p,v}^2$ or

$$\dot{W}_{p,v}^2\subset W_{p,v}^2 \text{ and } \dot{W}_{p,v}^2\neq W_{p,v}^2, \tag{4}$$

respectively.

In the paper, we study inequality (3) under condition (4) for a certain class of integral operators. Note that in the case when $\mathscr{K}$ is the identity operator $\mathscr{I}$, inequalities of form (3) have been studied in many papers. Some results with proofs and a survey of other results with comments are given in Chapter 4 of the book [3]. Our work is related to the works [5, 6], in which inequality (3) with $\mathscr{K}\equiv\mathscr{I}$ was studied under various zero boundary conditions for $f\in W_{p,v}^2$.

The boundedness of integral operators in form (2) from a first order weighted Sobolev space to a weighted Lebesgue space has been investigated in the series of papers (see, e.g., [1, 2] and references given therein).

The paper is organized as follows. In Section 2, we present definitions and statements required to prove the main results. In Section 3, we present and prove the main results, especially we obtain necessary and sufficient conditions for the validity of inequality (3). In Section 4, we present corollaries that follow from the results of Section 3.

2. Axillary Definitions and Statements

Let $-\infty \leq a < b \leq \infty$. In the paper, $\chi_{(a,b)}(\cdot)$ is the characteristic function of the interval (a,b). Moreover, the notation $A \ll B$ means $A \leq cB$ and $A \approx B$ means $A \ll B \ll A$.

From the book [3], we have the following theorem.

Theorem 1. *Let $1 < p \leq q < \infty$.*

(i) The inequality

$$\left(\int_a^b \left| u(t) \int_a^t f(s)ds \right|^q dt\right)^{1/q} \leq C\left(\int_a^b |v(t)f(t)|^p dt\right)^{1/p}, \quad (5)$$

holds if and only if

$$A = \sup_{z\in(a,b)} \left(\int_z^b u^q(t)dt\right)^{1/q} \left(\int_a^z v^{-p'}(s)ds\right)^{1/p'} < \infty. \quad (6)$$

In addition, $C \approx A$, where C is the best constant in (5).

(ii) The inequality

$$\left(\int_a^b \left| u(t) \int_t^b f(s)ds \right|^q dt\right)^{1/q} \leq C\left(\int_a^b |v(t)f(t)|^p dt\right)^{1/p}, \quad (7)$$

holds if and only if

$$A^* = \sup_{z\in(a,b)} \left(\int_a^z u^q(t)dt\right)^{1/q} \left(\int_z^b v^{-p'}(s)ds\right)^{1/p'} < \infty. \quad (8)$$

In addition, $C \approx A^*$, where C is the best constant in (7).

The following definitions and statements are from the paper [7].

Definition 2. Let $K(x,s)$ be a nonnegative function measurable on the set $\Omega\{(x,s): a < s \leq x < b\}$ and nonincreasing in the second argument. We say that the function $K(x,s)$ belongs to the class $\mathcal{O}_1^-(\Omega)$ if there exist nonnegative functions $w(x)$ and $K_{0,1}(t,s)$ measurable on Ω such that

$$K(x,s) \approx K(x,t) + w(x)K_{0,1}(t,s), \quad (9)$$

for $a < s \leq t \leq x < b$; moreover, the equivalence coefficients in (9) do not depend on s, t, and x.

Definition 3. Let $K(x,s)$ be a nonnegative function measurable on the set Ω and nonincreasing in the second argument. We say that the function $K(x,s)$ belongs to the class $\mathcal{O}_2^-(\Omega)$ if there exist $K_1(x,t) \in \mathcal{O}_1^-(\Omega)$ and nonnegative functions $w(x)$, $K_{0,2}(t,s)$, and $K_{1,2}(t,s)$ measurable on Ω such that

$$K(x,s) \approx K(x,t) + K_1(x,t)K_{1,2}(t,s) + w(x)K_{0,2}(t,s), \quad (10)$$

for $a < s \leq t \leq x < b$; moreover, the equivalence coefficients in (10) do not depend on s, t, and x.

Definition 4. Let $K(x,s)$ be a nonnegative function measurable on the set Ω and nonincreasing in the second argument. We say that the function $K(x,s)$ belongs to the class $\mathcal{O}_3^-(\Omega)$ if there exist $K_1(x,t) \in \mathcal{O}_1^-(\Omega)$, $K_2(x,t) \in \mathcal{O}_2^-(\Omega)$, and nonnegative functions $w(x)$, $K_{0,3}(t,s)$, $K_{1,3}(t,s)$, and $K_{2,3}(t,s)$ measurable on Ω such that

$$K(x,s) \approx K(x,t) + K_2(x,t)K_{2,3}(t,s) + K_1(x,t)K_{1,3}(t,s) + w(x)K_{0,3}(t,s), \quad (11)$$

for $a < s \leq t \leq x < b$; moreover, the equivalence coefficients in (11) do not depend on s, t, and x.

Let

$$B_{1,1}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z v^{-p'}(s)ds\right)^{1/p'}, \quad (12)$$

$$B_{1,0}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b u^q(x)w^q(x)dx\right)^{1/q} \left(\int_a^z K_{0,1}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \quad (13)$$

$$B_1(a,b) = \max\left\{B_{1,1}(a,b), B_{1,0}(a,b)\right\}. \quad (14)$$

Theorem 5. *Let $1 < p \leq q < \infty$. Let the kernel of operator (2) belong to the class $\mathcal{O}_1^-$. Then, the inequality*

$$\left(\int_a^b \left| u(x) \int_a^x K(x,s)f(s)ds \right|^q dx\right)^{1/q} \leq C\left(\int_a^b |v(t)f(t)|^p dt\right)^{1/p}, \quad (15)$$

holds if and only if $B_1(a,b) < \infty$. In addition, $C \approx B_1(a,b)$, where C is the best constant in (15).

Let

$$B_{2,2}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z v^{-p'}(s)ds\right)^{1/p'}, \quad (16)$$

$$B_{2,1}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K_1^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z K_{1,2}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \quad (17)$$

$$B_{2,0}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b u^q(x)w^q(x)dx\right)^{1/q} \left(\int_a^z K_{0,2}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \quad (18)$$

$$B_2(a,b) = \max\left\{B_{2,2}(a,b), B_{2,1}(a,b), B_{2,0}(a,b)\right\}. \quad (19)$$

Theorem 6. *Let $1 < p \le q < \infty$. Let the kernel of operator (2) belong to the class $\mathcal{O}_2^-(\Omega)$. Then, inequality (15) holds if and only if $B_2(a,b) < \infty$. In addition, $C \approx B_2(a,b)$, where C is the best constant in (15).*

Let

$$B_{3,3}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z v^{-p'}(s)ds\right)^{1/p'}, \tag{20}$$

$$B_{3,2}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K_2^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z K_{2,3}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \tag{21}$$

$$B_{3,1}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b K_1^q(x,z)u^q(x)dx\right)^{1/q} \left(\int_a^z K_{1,3}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \tag{22}$$

$$B_{3,0}(a,b) = \sup_{z\in(a,b)} \left(\int_z^b u^q(x)w^q(x)dx\right)^{1/q} \left(\int_a^z K_{0,3}^{p'}(z,s)v^{-p'}(s)ds\right)^{1/p'}, \tag{23}$$

$$B_3(a,b) = \max\left\{B_{3,3}(a,b), B_{3,2}(a,b), B_{3,1}(a,b), B_{3,0}(a,b)\right\}. \tag{24}$$

Theorem 7. *Let $1 < p \le q < \infty$. Let the kernel of operator (2) belong to the class $\mathcal{O}_3^-(\Omega)$. Then, inequality (15) holds if and only if $B_3(a,b) < \infty$. In addition, $C \approx B_3(a,b)$, where C is the best constant in (15).*

For $f \in W_{p,v}^2$, we assume that $\lim_{t\longrightarrow 0+} f(t) = f(0)$, $\lim_{t\longrightarrow 0+} f'(t) = f'(0)$, $\lim_{t\longrightarrow\infty} f(t) = f(\infty)$, and $\lim_{t\longrightarrow\infty} f'(t) = f'(\infty)$ regardless of whether they are finite or infinite.

The following statement is from the paper [4].

Theorem 8. *Let $1 < p < \infty$. If the conditions*

$$\int_0^\infty v^{-p'}(s)ds < \infty \text{and} \int_1^\infty t^{p'} v^{-p'}(t)dt = \infty, \tag{25}$$

hold; then, for $f \in W_{p,v}^2$, there exist the finite values $f(0)$, $f'(0)$, and $f'(\infty)$ such that

$$\dot{W}_{p,v}^2 = \left\{f \in W_{p,v}^2 : f(0) = f'(0) = f'(\infty) = 0\right\}. \tag{26}$$

3. Main Results

First, we state some necessary lemmas. Some of them are new and of independent interest, and therefore proved in detail.

Lemma 9. *Let $K(x,s) \equiv K_1(x,s) \in \mathcal{O}_1^-(\Omega)$, where $K_1(x,s) \approx K_1(x,t) + w(x)K_{0,1}(t,s)$ for $a < s \le t \le x < b$. Then*

$$K_2(x,s) \equiv \int_s^x K_1(x,t)dt \in \mathcal{O}_2^-(\Omega), \tag{27}$$

$$\int_s^\tau K_1(x,t)(t-s)dt \approx K_1(x,\tau)(\tau-s)^2 + w(x)\int_s^\tau K_{0,1}(\tau,t)(t-s)dt, \tag{28}$$

for $a < s \le \tau \le x < b$;

$$K_3(x,s) \equiv \int_s^x K_1(x,t)(t-s)dt \in \mathcal{O}_3^-(\Omega). \tag{29}$$

Proof.

(i) For $a < s \le \tau \le x < b$, we have

$$\begin{aligned} K_2(x,s) &= \int_s^x K_1(x,t)dt = \int_\tau^x K_1(x,t)dt + \int_s^\tau K_1(x,t)dt \\ &\approx K_2(x,\tau) + K_1(x,\tau)(\tau-s) + w(x)\int_s^\tau K_{0,1}(\tau,t)dt. \end{aligned} \tag{30}$$

Therefore, by (10), we get that $K_2(x,s) \in \mathcal{O}_2^-(\Omega)$.

(ii) For $a < s \le \tau \le x < b$, it easily follows that

$$\begin{aligned} \int_s^\tau K_1(x,t)(t-s)dt &\approx \int_s^\tau K_1(x,\tau)(t-s)dt + \int_s^\tau w(x)K_{0,1}(\tau,t)(t-s)dt \\ &\approx K_1(x,\tau)(\tau-s)^2 + w(x)\int_s^\tau K_{0,1}(\tau,t)(t-s)dt. \end{aligned} \tag{31}$$

(iii) Using (28), for $a < s \le \tau \le x < b$, we have

$$\begin{aligned} K_3(x,s) &= \int_s^x K_1(x,t)(t-s)dt = \int_\tau^x K_1(x,t)(t-s)dt + \int_s^\tau K_1(x,t)(t-s)dt \\ &\approx \int_\tau^x K_1(x,t)(t-\tau)dt + \int_\tau^x K_1(x,t)dt\,(\tau-s) + K_1(x,\tau)(\tau-s)^2 \\ &\quad + w(x)\int_s^\tau K_{0,1}(\tau,t)(t-s)dt = K_3(x,\tau) + K_2(x,\tau)(\tau-s) \\ &\quad + K_1(x,\tau)(\tau-s)^2 + w(x)\int_s^\tau K_{0,1}(\tau,t)(t-s)dt. \end{aligned} \tag{32}$$

Then, in view of (11), we obtain that $K_3(x,s) \in \mathcal{O}_3^-(\Omega)$. The proof is complete.

Let $a = 0$ and $b = \infty$. Assume that

$$B_{3,3}^{-}(\tau) = \sup_{0<z<\tau}\left(\int_z^\tau\left(\int_z^x K(x,t)(t-z)dt\right)^q u^q(x)dx\right)^{1/q}\left(\int_0^z v^{-p'}(s)ds\right)^{1/p'}, \tag{33}$$

$$B_{3,2}^{-}(\tau) = \sup_{0<z<\tau}\left(\int_z^\tau\left(\int_z^x K(x,t)dt\right)^q u^q(x)dx\right)^{1/q}\left(\int_0^z (z-s)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{34}$$

$$B_{3,1}^{-}(\tau) = \sup_{0<z<\tau}\left(\int_z^\tau K^q(x,z)u^q(x)dx\right)^{1/q}\left(\int_0^z (z-s)^{2p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{35}$$

$$B_{3,0}^{-}(\tau) = \sup_{0<z<\tau}\left(\int_z^\tau u^q(x)w^q(x)dx\right)^{1/q}\left(\int_0^z\left(\int_s^z K_{0,1}(z,t)(t-s)dt\right)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{36}$$

$$B_3^{-}(\tau) = \max\left\{B_{3,3}^{-}(\tau), B_{3,2}^{-}(\tau), B_{3,1}^{-}(\tau), B_{3,0}^{-}(\tau)\right\}. \tag{37}$$

□

By using part (iii) of Lemma 9 and Theorem 7, we have one more lemma.

Lemma 10. *Let $1 < p \le q < \infty$ and $K(x,s) \in \mathcal{O}_1^{-}(\Omega)$. Then, the inequality*

$$\left(\int_0^\tau\left|u(x)\int_0^x\left(\int_s^x K(x,t)(t-s)dt\right)g(s)ds\right|^q dx\right)^{1/q} \le C_1^{-}\left(\int_0^\tau |v(t)g(t)|^p dt\right)^{1/p}, \tag{38}$$

holds if and only if $B_3^{-}(\tau) < \infty$. In addition, $C_1^{-} \approx B_3^{-}(\tau)$, where C_1^{-} is the best constant in (38).

Let

$$B_{1,1}^{-}(\tau) = \left(\int_\tau^\infty K^q(x,\tau)u^q(x)dx\right)^{1/q}\left(\int_0^\tau (\tau-s)^{2p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{39}$$

$$B_{1,0}^{-}(\tau) = \left(\int_\tau^\infty u^q(x)w^q(x)dx\right)^{1/q}\left(\int_0^\tau\left(\int_s^\tau K_{0,1}(\tau,t)(t-s)dt\right)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{40}$$

$$B_1^{-}(\tau) = \max\left\{B_{1,1}^{-}(\tau), B_{1,0}^{-}(\tau)\right\}. \tag{41}$$

Using (28) and the inverse Hölder's inequality, by Theorem 5, we have the following lemma.

Lemma 11. *Let $1 < p \le q < \infty$ and $K(x,s) \in \mathcal{O}_1^{-}(\Omega)$. Then, the inequality*

$$\left(\int_\tau^\infty\left|u(x)\int_0^\tau\left(\int_s^\tau K(x,t)(t-s)dt\right)g(s)ds\right|^q dx\right)^{1/q} \le C_2^{-}\left(\int_0^\tau |v(t)g(t)|^p dt\right)^{1/p}, \tag{42}$$

holds if and only if $B_1^{-}(\tau) < \infty$. In addition, $C_2^{-} \approx B_1^{-}(\tau)$, where C_2^{-} is the best constant in (42).

Let

$$A^{-}(\tau) = \left(\int_\tau^\infty\left(\int_\tau^x K(x,t)dt\right)^q u^q(x)dx\right)^{1/q}\left(\int_0^\tau (\tau-s)^{p'} v^{-p'}(s)ds\right)^{1/p'}. \tag{43}$$

From part (i) of Theorem 1, we can state the following lemma.

Lemma 12. *Let $1 < p \le q < \infty$. Then, the inequality*

$$\left(\int_\tau^\infty\left|u(x)\left(\int_\tau^x K(x,t)dt\right)\int_0^\tau (\tau-s)g(s)ds\right|^q dx\right)^{1/q} \le C_3^{-}\left(\int_0^\tau |v(t)g(t)|^p dt\right)^{1/p}, \tag{44}$$

holds if and only if $A^{-}(\tau) < \infty$. In addition, $C_3^{-} \approx A^{-}(\tau)$, where C_3^{-} is the best constant in (44).

Assume that

$$B_{2,2}^{+}(\tau) = \sup_{z>\tau}\left(\int_z^\infty\left(\int_z^x K(x,t)dt\right)^q u^q(x)dx\right)^{1/q}\left(\int_\tau^z (s-\tau)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{45}$$

$$B_{2,1}^{+}(\tau) = \sup_{z>\tau}\left(\int_z^\infty K^q(x,z)u^q(x)dx\right)^{1/q}\left(\int_\tau^z (z-s)^{p'}(s-\tau)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{46}$$

$$B_{2,0}^{+}(\tau) = \sup_{z>\tau}\left(\int_z^\infty u^q(x)w^q(x)dx\right)^{1/q}\left(\int_\tau^z\left(\int_s^z K_{0,1}(z,t)dt\right)^{p'}(s-\tau)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{47}$$

$$B_2^{+}(\tau) = \max\left\{B_{2,2}^{+}(\tau), B_{2,1}^{+}(\tau), B_{2,0}^{+}(\tau)\right\}. \tag{48}$$

By using part (i) of Lemma 9 and Theorem 6, we get the following statement.

Lemma 13. *Let $1 < p \le q < \infty$ and $K(x,s) \in \mathcal{O}_1^{-}(\Omega)$. Then, the inequality*

$$\left(\int_\tau^\infty\left|u(x)\int_\tau^x\left(\int_s^x K(x,t)dt\right)(s-\tau)g(s)ds\right|^q dx\right)^{1/q} \le C_1^{+}\left(\int_\tau^\infty |v(t)g(t)|^p dt\right)^{1/p}, \tag{49}$$

holds if and only if $B_2^{+}(\tau) < \infty$. In addition, $C_1^{+} \approx B_2^{+}(\tau)$, where C_1^{+} is the best constant in (49).

Let

$$B_{1,1}^{+}(\tau) = \sup_{z>\tau}\left(\int_z^\infty K^q(x,z)u^q(x)dx\right)^{1/q}\left(\int_\tau^z (s-\tau)^{2p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{50}$$

$$B_{1,0}^{+}(\tau) = \sup_{z>\tau}\left(\int_z^\infty u^q(x)w^q(x)dx\right)^{1/q}\left(\int_\tau^z K_{0,1}^{p'}(z,s)(s-\tau)^{2p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{51}$$

$$A^{+}(\tau)=\sup_{z>\tau}\left(\int_{z}^{\infty}u^{q}(x)w^{q}(x)dx\right)^{1/q}\left(\int_{\tau}^{z}\left(\int_{\tau}^{s}K_{0,1}(s,t)(t-\tau)dt\right)^{p'}v^{-p'}(s)ds\right)^{1/p'}, \quad (52)$$

$$(B_{1}A)^{+}(\tau)=\max\left\{B_{1,1}^{+}(\tau),B_{1,0}^{+}(\tau),A^{+}(\tau)\right\}. \quad (53)$$

Lemma 14. *Let $1<p\le q<\infty$ and $K(x,s)\in\mathcal{O}_{1}^{-}(\Omega)$. Then, the inequality*

$$\left(\int_{\tau}^{\infty}\left|u(x)\int_{\tau}^{x}\left(\int_{\tau}^{s}K(x,t)(t-\tau)dt\right)g(s)ds\right|^{q}dx\right)^{1/q}\le C_{2}^{+}\left(\int_{\tau}^{\infty}|v(t)g(t)|^{p}dt\right)^{1/p}, \quad (54)$$

holds if and only if $(B_{1}A)^{+}(\tau)<\infty$. In addition, $C_{2}^{+}\approx(B_{1}A)^{+}(\tau)$, where C_{2}^{+} is the best constant in (54).

Proof. Since $K(x,s)\in\mathcal{O}_{1}^{-}(\Omega)$, by Lemma 9, we have that

$$\int_{\tau}^{s}K(x,t)(t-\tau)dt\approx K(x,s)(s-\tau)^{2}+w(x)\int_{\tau}^{s}K_{0,1}(s,t)(t-\tau)dt. \quad (55)$$

Hence, inequality (54) is equivalent to simultaneous fulfilment of the following inequalities:

$$\left(\int_{\tau}^{\infty}\left|u(x)\int_{\tau}^{x}K(x,s)(s-\tau)^{2}g(s)ds\right|^{q}dx\right)^{1/q}\le C_{2,2}^{+}\left(\int_{\tau}^{\infty}|v(t)g(t)|^{p}dt\right)^{1/p}, \quad (56)$$

$$\left(\int_{\tau}^{\infty}\left|u(x)w(x)\int_{\tau}^{x}\left(\int_{\tau}^{s}K_{0,1}(s,t)(t-\tau)dt\right)g(s)ds\right|^{q}dx\right)^{1/q}\le C_{2,1}^{+}\left(\int_{\tau}^{\infty}|v(t)g(t)|^{p}dt\right)^{1/p}. \quad (57)$$

In addition, $\max\{C_{2,1}^{+},C_{2,2}^{+}\}\approx C_{2}^{+}$, where $C_{2,1}^{+}$ and $C_{2,2}^{+}$ are the best constants in (56) and (57), respectively. By Theorem 5, inequality (56) holds if and only if $\max\{B_{1,1}^{+}(\tau),B_{1,0}^{+}(\tau)\}<\infty$, and in addition, $C_{2,2}^{+}\approx\max\{B_{1,1}^{+}(\tau),B_{1,0}^{+}(\tau)\}$. By part (i) of Theorem 1, inequality (57) holds if and only if $A^{+}(\tau)<\infty$, and in addition, $C_{2,1}^{+}\approx A^{+}(\tau)$. Then, inequality (54) holds if and only if $(B_{1}A)^{+}(\tau)<\infty$ and $C_{2}^{+}\approx(B_{1}A)^{+}(\tau)$. The proof is complete. □

Assume that

$$(A^{*})^{+}(\tau)=\sup_{z>\tau}\left(\int_{\tau}^{z}\left(\int_{\tau}^{x}K(x,t)(t-\tau)dt\right)^{q}u^{q}(x)dx\right)^{1/q}\left(\int_{x}^{\infty}v^{-p'}(s)ds\right)^{1/p'}. \quad (58)$$

By using part (ii) of Theorem 1, we have the following lemma.

Lemma 15. *Let $1<p\le q<\infty$. Then, the inequality*

$$\left(\int_{\tau}^{\infty}\left|u(x)\left(\int_{\tau}^{x}K(x,t)(t-\tau)dt\right)\int_{x}^{\infty}g(s)ds\right|^{q}dx\right)^{1/q}\le C_{3}^{+}\left(\int_{\tau}^{\infty}|v(t)g(t)|^{p}dt\right)^{1/p}, \quad (59)$$

holds if and only if $(A^{*})^{+}(\tau)<\infty$. In addition, $C_{3}^{+}\approx(A^{*})^{+}(\tau)$, where C_{3}^{+} is the best constant in (59).

Let infinitely differentiable functions φ and ψ be such that $1\ge\varphi\ge 0$, $1\ge\psi\ge 0$, $\varphi(t)=1$ for $0<t<1/2$, $\varphi(t)=0$ for $t\ge 3/2$, $\psi(t)=1$ for $t>3/2$ and $\psi(t)=0$ for $0<t\le 1/2$. Moreover, $\varphi(t)>0$, $\psi(t)>0$ for $1/2<t<3/2$ and $\varphi(t)+\psi(t)=1$ for all $t\in I$.

Assume that P^{-} and P^{+} are polynomials such that $P^{-}(t)=c_{0}+c_{1}t$ and $P^{+}(t)=c_{2}t$, where $c_{i}\in\mathbb{R}$, $i=0,1,2$. Denote by $\{P^{-}\}$ and $\{P^{+}\}$ the sets of polynomials in the form P^{-} and P^{+}, respectively.

Let the conditions of Theorem 8 hold. Then, from (26), we have

$$W_{p,v}^{2}=\dot{W}_{p,v}^{2}\oplus\varphi\chi_{(0,1)}\{P^{-}\}\oplus\psi\chi_{(1,\infty)}\{P^{+}\}, \quad (60)$$

where $\oplus$ means the direct sum.

From Theorem 8, it follows that (1) is equivalent to the norm

$$\|f\|_{W_{p,v}^{2}}=\left\|vf''\right\|_{p}+\left|f'(0)\right|+|f(0)|. \quad (61)$$

Therefore, for $f\in W_{p,v}^{2}$ we have

$$\|f\|_{W_{p,v}^{2}}=\left\|vf''\right\|_{p}. \quad (62)$$

First, using (60), we establish inequality (3) on the set $W_{p,v}^{2}$, which, due to (62), has the form:

$$\|u\mathcal{K}f\|_{q}\le C\left\|vf''\right\|_{p},f\in\dot{W}_{p,v}^{2}. \quad (63)$$

Assume that

$$E(\tau)=\max\{B_{3}^{-}(\tau),B_{1}^{-}(\tau),A^{-}(\tau)\},\ F(\tau)=\max\left\{B_{2}^{+}(\tau),(B_{1}A)^{+}(\tau),(A^{*})^{+}(\tau)\right\}, \quad (64)$$

$$EF=\inf_{\tau\in I}\max\{E(\tau),F(\tau)\}. \quad (65)$$

Our first main result reads.

Theorem 16. *Let $1<p\le q<\infty$ and $K(x,s)\in\mathcal{O}_{1}^{-}(\Omega)$. Let condition (25) hold. Then, inequality (63) holds if and only if $EF<\infty$. In addition, $C\approx EF$, where C is the best constant in (63).*

Proof. Sufficiency. From (25) by Theorem 8, it follows the validity of (26). As in Theorem 2.1 of [5], using (26), for $f\in W_{p,v}^{2}$, we get

$$f(t) = \chi_{(0,\tau)}(t)\int_0^t (t-s)f''(s)ds + \chi_{(\tau,\infty)}(t)\left[\int_0^\tau (\tau - s)f''(s)ds - \int_\tau^t (s-\tau)f''(s)ds - (t-\tau)\int_t^\infty f''(s)ds\right], \tag{66}$$

where $\tau \in I$. Assuming $f'' = g$ in (66), we have that $g \in L_{p,v}(I)$. Moreover, from (26), it follows that $\int_0^\infty g(s)ds = 0$.

Assume that $\tilde{L}_{p,v}(I) = \{g \in L_{p,v}(I)\colon \int_0^\infty g(s)ds = 0\}$. Then, in (66), the condition $f \in \dot{W}_{p,v}^2$ is equivalent to the condition $g \in \tilde{L}_{p,v}(I)$. Replacing (66) into the left-hand side of (63), we find that

$$\begin{aligned} F \equiv \|u\mathscr{K}f\|_q^q &= \int_0^\tau \left|u(x)\int_0^x K(x,t)f(t)dt\right|^q dx + \int_\tau^\infty \left|u(x)\int_0^x K(x,t)f(t)dt\right|^q dx = \int_0^\tau \left|u(x)\int_0^x K(x,t)\int_0^t (t-s)g(s)ds\right|^q dx \\ &+ \int_\tau^\infty \left|u(x)\int_0^\tau K(x,t)\int_0^t (t-s)g(s)dsdt + u(x)\int_\tau^x K(x,t)\int_0^\tau (\tau-s)g(s)dsdt - u(x)\int_\tau^x K(x,t)\int_\tau^t (s-\tau)g(s)dsdt - u(x)\int_\tau^x K(x,t)(t-\tau)\int_t^\infty g(s)dsdt\right|^q dx \\ &= \int_0^\tau \left|u(x)\int_0^x \left(\int_s^x K(x,t)(t-s)dt\right)g(s)ds\right|^q dx + \int_\tau^\infty \left|u(x)\int_0^\tau \left(\int_s^\tau K(x,t)(t-s)dt\right)g(s)ds + u(x)\int_\tau^x K(x,t)dt\int_0^\tau (\tau-s)g(s)ds - u(x)\right. \\ &\left.\int_\tau^x \left(\int_s^x K(x,t)dt\right)(s-\tau)g(s)ds - u(x)\int_\tau^x \left(\int_\tau^s K(x,t)(t-\tau)dt\right)g(s)ds - u(x)\left(\int_\tau^x K(x,t)(t-\tau)dt\right)\int_x^\infty g(s)ds\right|^q dx. \end{aligned} \tag{67}$$

Therefore, inequality (63) has the form

$$\begin{aligned} &\left(\int_0^\tau \left|u(x)\int_0^x \left(\int_s^x K(x,t)(t-s)dt\right)g(s)ds\right|^q dx + \int_\tau^\infty \left|u(x)\int_0^\tau \left(\int_s^\tau K(x,t)(t-s)dt\right)g(s)ds + u(x)\int_\tau^x K(x,t)dt\int_0^\tau (\tau-s)g(s)ds - u(x)\right.\right. \\ &\left.\left.\int_\tau^x \left(\int_s^x K(x,t)dt\right)(s-\tau)g(s)ds - u(x)\int_\tau^x \left(\int_\tau^s K(x,t)(t-\tau)dt\right)g(s)ds - u(x)\left(\int_\tau^x K(x,t)(t-\tau)dt\right)\int_x^\infty g(s)ds\right|^q dx\right)^{1/q} \le C\left(\int_0^\infty |v(s)g(s)|^p ds\right)^{1/p}, g \in \tilde{L}_{p,v}(I). \end{aligned} \tag{68}$$

In the left-hand side of (68), using the Minkowski's inequality for sums, then, applying Lemmas 10, 11, 12, 13, and 14 to each term, we get

$$\begin{aligned} F &\ll E(\tau)\left(\int_0^\tau |v(s)g(s)|^p ds\right)^{1/p} + F(\tau)\left(\int_\tau^\infty |v(s)g(s)|^p ds\right)^{1/p} \\ &\ll \max\{E(\tau), F(\tau)\}\left(\int_0^\infty |v(s)g(s)|^p ds\right)^{1/p}. \end{aligned} \tag{69}$$

Since the left-hand side of inequality (63) does not depend on $\tau \in I$, then, taking in the right-hand side of (69) infimum with respect to $\tau \in I$, we can conclude that

$$C \ll EF, \tag{70}$$

where C is the best constant in (63).

Necessity. By the conditions of Theorem 16, we have that $v^{-1} \in L_{p'}(I)$. Then, for any $\tau \in I$, there exists k_τ such that

$$\int_0^\tau v^{-p'}(t)dt = k_\tau \int_\tau^\infty v^{-p'}(t)dt, \tag{71}$$

in addition, k_τ increases in τ, $\lim_{\tau \longrightarrow 0^+} k_\tau = 0$ and $\lim_{\tau \longrightarrow \infty} k_\tau = \infty$.

Let us use the ideas in the paper [5]. For $\tau \in I$, we consider two sets $\mathscr{L}_1 = \{g \in L_{p,v}(0,\tau)\colon g \ge 0\}$ and $\mathscr{L}_2 = \{g \in L_{p,v}(\tau,\infty)\colon g \le 0\}$. For each $g_1 \in \mathscr{L}_1$ and $g_2 \in \mathscr{L}_2$, we, respectively, construct the functions $g_2 \in \mathscr{L}_2$ and $g_1 \in \mathscr{L}_1$ so that $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$ belongs to the set $\tilde{L}_{p,v}(I)$.

We define a strictly increasing function $\rho : (0,\tau) \longrightarrow (\tau,\infty)$ from the relation

$$\int_0^s v^{-p'}(t)dt = k_\tau \int_{\rho(s)}^\infty v^{-p'}(t)dt, s \in (0,\tau); \tag{72}$$

$$\int_0^{\rho^{-1}(s)} v^{-p'}(t)dt = k_\tau \int_s^\infty v^{-p'}(t)dt, s \in (\tau, \infty), \tag{73}$$

where ρ^{-1} is inverse to ρ. From (73), it follows that the functions ρ and ρ^{-1} are locally absolutely continuous, $\rho(\tau) = \tau$ and $\lim_{s \longrightarrow 0^+} \rho(s) = \infty$.Differentiating both relations in (73), we have

$$\begin{aligned} \frac{1}{k_\tau} &= \frac{v^{-p'}(\rho(s))}{v^{-p'}(s)} \left|\rho'(s)\right|, s \in (0,\tau)\,; k_\tau \\ &= \frac{v^{-p'}(\rho^{-1}(s))}{v^{-p'}(s)} \left|(\rho^{-1}(s))'\right|, s \in (\tau,\infty). \end{aligned} \tag{74}$$

Then, for $g_1 \in \mathscr{L}_1$, we construct

$$g_2(t) = -k_\tau g_1(\rho^{-1}(t)) \frac{v^{-p'}(t)}{v^{-p'}(\rho^{-1}(t))}, t > \tau, \tag{75}$$

while for $g_2 \in \mathscr{L}_2$, we construct

$$g_1(t) = -\frac{1}{k_\tau} g_2(\rho(t)) \frac{v^{-p'}(t)}{v^{-p'}(\rho(t))}, 0 < t \le \tau. \tag{76}$$

Changing the variables $\rho^{-1}(t) = s$ and using the first equality in (74), we find that

$$\begin{aligned} \int_\tau^\infty |v(t)g_2(t)|^p dt &= k_\tau^p \int_\tau^\infty \left| v(t) g_1(\rho^{-1}(t)) \frac{v^{-p'}(t)}{v^{-p'}(\rho^{-1}(t))} \right|^p dt \\ &= k_\tau^p \int_0^\tau \left| v^p(\rho(s)) g_1^p(s) \frac{v^{-pp'}(\rho(s))}{v^{-pp'}(s)} \right| \left|\rho'(s)\right| ds \\ &= k_\tau^p \int_0^\tau |v(s) g_1(s)|^p \frac{v^{-p'}(\rho(s))}{v^{-p'}(s)} \left|\rho'(s)\right| ds \\ &= k_\tau^{p-1} \int_0^\tau |v(s) g_1(s)|^p ds < \infty. \end{aligned} \tag{77}$$

Similarly, using the second equality in (74), we get

$$\int_0^\tau |v(t) g_1(t)|^p dt = k_\tau^{1-p} \int_\tau^\infty |v(s) g_2(s)|^p ds < \infty. \tag{78}$$

From (77) and (78), assuming that $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$, we have

$$\int_0^\infty |v(t)g(t)|^p dt = \left(1 + k_\tau^{p-1}\right) \int_0^\tau |v(t) g_1(t)|^p dt = \left(1 + k_\tau^{1-p}\right) \int_\tau^\infty |v(t) g_2(t)|^p dt < \infty, \tag{79}$$

i.e., $g \in L_{p,v}(I)$. For any $\tau \in I$ integrating both sides of (75) from τ to ∞ and (76) from 0 to τ, we find that

$$\int_\tau^\infty g(t)dt = -\int_0^\tau g(t)dt, \textit{i.e.}, \int_0^\infty g(t)dt = 0. \tag{80}$$

Hence, constructed from the functions $g_1 \in \mathscr{L}_1$ and $g_2 \in \mathscr{L}_2$, the function g belongs to $\tilde{L}_{p,v}(I)$. Replacing it into (68), we get

$$\begin{aligned} &\left(\int_0^\tau \left(u(x) \int_0^x \left(\int_s^x K(x,t)(t-s)dt \right) g_1(s)ds \right)^q dx \right. \\ &+ \int_\tau^\infty \left(u(x) \int_0^\tau \left(\int_s^\tau K(x,t)(t-s)dt \right) g_1(s)ds \right. \\ &+ u(x) \int_\tau^x K(x,t)dt \int_0^\tau (\tau - s) g_1(s)ds \\ &+ u(x) \int_\tau^x \left(\int_s^x K(x,t)dt \right) (s-\tau) |g_2(s)| ds \\ &+ u(x) \int_\tau^x \left(\int_\tau^s K(x,t)(\tau - s)dt \right) |g_2(s)| ds \\ &\left. \left. + u(x) \left(\int_\tau^x K(x,t)(t-\tau)dt \right) \int_x^\infty |g_2(s)| ds \right)^q dx \right)^{1/q} \\ &\le C \left(\int_0^\infty |v(s) g(s)|^p ds \right)^{1/p}, \end{aligned} \tag{81}$$

where all terms in the left-hand side are nonnegative.

Let the function $g \in \tilde{L}_{p,v}(I)$ constructed from the function $g_1 \in \mathscr{L}_1$. Then, from (81) and (79), we have

$$\left(\int_0^\tau \left(u(x) \int_0^x \left(\int_s^x K(x,t)(t-s)dt \right) g_1(s)ds \right)^q dx + \int_\tau^\infty \left(u(x) \int_0^\tau \left(\int_s^\tau K(x,t)(t-s)dt \right) g_1(s)ds + u(x) \int_\tau^x K(x,t)dt \int_0^\tau (\tau - s) g_1(s) \right)^q dx \right)^{1/q} \le C \left(1 + k_\tau^{p-1}\right)^{1/p} \left(\int_0^\tau |v(s) g_1(s)|^p ds \right)^{1/p}. \tag{82}$$

Due to arbitrariness of $g_1 \in L_{p,v}(0,\tau)$, by Lemmas 10, 11, and 12, the latter gives that

$$E(\tau) \ll \left(1 + k_\tau^{p-1}\right)^{1/p} C. \tag{83}$$

Similarly, due to (81) and (79), for the function $g \in \tilde{L}_{p,v}(I)$ constructed from the function $g_2 \in \mathscr{L}_2$, we obtain

$$F(\tau) \ll \left(1 + k_\tau^{1-p}\right)^{1/p} C. \tag{84}$$

From (83) and (84), we find that

$$EF = \inf_{\tau \in I} \max\{E(\tau), F(\tau)\} \ll C \inf_{\tau \in I}\left[\max\left\{\left(1 + k_\tau^{p-1}\right)\left(1 + k_\tau^{1-p}\right)\right\}\right]^{1/p} \le 4^{1/p} C. \tag{85}$$

Therefore, $EF \ll C$, which, together with (70), yields that $EF \approx C$, where C is the best constant in (63). The proof is complete. □

Let

$$G = \max\left\{\left(\int_0^\infty \left|u(x)\int_0^x K(x,t)\,dt\right|^q dx\right)^{1/q}, \left(\int_0^\infty \left|u(x)\int_0^x K(x,t)\,t\,dt\right|^q dx\right)^{1/q}\right\}. \tag{86}$$

Our main result concerning Hardy-type inequality (3) reads.

Theorem 17. *Let $1 < p \le q < \infty$ and $K(x,s) \in \mathcal{O}_1^-(\Omega)$. Let conditions in (25) hold. Then, inequality (3) holds if and only if $\max\{G, EF\} < \infty$. In addition, $C \approx \max\{G, EF\}$, where C is the best constant in (3).*

Proof. Due to (60), we consider inequality (3) on the set

$$\varphi\chi_{(0,1)}\{P^-\} + \psi\chi_{(1,\infty)}\{P^+\} = H. \tag{87}$$

The function $f \in H$ has the form

$$f(t) = \varphi(t)\chi_{(0,1)}(t)(c_0 + c_1 t) + \psi(t)\chi_{(1,\infty)}(t) c_2 t. \tag{88}$$

Hence, $f''(t) = 0$ almost everywhere on $t \in I$. Therefore, on the basis of (61), we have

$$\|u\mathscr{K}f\|_q \le C\left(\left|f'(0)\right| + |f(0)|\right), f \in H. \tag{89}$$

Let $f_0 \in H$ be such that

$$f_0(t) = \varphi(t)\chi_{(0,1)}(t) + \psi(t)\chi_{(1,\infty)}(t) t. \tag{90}$$

Then, from (89), we obtain

$$C \ge \left(\int_0^\infty \left|u(x)\int_0^x K(x,t) f_0(t)\,dt\right|^q dx\right)^{1/q} \ge \min\{\varphi(1), \psi(1)\} \times \left(\int_0^1 \left|u(x)\int_0^x K(x,t)dt\right|^q dx + \int_1^\infty \left|u(x)\int_0^1 K(x,t)\,dt + u(x)\int_1^x K(x,t)\,t\,dt\right|^q dx\right)^{1/q}, \tag{91}$$

which implies that $C \gg G$.

Let $\max\{|c_0| + |c_1|, |c_2|\} = L$. Then, $|f(t)| \le L(\chi_{(0,1)}(t) + \chi_{(1,\infty)}(t) t)$. Replacing the function f into the left-hand side of (89), we get

$$\left(\int_0^\infty \left|u(x)\int_0^x K(x,t) f(t)\,dt\right|^q dx\right)^{1/q} \le L\left(\int_0^1 \left|u(x)\int_0^x K(x,t)\,dt\right|^q dx + \int_1^\infty \left|u(x)\int_0^1 K(x,t)\,dt + u(x)\int_1^x K(x,t)\,t\,dt\right|^q dx\right)^{1/q} \ll LG. \tag{92}$$

The latter, together with $C \gg G$, gives that $C \approx G$. Then, by Theorem 16, it follows that $C \approx \max\{G, EF\}$, where C is the best constant in (3). The proof is complete. □

4. Corollaries

Assume that the kernel $K(x,s)$ of operator (2) satisfies the Oinarov condition

$$K(x,s) \approx K(x,t) + K(t,s), 0 < s \le t \le x < \infty, \tag{93}$$

which is often applied for integral operators. Then, in $B_3^-(\tau)$, the expression $B_{3,0}^-(\tau)$ turns to the expression

$$B_{3,0}^-(\tau) = \sup_{0<z<\tau}\left(\int_z^\tau u^q(x)dx\right)^{1/q}\left(\int_0^z\left(\int_s^z K(z,t)(t-s)dt\right)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{94}$$

in $B_1^-(\tau)$ the expression $B_{1,0}^-(\tau)$ turns to the expression

$$B_{1,0}^-(\tau) = \left(\int_\tau^\infty u^q(x)dx\right)^{1/q}\left(\int_0^\tau\left(\int_s^\tau K(\tau,t)(t-s)dt\right)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{95}$$

in $B_2^+(\tau)$ the expression $B_{2,0}^+(\tau)$ turns to the expression

$$B_{2,0}^+(\tau) = \sup_{z>\tau}\left(\int_z^\infty u^q(x)dx\right)^{1/q}\left(\int_\tau^z\left(\int_s^z K(z,t)dt\right)^{p'} (s-\tau)^{p'} v^{-p'}(s)ds\right)^{1/p'}, \tag{96}$$

and in $(B_1 A)^+(\tau)$, the expressions $B_{1,0}^+(\tau)$, $A^+(\tau)$, respectively, turn to the expressions

$$B_{1,0}^+(\tau) = \sup_{z>\tau}\left(\int_z^\infty u^q(x)\,dx\right)^{1/q}\left(\int_\tau^z K^{p'}(z,s)(s-\tau)^{2p'} v^{-p'}(s)\,ds\right)^{1/p'}, \tag{97}$$

$$A^{+}(\tau)=\sup_{z>\tau}\left(\int_{z}^{\infty}u^{q}(x)\,dx\right)^{1/q}\left(\int_{\tau}^{z}\left(\int_{\tau}^{s}K(s,t)(t-\tau)\,dt\right)^{p'}v^{-p'}(s)ds\right)^{1/p'}. \tag{98}$$

After these changes, we denote $E(\tau)$ by $\tilde{E}(\tau)$, $F(\tau)$ by $\tilde{F}(\tau)$, and EF by $\tilde{E}\tilde{F}$ and get the following statement.

Corollary 18. *Let $1<p\leq q<\infty$ and the kernel of (2) satisfy condition (93). Let conditions (25) hold.*

(i) Inequality (63) holds if and only if $\tilde{E}\tilde{F}<\infty$. In addition, $C\approx\tilde{E}\tilde{F}$, where C is the best constant in (63)

(ii) Inequality (3) holds if and only if $\max\{G,\tilde{E}\tilde{F}\}<\infty$. In addition, $C\approx\max\{G,\tilde{E}\tilde{F}\}$, where C is the best constant in (3)

Let $n\geq 3$. Instead of operator (2), we consider the operator of Riemann-Liouville I_{n-2}, defined by

$$I_{n-2}f(x)=\frac{1}{(n-3)!}\int_{0}^{x}(x-t)^{n-3}f(t)\,dt. \tag{99}$$

The kernel $(x-t)^{n-3}$ of the operator I_{n-2} satisfies condition (93), and therefore, it belongs to the class $\mathcal{O}_{1}^{-}(\Omega)$. In this case, we replace $K(x,t)$ and $K_{0,1}(x,t)$ by $(x-t)^{n-3}$ and assume that $w(x)\equiv 1$. For the kernel $(x-t)^{n-3}$ inequality, (38) has the form

$$\left(\int_{0}^{\tau}\left|u(x)\int_{0}^{x}(x-s)^{n-1}g(s)ds\right|^{q}dx\right)^{1/q}\leq C_{1}^{-}\left(\int_{0}^{\tau}|v(s)g(s)|^{p}ds\right)^{1/p}. \tag{100}$$

Then, according to Theorem 5, we have $C_{1}^{-}\approx\mathscr{B}_{1}^{-}(\tau)=\max\{\mathscr{B}_{1,1}^{-}(\tau),\mathscr{B}_{1,2}^{-}(\tau)\}$, where

$$\mathscr{B}_{1,1}^{-}(\tau)=\sup_{0<z<\tau}\left(\int_{z}^{\tau}(x-z)^{q(n-1)}u^{q}(x)dx\right)^{1/q}\left(\int_{0}^{z}v^{-p'}(s)ds\right)^{1/p'}, \tag{101}$$

$$\mathscr{B}_{1,0}^{-}(\tau)=\sup_{0<z<\tau}\left(\int_{z}^{\tau}u^{q}(x)dx\right)^{1/q}\left(\int_{0}^{z}(z-s)^{p'(n-1)}v^{-p'}(s)ds\right)^{1/p'}. \tag{102}$$

For the sum of kernels of the operators in (42) and (44), we deduce that

$$\int_{s}^{\tau}(x-t)^{n-3}(t-s)dt+\int_{\tau}^{x}(x-t)^{n-3}dt\,(\tau-s)\approx(x-\tau)^{n-3}(\tau-s)^{2}+(\tau-s)^{n-1}+(x-\tau)^{n-2}(\tau-s)\approx(\tau-s)^{n-1}+(x-\tau)^{n-2}(\tau-s). \tag{103}$$

Then instead of inequalities (42) and (44), we, respectively, have

$$\left(\int_{\tau}^{\infty}\left|u(x)\int_{0}^{\tau}(\tau-s)^{n-1}g(s)ds\right|^{q}dx\right)^{1/q}\leq C_{2}^{-}\left(\int_{0}^{\tau}|v(s)g(s)|^{p}ds\right)^{1/p}, \tag{104}$$

$$\left(\int_{\tau}^{\infty}\left|u(x)(x-\tau)^{n-2}\int_{0}^{\tau}(\tau-s)g(s)ds\right|^{q}dx\right)^{1/q}\leq C_{3}^{-}\left(\int_{0}^{\tau}|v(s)g(s)|^{p}ds\right)^{1/p}. \tag{105}$$

By part (i) of Theorem 1, this yields that

$$C_{2}^{-}\approx\left(\int_{\tau}^{\infty}u^{q}(x)dx\right)^{1/q}\left(\int_{0}^{\tau}(\tau-s)^{p'(n-1)}v^{-p'}(s)ds\right)^{1/p'}=\mathscr{A}_{1}^{-}(\tau), \tag{106}$$

$$C_{3}^{-}\approx\left(\int_{\tau}^{\infty}(x-\tau)^{q(n-2)}u^{q}(x)dx\right)^{1/q}\left(\int_{0}^{\tau}(\tau-s)^{p'}v^{-p'}(s)ds\right)^{1/p'}=\mathscr{A}_{2}^{-}(\tau). \tag{107}$$

Assume that $\mathscr{A}^{-}(\tau)=\max\{\mathscr{A}_{1}^{-}(\tau),\mathscr{A}_{2}^{-}(\tau)\}$. Now, for the sum of kernels of the operators in (49) and (54), we get

$$\int_{s}^{x}(x-t)^{n-3}dt\,(s-\tau)+\int_{\tau}^{s}(x-t)^{n-3}(t-\tau)dt\approx(x-s)^{n-2}(s-\tau)+(x-s)^{n-3}(s-\tau)^{2}+(s-\tau)^{n-1}\approx(x-s)^{n-2}(s-\tau)+(s-\tau)^{n-1}\approx(x-\tau)^{n-2}(s-\tau). \tag{108}$$

Then instead of inequalities (49) and (54), we obtain

$$\left(\int_{\tau}^{\infty}\left|u(x)(x-\tau)^{n-2}\int_{\tau}^{x}(s-\tau)g(s)ds\right|^{q}dx\right)^{1/q}\leq C_{1}^{+}\left(\int_{\tau}^{\infty}|v(s)g(s)|^{p}ds\right)^{1/p}. \tag{109}$$

Hence, by part (i) of Theorem 1 we have

$$C_{1}^{+}\approx\sup_{z>\tau}\left(\int_{z}^{\infty}(x-\tau)^{q(n-2)}u^{q}(x)dx\right)^{1/q}\left(\int_{\tau}^{z}(s-\tau)^{p'}v^{-p'}(s)ds\right)^{1/p'}=\mathscr{A}^{+}(\tau). \tag{110}$$

For the kernel $(x-t)^{n-3}$ *inequality (59) can be written as follows:*

$$\left(\int_\tau^\infty \left| u(x)(x-\tau)^{n-1} \int_x^\infty g(s)ds \right|^q dx\right)^{1/q} \le C_2^+ \left(\int_\tau^\infty |v(s)g(s)|^p ds\right)^{1/p}. \quad (111)$$

Therefore, by using part (ii) of Theorem 1, we find

$$C_2^+ \approx \left(\int_\tau^z (x-\tau)^{q(n-1)} u^q(x) dx\right)^{1/q} \left(\int_z^\infty v^{-p'}(s) ds\right)^{1/p'} = (\mathscr{A}^*)^+(\tau). \quad (112)$$

Assume that

$$\mathscr{E}(\tau) = \max\left\{\mathscr{B}_1^-(\tau), \mathscr{A}^-(\tau)\right\}, \mathscr{F}(\tau) = \max\left\{\mathscr{A}^+(\tau), (\mathscr{A}^*)^+(\tau)\right\}, \quad (113)$$

$$\mathscr{E}\mathscr{F} = \inf_{\tau\in I} \max\left\{\mathscr{E}(\tau), \mathscr{F}(\tau)\right\}. \quad (114)$$

Thus, for inequality (63) with operator (99)

$$\|uI_{n-2}f\|_q \le C\|vf''\|_p, \ f \in W_{p,v}^2, \quad (115)$$

we can conclude the following statement.

Corollary 19. *Let* $1 < p \le q < \infty$ *and conditions (25) hold. Then, inequality (115) holds if and only if* $\mathscr{E}\mathscr{F} < \infty$*. In addition,* $C \approx \mathscr{E}\mathscr{F}$*, where C is the best constant in (115).*

Assume that $I_{n-2}f(t) = g(t)$ in (115). Then, $g^{(n-2)}(t) = f(t)$ and $g^{(i)}(0) = 0$, $i = 0, 1, \cdots, n-3$. Moreover, $g^{(n)}(t) = f''(t)$ and inequality (115) turns to the inequality

$$\|ug\|_q \le C\left\|vg^{(n)}\right\|_p, \quad (116)$$

with conditions

$$g^{(i)}(0) = 0, i = 0, 1, \cdots, n-1, g^{(n-1)}(\infty) = 0. \quad (117)$$

From Corollary 19, we get one more corollary.

Corollary 20. *Let* $1 < p \le q < \infty$ *and conditions (25) hold. Then, inequality (116) with conditions (117) holds if and only if* $\mathscr{E}\mathscr{F} < \infty$*. In addition,* $C \approx \mathscr{E}\mathscr{F}$*, where C is the best constant in (116).*

The statement of Corollary 20 gives one of the results of the work [6].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that she has no conflicts of interest.

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References

[1] A. Kalybay and R. Oinarov, "Kernel operators and their boundedness from weighted Sobolev space to weighted Lebesgue space," *Turkish Journal of Mathematics*, vol. 43, no. 1, pp. 301–315, 2019.

[2] A. Kalybay and R. Oinarov, "Boundedness of Riemann-Liouville operator from weighted Sobolev space to weighted Lebesgue space," *Eurasian Mathematical Journal*, vol. 12, no. 1, pp. 39–48, 2021.

[3] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey, 2nd edition, 2017.

[4] P. I. Lizorkin, "On the closure of the set of compactly supported functions in the weighted space," *Doklady Akademii Nauk SSSR*, vol. 239, no. 4, pp. 789–792, 1978.

[5] M. Nasyrova and V. D. Stepanov, "On weighted Hardy inequalities on semiaxis for functions vanishing at the endpoints," *Journal of Inequalities and Applications*, vol. 1997, no. 3, Article ID 291238, 1997.

[6] M. Nasyrova, *Weighted Inequalities Involving Hardy-Type and Limiting Geometric Mean Operators*, Luleå University of Technology, Sweden, [Ph.D. thesis], 2002.

[7] R. Oinarov, "Boundedness and compactness of Volterra type integral operators," *Siberian Mathematical Journal*, vol. 48, no. 5, pp. 884–896, 2007.