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Research Article

On Pentagonal Controlled Fuzzy Metric Spaces with an Application to Dynamic Market Equilibrium

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In this manuscript, we coined pentagonal controlled fuzzy metric spaces and fuzzy controlled hexagonal metric space as generalizations of fuzzy triple controlled metric spaces and fuzzy extended hexagonal b-metric spaces. We use a control function in fuzzy controlled hexagonal metric space and introduce five noncomparable control functions in pentagonal controlled fuzzy metric spaces. In the scenario of pentagonal controlled fuzzy metric spaces, we prove the Banach fixed point theorem, which generalizes the Banach fixed point theorem for the aforementioned spaces. An example is offered to support our main point. We also presented an application to dynamic market equilibrium.

1. Introduction

Fuzzy notions are used to describe the degrees of possession of a certain property. The ability of fuzzy set (FS) theory to address circumstances that fixed point theory finds problematic originates from its attractiveness in tackling control problems. FSs are used to govern ill-defined, convoluted, and nonlinear systems The fabulous idea of FSs was presented by Zadeh [1] in his research paper. A FS extends the concept of a crisp set by associating all elements with membership values in the range of [0,1]. The FS theory has been widely employed in mathematics since then. Schweizer and Sklar [2] presented continuous t-norms (CTNs). Itoh [3] proved fixed point theorems with an application to random differential equations in Banach spaces. Kramosil and Michálek [4] presented the fuzzy metric space (FMS) approach. George and Veeramani [5] modify the notion of FMS and presented the Hausdorff topology in FMS. Grabiec [6] proved the Banach contraction theorem in fuzzy version, and also, he proved the Edelstein theorem in FMS. Han [7] proved the Banach fixed point theorem from the view point of digital topology. Uddin et al. [8] gave a solution of the Fredholm integral inclusions via Suzuki-type fuzzy contractions. Kamran et al. [9] presented the approach of extended metric space and proved several fixed point results for contraction mappings. Mehmood et al. [10] presented fuzzy rectangular b-metric spaces and proved fixed point theorems. Saleem et al. [11] coined the notion of fuzzy double controlled metric spaces and proved several fixed point results. Badshah-e-Rome and Sarwar [12] presented the approach of extended fuzzy rectangular b-metric spaces and proved fixed point results for contraction mappings via α-admissibility. Furqan et al. [13] presented the notion of fuzzy triple controlled metric spaces (FTCMSs) as a generalization of various spaces. Zubair et al. [14] presented fuzzy extended hexagonal b-metric spaces (FEHBMSs) and proved several fixed point results.

In this manuscript, we generalized the ideas of FTCMSs and FEHBMSs and present the approaches of pentagonal controlled fuzzy metric spaces (PCFMSs) and fuzzy controlled hexagonal metric spaces (FCHMSs). We extend the Banach contraction principle in the setting of FCHMSs. At the end, an application to dynamic market equilibrium is given to validate the main result.

2. Preliminaries

This section contains some important definitions that aid comprehension of the main section.

Definition 1 (see [2]). A binary operation $*: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a CTN if

(1) $\tau * \theta = \theta * \tau, (\forall)\, \tau, \theta \in [0, 1]$

(2) $*$ is continuous

(3) $\tau * 1 = \tau, (\forall)\, \tau \in [0, 1]$

(4) $(\tau * \theta) * \rho = \tau * (\theta * \rho), (\forall)\, \tau, \theta, \rho \in [0, 1]$

(5) If $\tau \leq \rho$ and $\theta \leq \sigma$, with $\tau, \theta, \rho, \sigma \in [0, 1]$, then $\tau * \theta \leq \rho * \sigma$

Definition 2 (see [13]). Let X be a nonempty set. A 3-tuple $(X, N, *)$ is named a FTCMS if $*$ is a CTN, N is a FS on $X \times X \times [0,\infty)$, and $Q, W, E : X \times X \longrightarrow [1,\infty)$ are noncomparable and fulfill the following assertions for all $z, d, e, \mathscr{f} \in X, z \neq e, e \neq \mathscr{f}, \mathscr{f} \neq d$, and $\alpha, \beta, \gamma \geq 0$; the following circumstances are fulfilled:

(S1) $N(z, d, 0) = 0$

(S2) $N(z, d, \alpha) = 1$ implies $z = d$

(S3) $N(z, d, \alpha) = M(d, z, \alpha)$

(S4) $N(z, d, \alpha + \beta + \gamma) \geq N(z, e, \alpha/Q(z, e)) * N(e, \mathscr{f}, \beta/W(e, \mathscr{f})) * N(\mathscr{f}, g, \gamma/E(\mathscr{f}, d))$

(S5) $N(z, d,.)$: $[0,\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{\alpha \longrightarrow \infty} N(z, d, \alpha) = 1$

Definition 3 (see [14]). Let X be a nonempty set. A 4-tuple $(X, L, *)$ is a FEHBMS, if $*$ is a CTN, L is a FS on $X \times X \times [0,\infty)$, and $Q : X \times X \longrightarrow [1,\infty)$ fulfills the following assertions for all $z, d, e, \mathscr{f}, g, \hbar \in X, z \neq e, e \neq \mathscr{f}, \mathscr{f} \neq g, g \neq \hbar, \hbar \neq d$, and $\alpha, \beta, \gamma, \delta, w \geq 0$; the following circumstances are fulfilled:

(F1) $L(z, d, 0) = 0$

(F2) $L(z, d, \alpha) = 1$ implies $z = d$

(F3) $L(z, d, \alpha) = L(d, z, \alpha)$

(F4) $L(z, d, Q(z, d)(\alpha + \beta + \gamma + \delta + w)) \geq L(z, e, \alpha) * L(e, \mathscr{f}, \beta) * L(\mathscr{f}, g, \gamma) * L(g, \hbar, \delta) * L(\hbar, d, w)$

(F5) $L(z, d,.)$: $[0,\infty) \longrightarrow [0, 1]$ is left continuous

3. Main Results

The definitions of FCHMS and PCFMS are presented in this section, as well as proofs of fixed point findings.

Definition 4. Let X be a nonempty set. A 4-tuple $(X, H, *, Q)$ is a FCHMS if $*$ is a CTN, H is a FS on $X \times X \times [0,\infty)$, and $Q : X \times X \longrightarrow [1,\infty)$ fulfills the following assertions for all $z, d, e, \mathscr{f}, g, \hbar \in X, z \neq e, e \neq \mathscr{f}, \mathscr{f} \neq g, g \neq \hbar, \hbar \neq d$, and $\alpha, \beta, \gamma, \delta, w \geq 0$; the following circumstances are fulfilled:

(T1) $H(z, d, 0) = 0$

(T2) $H(z, d, \alpha) = 1$ implies $z = d$

(T3) $H(z, d, \alpha) = M(d, z, \alpha)$

(T4) $H(z, d, \alpha + \beta + \gamma + \delta + w) \geq H(z, e, \alpha/Q(z, e)) * H(e, \mathscr{f}, \beta/Q(e, \mathscr{f})) * H(\mathscr{f}, g, \gamma/Q(\mathscr{f}, g)) * H(g, \hbar, \delta/Q(g, \hbar)) * H(\hbar, d, w/Q(\hbar, d))$

(T5) $H(z, d,.)$: $[0,\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{\alpha \longrightarrow \infty} H(z, d, \alpha) = 1$

Example 5. Let $X = \{1, 2, 3, 4, 5, 6\}$. Define $H : X \times X \times [0,\infty) \longrightarrow [0, 1]$ as

$$H(z, d, \alpha) = \left[e^{|z-d|^2/\alpha}\right]^{-1} \quad \text{for all } \alpha > 0, \tag{1}$$

with the CTN $*$ such that $\alpha_1 * \alpha_2 = \alpha_1 \alpha_2$. Then, $(X, M, *)$ is FCHMS with control functions $Q(z, d) = 1 + z + d$.

Definition 6. Let X be a nonempty set. A triplet $(X, M, *)$ is a PCFMS if $*$ is a CTN, M is a fuzzy set on $X \times X \times [0,\infty)$, and $Q, W, E, R, T : X \times X \longrightarrow [1,\infty)$ are five noncomparable functions that fulfill the following assertions for all $z, d, e, \mathscr{f}, g, \hbar \in X, z \neq e, e \neq \mathscr{f}, \mathscr{f} \neq g, g \neq \hbar, \hbar \neq d$, and $\alpha, \beta, \gamma, \delta, w \geq 0$; the following circumstances are fulfilled:

(A1) $M(z, d, 0) = 0$

(A2) $M(z, d, \alpha) = 1$ implies $z = d$

(A3) $M(z, d, \alpha) = M(d, z, \alpha)$

(A4) $M(z, d, \alpha + \beta + \gamma + \delta + w) \geq M(z, e, \alpha/Q(z, e)) * M(e, \mathscr{f}, \beta/W(e, \mathscr{f})) * M(\mathscr{f}, g, \gamma/E(\mathscr{f}, g)) * M(g, \hbar, \delta/R(g, \hbar)) * M(\hbar, d, w/T(\hbar, d))$

(A5) $M(z, d,.)$: $[0,\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{\alpha \longrightarrow \infty} M(z, d, \alpha) = 1$

Example 7. Let $X = \{1, 2, 3, 4, 5, 6\}$. Define $M : X \times X \times [0,\infty) \longrightarrow [0, 1]$ as

$$M(z, d, \alpha) = \frac{\alpha}{\alpha + |z - d|^6} \quad \text{for all } \alpha > 0, \tag{2}$$

with the CTN $*$ such that $\alpha_1 * \alpha_2 = \alpha_1\alpha_2$. Then, $(X, M, *)$ is a PCFMS with noncomparable control functions $Q(z, d) = 1 + z + d$,

$$W(z, d) = 1 + z^2 + d^2, E(z, d) = 1 + \frac{z}{d}, R(z, d) = 1 + \frac{d}{z}, T(z, d) = 1 + z^2 + d. \tag{3}$$

Remark 8. From the definition of PCFMS,

(i) If we take $W(e, f) = Q(e, f), E(f, g) = Q(f, g), R(g, h) = Q(g, h), T(h, d) = Q(h, d)$, then it will becomes the definition of FCHMS

(ii) If we take $Q(z, e) = W(e, f) = E(f, g) = R(g, h) = T(h, d) = b(z, d)$, then it will becomes the definition of FEHBMS in [14]

(iii) If $g = h = d$ and $\gamma = \delta = w = r'$ in (A4), then it will becomes FTCMS in [13]

Definition 9. Let $(X, M, *)$ be a PCFMS and $\{z_n\}$ be a sequence in X, then $\{z_n\}$ is named to be

(i) a convergent, if there exists $z \in X$ such that

$$\lim_{n \longrightarrow \infty} M(z_n, z, \alpha) = 1 \quad \text{for all } \alpha > 0 \tag{4}$$

(ii) a Cauchy, if and only if for each $\omega > 0, \alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(z_n, z_m, \alpha) \geq 1 - \omega, \quad \text{for all } n, m \geq n_0 \tag{5}$$

If every Cauchy sequence is convergent in X, then $(X, M, *)$ is a complete PCFMS.

Definition 10. Let $(X, M, *)$ be a PCFMS, then we define an open ball $B(z, r, \alpha)$ with centre z, radius $r, 0 < r < 1$, and $\alpha > 0$ as follows:

$$B(z, r, \alpha) = \{d \in X : M(z, d, \alpha) > 1 - r\}, \tag{6}$$

and the topology that corresponds to it is defined as

$$\tau_{pz} = \{D \subset X : B(z, r, \alpha) \subset D\}. \tag{7}$$

Theorem 11. *Let $(X, M, *)$ be a complete PCFMS and $Q, W, E, R, T : X \times X \longrightarrow [1, \infty)$ such that*

$$\lim_{\alpha \longrightarrow \infty} M(z, d, \alpha) = 1, \quad \text{for all } \alpha > 0, z, d \in X. \tag{8}$$

Let $F : X \longrightarrow X$ be a mapping satisfying

$$M(Fz, Fd, q\alpha) \geq M(z, d, \alpha), \quad \text{for all } \alpha > 0, z, d \in X, \tag{9}$$

where $0 < p < 1$. Furthermore, if, for $z_0 \in X$ and $n, q \in \{1, 2, 3, \cdots\}$, it holds $b(z_n, z_{n+q}) < 1/p$ where $z_n = F^n z_0$, then F has a unique fixed point.

Proof. Assume $z_0 \in X$ and construct a sequence $\{z_n\}$ by

$$z_n = Fz_{n-1} \quad \text{for all } n \in \{1, 2, 3, \cdots\}. \tag{10}$$

Without restricting generality, suppose that $z_n \neq z_{n+1}$ for all $n \in \{0, 1, 2, 3, \cdots\}$. With the help of (9), we deduce

$$M(z_n, z_{n+1}, \alpha) = M(Fz_{n-1}, Fz_n, \alpha) \geq M\left(z_{n-1}, z_n, \frac{\alpha}{p}\right) \vdots \geq M\left(z_0, z_1, \frac{\alpha}{p^n}\right). \tag{11}$$

Continuing in this way, we obtain

$$M(z_n, z_{n+2}, \alpha) \geq M\left(z_0, z_2, \frac{\alpha}{p^n}\right), \tag{12}$$

$$M(z_n, z_{n+3}, \alpha) \geq M\left(z_0, z_3, \frac{\alpha}{p^n}\right), \tag{13}$$

$$M(z_n, z_{n+4}, \alpha) \geq M\left(z_0, z_4, \frac{\alpha}{p^n}\right). \tag{14}$$

It implies, if $m = 1, 2, 3, \cdots$,

$$M(z_n, z_{n+4m+1}, \alpha) \geq M\left(z_0, z_{4m+1}, \frac{\alpha}{p^n}\right), \tag{15}$$

$$M(z_n, z_{n+4m+2}, \alpha) \geq M\left(z_0, z_{4m+2}, \frac{\alpha}{p^n}\right), \tag{16}$$

$$M(z_n, z_{n+4m+3}, \alpha) \geq M\left(z_0, z_{4m+3}, \frac{\alpha}{p^n}\right), \tag{17}$$

$$M(z_n, z_{n+4m+4}, \alpha) \geq M\left(z_0, z_{4m+4}, \frac{\alpha}{p^n}\right). \tag{18}$$

Expressing $\alpha = \alpha/5 + \alpha/5 + \alpha/5 + \alpha/5 + \alpha/5$ and by using

(11) and (A1), we obtain

$$\begin{aligned}
M(z_0, z_5, \alpha) \geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) \\
&* M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5Rb(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{5Tb(z_4, z_5)}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^2E(z_2, z_3)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^3R(z_3, z_4)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^4T(z_4, z_5)}\right).
\end{aligned} \tag{19}$$

In similar manner, we can deduce

$$\begin{aligned}
M(z_0, z_9, \alpha) \geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) \\
&* M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_9, \frac{\alpha}{5T(z_4, z_9)}\right) \\
\geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) \\
&* M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{(5)^2T(z_4, z_9)Q(z_4, z_5)}\right) \\
&* M\left(z_5, z_6, \frac{\alpha}{(5)^2T(z_4, z_9)W(z_5, z_6)}\right) \\
&* M\left(z_6, z_7, \frac{\alpha}{(5)^2T(z_4, z_9)E(z_6, z_7)}\right) \\
&* M\left(z_7, z_8, \frac{\alpha}{(5)^2T(z_4, z_9)R(z_7, z_8)}\right) \\
&* M\left(z_8, z_9, \frac{\alpha}{(5)^2T(z_4, z_9)T(z_8, z_9)}\right) \\
\geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^2E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^3R(z_3, z_4)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^4T(z_4, z_9)Q(z_4, z_5)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^5T(z_4, z_9)W(z_5, z_6)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^6T(z_4, z_9)E(z_6, z_7)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^7T(z_4, z_9)R(z_7, z_8)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^8T(z_4, z_9)T(z_8, z_9)}\right).
\end{aligned} \tag{20}$$

We obtain for each $m = 1, 2, 3, \cdots$,

$$\begin{aligned}
M(z_0, z_{4m+1}, \alpha) \geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_{4m+1}, \frac{\alpha}{5T(z_4, z_{4m+1})}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})Q(z_4, z_5)}\right) * M\left(z_5, z_6, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})W(z_5, z_6)}\right) \\
&* M\left(z_6, z_7, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})E(z_6, z_7)}\right) * M\left(z_7, z_8, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})R(z_7, z_8)}\right) \\
&* M\left(z_8, z_{4m+1}, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})T(z_8, z_{4m+1})}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})Q(z_4, z_5)}\right) * M\left(z_5, z_6, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})W(z_5, z_6)}\right) \\
&* M\left(z_6, z_7, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})E(z_6, z_7)}\right) * M\left(z_7, z_8, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})R(z_7, z_8)}\right) \\
&* M\left(z_8, z_9, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})Q(z_8, z_9)}\right) \\
&* M\left(z_9, z_{10}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})W(z_9, z_{10})}\right) \\
&* M\left(z_{10}, z_{11}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})E(z_{10}, z_{11})}\right) \\
&* M\left(z_{11}, z_{12}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})R(z_{11}, z_{12})}\right) \\
&* M\left(z_{12}, z_{4m+1}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})T(z_{12}, z_{4m+1})}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})Q(z_4, z_5)}\right) * M\left(z_5, z_6, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})W(z_5, z_6)}\right) \\
&* M\left(z_6, z_7, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})E(z_6, z_7)}\right) * M\left(z_7, z_8, \frac{\alpha}{(5)^2T(z_4, z_{4m+1})R(z_7, z_8)}\right) \\
&* M\left(z_8, z_9, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})Q(z_8, z_9)}\right) \\
&* M\left(z_9, z_{10}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})W(z_9, z_{10})}\right) \\
&* M\left(z_{10}, z_{11}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})E(z_{10}, z_{11})}\right) \\
&* M\left(z_{11}, z_{12}, \frac{\alpha}{(5)^3T(z_4, z_{4m+1})T(z_8, z_{4m+1})R(z_{11}, z_{12})}\right) \\
&* M\left(z_{12}, z_{13}, \frac{\alpha}{(5)^4T(z_4, z_{4m+1})T(z_8, z_{4m+1})T(z_{12}, z_{4m+1})Q(z_{12}, z_{13})}\right) * \vdots \\
&* M\left(z_{4m}, z_{4m+1}, \frac{\alpha}{(5)^mT(z_4, z_{4m+1})T(z_8, z_{4m+1})T(z_{12}, z_{4m+1}) \cdots T(z_{4m-4}, z_{4m+1})T(z_{4m}, z_{4m+1})}\right) \\
\geq{}& M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^2E(z_2, z_3)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^3R(z_3, z_4)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2p^4T(z_4, z_{4m+1})Q(z_4, z_5)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^5T(z_4, z_{4m+1})W(z_5, z_6)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2p^6T(z_4, z_{4m+1})E(z_6, z_7)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2p^7T(z_4, z_{4m+1})R(z_7, z_8)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^3p^8T(z_4, z_{4m+1})T(z_8, z_{4m+1})Q(z_8, z_9)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3p^9T(z_4, z_{4m+1})T(z_8, z_{4m+1})W(z_9, z_{10})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3p^{10}T(z_4, z_{4m+1})T(z_8, z_{4m+1})E(z_{10}, z_{11})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3p^{11}T(z_4, z_{4m+1})T(z_8, z_{4m+1})R(z_{11}, z_{12})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4p^{12}T(z_4, z_{4m+1})T(z_8, z_{4m+1})T(z_{12}, z_{4m+1})Q(z_{12}, z_{13})}\right) * \vdots \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^mp^{4m}T(z_4, z_{4m+1})T(z_8, z_{4m+1})T(z_{12}, z_{4m+1}) \cdots T(z_{4m-4}, z_{4m+1})T(z_{4m}, z_{4m+1})}\right).
\end{aligned} \tag{21}$$

Now, using (15), we deduce that

$$\begin{aligned}
M(z_n, z_{n+4m+1}, \alpha) &\geq M\left(z_0, z_{4m+1}, \frac{\alpha}{p^n}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5p^n Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+1} W(z_1, z_2)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^{n+2} E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+3} R(z_3, z_4)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+4} T(z_4, z_{4m+1}) Q(z_4, z_5)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+5} T(z_4, z_{4m+1}) W(z_5, z_6)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+6} T(z_4, z_{4m+1}) E(z_6, z_7)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+7} T(z_4, z_{4m+1}) R(z_7, z_8)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+8} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) Q(z_8, z_9)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+9} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) W(z_9, z_{10})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+10} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) E(z_{10}, z_{11})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+11} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) R(z_{11}, z_{12})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4 p^{n+12} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) T(z_{12}, z_{4m+1}) Q(z_{12}, z_{13})}\right) * \vdots \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^m p^{n+4m} T(z_4, z_{4m+1}) T(z_8, z_{4m+1}) T(z_{12}, z_{4m+1}) \cdots T(z_{4m-4}, z_{4m+1}) T(z_{4m}, z_{4m+1})}\right).
\end{aligned} \tag{22}$$

Furthermore, from (11) and (12), we can obtain

$$\begin{aligned}
M(z_0, z_6, \alpha) &\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) \\
&* M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_6, \frac{\alpha}{5T(z_4, z_6)}\right) \\
&\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^2 E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^3 R(z_3, z_4)}\right) \\
&* M\left(z_0, z_2, \frac{\alpha}{5p^4 T(z_4, z_6)}\right).
\end{aligned} \tag{23}$$

In similar manner, we can deduce

$$\begin{aligned}
M(z_0, z_{10}, \alpha) &\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_{10}, \frac{\alpha}{5T(z_4, z_{10})}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) * M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) \\
&* M\left(z_4, z_5, \frac{\alpha}{(5)^2 T(z_4, z_{10}) Q(z_4, z_5)}\right) * M\left(z_5, z_6, \frac{\alpha}{(5)^2 T(z_4, z_{10}) W(z_5, z_6)}\right) \\
&* M\left(z_6, z_7, \frac{\alpha}{(5)^2 T(z_4, z_{10}) E(z_6, z_7)}\right) * M\left(z_7, z_8, \frac{\alpha}{(5)^2 T(z_4, z_{10}) R(z_7, z_8)}\right) \\
&* M\left(z_8, z_{10}, \frac{\alpha}{(5)^2 T(z_4, z_9) T(z_8, z_{10})}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^2 E(z_2, z_3)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^3 R(z_3, z_4)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^4 T(z_4, z_9) Q(z_4, z_5)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^5 T(z_4, z_9) W(z_5, z_6)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^6 T(z_4, z_9) E(z_6, z_7)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^7 T(z_4, z_9) R(z_7, z_8)}\right) * M\left(z_0, z_2, \frac{\alpha}{(5)^2 p^8 T(z_4, z_9) T(z_8, z_{10})}\right).
\end{aligned} \tag{24}$$

We obtain for each $m = 1, 2, 3, \cdots$,

$$\begin{aligned}
M(z_0, z_{4m+2}, \alpha) &\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_{4m+2}, \frac{\alpha}{5T(z_4, z_{4m+2})}\right) \\
&\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_5, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) Q(z_4, z_5)}\right) \\
&* M\left(z_5, z_6, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) W(z_5, z_6)}\right) * M\left(z_6, z_7, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) E(z_6, z_7)}\right) \\
&* M\left(z_7, z_8, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) R(z_7, z_8)}\right) * M\left(z_8, z_{4m+2}, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) T(z_8, z_{4m+2})}\right) \\
&\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_5, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) Q(z_4, z_5)}\right) \\
&* M\left(z_5, z_6, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) W(z_5, z_6)}\right) * M\left(z_6, z_7, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) E(z_6, z_7)}\right) \\
&* M\left(z_7, z_8, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) R(z_7, z_8)}\right) * M\left(z_8, z_9, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) Q(z_8, z_9)}\right) \\
&* M\left(z_9, z_{10}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) W(z_9, z_{10})}\right) \\
&* M\left(z_{10}, z_{11}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) E(z_{10}, z_{11})}\right) \\
&* M\left(z_{11}, z_{12}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) R(z_{11}, z_{12})}\right) \\
&* M\left(z_{12}, z_{4m+2}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2})}\right) \\
&\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_1, z_2, \frac{\alpha}{5W(z_1, z_2)}\right) * M\left(z_2, z_3, \frac{\alpha}{5E(z_2, z_3)}\right) \\
&* M\left(z_3, z_4, \frac{\alpha}{5R(z_3, z_4)}\right) * M\left(z_4, z_5, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) Q(z_4, z_5)}\right) \\
&* M\left(z_5, z_6, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) W(z_5, z_6)}\right) * M\left(z_6, z_7, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) E(z_6, z_7)}\right) \\
&* M\left(z_7, z_8, \frac{\alpha}{(5)^2 T(z_4, z_{4m+2}) R(z_7, z_8)}\right) * M\left(z_8, z_9, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) Q(z_8, z_9)}\right) \\
&* M\left(z_9, z_{10}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) W(z_9, z_{10})}\right) \\
&* M\left(z_{10}, z_{11}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) E(z_{10}, z_{11})}\right) \\
&* M\left(z_{11}, z_{12}, \frac{\alpha}{(5)^3 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) R(z_{11}, z_{12})}\right) \\
&* M\left(z_{12}, z_{13}, \frac{\alpha}{(5)^4 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) Q(z_{12}, z_{13})}\right) * \vdots \\
&* M\left(z_{4m}, z_{4m+2}, \frac{\alpha}{(5)^m T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) \cdots T(z_{4m-4}, z_{4m+2}) T(z_{4m}, z_{4m+2})}\right) \\
&\geq M\left(z_0, z_1, \frac{\alpha}{5Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5pW(z_1, z_2)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^2 E(z_2, z_3)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{5p^3 R(z_3, z_4)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^4 T(z_4, z_{4m+2}) Q(z_4, z_5)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^5 T(z_4, z_{4m+2}) W(z_5, z_6)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^6 T(z_4, z_{4m+2}) E(z_6, z_7)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^7 T(z_4, z_{4m+2}) R(z_7, z_8)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^8 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) Q(z_8, z_9)}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^9 T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) W(z_9, z_{10})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{10} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) E(z_{10}, z_{11})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{11} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) R(z_{11}, z_{12})}\right) \\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4 p^{12} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) Q(z_{12}, z_{13})}\right) * \vdots \\
&* M\left(z_0, z_2, \frac{\alpha}{(5)^m p^{4m} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) \cdots T(z_{4m-4}, z_{4m+2}) T(z_{4m}, z_{4m+2})}\right).
\end{aligned} \tag{25}$$

Now, using (16), we deduce that

$$\begin{aligned}M(z_n, z_{n+4m+2}, \alpha) &\geq M\left(z_0, z_{4m+2}, \frac{\alpha}{p^n}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5p^n Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+1} W(z_1, z_2)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{5p^{n+2} E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+3} R(z_3, z_4)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+4} T(z_4, z_{4m+2}) Q(z_4, z_5)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+5} T(z_4, z_{4m+2}) W(z_5, z_6)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+6} T(z_4, z_{4m+2}) E(z_6, z_7)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+7} T(z_4, z_{4m+2}) R(z_7, z_8)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+8} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) Q(z_8, z_9)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+9} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) W(z_9, z_{10})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+10} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) E(z_{10}, z_{11})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+11} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) R(z_{11}, z_{12})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4 p^{n+12} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) Q(z_{12}, z_{13})}\right) * \vdots\\
&* M\left(z_0, z_2, \frac{\alpha}{(5)^m p^{n+4m} T(z_4, z_{4m+2}) T(z_8, z_{4m+2}) T(z_{12}, z_{4m+2}) \cdots T(z_{4m-4}, z_{4m+2}) T(z_{4m}, z_{4m+2})}\right).\end{aligned} \tag{26}$$

Accordingly, from (27) and (28) can be deduced as

$$\begin{aligned}M(z_n, z_{n+4m+3}, \alpha) &\geq M\left(z_0, z_{4m+3}, \frac{\alpha}{p^n}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5p^n Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+1} W(z_1, z_2)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{5p^{n+2} E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+3} R(z_3, z_4)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+4} T(z_4, z_{4m+3}) Q(z_4, z_5)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+5} T(z_4, z_{4m+3}) W(z_5, z_6)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+6} T(z_4, z_{4m+3}) E(z_6, z_7)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+7} T(z_4, z_{4m+3}) R(z_7, z_8)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+8} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) Q(z_8, z_9)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+9} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) W(z_9, z_{10})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+10} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) E(z_{10}, z_{11})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+11} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) R(z_{11}, z_{12})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4 p^{n+12} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) T(z_{12}, z_{4m+3}) Q(z_{12}, z_{13})}\right) * \vdots\\
&* M\left(z_0, z_3, \frac{\alpha}{(5)^m p^{n+4m} T(z_4, z_{4m+3}) T(z_8, z_{4m+3}) T(z_{12}, z_{4m+3}) \cdots T(z_{4m-4}, z_{4m+3}) T(z_{4m}, z_{4m+3})}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}M(z_n, z_{n+4m+4}, \alpha) &\geq M\left(z_0, z_{4m+4}, \frac{\alpha}{p^n}\right) \geq M\left(z_0, z_1, \frac{\alpha}{5p^n Q(z_0, z_1)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+1} W(z_1, z_2)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{5p^{n+2} E(z_2, z_3)}\right) * M\left(z_0, z_1, \frac{\alpha}{5p^{n+3} R(z_3, z_4)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+4} T(z_4, z_{4m+4}) Q(z_4, z_5)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+5} T(z_4, z_{4m+4}) W(z_5, z_6)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+6} T(z_4, z_{4m+4}) E(z_6, z_7)}\right) * M\left(z_0, z_1, \frac{\alpha}{(5)^2 p^{n+7} T(z_4, z_{4m+4}) R(z_7, z_8)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+8} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) Q(z_8, z_9)}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+9} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) W(z_9, z_{10})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+10} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) E(z_{10}, z_{11})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^3 p^{n+11} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) R(z_{11}, z_{12})}\right)\\
&* M\left(z_0, z_1, \frac{\alpha}{(5)^4 p^{n+12} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) T(z_{12}, z_{4m+4}) Q(z_{12}, z_{13})}\right) * \vdots\\
&* M\left(z_0, z_4, \frac{\alpha}{(5)^m p^{n+4m} T(z_4, z_{4m+4}) T(z_8, z_{4m+4}) T(z_{12}, z_{4m+4}) \cdots T(z_{4m-4}, z_{4m+4}) T(z_{4m}, z_{4m+4})}\right).\end{aligned} \tag{28}$$

Therefore, for each q and from inequalities (22)–(28), we obtain

$$\lim_{n \longrightarrow \infty} M(z_n, z_{n+q}, \alpha) = 1 * 1 * 1 * \cdots * 1 = 1, \tag{29}$$

as $b(z_n, z_{n+q}) < 1/p$ for all $n, q \in \mathbb{N}$ and $p \in (0, 1)$, i.e., z_n is a Cauchy sequence in X. Since $(X, M, *)$ is complete, there exists $z \in X$ such that $z_n \longrightarrow z$ as $n \longrightarrow \infty$. We assert that z is the fixed point of F. From equation (29) and condition (A4), we obtain

$$\begin{aligned}M(z, Fz, \alpha) &\geq M\left(z, z_n, \frac{\alpha}{5Q(z, z_n)}\right) * M\left(Fz_{n-1}, Fz_n, \frac{\alpha}{5W(z_n, z_{n+1})}\right)\\
&* M\left(Fz_n, Fz_{n+1}, \frac{\alpha}{5E(z_{n+1}, z_{n+2})}\right)\\
&\geq M\left(z, z_n, \frac{\alpha}{5Q(z, z_n)}\right) * M\left(z_{n-1}, z_n, \frac{\alpha}{5pW(z_n, z_{n+1})}\right)\\
&* M\left(z_n, z_{n+1}, \frac{\alpha}{5pE(z_{n+1}, z_{n+2})}\right) * M\left(z_{n+1}, z_{n+2}, \frac{\alpha}{5pR(z_{n+2}, z_{n+3})}\right)\\
&* M\left(z_{n+2}, z, \frac{\alpha}{5pT(z_{n+3}, Fz)}\right).\end{aligned} \tag{30}$$

Letting $n \longrightarrow \infty$ in the above inequality, we get $Fz = z$, i.e., z is a fixed point of F. By using the inequality (9), we can easily examine that z is a unique fixed point of F. □

Theorem 12. *Let $(X, M, *, Q)$ be a complete FCHMS and $Q : X \times X \longrightarrow [1, \infty)$ such that*

$$\lim_{\alpha \longrightarrow \infty} M(z, d, \alpha) = 1, \quad \text{for all } \alpha > 0, z, d \in X. \tag{31}$$

Let $F : X \longrightarrow X$ be a mapping satisfying

$$M(Fz, Fd, q\alpha) \geq M(z, d, \alpha), \quad \text{for all } \alpha > 0, z, d \in X, \tag{32}$$

where $0 < p < 1$. Furthermore, if, for $z_0 \in X$ and $n, q \in \{1, 2, 3, \cdots\}$, it holds $b(z_n, z_{n+q}) < 1/p$ where $z_n = F^n z_0$, then F has a unique fixed point.

Proof. It is easy to prove on the lines of Theorem 11. □

Example 13. Let $X = [0, 1]$. Define $M : X \times X \times [0, \infty) \longrightarrow [0, 1]$ as

$$M(z, d, \alpha) = e^{-\left(|z-d|^{\hbar}/\alpha\right)} \quad \text{for all } \alpha > 0, \hbar \geq 1, \tag{33}$$

with the CTN $*$ such that $\alpha_1 * \alpha_2 = \alpha_1 \alpha_2$. Then, $(X, M, *)$ is a complete PCFMS with noncomparable control functions $Q(z, d) = 1 + z + d$, $W(z, d) = 1 + z^2 + d^2$, $E(z, d) = 1 + (z/d)$, $R(z, d) = 1 + d/z$, and $T(z, d) = 1 + z^2 + d$.

Define $F : X \longrightarrow X$ by $Fz = z/6$. Consider

$$\begin{aligned}M(Fz, Fd, q\alpha) &= M\left(\frac{z}{6}, \frac{d}{6}, q\alpha\right) = e^{-\left(|z/6 - d/6|^{\hbar}/\alpha\right)} = e^{-\left(|z-d|^{\hbar}/6^{\hbar}\alpha\right)}\\
&\geq e^{-\left(|z-d|^{\hbar}/\alpha\right)} = M(z, d, \alpha).\end{aligned} \tag{34}$$

Hence, by Theorem 11, F has unique fixed point, which is 0.

4. Application

Now, we show how our established result can be used to find the unique solution to an integral equation in dynamic market equilibrium economics. For supply Q_β and demand Q_d, in many markets, current prices and pricing trends (whether prices are rising or dropping and whether they are rising or falling at an increasing or decreasing rate) have an impact. The economist, therefore, wants to know what the current price is $P(\alpha)$, the first derivative $dP(\alpha)/d\alpha$, and the second derivative $d^2P(\alpha)/d\alpha^2$. Assume

$$\begin{aligned} Q_\beta &= g_1 + \gamma_1 P(\alpha) + e_1 \frac{dP(\alpha)}{d\alpha} + z_1 \frac{d^2P(\alpha)}{d\alpha^2}, \\ Q_d &= g_2 + \gamma_2 P(\alpha) + e_2 \frac{dP(\alpha)}{d\alpha} + z_2 \frac{d^2P(\alpha)}{d\alpha^2}. \end{aligned} \tag{35}$$

$g_1, g_2, \gamma_1, \gamma_2, e_1$, and e_2 are constants. If pricing clears the market at each point in time, we comment on the dynamic stability of the market. In equilibrium, $Q_\beta = Q_d$. So,

$$g_1 + \gamma_1 P(\alpha) + e_1 \frac{dP(\alpha)}{d\alpha} + z_1 \frac{d^2P(\alpha)}{d\alpha^2} = g_2 + \gamma_2 P(\alpha) + e_2 \frac{dP(\alpha)}{d\alpha} + z_2 \frac{d^2P(\alpha)}{d\alpha^2}. \tag{36}$$

Since

$$(z_1 - z_2)\frac{d^2P(\alpha)}{d\alpha^2} + (e_1 - e_2)d\frac{dP(\alpha)}{d\alpha} + (\gamma_1 - \gamma_2)P(\alpha) = -(g_1 - g_2). \tag{37}$$

Letting $z = z_1 - z_2, e = e_1 - e_2, \gamma = \gamma_1 - \gamma_2$, and $g = g_1 - g_2$ in above, we have

$$z\frac{d^2P(\alpha)}{d\alpha^2} + e\frac{dP(\alpha)}{d\alpha} + \gamma P(\alpha) = -g. \tag{38}$$

Dividing through by z, $P(\alpha)$ is governed by the following initial value problem

$$\begin{cases} P'' + \dfrac{e}{z}P'' + \dfrac{\gamma}{z}P(\alpha) = -\dfrac{g}{z}, \\ P(0) = 0, \\ P'(0) = 0, \end{cases} \tag{39}$$

where $e^2/z = 4\gamma/z$ and $\gamma/e = \mu$ is a continuous function. It is easy to show that problem (39) is equivalent to the integral equation:

$$P(\alpha) = \int_0^T \xi(\alpha, r) F(\alpha, r, P(r))\, dr, \tag{40}$$

where $\xi(\alpha, r)$ is Green's function given by

$$\xi(\alpha, r) = \begin{cases} re^{(\mu/2)(\alpha - r)} & \text{if } 0 \le r \le \alpha \le T, \\ \alpha e^{(\mu/2)(r-\alpha)} & \text{if } 0 \le \alpha \le r \le \alpha \le T. \end{cases} \tag{41}$$

We will show the existence of a solution to the integral equation:

$$P(\alpha) = \int_0^T G(\alpha, r, P(r)) dr. \tag{42}$$

Let $X = C([0, T])$ set of real continuous functions defined on $[0, T]$ for $\alpha > 0$, we define

$$M(\hbar, z, \alpha) = \begin{cases} 0, & \text{if } \alpha = 0, \\ \sup_{\alpha \epsilon [0,T]} \dfrac{\min\{\hbar, z\} + \alpha}{\max\{\hbar, z\} + \alpha}, & \text{otherwise.} \end{cases} \tag{43}$$

For all $\hbar, z \,\epsilon\, X$ with the CTN " $*$" such that $\alpha_1 * \alpha_2 = \alpha_1\alpha_2$, define $Q, W, E, R, T : X \times X \longrightarrow [1,\infty)$as

$$Q(z, d) = 1 + z + d,$$

$$\begin{aligned} W(z, d) &= 1 + z^2 + d^2, E(z, d) = 1 + \frac{z}{d}, R(z, d) \\ &= 1 + \frac{d}{z}, T(z, d) = 1 + z^2 + d. \end{aligned} \tag{44}$$

It is easy to prove that $(X, M, *)$ is a complete PCFMS, and $F : X \longrightarrow X$ is defined by

$$FP(\alpha) = \int_0^T G(\alpha, r, P(r) dr. \tag{45}$$

Theorem 14. *Consider equation (42) and suppose that*

(i) $G : [0, T] \times [0, T] \longrightarrow R^+$ *is a continuous function*

(ii) *There exists a continuous function* $\xi : [0, T] \times [0, T] \longrightarrow R^+$ *such that* $\sup_{\alpha \epsilon [0,T]} \int_0^T \xi(\alpha, r) dr \ge 1$

(iii) $\max\{G(\alpha, r, \hbar(r)) - G(\alpha, r, z(r))\} \ge \xi(\alpha, r) \max\{\hbar(r), z(r)\}$ *and* $\min\{G(\alpha, r, \hbar(r)) - G(\alpha, r, z(r))\} \ge \xi(\alpha, r) \min\{\hbar(r), z(r)\}$

Then, the integral equation (42) has a unique solution.

Proof. For $\hbar, z \in X$, by use of assumptions (i) to (iii), we have

$$\begin{aligned} M(F\hbar, Fz, \alpha) &= \sup_{\alpha\epsilon[0,T]} \frac{\min\left\{\int_0^T G(\alpha, r, \hbar(r))\,dr, \int_0^T G(\alpha, r, z(r))\,dr\right\} + \alpha}{\max\left\{\int_0^T G(\alpha, r, \hbar(r))\,dr, \int_0^T G(\alpha, r, z(r))\,dr\right\} + \alpha} \\ &= \sup_{\alpha\epsilon[0,T]} \frac{\int_0^T \min\{G(\alpha, r, \hbar(r)), G(\alpha, r, z(r))\}dr + \alpha}{\int_0^T \max\{G(\alpha, r, \hbar(r)), G(\alpha, r, z(r))\}dr + \alpha} \\ &\geq \sup_{\alpha\epsilon[0,T]} \frac{\int_0^T \xi(\alpha, r)\min\{\hbar(r), z(r)\}dr + \alpha}{\int_0^T \xi(\alpha, r)\max\{\hbar(r), z(r)\}dr + \alpha} \\ &\geq \sup_{\alpha\epsilon[0,T]} \frac{\min\{\hbar(r), z(r)\}\int_0^T \xi(\alpha, r)dr + \alpha}{\max\{\hbar(r), z(r)\}\int_0^T \xi(\alpha, r)dr + \alpha} \\ &\geq \left(\frac{\min\{\hbar(r), z(r)\} + \alpha}{\max\{\hbar(r), z(r)\} + \alpha}\right) = M(\hbar, z, \alpha). \end{aligned} \tag{46}$$

Thus, $M(F\hbar, Fz, q\alpha) \geq M(\hbar, z, \alpha)$ for all $\hbar, z \in X$, and all conditions of Theorem 11 are satisfied. Therefore, equation (42) has a unique fixed point. □

5. Conclusion

The aim of this study is to present the notions of FEHMSs and PCFMS and prove Banach fixed point theorems in these spaces; nontrivial examples and an application to integral equation are also given to support our results. Due to a diverse range of applications of metric fixed point theory in mathematics, science, and economics, it is researched widely. Different types of fixed point results for single- and multivalued mappings can be proven in the sense of the above defined notions in this manuscript.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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