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Research Article

A New Iterative Method for the Approximate Solution of Klein-Gordon and Sine-Gordon Equations

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This article presents a new iterative method (NIM) for the investigation of the approximate solution of the Klein-Gordon and sine-Gordon equations. This approach is formulated on the combination of the Mohand transform and the homotopy perturbation method. Mohand transform (MT) is capable to handle the linear terms only, thus we introduce homotopy perturbation method (HPM) to tackle the nonlinear terms. This NIM derives the results in the form of a series solution. The proposed method emphasizes the stability of the derived solutions without any linearization, discretization, or hypothesis. Graphical representation and absolute error demonstrate the efficiency and authenticity of this scheme. Some numerical models are illustrated to show the compactness and reliability of this strategy.

1. Introduction

Many linear and nonlinear phenomena appear in several areas of scientific fields like physics, chemistry and biology can be modeled by different type of partial differential equation [1–4]. A broad class of analytical methods and numerical methods have been introduced such as (G'/G)-expansion method [5], Exp-function method [6], Homotopy perturbation method [7], Homotopy analysis method [8], Laplace transform [9], Residual power series [10], Quasi wavelet method [11], Fourier series [12], Chebyshev-Tau method [13], Haar wavelets method [14], trial equation method [15] and Two scale approach [16] to handle these linear and nonlinear PDEs but to reach exact solutions is not an easy way. In past few decades, The Klein-Gordon and sine-Gordon equations are a type of hyperbolic partial differential equation which are often used to describe and simulate the physical phenomena in a variety of fields of engineering and science, i.e., physics, fluid dynamics, mathematical biology and quantum mechanics. Let us consider the Klein-Gordon and sine-Gordon [17],

$$\mathfrak{J}_{\eta\eta}(\xi,\eta) - \mathfrak{J}_{\xi\xi}(\xi,\eta) + c_1\mathfrak{J}(\xi,\eta) + c_2 G(\mathfrak{J}(\xi,\eta)) = f_1(\xi,\eta), \tag{1}$$

$$\mathfrak{J}_{\eta\eta}(\xi,\eta) - \mathfrak{J}_{\xi\xi}(\xi,\eta) + c_3 \sin(\mathfrak{J}(\xi,\eta)) = f_2(\xi,\eta), \tag{2}$$

where $\mathfrak{J}$ is a function of ξ and η; G is a nonlinear function, f_1 and f_2 are known analytic functions whereas c_1, c_2 and c_3 are constants.

In recent years, The Klein-Gordon and sine-Gordon equations have attracted more attention from the scientists due to its applications in plasma, nonlinear wave equations, studying the solutions and condensed matter physics and relativistic physics as a model of dispersive phenomena. Yousif and Mahmood [18] used variational iteration method coupled with homotopy perturbation method to investigate the approximate solution of Klein-Gordon an sine-Gordon equations. Nadeem and Li [17] applied the He-Laplace method to obtain the analytical solution of Klein-Gordon an sine-Gordon equations. Liu.et al. [19] employed Yang transformation for the approximate solution of Klein-Gordon an sine-Gordon equations. Agom and Ogunfiditimi [20] utilized modified Adomian decomposition method for nonlinear Klein-Gordon equations with quadratic nonlinearity. Ikram et. al [21] obtained the approximate solution of linear Klein-Gordon equations using Haar wavelet approach. Lotfi and Alipanah [22] used Legendre spectral element method for solving sine-Gordon equation. Lu [23] applied modified homotopy perturbation method for the solution of sine-Gordon equation. Many authors applied various approaches to investigate the approximate solution of the Klein-Gordon and sine-Gordon equations [24–26].

The homotopy perturbation method (HPM) was first developed by a Chinese mathematician He [27, 28] to present the analytical solution of linear and nonlinear partial differential equation. Later, Nadeem and Li [29] combined HPM with Laplace transform for solving nonlinear vibration systems and nonlinear wave equations to show the accuracy and validity of HPM. Khan and Qingbiao [30] used HPM using He's polynomials for the solution of nonlinear equations. Many authors have performed the accuracy of HPM for different system of PDEs. [31–34].

The main purpose of this paper is to develop a new iterative method (NIM) where Mohand transform is combined with homotopy perturbation method for obtaining the approximate solution of Klein-Gordon and sine-Gordon equations. This scheme derive the results in aspect of series without any linearization, variation and limiting expectations. In addition, this study is organized as follow: In Section 2, we present some basic definitions of Mohand transform. In Section 3, we formulate the idea of new iterative method (NIM) for obtaining the solution of illustrated problems. In Section 4, we executed NIM for finding the approximate solution of the problems to show the accuracy and validity of this approach. Finally, we present some results and discussion in Section 5 and conclusion in Section 6.

2. Fundamentals Concepts of Mohand Transform

In this section, we introduce some basic definitions and preliminaries concepts of Mohand transform which reveals the idea of its implementations to functions.

Definition 1. Let $\mathfrak{I}(\eta)$ be a function precise for $\eta \geq 0$ [17], then

$$\mathscr{L}[\mathfrak{I}(\eta)] = V(\theta) = \int_0^\infty \mathfrak{I}(\eta) e^{-\theta\eta} d\eta, \tag{3}$$

is said to be Laplace transform, where η is function (i.e. a function of time domain), defined on $[0, \infty)$ to a function of θ (i.e. of frequency domain).

Definition 2. If $V(\theta)$ symbolizes the Laplace transform of $\mathfrak{I}(\eta)$, then

$$\mathfrak{I}(\eta) = \mathscr{L}^{-1} V(\theta), \tag{4}$$

is termed as inverse Laplace transform of $V(\theta)$.

Definition 3. Mohand and Mahgoub [35] presented a new scheme Mohand transform $M(.)$ in order to gain the results of ordinary differential equations and is defined as

$$M\{\mathfrak{I}(\eta)\} = R(\theta) = \theta^2 \int_0^\infty \mathfrak{I}(\eta) e^{-\theta\eta} dt, k_1 \leq \theta \leq k_2. \tag{5}$$

On the other hand, if $R(\theta)$ is the Mohand transform of a function $\mathfrak{I}(\eta)$, then $\mathfrak{I}(\eta)$ is the inverse of $R(\theta)$ such as

$$M^{-1}\{R(\theta)\} = \mathfrak{I}(\eta), M^{-1} \text{ is inverse Mohand operator.} \tag{6}$$

Definition 4. If $\mathfrak{I}(\eta) = \eta^m$,

$$R(\theta) = \frac{m!}{\theta^{m-1}}. \tag{7}$$

Definition 5. If $M\{\mathfrak{I}(\eta)\} = R(\theta)$ *,then it has the following* differential properties [36]

(i) $M\{\mathfrak{I}'(\eta)\} = \theta R(\theta) - \theta^2 \mathfrak{I}(0)$

(ii) $M\{\mathfrak{I}''(\eta)\} = \theta^2 R(\theta) - \theta^3 F(0) - \theta^2 \mathfrak{I}'(0)$

(iii) $M\{\mathfrak{I}^m(\eta)\} = \theta^m R(\theta) - \theta^{m+1} \mathfrak{I}(0) - \theta^m \mathfrak{I}'(0) - \cdots - \theta^m \mathfrak{I}^{m-1}(0)$

3. Formulation of New Iterative Method (NIM)

This segment presents the construction of new iterative method (NIM) for obtaining the approximate solution of Klein-Gordon and sine-Gordon equations. Let us consider a nonlinear second order differential equation of the form,

$$\mathfrak{I}_{\eta\eta}(\xi, \eta) + \mathfrak{I}_\eta(\xi, \eta) + \mathfrak{I}(\xi, \eta) + g(\mathfrak{I}) = g(\xi, \eta), \tag{8}$$

with the following conditions

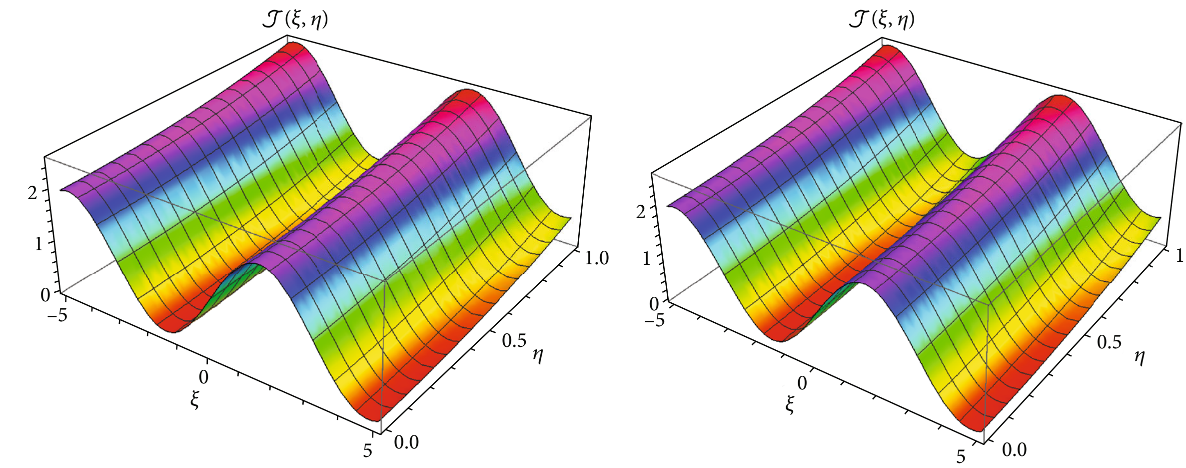
$$\mathfrak{I}(\xi, 0) = a_1, \mathfrak{I}_\eta(\xi, 0) = a_2, \tag{9}$$

where $\mathfrak{I}$ is a function in time domain η, $g(\mathfrak{I})$ represents nonlinear term, $g(\eta)$ is a source term whereas a_1 and a_2 are constants. Rewrite Eq. (8) again

$$\mathfrak{I}_{\eta\eta}(\xi, \eta) + \mathfrak{I}_\eta(\xi, \eta) = -\mathfrak{I}(\xi, \eta) - g(\mathfrak{I}) + g(\xi, \eta). \tag{10}$$

Now, taking MT on both sides of Eq. (10), we obtain

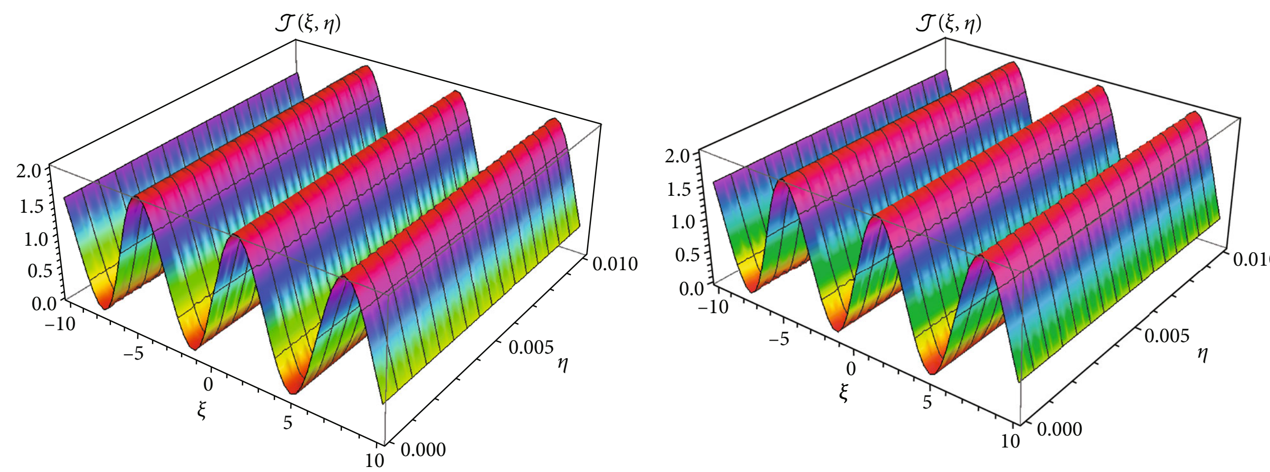
$$M\left[\mathfrak{I}_{\eta\eta}(\xi, \eta) + \mathfrak{I}_\eta(\xi, \eta)\right] = M[-\mathfrak{I}(\eta) - g(\mathfrak{I}) + g(\eta)]. \tag{11}$$



(a) Surface plot for exact solution

(b) Surface plot for approximate solution

Figure 1: Surfaces plots for the linear Klein-Gordon equation.



(a) urface plot for exact solution

(b) Surface plot for approximate solution

Figure 2: Surfaces plots for the nonlinear Klein-Gordon equation.

Table 1: The absolute error of $\mathfrak{I}(\xi,\eta)$ for different values of ξ at $\eta = 0.01$.

ξ	Exact solution	Approximate solution	Absolute error
0.5	1.47948	1.47948	0.0000
1	1.84152	1.84152	0.0000
1.5	1.99754	1.99754	0.0000
2	1.90935	1.90935	0.0000
2.5	1.59852	1.59852	0.0000
3	1.14117	1.14117	0.0000
3.5	0.649267	0.649267	0.0000
4	0.243248	0.243248	0.0000
4.5	0.0225199	0.0225199	0.0000
5	0.0411257	0.0411257	0.0000

Table 2: The absolute error of $\mathfrak{I}(\xi,\eta)$ for different values of ξ at $\eta = 0.01$.

ξ	Exact solution	Approximate solution	Absolute error
0.5	1.47929	1.47951	0.00022
1	1.84126	1.8416	0.00034
1.5	1.99725	1.99764	0.00039
2	1.90907	1.90943	0.00036
2.5	1.59831	1.59857	0.00026
3	1.14105	1.14118	0.00013
3.5	0.649213	0.649255	0.000042
4	0.243232	0.243238	6×10^{-6}
4.5	0.0225187	0.0225188	1×10^{-7}
5	0.0411236	0.0411238	2×10^{-7}

TABLE 3: The absolute error of $\mathfrak{I}(\xi, \eta)$ for different values of η.

η	Exact solution	Approximate solution	Absolute error
0.1	1.57580	1.57582	0.00022
0.2	1.5908	1.59088	0.00034
0.3	1.61579	1.61595	0.00039
0.4	1.65078	1.65094	0.00036
0.5	1.69573	1.69569	0.00026
0.6	1.7506	1.74993	0.00067
0.7	1.81531	1.81327	0.00204
0.8	1.88971	1.88515	0.00456
0.9	1.9736	1.96485	0.00875
1.0	2.06668	2.05142	0.00875

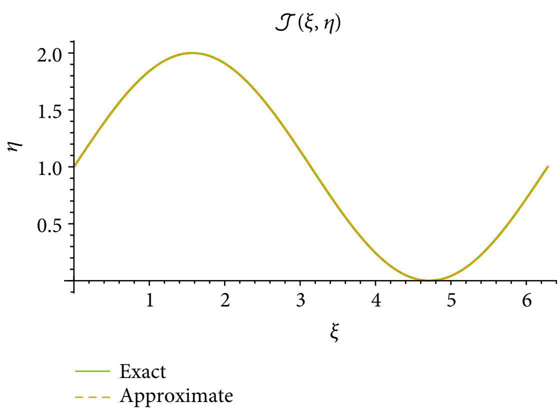


FIGURE 3: 2D Plot for $\mathfrak{I}(\eta)$ with various parameter of η.

Applying the differential properties of MT, we get

$$\theta^2 R[\theta] - \theta^3 \mathfrak{I}(\xi, 0) - \theta^2 \mathfrak{I}_\eta(\xi, 0) + \theta R[\theta] - \theta^2 \mathfrak{I}(\xi, 0) = M[-\mathfrak{I}(\xi, \eta) - g(\mathfrak{I}) + g(\xi, \eta)]. \tag{12}$$

Thus $R(\theta)$ can be obtained from Eq. (12) such as

$$R[\theta] = \frac{\theta^2 a_1 + \theta^2 a_2 + \theta^3 a_1}{(\theta + \theta^2)} - \frac{M[\mathfrak{I}(\xi, \eta) + g(\mathfrak{I}) - g(\xi, \eta)]}{(\theta + \theta^2)}. \tag{13}$$

Operating inverse Mohand transform, on Eq. (13), we get

$$\mathfrak{I}(\xi, \eta) = G(\xi, \eta) - M^{-1}\left[\frac{M[\mathfrak{I}]}{(\theta + \theta^2)} + \frac{M[g(\mathfrak{I})]}{(\theta + \theta^2)}\right], \tag{14}$$

where

$$G(\xi, \eta) = M^{-1}\left[\frac{\theta^2 a_1 + \theta^2 a_2 + \theta^3 a_1}{(\theta + \theta^2)} + M\left[\frac{g(\xi, \eta)}{(\theta + \theta^2)}\right]\right]. \tag{15}$$

Now, we apply HPM on Eq. (14). Let

$$\mathfrak{I}(\xi, \eta) = \sum_{i=0}^{\infty} p^i \mathfrak{I}_i(m) = \mathfrak{I}_0 + p^1 \mathfrak{I}_1 + p^2 \mathfrak{I}_2 + \cdots, \tag{16}$$

and nonlinear terms $g(\mathfrak{I})$ can be calculated by using formula,

$$g(\mathfrak{I}) = \sum_{i=0}^{\infty} p^i H_i(\mathfrak{I}) = H_0 + p^1 H_1 + p^2 H_2 + \cdots, \tag{17}$$

where $Hm's$ is the He's polynomial, which may be computed using the following procedure.

$$H_m(\mathfrak{I}_0 + \mathfrak{I}_1 + \cdots + \mathfrak{I}_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left(g\left(\sum_{i=0}^{\infty} p^i \mathfrak{I}_i\right)\right)_{p=0}, m = 0, 1, 2, \cdots \tag{18}$$

Put Eqs. (16), (17) and (18) in Eq. (14) and comparing the similar factors of p, we get the following consecutive elements

$$\begin{aligned}
p^0 &: \mathfrak{I}_0(\xi, \eta) = G(\xi, \eta),\\
p^1 &: \mathfrak{I}_1(\xi, \eta) = -M^{-1}\left[\frac{1}{(\theta + \theta^2)} M\{\mathfrak{I} + H_0(\mathfrak{I})\}\right],\\
p^2 &: \mathfrak{I}_1(\xi, \eta) = -M^{-1}\left[\frac{1}{(\theta + \theta^2)} M\{\mathfrak{I} + H_1(\mathfrak{I})\}\right],\\
p^3 &: \mathfrak{I}_1(\xi, \eta) = -M^{-1}\left[\frac{1}{(\theta + \theta^2)} M\{\mathfrak{I} + H_2(\mathfrak{I})\}\right],\\
&\vdots
\end{aligned} \tag{19}$$

on continuing the similar process, we can summarize this series to get the approximate solution such as

$$\mathfrak{I}(\xi, \eta) = \mathfrak{I}_0(\xi, \eta) + p^1 \mathfrak{I}_1(\xi, \eta) + p^2 \mathfrak{I}_2(\xi, \eta) + + p^3 \mathfrak{I}_3(\xi, \eta) + \cdots. \tag{20}$$

Let $p = 1$ in above equation, thus the analytical solution of Eq. (8) is as follows

$$\mathfrak{I}(\xi, \eta) = \mathfrak{I}_0 + \mathfrak{I}_1 + \mathfrak{I}_2 + \cdots = \sum_{i=0}^{\infty} \mathfrak{I}_i. \tag{21}$$

Thus, Eq. (21) is considered as an approximate solution of nonlinear differential equation (8).

4. Numerical Examples

In this part, we test two examples for the authenticity and validity of MHPTM. We also demonstrate 2D plots for a better understanding of this strategy where we see that the solution graphs of the approximate solution and the exact solution coincide with each other only after a few iterations.

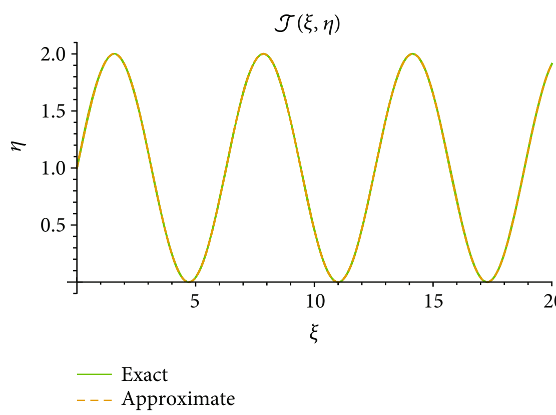


FIGURE 4: 2D Plot for $\mathfrak{J}(\eta)$ with various parameter of η.

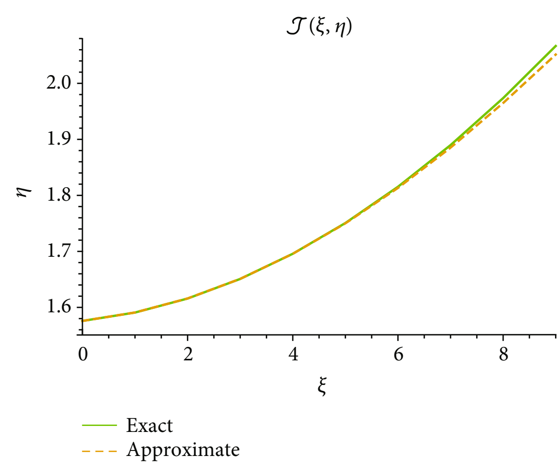


FIGURE 5: 2D Plot for $\mathfrak{J}(\eta)$ with various parameter of η.

4.1. Example 1. Consider a linear Klein-Gordon equation

$$\frac{\partial^2 \mathfrak{J}}{\partial \eta^2} - \frac{\partial^2 \mathfrak{J}}{\partial \xi^2} = \mathfrak{J}, \tag{22}$$

with the initial condition

$$\mathfrak{J}(\xi, 0) = 1 + \sin(\xi), \mathfrak{J}_\eta(\xi, 0) = 0, \tag{23}$$

Applying MT on Eq. (22) together with the differential property as defined in Eq. (7), we get

$$\theta^2 R(\theta) - \theta^3 \mathfrak{J}(0) - \theta^2 \mathfrak{J}'(0) = M\left[\mathfrak{J} + \frac{\partial^2 \mathfrak{J}}{\partial \eta^2}\right]. \tag{24}$$

Using Eq. (23) into Eq. (24) for solving $R(\theta)$, it yields

$$R(\theta) = \theta(1 + \sin(\xi)) + M\left[\mathfrak{J} + \frac{\partial^2 \mathfrak{J}}{\partial \eta^2}\right]. \tag{25}$$

Using inverse Mohand transform on Eq. (25), we get

$$\mathfrak{J}(\xi, \eta) = 1 + \sin(\xi) + M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{J} + \frac{\partial^2 \mathfrak{J}}{\partial \eta^2}\right\}\right] \tag{26}$$

Applying MHPTM to get the He's polynomials

$$\sum_{i=0}^{\infty} p^i \mathfrak{J}_i(m) = 1 + \sin(\xi) + M^{-1}\left[\frac{1}{\theta^2} M\left\{\sum_{i=0}^{\infty} p^i \mathfrak{J}_i + \sum_{i=0}^{\infty} p^i \left(\frac{\partial^2 \mathfrak{J}}{\partial \eta^2}\right)\right\}\right]. \tag{27}$$

Observing the similar powers of p, we get

$$\begin{aligned}
p^0 &: \mathfrak{J}_0(\xi, \eta) = 1 + \sin(\xi), \\
p^1 &: \mathfrak{J}_1(\eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{J}_0 + \frac{\partial^2 \mathfrak{J}_0}{\partial \eta^2}\right\}\right] = \frac{\eta^2}{2}, \\
p^2 &: \mathfrak{J}_2(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{J}_0 + \frac{\partial^2 \mathfrak{J}_0}{\partial \eta^2}\right\}\right] = 2\frac{\eta^4}{4!}, \\
p^3 &: \mathfrak{J}_3(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{J}_0 + \frac{\partial^2 \mathfrak{J}_0}{\partial \eta^2}\right\}\right] = 2\frac{\eta^6}{6!}, \\
p^4 &: \mathfrak{J}_4(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{J}_0 + \frac{\partial^2 \mathfrak{J}_0}{\partial \eta^2}\right\}\right] = 2\frac{\eta^8}{8!}, \\
&\vdots .
\end{aligned} \tag{28}$$

On continuing this process, the results of obtained series cab be summarized as,

$$\begin{aligned}
\mathfrak{J}(\xi, \eta) &= \mathfrak{J}_0(\xi, \eta) + \mathfrak{J}_1(\xi, \eta) + \mathfrak{J}_2(\xi, \eta) + \mathfrak{J}_3(\xi, \eta) + \mathfrak{J}_4(\xi, \eta) + \cdots, \\
&= 1 + \sin(\xi) + \frac{\eta^2}{2} + 2\frac{\eta^4}{4!} + 2\frac{\eta^6}{6!} + 2\frac{\eta^8}{8!} + \cdots.
\end{aligned} \tag{29}$$

This series converges to the exact solution

$$\mathfrak{J}(\xi, \eta) = \sin(\xi) + \cosh(\eta). \tag{30}$$

4.2. Example 2. Consider a nonlinear Klein-Gordon equation

$$\frac{\partial^2 \mathfrak{J}}{\partial \eta^2} - \frac{\partial^2 \mathfrak{J}}{\partial \xi^2} = \mathfrak{J}^2, \tag{31}$$

with the initial condition

$$\mathfrak{J}(\xi, 0) = 1 + \sin(\xi), \mathfrak{J}_\eta(\xi, 0) = 0. \tag{32}$$

Applying MT on Eq. (18) together with the differential property as defined in Eq. (2), we get

$$\theta^2 R(\theta) - \theta^3 \mathfrak{I}(0) - \theta^2 \mathfrak{I}'(0) = M\left[\mathfrak{I}^2 + \frac{\partial^2 \mathfrak{I}}{\partial \eta^2}\right]. \quad (33)$$

Using Eq. (21) into Eq. (22) for solving $R(\theta)$, it yields

$$R(\theta) = \theta(1 + \sin(\xi)) + M\left[\mathfrak{I}^2 + \frac{\partial^2 \mathfrak{I}}{\partial \eta^2}\right]. \quad (34)$$

Using inverse Mohand transform on Eq. (18), we get

$$\mathfrak{I}(\xi, \eta) = 1 + \sin(\xi) + M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{I}^2 + \frac{\partial^2 \mathfrak{I}}{\partial \eta^2}\right\}\right]. \quad (35)$$

Applying MHPTM to get the He's polynomials

$$\sum_{i=0}^{\infty} p^i \mathfrak{I}_i(m) = 1 + \sin(\xi) + M^{-1}\left[\frac{1}{\theta^2} M\left\{\sum_{i=0}^{\infty} p^i \mathfrak{I}_i^2 + \sum_{i=0}^{\infty} p^i \frac{\partial^2 \mathfrak{I}}{\partial \eta^2}\right)\right\}\right]. \quad (36)$$

Observing the similar powers of p, we get

$$\begin{aligned}
p^0 &: \mathfrak{I}_0(\xi, \eta) = 1 + \sin(\xi),\\
p^1 &: \mathfrak{I}_1(\eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{I}_0^2 + \frac{\partial^2 \mathfrak{I}_0}{\partial \eta^2}\right\}\right] = \frac{\eta^2}{2},\\
p^2 &: \mathfrak{I}_2(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{2\mathfrak{I}_0\mathfrak{I}_1 + \frac{\partial^2 \mathfrak{I}_1}{\partial \eta^2}\right\}\right] = 2\frac{\eta^4}{4!},\\
p^3 &: \mathfrak{I}_3(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\mathfrak{I}_1^2 + 2\mathfrak{I}_0\mathfrak{I}_2 + \frac{\partial^2 \mathfrak{I}_2}{\partial \eta^2}\right\}\right] = 2\frac{\eta^6}{6!},\\
p^4 &: \mathfrak{I}_4(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{2\mathfrak{I}_1\mathfrak{I}_2 + 2\mathfrak{I}_0\mathfrak{I}_3 + \frac{\partial^2 \mathfrak{I}_3}{\partial \eta^2}\right\}\right] = 2\frac{\eta^8}{8!},\\
&\vdots\\
&\mathfrak{I}_0(\xi, \eta) = 1 + \sin(\xi),\\
&\mathfrak{I}_1(\xi, \eta) = \left(1 + \sin(\xi) + \sin^2(\xi)\right)\frac{\eta^2}{2},\\
&\mathfrak{I}_2(\xi, \eta) = -(-8 - 9\sin(\xi) + \sin(3\xi))\frac{\eta^4}{48},\\
&\mathfrak{I}_3(\xi, \eta) = (119 - 68\cos(2\xi) + 5\cos(4\xi) + 134\sin(\xi) + 2\sin(3\xi))\frac{\eta^6}{2880},\\
&\mathfrak{I}_4(\xi, \eta) = (681 - 67\cos(\xi) - 404\cos(2\xi) - 27\cos(3\xi) + 19\cos(4\xi) + 1007\sin(\xi) - 272\sin(2\xi) - 147\sin(3\xi) + 160\sin(4\xi) + 10\sin(5\xi))\frac{\eta^8}{80640},\\
&\vdots
\end{aligned} \quad (37)$$

On continuing this process, the approximate solution cab be summarized as,

$$\mathfrak{I}(\xi, \eta) = \mathfrak{I}_0(\xi, \eta) + \mathfrak{I}_1(\xi, \eta) + \mathfrak{I}_2(\xi, \eta) + \mathfrak{I}_3(\xi, \eta) + \mathfrak{I}_4(\xi, \eta) + \cdots, \quad (38)$$

which is in full agreement with [17, 18].

4.3. Example 3. Consider a nonlinear sine-Gordon equation

$$\frac{\partial^2 \mathfrak{I}}{\partial \eta^2} - \frac{\partial^2 \mathfrak{I}}{\partial \xi^2} = \sin(\mathfrak{I}), \quad (39)$$

with the initial condition

$$\mathfrak{I}(\xi, 0) = \frac{\pi}{2}, \mathfrak{I}_\eta(\xi, 0) = 0. \quad (40)$$

Let $\sin(\mathfrak{I}) = \mathfrak{I} - (\mathfrak{I}^3/6) + (\mathfrak{I}^5/120)$, Thus above equation becomes as

$$\frac{\partial^2 \mathfrak{I}}{\partial \eta^2} - \frac{\partial^2 \mathfrak{I}}{\partial \xi^2} = \mathfrak{I} - \frac{\mathfrak{I}^3}{6} + \frac{\mathfrak{I}^5}{120}. \quad (41)$$

Applying MT on Eq. (41) together with the differential property as defined in Eq. (7), we get

$$\theta^2 R(\theta) - \theta^3 \mathfrak{I}(0) - \theta^2 \mathfrak{I}'(0) = M\left[\frac{\partial^2 \mathfrak{I}}{\partial \xi^2} + \mathfrak{I} - \frac{\mathfrak{I}^3}{6} + \frac{\mathfrak{I}^5}{120}\right]. \tag{42}$$

Using Eq. (40) into Eq. (42) for solving $R(\theta)$, it yields

$$R(\theta) = \theta\left(\frac{\pi}{2}\right) + M\left[\frac{\partial^2 \mathfrak{I}}{\partial \xi^2} + \mathfrak{I} - \frac{\mathfrak{I}^3}{6} + \frac{\mathfrak{I}^5}{120}\right]. \tag{43}$$

Using inverse Mohand transform on Eq. (43), we get

$$\mathfrak{I}(\xi, \eta) = \frac{\pi}{2} + M^{-1}\left[\frac{1}{\theta^2} M\left\{\frac{\partial^2 \mathfrak{I}}{\partial \xi^2} + \mathfrak{I} - \frac{\mathfrak{I}^3}{6} + \frac{\mathfrak{I}^5}{120}\right\}\right]. \tag{44}$$

Applying MHPTM to get the He's polynomials

$$\sum_{i=0}^{\infty} p^i \mathfrak{I}_i(m) = \frac{\pi}{2} + M^{-1}\left[\frac{1}{\theta^2} M\left\{\sum_{i=0}^{\infty} p^i \frac{\partial^2 \mathfrak{I}_i}{\partial \xi^2} + \sum_{i=0}^{\infty} p^i \mathfrak{I}_i - \sum_{i=0}^{\infty} p^i \frac{\mathfrak{I}_i^3}{6} + \sum_{i=0}^{\infty} p^i \frac{\mathfrak{I}_i^5}{120}\right)\right\}\right]. \tag{45}$$

Observing the similar powers of p, we get

$$\begin{aligned}
p^0 &: \mathfrak{I}_0(\xi, \eta) = \frac{\pi}{2}, \\
p^1 &: \mathfrak{I}_1(\eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\frac{\partial^2 \mathfrak{I}_0}{\partial \eta^2} + \mathfrak{I}_0 - \frac{\mathfrak{I}_0^3}{6} + \frac{\mathfrak{I}_0^5}{120}\right\}\right], \\
p^2 &: \mathfrak{I}_2(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\frac{\partial^2 \mathfrak{I}_1}{\partial \eta^2} + \mathfrak{I}_1 - \frac{1}{2}\mathfrak{I}_0^2 \mathfrak{I}_1 + \frac{1}{24}\mathfrak{I}_0^4 \mathfrak{I}_1\right\}\right], \\
p^3 &: \mathfrak{I}_3(\xi, \eta) = M^{-1}\left[\frac{1}{\theta^2} M\left\{\frac{\partial^2 \mathfrak{I}_2}{\partial \eta^2} + \frac{1}{2}\mathfrak{I}_0 \mathfrak{I}_1^2 + \frac{1}{12}\mathfrak{I}_0^3 \mathfrak{I}_1^2 + \mathfrak{I}_2 - \frac{1}{2}\mathfrak{I}_0^2 \mathfrak{I}_2 + \frac{1}{24}\mathfrak{I}_0^4 \mathfrak{I}_2\right\}\right], \\
&\vdots .
\end{aligned} \tag{46}$$

$$\begin{aligned}
\mathfrak{I}_0(\xi, \eta) &= \frac{\pi}{2}, \\
\mathfrak{I}_1(\xi, \eta) &= \left(\frac{\pi}{2} - \frac{\pi^3}{48} + \frac{\pi^5}{3840}\right)\frac{\eta^2}{2}, \\
\mathfrak{I}_2(\xi, \eta) &= \left(\frac{\pi}{2} - \frac{\pi^3}{12} + \frac{\pi^5}{240} - \frac{\pi^7}{11520} + \frac{\pi^9}{1474560}\right)\frac{\eta^5}{5!}, \\
\mathfrak{I}_3(\xi, \eta) &= -\left(\frac{3\pi^3}{8} - \frac{3\pi^5}{64} + \frac{3\pi^7}{1280} - \frac{11\pi^9}{184320} + \frac{23\pi^{11}}{707788800} - \frac{\pi^{13}}{235929600}\right)\frac{\eta^5}{5!} + \left(\frac{\pi}{2} - \frac{7\pi^3}{48} + \frac{61\pi^5}{3840} - \frac{19\pi^7}{23040} + \frac{6451\pi^9}{2264833200} - \frac{23347\pi^{11}}{21742387200} + \frac{\pi^{13}}{56620800}\right)\frac{\eta^6}{6!}, \\
&\vdots .
\end{aligned} \tag{47}$$

On continuing this process, the approximate solution cab be summarized as,

$$\mathfrak{I}(\xi, \eta) = \mathfrak{I}_0(\xi, \eta) + \mathfrak{I}_1(\xi, \eta) + \mathfrak{I}_2(\xi, \eta) + \mathfrak{I}_3(\xi, \eta) + \mathfrak{I}_4(\xi, \eta) + \cdots, \tag{48}$$

which is in full agreement with [17, 18].

5. Results and Discussion

In this segment, we demonstrate the validity and the accuracy of NIM though the graphical representations. Figure 1 shows the surface solution of linear Klein-Gordon equation for $-5 \leq \xi \leq 5$ at $\eta = 1$ and Figure 2 shows the surface solution of nonlinear Klein-Gordon equation for $-10 \leq \xi \leq 10$ at $\eta = 0.01$. The absolute errors in Tables 1–3 show the comparison between other approaches and the approximate solution obtained by NIM. We also compare the NIM results in Figures 3, 4 and 5 to show the accuracy of the present approach at $\xi = \pi$ and $\xi = 20$ with different values of η. These results show the high accuracy and validity of this approach. All the computations and graphical representations are made with wolfram Mathematica software. These plot distribution and absolute error show that NIM is powerful, straight forward and easy to implement for such kind of linear and nonlinear partial differential equations. We observe that the approximate of sine-Gordon Eq. (39) is independent of ξ variable due to its independent of initial condition in Eq. (40). Thus, it appropriate solution obtained by NIM is also independent of ξ variable.

6. Conclusion

In this study, we have successfully employed the new iterative method (NIM) to obtain the approximate solution of Klein-Gordon and sine-Gordon equations. The obtained results are derived in the form of series and all are in full agreement which shows that NIM is a very simple and straightforward approach for linear and nonlinear problems. The Mohand transform has been used directly without any perturbation theory and recurrence relation which ruins the physical nature of the problem. We also demonstrate the absolute error and 2D plot distribution with various time parameters. The solution graphs and absolute errors have confirmed the validity and reliability of NIM toward the solutions of other nonlinear partial differential equations in science and engineering.

Data Availability

All the data are available within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

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