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Research Article

On p-Laplace Equations with Singular Nonlinearities and Critical Sobolev Exponent

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In this paper, we deal with p-Laplace equations with singular nonlinearities and critical Sobolev exponent. By using the Nehari manifold, Mountain Pass theorem, and Maximum principle theorem, we prove the existence of at least four distinct nontrivial solutions.

1. Introduction

Let Ω a bounded smooth domain in $\mathbb{R}^N (N \geq 3)$, and consider the following p-Laplace equations with singular nonlinearities

$$\begin{cases} \Delta_p u = |u|^{q-1}u + \lambda \dfrac{|u|^{-1-\beta}}{|x|^{\alpha}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where() $1 < p < N$, $0 \in \Omega, \lambda > 0$, and $q + 1 = p^*$ with $p^* := pN/(N-p)$ is the critical Sobolev exponent, $0 \leq \alpha < N(q+\beta)/q$, $0 < \beta < 1$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator which is degenerate if $p > 2$ and singular if $p < 2$.

In recent years, researchers have been interested in studying problems of the type:

$$\begin{cases} \Delta_p u = |u|^{q-1}u + \lambda f(x,u), & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 3)$, $0 \in \Omega$, $\lambda > 0$, and $q + 1 = p^*$ with $p^* := pN/(N-p)$ is the critical Sobolev exponent and f is a suitable function containing singularities on x (see [1–4] and references therein). For $p = 2$ and after the work of Brézis and Nirenberg [5], Problem (2) has studied by many authors (see, e.g.,[6–18]). Problem (2) becomes the well-known Brézis and Nirenberg problem and is studied extensively in [19]. Ding and Tang in [20] studied the existence of positive solutions with $N \geq 3$ and $f(x,u)$ satisfying (AR) condition in the case $\lambda = 1$. Very recently, M. E. O. El Mokhtar et al. [21] considered Problem (1) with $p = 2$.

The term represented by the function $f(x,u) := |x|^{-\alpha}u|u|^{-\beta}$ with $0 < \beta < 1$ is the key to this famous work because we will allow us to combine the perturbation with the variational methods to overcome shortcomings in the form of singularity. He is well known in the scientific literature that the problems dealt with in applied mathematics have their origins in different fields we will cite as example: heterogeneous chemical catalysis, kinetic chemical catalysis, heat induction or electrical induction, non-Newtonian fluid theory, and viscous fluid theory (see, e.g., [22–26]).

We encounter Problem (1) in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mapping, in the generalized reaction-diffusion theory, in the turbulent flow of a gas in a porous medium, and in the non-Newtonian fluid theory (see [27–30]).

Before giving our main results, we state here some definitions, notations, and known results.

We denote by $\mathscr{H} = \mathscr{W}_0^{1,p}(\Omega \setminus \{0\})$ with respect to the norms

$$\|u\| = \left(\int_{\Omega} (|\nabla u|^p) dx \right)^{1/p}. \tag{3}$$

We consider the following approximation equation:

$$\begin{cases} \Delta_p u = |u|^{q-1}u + \dfrac{\lambda}{|x|^\alpha (u+\theta)^\beta} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

for any $\theta > 0$. The energy functional of (4) E_θ is defined by

$$E_\theta(u) \coloneqq \frac{1}{p}\|u\|^p - \frac{1}{q+1}\int_\Omega (u^+)^{q+1}dx - \frac{\lambda}{1-\beta}\int_\Omega \frac{(u^+ + \theta)^{1-\beta} - \theta^{1-\beta}}{|x|^\alpha}dx. \quad (5)$$

for all $u \in \mathscr{H}$, where $u^\pm = \max\{u^\pm, 0\}$.

We know that E_θ is a C^1 – function on $\mathscr{H} = \mathscr{H}_0^1(\Omega)$.

A point $u \in \mathscr{H}$ is a weak solution of Equation (1) if it satisfies

$$\left\langle E_\theta'(u), \varphi \right\rangle \coloneqq \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx - \int_\Omega (u^+)^{q-1} u\varphi dx - \lambda\int_\Omega \frac{\varphi}{(u^+ + \theta)^\beta |x|^\alpha}dx = 0, \text{for all } \varphi \in \mathscr{H}. \quad (6)$$

Here .,. denotes the product in the duality $\mathscr{H}'$, $\mathscr{H}$($\mathscr{H}'$ dual of $\mathscr{H}$).

Let

$$S \coloneqq \inf_{u\in\mathscr{H}\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |u|^{q+1}dx\right)^{p/(q+1)}}. \quad (7)$$

From [3], S is achieved.

Let λ_* and λ_{**} be positive numbers such that

$$\lambda_* \coloneqq \frac{q+1-p}{(q+\beta)\Psi}\left[\frac{p-1+\beta}{q+\beta}S^{(q+1)/p}\right]^{(p-1+\beta)/(q+1-p)} \quad (8)$$

and

$$\lambda_{**} \coloneqq \frac{q-1}{(q+\beta)\Psi}\left(\frac{1-\beta}{p}\right)^{1/p}\left(\frac{p-1+\beta}{q+\beta}\right)^{(p-1)/(q-1)} S^{(p(q+1)-(q-1)(1+\beta))/(q-1)} \quad (9)$$

where,

$$\Psi = \left[\frac{2\pi^{N/2}(q+\beta)}{N\Gamma(N/2)(q+\beta)-\alpha(q+1)}\right]^{(q+\beta)/(q+1)} R_0^{(N/(q+1))(q+\beta)-\alpha} > 0, \quad (10)$$

with

$$0 \le \alpha < \frac{N}{q+1}(q+\beta). \quad (11)$$

The main results are concluded as the following theorems.

Theorem 1. *Assume that $N \ge 3$, $0 \le \alpha < (N/(q+1))(q+\beta)$, $0 < \beta < 1$, and λ verifying $0 < \lambda < \lambda_*$; then, the system (1) has at least one positive solution.*

Theorem 2. *In addition to the assumptions of the Theorem 1, there exists $\Lambda_* \in (0, \lambda_{**})$ such that if λ satisfying $0 < \lambda < \Lambda_*$, then (1) has at least two positive solutions.*

Theorem 3. *Under the assumptions of Theorem 2 then, there exists a positive real λ^{**} such that if λ satisfies $0 < \lambda < \lambda^{**}$, then (1) has at least four nontrivial solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Sections 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last section, we prove Theorem 3.

2. Preliminaries

Definition 4 (see [31]). Let $c \in \mathbb{R}M$ be a Banach space and $E \in C^1(M, \mathbb{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in M for E

$$E(u_n) = c + o_n(1) \text{ and } E'(u_n) = o_n(1), \quad (12)$$

where $o_n(1)$ tends to 0 as n goes at infinity

(ii) We say that E satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in M for E has a convergent subsequence

2.1. Nehari Manifold [32, 33]. It is well known that E is of class C^1 in $\mathscr{H}$ and the solutions of (1) are the critical points of E which is not bounded below on $\mathscr{H}$. Consider the following Nehari manifold:

$$\mathscr{W} = \left\{u \in \mathscr{H}\setminus\{0\}: E'(u), u = 0\right\}, \quad (13)$$

Note that $\mathscr{W}$ contains every nontrivial solution of the problem (1). Thus, $u \in \mathscr{W}$ if and only if $u \in \mathscr{H}\setminus\{0\}$ and

$$\|u\|^p - \int_\Omega (u^+)^{q+1}dx - \lambda\int_\Omega \frac{(u^+ + \theta)^{1-\beta} - \theta^{1-\beta}}{|x|^\alpha}dx = 0. \quad (14)$$

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1) on the constraint defined by the Nehari manifold.

In order to obtain the first positive solution, we give the following important lemmas.

Lemma 5. *E_θ is coercive and bounded from below on $\mathscr{W}$.*

Proof. Let $R_0 > 0$ such that $\Omega \subset B(0, R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. If $u \in \mathscr{W}$, then by (14) and the Hölder inequality, we obtain

$$\int_\Omega \frac{(u^+)^{1-\beta}}{|x|^\alpha} dx \le C_1 \|u\|^{1-\beta} \tag{15}$$

$$\text{with } C_1 = \Psi S_1^{-(1-\beta)/p}, \tag{16}$$

where,

$$S_1 := \inf_{u \in \mathscr{H}\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega \left(|u|^{1-\beta}/|x|^\alpha\right) dx\right)^{p/(1-\beta)}}. \tag{17}$$

Therefore, we obtain that

$$\begin{aligned} E_\theta(u) &= ((q+1-p)/p(1+q))\|u\|^p + -\lambda((q+\beta)/(q+1) \\ &\quad \cdot (1-\beta)) \int_\Omega \frac{(u^+)^{1-\beta}}{|x|^\alpha} dx \\ &\ge ((q+1-p)/p(q+1))\|u\|^p + -\lambda((q+\beta)/(q+1) \\ &\quad \cdot (1-\beta))\Psi \|u\|^{1-\beta} S_1^{-(1-\beta)/p}, \end{aligned} \tag{18}$$

for $0 \le \alpha < (N/(q+1))(q+\beta)$.

Thus, E_θ is coercive and bounded from below on $\mathscr{W}$. □

Define

$$\phi(u) = \left\langle E'_\theta(u), u \right\rangle. \tag{19}$$

Then, for $u \in \mathscr{W}$

$$\begin{aligned} \left\langle \phi'(u), u \right\rangle &= p\|u\|^p - (q+1)\int_\Omega^{q+1} (u^+)^{q+1} dx + -\lambda(1-\beta)\int_\Omega \frac{(u^+ + \theta)^{1-\beta} - \theta^{1-\beta}}{|x|^\alpha} dx \\ &= (p-1+\beta)\|u\|^p - (q+\beta)\int_\Omega (u^+)^{q+1} dx \\ &= \lambda(q+\beta)\int_\Omega \frac{(u^+ + \theta)^{1-\beta} - \theta^{1-\beta}}{|x|^\alpha} dx - (q+1-p)\|u\|^p. \end{aligned} \tag{20}$$

Splitting $\mathscr{W}$ in three parts, we set

$$\begin{aligned} \mathscr{W}^+ &= \left\{u \in \mathscr{W} : \phi'(u), u > 0\right\} \\ \mathscr{W}^0 &= \left\{u \in \mathscr{W} : \phi'(u), u = 0\right\}. \\ \mathscr{W}^- &= \left\{u \in \mathscr{W} : \phi'(u), u < 0\right\} \end{aligned} \tag{21}$$

We have the following results.

Lemma 6. *Suppose that u_0 is a local minimizer for E_θ on $\mathscr{W}$.If $u_0 \notin \mathscr{W}^0$, then u_0 is a critical point of E_θ.*

Proof. If u_0 is a local minimizer for E_θ on $\mathscr{W}$, then u_0 is a solution of the optimization problem:

$$\min_{\{u \in \mathscr{H}\setminus\{0\}/\phi(u)=0\}} E_\theta(u). \tag{22}$$

Hence, there exists a Lagrange multipliers $\mu \in \mathbb{R}$ such that

$$E'_\theta(u_0) = \mu\phi'(u_0) \text{in } \mathscr{H}' \tag{23}$$

Thus,

$$\left\langle E'_\theta(u_0), u_0 = \mu\phi'(u_0), u_0 \right\rangle. \tag{24}$$

But $\phi'(u_0), u_0 \ne 0$, since $u_0 \notin \mathscr{W}^0$. Hence, $\mu = 0$. This completes the proof. □

Lemma 7. *There exists a positive number λ_* such that for all λ verifying*

$$0 < \lambda < \lambda_*, \tag{25}$$

we have $\mathscr{W}^0 = \varnothing$.

Proof. Let us reason by contradiction.

Suppose that $\mathscr{W}^0 \ne \varnothing$ for all λ such that $0 < \lambda < \lambda_*$. Then, by (20) and for $u \in \mathscr{W}^0$, we have

$$\begin{aligned} (p-1+\beta)\|u\|^p - (q+\beta)\int_\Omega (u^+)^{q+1} dx = 0, \\ \lambda(q+\beta)\int_\Omega \frac{(u^+ + \theta)^{1-\beta} - \theta^{1-\beta}}{|x|^\alpha} dx - (q+1-p)\|u\|^p = 0. \end{aligned} \tag{26}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\| \ge S^{((q+1))/(p(q+1-p))} \left[\frac{(q+\beta)}{(p-1+\beta)}\right]^{1/(q+1-p)}, \tag{27}$$

$$\|u\| \le \left[\lambda\left(\frac{q+\beta}{q+1-p}\right)\Psi\right]^{(1/((p-1+\beta)))} S_1^{(\beta-1)/(p(p-1+\beta))}. \tag{28}$$

From (26) and (27), we obtain $\lambda \ge \lambda_*$, which contradicts our hypothesis. □

As $\mathscr{W}^0 = \emptyset$ then $\mathscr{W} = \mathscr{W}^+ \cup \mathscr{W}^-$. Define

$$c := \inf_{u\in\mathscr{W}} E_\theta(u), c^+ := \inf_{u\in\mathscr{W}^+} E_\theta(u) \text{ and } c^- := \inf_{u\in\mathscr{W}^-} E_\theta(u). \quad (29)$$

For the sequel, we need the following Lemma.

Lemma 8.
(i) For all λ such that $0 < \lambda < \lambda_*$, one has $c \le c^+ < 0$
(ii) For all λ such that $0 < \lambda < \lambda_{**}$ there exists $C_0 > 0$ such hat $c^- > -\lambda^{p/(p-1+\beta)} C_0$

Proof.

(i) Let $u \in \mathscr{W}^+$. By (20), we have

$$\frac{p-1+\beta}{q+\beta} \|u\|^p > \int_\Omega (u^+)^{q+1} dx \quad (30)$$

and so

$$\begin{aligned} E_\theta(u) &= -\frac{p+\beta-1}{p(1-\beta)} \|u\|^p + \frac{q+\beta}{(q+1)(1-\beta)} \int_\Omega (u^+)^{q+1} dx \\ &< \frac{(p+\beta-1)(p-q-1)}{p(1-\beta)(q+1)} \|u\|^p < 0, \end{aligned} \quad (31)$$

since $p+1>2$ and $0<\beta<1$. Then, we conclude that $c \le c^+ < 0$.

(ii) Let $u \in \mathscr{W}^-$. By(20), we get

$$\frac{p-1+\beta}{q+\beta} \|u\|^p < \int_\Omega (u^+)^{q+1} dx. \quad (32)$$

By Sobolev embedding theorem, we obtain

$$\int_\Omega (u^+)^{q+1} dx \le S^{-(q+1)/p} \|u\|^{q+1}. \quad (33)$$

This implies

$$\|u\| > S^{(q+1)/(p(q+1-p))} \left[\frac{p-1+\beta}{q+\beta}\right]^{1/(q+1-p)}, \text{ for all } u \in \mathscr{W}^-. \quad (34)$$

By the proof of Lemma 5, we have

$$\begin{aligned} E_\theta(u) &\ge \|u\|^p \frac{q-1}{p(q+1)} + -\lambda \|u\|^{1-\beta} \frac{q+\beta}{(q+1)(1-\beta)} \Psi(S)^{(1-\beta)/p} \\ &\ge \frac{q-p}{pq} \left[\frac{p-1+\beta}{(q+\beta)}\right]^{p/(q-1)} S^{(q+1)/(q-1)} \\ &\quad - \lambda^p \frac{(q+\beta)^p}{(q+1)(q-1)(1-\beta)} \Psi^p S^{(1-\beta)/p}. \end{aligned} \quad (35)$$

Thus, for all λ such that $0 < \lambda < \lambda_{**}$ we have $E_\theta(u) \ge -\lambda^{p/(p-1+\beta)} C_0$ with

$$C_0 = \left(\frac{1}{\beta-1} - \frac{1}{q+1}\right)\left(\frac{p-1+\beta}{p}\Psi\right)\left[\frac{q+\beta}{q-p+1}\right]^{(1-\beta)/(p-1+\beta)}. \quad (36)$$

As in [34] we have the following result. □

Proposition 9.
(i) For all λ such that $0 < \lambda < \lambda_*$, there exists a $(PS)_{c^+}$ sequence in $\mathscr{W}^+$
(ii) For all λ such that $0 < \lambda < \lambda_{**}$, there exists a $(PS)_{c^-}$ sequence in $\mathscr{W}^-$ and for each $u \in \mathscr{H} \setminus \{0\}$.
Define

$$t_M := t_{\max}(u) = \left[\frac{(p-1+\beta)\|u\|^p}{(q+p)\int_\Omega (u^+)^{q+1} dx}\right]^{1/(q+1-p)} > 0. \quad (37)$$

Lemma 10. *Suppose that $0 < \lambda < \lambda_*$. For each $u \in \mathscr{H} \setminus \{0\}$, there exists unique t^+ and t^- such that $0 < t^+ < t_M < t^-$, $(t^+u) \in \mathscr{W}^+$, and $(t^-u) \in \mathscr{W}^-$,*

$$\begin{aligned} E_\theta(t^+u) &= \inf \quad E(tu) \text{ for } 0 \le t \le t_M, \\ E_\theta(t^-u) &= \sup \quad E(tu) \text{ for } t \ge 0. \end{aligned} \quad (38)$$

Proof. With minor modifications, we refer the reader to [34]. □

3. Proof of Theorem 1

Now, taking as a starting point the work of Tarantello [35], we establish the existence of a local minimum for E_θ on $\mathscr{W}^+$.

Proposition 11. *For all λ such that $0 < \lambda < \lambda_*$, the functional E_θ has a minimizer $u_0^+ \in \mathscr{W}^+$, and it satisfies*

(i) $E_\theta(u_0^+) = c^+$

(ii) (u_0^+) is a nontrivial solution of (1)

Proof. If $0 < \lambda < \lambda_*$, then by Proposition 9, (i) there exists a $(u_n)(PS)_{c^+}$ sequence in ;, thus, it bounded by Lemma 5.

Then, there exists $u_0^+ \in \mathscr{H}$, and we can extract a subsequence which will denoted by (u_n) such that

$$\begin{aligned} u_n &\longrightarrow u_0^+ \text{weakly in } \mathscr{H} \\ u_n &\longrightarrow u_0^+ \text{ strongly in } L^{1-\beta}(\Omega, |x|^{-\alpha}). \\ u_n &\longrightarrow u_0^+ \text{a.e in } \Omega \end{aligned} \quad (39)$$

By (15) and (39), we have

$$\lim_{n\longrightarrow\infty} \int_\Omega \frac{|u_n+\theta|^{1-\beta}}{|x|^\alpha} dx = \int_\Omega \frac{|u_0^+ + \theta|^{1-\beta}}{|x|^\alpha} dx + o(1). \quad (40)$$

Thus, by (39), u_0^+ is a weak nontrivial solution of (1). Now, we show that (u_n) converges to u_0^+ strongly in $\mathscr{H}$. Suppose otherwise. By the lower semi-continuity of the norm, if $u_n(1/2)u_0^+$ we have $\|u_0^+\| < \liminf_{n\longrightarrow\infty} \|u_n\|$, and we obtain

$$\begin{aligned} c &\le E_\theta(u_0^+) - \frac{1}{q+1} E'_\theta \\ &= \frac{q+1-p}{p(q+1)} / \|u_0^+\|^p + -\lambda \frac{q+\beta}{(q+1)(1-\beta)} \int_\Omega \frac{|u_0^+ + \theta|^{1-\beta}}{x^\alpha} dx \\ &< \liminf_{n\longrightarrow\infty} E_\theta(u_n) = c. \end{aligned} \quad (41)$$

We get a contradiction. Therefore, (u_n) converge to u_0^+ strongly in $\mathscr{H}$. Moreover, we have $u_0^+ \in \mathscr{W}^+$. If not, then by Lemma 10, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+ u_0^+) \in \mathscr{W}^+$ and $(t^- u_0^+) \in \mathscr{W}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\begin{aligned} \frac{d}{dt} E_\theta(tu_0^+)_{\gg t=t_0^+} &= 0, \\ \frac{d^2}{dt^2} E_\theta(tu_0^+)_{\gg t=t_0^+} &> 0, \end{aligned} \quad (42)$$

there exists $t_0^- < t^- \le t_0^+$ such that $E_\theta(t_0^- u_0^+) < E_\theta(t^+ u_0^+)$. Then, we get

$$E_\theta(t_0^- u_0^+) < E_\theta(t^- u_0^+) < E_\theta(t_0^+ u_0^+) = E_\theta(u_0^+), \quad (43)$$

which contradicts the fact that $E_\theta(u_0^+) = c^+$. Since $E_\theta(u_0^+) = E_\theta(|u_0^+|)$ and $|u_0^+| \in \mathscr{W}^+$, then by Lemma 6, we may assume that u_0^+ is a nontrivial nonnegative solution of (1). By the Harnack inequality, we conclude that $u_0^+ > 0$ (see, e.g., [29]). □

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for E_θ on $\mathscr{W}^-$. For this, we require the following Lemmas.

Lemma 12. *Let (u_n) be $(PS)_c$ sequence for E_θ for some $c \in \mathbb{R}$ with $u_n \longrightarrow u$ in $\mathscr{H}$.*

Then, $E'_\theta(u) = 0$ and $E_\theta(u) \ge -\lambda^{p/(1+\beta)} C(q, \beta, \Psi, S)$, with $C(q, \beta, \Psi, S) > 0$.

Proof. Let $(u_n) \subset S_u$ be a minimizing sequence for E_θ with S_u is the unit sphere. By Ekeland' s variational principle [12], we may assume $E'_\theta(u_n) \longrightarrow 0$. So (u_n) is a $(PS)_c$ sequence and therefore $u_n \longrightarrow u$ after passing to a subsequence. Hence $E_\theta(u) = c$ and $\langle E'_\theta(u) \rangle = 0$, which implies that $\langle E'_\theta(u), u \rangle = 0$, and

$$\int_\Omega (u^+)^{q+1} dx = \|u\|^p - \lambda \int_\Omega \frac{(u^+ + \theta)^{1-\beta}}{|x|^\alpha} dx. \quad (44)$$

Therefore,

$$E_\theta(u) = \frac{q-1}{p(p+1)} \|u\|^p + -\lambda \frac{q+\beta}{(q+1)(p+1-\beta)} \int_\Omega \frac{(u^+ + \theta)^{1-\beta}}{|x|^\alpha} dx. \quad (45)$$

From (15) and considering ||u|| small enough, we get

$$\int_\Omega \frac{(u^+ + \theta)^{1-\beta}}{|x|^\alpha} dx \le \left[\frac{2\pi^{N/2}(q+\beta)}{N\Gamma(N/2)(q+\beta) - \alpha(q+1)} \right]^{q+\beta/q+1} R_0^{(N/(q+1))(q+\beta)-\alpha} \|u\|^{1-\beta} S^{-(1-\beta)/p}, \quad (46)$$

which implies that

$$E(u) \ge \frac{q-1}{p(q+1)} \|u\|^p - \lambda \frac{q+\beta}{(q+1)(p+1-\beta)} S^{-(1-\beta)/p} \Psi \|u\|^{1-\beta}, \quad (47)$$

with

$$\Psi = \left[\frac{2\pi^{N/2}(q+\beta)}{N\Gamma(N/2)(q+\beta) - \alpha(q+1)} \right]^{(q+\beta)/(q+1)} R_0^{N/(q+1)(q-1+\beta)-\alpha}. \quad (48)$$

Set $f(t) = Dt^p - \lambda E t^{1-\beta}$ for all $t > 0$, with

$$\begin{aligned} D &= \frac{q-1}{p(q+1)}, \\ E &= \frac{q+\beta}{(q+1)(1-\beta)} S^{(-(1-\beta))/p} \Psi. \end{aligned} \quad (49)$$

Using (46), we obtain that

$$f(t) \geq -\lambda^{p/(p-1+\beta)} C(p, q, \beta, \Psi, S) \text{ for all } t > 0 \text{ small enough}, \quad (50)$$

where

$$C(p, q, \beta, \Psi, S) = D\left[\frac{(1-\beta)E}{pD}\right]^{p/(1+\beta)} \frac{\beta}{(1-\beta)}, \quad (51)$$

We have $C(p, q, \beta, \Psi, S) > 0$ since $0 < \beta < 1$. Then, we conclude that

$$E(u) \geq -\lambda^{p/(p-1+\beta)} C(p, q, \beta, \Psi, S). \quad (52)$$

□

Lemma 13. *Let $\lambda \in (0,\lambda_{**})$; then, the functional E_θ satisfies the $(PS)_c$ condition in $\mathscr{H}$ with $c < c^*$, where*

$$c^* = \frac{(q+1-p)}{p(q+1)} S^{(q+1)/(q+1-p)} - \lambda^{p/(p-1+\beta)} C(p, q, \beta, \Psi, S). \quad (53)$$

Proof. If $0 < \lambda < \lambda_{**}$, then by Proposition 9, (ii) there exists a (u_n), $(PS)_c$ sequence in $\mathscr{W}$; thus, it bounded by Lemma 5. Then, there exists $u \in \mathscr{H}$, and we can extract a subsequence which will denoted by (u_n) such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } \mathscr{H} \\ &u_n \rightharpoonup u \text{ weakly in } L^{q+1}(\Omega) \\ &u_n \longrightarrow u \text{ a.e in } \Omega. \end{aligned} \quad (54)$$

Then, u is a weak solution of (1). Let $v_n = u_n - u$; then, by Brézis-Lieb [36], we obtain

$$\|v_n\|^p = \|u_n\|^p - \|u\|^p + o_n(1), \quad (55)$$

$$\int_\Omega |v_n|^{q+1} dx = \int_\Omega |u_n|^{q+1} - dx \int_\Omega |u|^{q+1} dx + o_n(1). \quad (56)$$

Since

$$E_\theta(u_n) = c + o_n(1), E'_\theta(u_n) = o_n(1), \quad (57)$$

and by (55) and (56), we deduce that

$$\begin{aligned} \frac{1}{p}\|v_n\|^p - \frac{1}{q+1}\int_\Omega |v_n|^{q+1} dx &= c - E_\theta(u) + o_n(1), \\ \|v_n\|^p - \int_\Omega |v_n|^{q+1} dx &= o_n(1). \end{aligned} \quad (58)$$

Hence, we may assume that

$$\|v_n\|^p \longrightarrow l, \int_\Omega |v_n|^{q+1} dx \longrightarrow l. \quad (59)$$

Moreover, by Sobolev inequality, we have

$$\|v_n\|^p \geq S\int_\Omega |v_n|^{q+1} dx. \quad (60)$$

Combining (60) and (59), we obtain

$$l \geq l^{p/(q+1)} S. \quad (61)$$

Either,

$$l = 0 \text{or} l \geq S^{(q+1)/(q-1)}. \quad (62)$$

Then from (58), (59), Lemma 13 and Lemma 12, we obtain

$$c \geq \frac{q-1}{p(q+1)} l + E_\theta(u_n) \geq c^*. \quad (63)$$

which is a contradiction. Therefore, $l = 0$, and we conclude that (u_n) converges to u strongly in $\mathscr{H}$.

Thus,

$$E_\theta(u_n) \text{ converges to } E_\theta(u) = c \text{ as } n \text{ tends to} +\infty. \quad (64)$$

□

Lemma 14. *There exists $v \in \mathscr{H}$ and $\Lambda_* > 0$ such that for all $\lambda \in (0, \Lambda_*)$, one has*

$$\sup_{t\geq 0} E_\theta(tv) < c^*. \quad (65)$$

In particular, $c < c^*$ for all $\lambda \in (0, \Lambda_*)$.

Proof. Let $\varphi_\varepsilon(x)$ satisfies (4). Then, we have

$$\frac{\lambda}{1-\beta}\int_\Omega \frac{|\varphi_\varepsilon + \theta|^{1-\beta}}{|x|^\alpha} dx > 0. \quad (66)$$

We consider the two functions:

$$f(t) := E_\theta(t\varphi_\varepsilon) \text{and} g(t) = \frac{t^p}{p}\|\varphi_\varepsilon\|^p - \frac{t^{q+1}}{q+1}\int_\Omega |\varphi_\varepsilon|^{q+1} dx. \quad (67)$$

Then, for all for all $\lambda \in (0,\lambda_{**})$,

$$f(0) = 0 < c^*. \quad (68)$$

By the continuity of f, there exists $t_0 > 0$ such that

$$f(t) < c^*, \forall t \in (0, t_0). \quad (69)$$

On the other hand we have

$$\max_{t\geq 0} g(t) = \frac{(q-1)(q+\beta)}{p(1+\beta)(q+1)} S^{(q+1)/(q-1)}. \quad (70)$$

Then, we obtain

$$\sup_{t\geq 0} E_\theta(t\varphi_\varepsilon) < \frac{(q-1)(q+\beta)}{p(1+\beta)(q+1)} S^{(q+1)/(q-1)} - \lambda^{p/(1+\beta)} C(p,q,\beta,\Psi,S). \tag{71}$$

Now, taking $\lambda > 0$ such that

$$-\frac{\lambda t_0^{1-\beta}}{1-\beta}\int_\Omega \frac{|\varphi_\varepsilon|^{1-\beta}}{|x|^\alpha}dx < -\lambda^{p/(1+\beta)} C(p,q,\beta,\Psi,S), \tag{72}$$

we obtain

$$0 < \lambda < \frac{t_0^{1+\beta}}{[(1-\beta)C(p,q,\beta,\Psi,S)]^{(1+\beta)/(1-\beta)}} \cdot \left[\int_\Omega \frac{|\varphi_\varepsilon+\theta|^{1-\beta}}{|x|^\alpha}dx\right]^{(1+\beta)/(1-\beta)} = \Lambda_1. \tag{73}$$

Set

$$\Lambda_* = \min \quad \{\lambda_{**}, \Lambda_1\}. \tag{74}$$

We deduce that $c^- < c^*$ for all $\lambda \in (0, \Lambda_*)$; then, there exists $t_n > 0$ such that $(t_n w_n) \in \mathscr{W}^-$ with w_n satisfying (4) and for all $\lambda \in (0, \Lambda_*)$,

$$c^- \leq E_\theta(t_n w_n) \leq \sup_{t\geq 0} \quad E(t w_n) < c^*. \tag{75}$$

□

Lemma 15. *For all λ such that $0 < \lambda < \Lambda_* = \min\{\lambda_{**}, \Lambda_1\}$, the functional E_θ has a minimizer u_0^- in $\mathscr{W}^-$, and it satisfies*

$$E_\theta(u_0^-) = c^- > 0, \tag{76}$$

u_0^- is a nontrivial solution of (1) in $\mathscr{H}$.

Proof. By Proposition 9 (ii), there exists a $(PS)_{c^-}$ sequence (u_n) for E, in $\mathscr{W}^-$ for all $\lambda \in (0,\lambda_{**})$. From Lemmas 13, 18, and 8(ii), for $\lambda \in (0, \Lambda_1)$, E_θ satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that (u_n) is bounded in $\mathscr{H}$. Therefore, there exist a subsequence of (u_n) still denoted by (u_n) and $u_0^- \in \mathscr{W}^-$ such that (u_n) converges to u_0^- strongly in $\mathscr{H}$ and $E_\theta(u_0^-) = c^- > 0$ for all $\lambda \in (0, \Lambda_*)$.

Finally, by using the same arguments as in the proof of Proposition 11 for all $\lambda \in (0, \lambda_*)$, we have that u_0^- is a solution of (1). □

Now, we complete the proof of Theorem 2. By Proposition 11 and Lemma 15, we obtain that (1) has two positive solutions $u_0^+ \in \mathscr{W}^+$ and $u_0^- \in \mathscr{W}^-$. Since $\mathscr{W}^+ \cap \mathscr{W}^- = \varnothing$, then, u_0^+ and u_0^- are distinct.

5. Proof of Theorem 3

Now, we consider the following Nehari submanifold of $\mathscr{W}$:

$$\mathscr{W}_\rho = \left\{u \in \mathscr{H}/\{0\}\colon E'_\theta(u), u = 0 \text{ and } \|u\| \geq \rho > 0\right\}. \tag{77}$$

Thus, $u \in \mathscr{W}_\rho$ if and only if

$$\begin{gathered} \|u\|^p - \int_\Omega (u^+)^{q+1}dx - \lambda\int_\Omega \frac{(u^+ + \theta)^{1-\beta}}{|x|^\alpha}dx = 0, \\ \|u\| \geq \rho > 0. \end{gathered} \tag{78}$$

Firstly, we need the following Lemmas.

Lemma 16. *Under the hypothesis of theorem 3, there exist $\Lambda_2 > 0$ such that $\mathscr{W}_\rho$ is nonempty for any $\lambda \in (0, \Lambda_2)$.*

Proof. Fix $u_0 \in \mathscr{H} \setminus \{0\}$, and let

$$\begin{aligned} g(t) &= \left\langle E'_\theta(tu_0), tu_0\right\rangle \\ &= t^p\|u_0\|^p - t^{q+1}\int_\Omega |u_0|^{q+1}dx - t^{1-\beta}\lambda\int_\Omega \frac{|u_0+\theta|^{1-\beta}}{|x|^\alpha}dx. \end{aligned} \tag{79}$$

Clearly $g(0) = 0$ and $g(t) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Moreover, we have

$$\begin{aligned} g(1) &= \|u_0\|^p - \int_\Omega |u_0|^{q+1}dx - \lambda\int_\Omega \frac{|u_0+\theta|^{1-\beta}}{|x|^\alpha}dx \\ &\geq \|u_0\|^{2-\beta}\left[\|u_0\|^\beta - (S_\mu)^{-(q+1)/p}\|u_0\|^{(q+\beta)} - \lambda\Psi(S)^{(\beta-1)/p}\right]. \end{aligned} \tag{80}$$

for $t \geq 0$, put $\varphi(t) = t^\beta - (S)^{-(q+1)/p}t^{q+\beta}$; then, we obtain $\max_{t\geq 0} \varphi(t) = \varphi(t_1) > 0$ since $q > 2$ with $t_1 = (\beta/(q-+\beta))^{(q+1)/(q-1)}(S)^{(q+1)/(p(q-1))}$. Thus, we obtain

$$g(1) \geq \|u_0\|^{1-\beta}\left[\varphi(t_1) - \lambda\Psi(S)^{(\beta-1)/p}\right] > 0, \tag{81}$$

if $\lambda < (((S)^{(1-\beta)/p})/\Psi)\varphi(t_1) \coloneqq \Lambda_2$.

Then, there exists $t_0 > 0$ such that $g(t_0) = 0$. Thus, $(t_0 u_0) \in \mathscr{W}_\rho$ and $\mathscr{W}_\rho$ is nonempty for any $\lambda \in (0, \Lambda_2)$. □

Lemma 17. *There exist δ,λ^* positive real numbers such that $\phi'(u), u? < -? < 0$, for $u \in \mathscr{W}_\rho$ and any λ verifying*

$$0 < \lambda < \lambda^*. \tag{82}$$

Proof. Let $u \in \mathscr{W}_\rho$, then by (14), (20) and the Holder inequality, it allows us to write

$$\begin{aligned}\phi'(u), u &= \lambda(q+\beta)\Psi(S)^{(\beta-1)/p}u^{1-\beta} - (q-1)u^p \\ &\le u^{1-\beta}\left[\lambda(q+\beta)\Psi S^{(\beta-1)/p} - (q-1)u^\beta\right] \\ &\le u^{1-\beta}\left[\lambda(q+\beta)\Psi S^{(\beta-1)/p} - (q-1)\rho^\beta\right].\end{aligned} \tag{83}$$

Thus, if

$$0 < \lambda < \Lambda_3 = \left[\frac{(q-1)\rho^\beta}{(q+\beta)\Psi}\right] S^{(\beta-1)/p}, \tag{84}$$

and choosing $\lambda^* \coloneqq \min(\Lambda_2, \Lambda_3)$ with Λ_2 defined in Lemma 16, then we obtain that

$$\phi'(u), u < 0, \text{forany} u \in \mathscr{W}_\rho. \tag{85}$$

□

Lemma 18. *Suppose $q > 2, \beta \in (0,1)$ and $0 < \lambda < \min(\Lambda_2, \Lambda_3, \Lambda_4)$ when*

$$\Lambda_4 = \left[\frac{(q-1)(1-\beta)}{pq(q+\beta)\Psi}\left(\frac{1-\beta}{p}\right)^{(\beta+1)/\beta}\right](S)^{(1-q)/p}. \tag{86}$$

Then, there exist ε and η positive constants such that

(i) We have

$$E_\theta(u) \ge \eta > 0 \, for \, \|u\| = \varepsilon. \tag{87}$$

(ii) There exists $v \in \mathscr{W}_\rho$ when $\|v\| > \varepsilon$, with $\varepsilon = \|u\|$, such that $E_\theta(v) \le 0$

Proof. We can suppose that the minima of E_θ are realized by (u_0^+) and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have the following:

(i) By (20) and (85), we get

$$\begin{aligned}E_\theta(u) &= (q-1)/p(q+1)\|u\|^p \\ &\quad - \lambda(q+\beta)/(q+1)(1-\beta)\int_\Omega \frac{(u^+ + \theta)^{1-\beta}}{|x|^\alpha}dx \\ &\ge (q-1)/p(q+1)\|u\|^p \\ &\quad - \lambda(q+\beta)/(q+1)(1-\beta)\Psi(S)^{(\beta-1)/p}\|u\|^{1-\beta},\end{aligned} \tag{88}$$

By exploiting the function $\phi(t) = at^p - bt^{1-\beta}$ which achieve its maximum at the point $t_1 = (1-\beta/p)^{p/(\beta-1+p)}(a/b)^{(p-1)/(q-1)}$ such that $\max_{t\ge0}\phi(t) = \phi(t_1) > 0$ if

$$\lambda < \Lambda_4 = \left[\frac{(q-1)(1-\beta)}{p(q+\beta)\Psi}\left(\frac{1-\beta}{p}\right)^{\beta/(p-1+\beta)}\right](S)^{(1-q)/p}, \tag{89}$$

and the fact that, $q > 2, \beta \in (0,1)$ then, we obtain that

$$E_\theta(u) \ge \eta > 0 \text{when} \varepsilon = \|u\| \text{small}. \tag{90}$$

(ii) Let $t > 0$, then we have for all $\varphi \in \mathscr{W}_\rho$

$$E_\theta(t\varphi) \coloneqq \frac{t^p}{p}\|\varphi\|^p - \left(\frac{t^{q+1}}{q+1}\right)\int_\Omega |\varphi|^{q+1}dx - \lambda\left(\frac{t^{1-\beta}}{1-\beta}\right)\int_\Omega |x|^{-\alpha}\frac{|\varphi|^{1-\beta}}{|x|^\alpha}dx. \tag{91}$$

Letting $v = t\varphi$ for t large enough, we obtain $E_\theta(v) \le 0$. For t large enough, we can ensure $\|v\| > \varepsilon$. □

Let Γ and c defined by

$$\begin{aligned}\Gamma &\coloneqq \left\{\gamma : [0,1] \longrightarrow \mathscr{W}_\rho : \gamma(0) = u_0^- \text{and} \gamma(1) = u_0^+\right\}, \\ c &\coloneqq \inf_{\gamma\in\Pi}\max_{t\in[0,1]}(E_\theta(\gamma(t))).\end{aligned} \tag{92}$$

Proof of Theorem 19. If

$$0 < \lambda < \lambda^{**} \coloneqq \min(\lambda^*, \Lambda_4), \tag{93}$$

then, by the Lemma 5 and Proposition 9 (ii), E_θ verifying the Palais-Smale condition in $\mathscr{W}_\rho$. Moreover, from the Lemmas 6, 17, and 18, there exists u_c such that

$$E_\theta(u_c) = c \text{and} u_c \in \mathscr{W}_\rho. \tag{94}$$

Thus, u_c is the third solution of our system such that $u_c \ne u_0^+$ and $u_c \ne u_0^-$. Since (1) is odd with respect u, we obtain that $-u_c$ is also a solution of (1).

Finally, for every $\theta \in (0,1)$, problem (4) has solution $u_\theta \in \mathscr{H}$ such that $E_\theta(u_\theta) = 0$. Thus, there exist $\{\theta_n\} \subset (0,1)$ with $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, we get $u = \lim_{n\longrightarrow\infty} u_{\theta_n}$. □

6. Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem on the constraint defined by the Nehari manifold $\mathscr{W}$, which are solutions of our problem. Under some sufficient conditions on coefficients of equation of (1) such that $N \ge 3$, $0 \le \alpha < ((N(q+\beta))/((q+1)))$ and $\beta \in (0,1)$, we split $\mathscr{W}$ in two disjoint subsets $\mathscr{W}^+$ and $\mathscr{W}^-$; thus, we consider the minimization problems on $\mathscr{W}^+$ and $\mathscr{W}^-$, respectively. In Sections 3 and 4 we have proved the existence of at least two nontrivial solutions on $\mathscr{W}_\rho$ for all $0 < \lambda < \lambda^{**} \coloneqq \min(\lambda^*, \Lambda_4)$ if $N \ge 3$ and $\beta \in (0,1)$.

Data Availability

The functional analysis data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there is no conflicts of interest.

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References

[1] B. Abdellaoui, V. Felli, and I. Peral, “Existence and non-existence results for quasilinear elliptic equations involving the p-Laplacian,” *Bollettino dell'Unione Matematica Italiana*, vol. 9, pp. 445–484, 2006.

[2] M. Bhakta and A. Biswas, “Hardy-Sobolev type equations for p -Laplacian, $1 < p < 2$, in bounded domain,” *Differential and Integral Equations.*, vol. 24, no. 1/2, pp. 135–158, 2011.

[3] N. Ghoussoub and C. Yuan, “Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents,” *Transactions of the American Mathematical Society*, vol. 352, no. 12, pp. 5703–5743, 2000.

[4] P. Pucci and R. Servadei, “Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights,” *Annales de l'IHP Analyse non linéaire*, vol. 25, no. 3, pp. 505–537, 2008.

[5] H. Brezis and L. Nirenberg, “Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,” *Communications on Pure and Applied Mathematics*, vol. 36, no. 4, pp. 437–477, 1983.

[6] N. E. Alaa, F. Aqel, and L. Taourirte, “On singular quasilinear elliptic equations with data measures,” *Advances in Nonlinear Analysis*, vol. 10, no. 1, pp. 1284–1300, 2021.

[7] A. Bahri and J. M. Coron, “On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain,” *Communications on Pure and Applied Mathematics*, vol. 41, no. 3, pp. 253–294, 1988.

[8] D. Cao and S. Peng, “A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms,” *Journal of Differential Equations*, vol. 193, no. 2, pp. 424–434, 2003.

[9] J. Chen, “Existence of solutions for a nonlinear PDE with an inverse square potential,” *Journal of Differential Equations*, vol. 195, no. 2, pp. 497–519, 2003.

[10] J. Q. Chen, “Multiple positive solutions for a class of nonlinear elliptic equations,” *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 341–354, 2004.

[11] I. Ekeland and N. Ghoussoub, “Selected new aspects of the calculus of variations in the large,” *Bulletin of the American Mathematical Society*, vol. 39, no. 2, pp. 207–265, 2002.

[12] I. Ekeland, “On the variational principle,” *Journal of Mathematical Analysis and Applications*, vol. 47, no. 2, pp. 324–353, 1974.

[13] M. E. O. El Mokhtar and A. Matallah, “Existence of multiple positive solutions for Brezis–Nirenberg-type problems involving singular nonlinearities,” *Journal of Mathematics*, vol. 2021, Article ID 4709839, 8 pages, 2021.

[14] M. Ghergu and V. Radulescu, *Nonlinear PDEs : Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer Verlag, Heidelberg, 2011.

[15] C. S. Goodrich and A. Scapellato, “Partial regularity of minimizers of asymptotically convex functionals with p(x)-growth,” *Studia Mathematica*, vol. 264, no. 1, pp. 71–102, 2022.

[16] J. Lee, J.-M. Kim, Y.-H. Kim, and A. Scapellato, “On multiple solutions to a nonlocal fractional p(\cdot)-Laplacian problem with concave–convex nonlinearities,” *Advances in Continuous and Discrete Models*, vol. 2022, no. 1, pp. 1–14, 2022.

[17] N. S. Papageorgiou and A. Scapellato, “Nonlinear singular problems with convection,” *Journal of Difference Equations*, vol. 296, 511 pages, 2021.

[18] N. S. Papageorgiou and A. Scapellato, “Positive solutions for anisotropic singular (p, q)-equations,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 71, no. 5, p. 155, 2020.

[19] M. Schechter and W. M. Zou, “On the Brézis–Nirenberg problem,” *Archive for Rational Mechanics and Analysis*, vol. 197, no. 1, pp. 337–356, 2010.

[20] L. Ding and C. L. Tang, “Existence and multiplicity of solutions for semilinear elliptic equations with Hardy terms and Hardy-Sobolev critical exponents,” *Applied Mathematics Letters*, vol. 20, no. 12, pp. 1175–1183, 2007.

[21] M. E. O. El Mokhtar and Z. I. Almuhiameed, “On singular elliptic equation with singular nonlinearities,” *Hardy-Sobolev Critical Exponent and Weights, Differential Equations and Applications*, vol. 4, pp. 397–410, 2009.

[22] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, vol. I, Elliptic Equations, Pitman, London, 1985.

[23] X. Fang and J. Zhang, “Multiplicity of positive solutions for quasilinear elliptic equations involving critical nonlinearity,” *Advances in Nonlinear Analysis*, vol. 9, no. 1, pp. 1420–1436, 2020.

[24] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, “Nonlinear nonhomogeneous singular problems,” *Calculus of Variations and Partial Differential Equations*, vol. 59, no. 1, pp. 1–31, 2020.

[25] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, “Positive solutions for nonlinear parametric singular Dirichlet problems,” *Bulletin of Mathematical Sciences*, vol. 20, pp. 1–22, 2019.

[26] D. Kang and S. Peng, “Positive solutions for singular critical elliptic problems,” *Applied Mathematics Letters*, vol. 17, no. 4, pp. 411–416, 2004.

[27] J. R. Esteban and J. L. Vázquez, “On the equation of turbulent filtration in one- dimensional porous media,” *Nonlinear Analysis: Theory, Methods & Applications.*, vol. 10, no. 11, pp. 1303–1325, 1986.

[28] M. A. Herrero and J. L. Vazquez, “On the propagation properties of a nonlinear degenerate parabolic equation,” *Differential Equations*, vol. 7, no. 12, pp. 1381–1402, 1982.

[29] V. Mikljukov, “On the asymptotic properties of sub-solutions of quasilinear equations of elliptic type and mappings with bounded distortion,” *Sbornik Mathematics (N.S.)*, vol. 111, 1980(in Russian).

[30] K. Uhlenbeck, “Regularity for a class of non-linear elliptic systems,” *Acta Mathematica*, vol. 138, pp. 219–240, 1977.

[31] L. Jeanjean, “On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on$\mathbb{R}$N,” *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 129, no. 4, pp. 787–809, 1999.

[32] Z. Nehari, “On a class of nonlinear second-order differential equations,” *Transactions of the American Mathematical Society*, vol. 95, no. 1, pp. 101–123, 1960.

[33] Z. Nehari, “Characteristic values associated with a class of nonlinear second-order differential equations,” *Acta Math.*, vol. 105, no. 3-4, pp. 141–175, 1961.

[34] K. J. Brown and Y. Zhang, “The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function,” *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[35] G. Tarantello, “On nonhomogeneous elliptic equations involving critical Sobolev exponent,” *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[36] H. Brezis and E. Lieb, “A relation between pointwise convergence of functions and convergence of functionals,” *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.