

Retraction

Retracted: Double Weak Hopf Quiver and Its Path Coalgebra

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] M. N. Khan, M. Ahmed, M. Arshad, W. Almutiry, R. A. R. Bantan, and M. Elgarhy, "Double Weak Hopf Quiver and Its Path Coalgebra," *Journal of Function Spaces*, vol. 2022, Article ID 5421294, 9 pages, 2022.

Research Article

Double Weak Hopf Quiver and Its Path Coalgebra

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The main input of this research is the introduction of the concept of double weak Hopf quiver (DWHQ). In addition, the structures of weak Hopf algebra (WHA) are obtained through path coalgebra of the proposed quivers. Furthermore, the module and comodule structures on the said WHA are discussed. Moreover, the classification of the semilattice-graded WHA structures of the path coalgebra obtained from the so called DWHQ was presented. This contributes a step further in the development of the module and comodule structures of more general algebras. These structures are important in the physics for dynamic systems.

1. Introduction

Hopf quiver (HQ) was introduced by Cibils and Rosso, who also explained the framework and structures of the Hopf algebra (HA) which was obtained corresponding to the HQ [1]. By [2], the classes of HA are deliberated for the representation has tensor structures that are induced by the graded Hopf structures of kT . Huang and Tao [3] give a thorough list of graded coquasitriangular HA over the HQ. Ahmed and Li introduced the concept of the so-called weak HQ (WHQ) and discussed some structures of its corresponding weak HA (WHA) and weak Hopf modules (WHM) [4]. Some literatures that aid in better understanding these algebras are listed in [1, 2, 5–10]. Nichita [5] presented the Yang-Baxter equation with open problems. Virginia et al. [6] defined the quivers, Yang [7] worked on the weak Hopf algebras corresponding to Cartan matrices, Cheng [8] described the classes of weak Hopf algebra, Cao [9] discusses the semilattice-graded WHA along with its quasibicrossed product, Huang and Yang [2] gave the green ring of minimal HQ, and Hai [10] introduced the coquasitriangular Hopf algebras in braided categories. Furthermore,

refer to [3, 4, 11–18] for representation theory, bimodules, and characteristics of Hopf Algebra; Huang and Tao [3] also worked on coquasitriangular structures on Hopf quivers. Ahmed and Li [4] proposed the weak Hopf quivers. Auslander et al. [11] came with the representation theory of artin algebra, Chin and Montgomery [12] defined the basic coalgebras. Cibils [13] found the tensor products of Hopf bimodules on some group. Nakajima [14] introduced the quiver varieties for representation theorists and ring. Simson [15] studied the comodules, coalgebras, tame comodule type, and pseudocompact algebras. Woodcock [16] offered some remarks on the coalgebras' representation theory. Alvarez et al. [17] proposed quasigroupoids and weak Hopf quasigroups, and Rodriguez and Raposo [18] formulated the weak crossed products over weak Hopf algebras. Naseer Khan et al. [19] worked on the WHA and its quiver representation. Li [20] solved the quantum Yang-Baxter equation, Montgomery [21] showed the action of HA on rings, Radford [22] described the projection of structures of HA, Daele and Wang [23] defined the multipliers of HA, Swedler [24], Yang and Zhang [25], and Smith [26] also extended the theory of Hopf Algebra.

Dual weak HQ (DWHQ) is introduced in [27], where the structures of the path coalgebra of a dual weak Hopf quiver are discussed. This coalgebra equips the structure of a WHA.

Dual weak Hopf quiver and its path algebra which is a semilattice-graded weak Hopf algebra (SLGWHA). For $\bar{a}, \bar{b} \in H^*$, where $\bar{a} \in G^*_{\mu}$, and $\bar{b} \in G^*_{\lambda}$, the multiplication $\bar{a} \cdot \bar{b}$ in \bar{H} is given by $\bar{a} \cdot \bar{b} = \varphi^*_{\mu, \lambda \mu}(\bar{a}) \varphi^*_{\lambda, \lambda \mu}(\bar{b})$.

We introduce the double weak Hopf quiver and obtain a WHA structure from the path coalgebra of this quiver. The path coalgebra $k\bar{T}$ has the structure of cotensor coalgebra. We denote \bar{T} by $\bar{T}(S \times S^*, R)$ is called a double WHQ of a Clifford monoid. We have an algebra $\bar{H} = \bigoplus_{\mu, \lambda \in Y} H_{\mu} \otimes H_{\lambda}^* = kS \otimes (kS)^* \cong k\bar{T}(S \times S^*, R)$, where $H_{\mu} = kG_{\mu}$ and $H_{\lambda}^* = (kG_{\lambda})^*$.

The weak Hopf algebra \bar{H} is semilattice graded if $\bar{H} = \bigoplus_{\mu, \lambda \in Y} H_{\mu} \otimes H_{\lambda}^*$ is a semilattice-graded sum, where $H_{\mu, \lambda}$; $\mu, \lambda \in Y$ are subweak HAs which are HAs with antipodes the restriction $\bar{T}|_{H_{\mu, \lambda}}$ for every $\mu, \lambda \in Y$ [9]. There are weak Hopf algebra homomorphism $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_{\mu} \times G_{\lambda}^* \longrightarrow G_{\gamma} \times G_{\delta}^*$, where $\varphi^{\gamma, \delta}_{\mu, \lambda} = (\varphi_{\mu, \lambda}, \varphi^*_{\gamma, \delta})$ and $G_{\alpha} \times G_{\beta}$; $\alpha, \beta \in Y$ is a of $S \times S^*$. For two homomorphism $\varphi^{\gamma, \delta}_{\mu, \lambda}, \varphi^{\sigma, \rho}_{\gamma, \delta}$, their composition $\varphi^{\sigma, \rho}_{\gamma, \delta} \varphi^{\gamma, \delta}_{\mu, \lambda} = \varphi^{\sigma, \rho}_{\mu, \lambda}$ from $H_{\lambda} \otimes H_{\mu}$ to $H_{\sigma} \otimes H_{\rho}$ if $\lambda \mu = \mu$.

Let $u, v \in S \times S^*$ the multiplication $u \cdot v$ in \bar{H} is given by $u \cdot v = \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v)$, where $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_{\mu} \times G_{\lambda}^* \longrightarrow G_{\gamma} \times G_{\delta}^*$ is the mapping from the subgroup $G_{\mu} \times G_{\lambda}^*$ to $G_{\gamma} \times G_{\delta}^*$ for all $\mu \geq \gamma$ or $\lambda \leq \delta$; $\lambda, \mu, \gamma, \delta \in Y$.

$$u \cdot v = \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v). \quad (1)$$

A Clifford monoid $S \times S^*$ is a regular semigroup $S \times S^*$ with identity. Let $u, v \in S \times S^*$ where $u = (g_{\mu}, g^*_{\delta}) \in G_{\mu} \times G_{\delta}^*$ and $v = (g_{\gamma}, g^*_{\rho}) \in G_{\gamma} \times G_{\rho}^*$ then

$$u \cdot v = \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v). \quad (2)$$

The structure of comultiplication in $\bar{H} = H \otimes H^*$ is defined by

$$\Delta(a) = \Delta(x \otimes \bar{y}) = \sum_{\substack{x', x'' \in H \\ \bar{y}', \bar{y}'' \in H^*}} (x' \otimes \bar{y}') \otimes (x'' \otimes \bar{y}''), \quad (3)$$

for all $a = x \otimes \bar{y} \in \bar{H} = H \otimes H^*$, where $x \in H$ & $\bar{y} \in H^*$. Since the weak antipode T in H is an element of $H \text{om}_{\kappa}(H, H)$ and the weak antipode T^* exists in H^* which is an element of $H \text{om}_{\kappa}(H^*, H^*)$ so that \bar{T} exists in $\bar{H} = H \otimes H^*$ as an element of $H \text{om}_{\kappa}(\bar{H}, \bar{H})$. $id_{\bar{H}} * \bar{T} * id_{\bar{H}} = id_{\bar{H}}$ satisfying $\bar{T} * id_{\bar{H}} * \bar{T} = \bar{T}$. This makes \bar{H} is a WHA. If H is finite dimensional vectors space, so is H^* hence $\bar{H} = H \otimes H^*$ is also a

finite dimensional. Thus

$$:\bar{H}^* = \bar{H}. \quad (4)$$

Here, $k\bar{T}_0$ is a linearization of \bar{T}_0 having the natural structure of coalgebra. The vector space $k\bar{T}$ is said to be a $k\bar{T}_0$ bimodule and $k\bar{T}_0$ comodule with left and right comodule structure maps $\delta_L(u) = t(u) \otimes u$ and $\delta_R(u) = u \otimes s(u)$, respectively $\forall u \in k\bar{T}$. The self-cotensor product of $k\bar{T}$ is the kernel of

$$\delta_R \otimes 1 - 1 \otimes \delta_L : k\bar{T} \otimes k\bar{T} \longrightarrow k\bar{T} \otimes k\bar{T}_0 \otimes k\bar{T}. \quad (5)$$

The cotensor coalgebra over $k\bar{T}_0$ of the bicomodule $k\bar{T}$ is a path coalgebra.

For two vertices u and v of \bar{T} the (v, u) -isotypic component (IC) of a $k\bar{T}_0$ -bicomodule, \bar{M} is

$${}^v(\bar{M})^u = \{\eta \in \bar{M} \mid \delta_R(\eta) = \eta \otimes u, \delta_L(\eta) = v \otimes \eta\}. \quad (6)$$

Particularly, ${}^v(k\bar{T}_n)^u$ corresponds to the vector space that has n -paths to v from u , see [1].

A weak Hopf bimodule is a semigroup algebra provided that the classification of path coalgebra admits graded WHA structure. The representations of the WHA are called the weak Hopf bimodules (WHBM).

By Cibils and Rosso [1], we obtain that there is a one-to-one correspondence between the elements of WHBM with a complete list-graded WHA structures on the path coalgebra $k\bar{T}$ corresponding to the DWHQ \bar{T} .

Further, the group-like elements of the SLGWHA correspond to the vertices of the DWHQ \bar{T} . By [17, th. 4.5], the $G(\bar{H}) = G(kS \otimes (kS)^*)$ of SLGWHA \bar{H} is the Clifford monoid $S \times S^*$. Thus, $k(S \times S^*)$ becomes a SLGWHA.

The remaining paper is arranged such that Section 2 defines some basic concepts, Section 3 presents the main contributions of the paper, i.e., double weak Hopf quiver, its results, properties, and example. Section 4 concludes the paper.

2. Preliminaries

This section presents some basic concepts and definitions to the work such as algebra, coalgebra, bialgebra, module, comodule, and bimodule.

Definition 1. (see [20]). A k algebra H is a vector space over a field k . H is called an algebra if H has a multiplication $\eta : H \otimes H \longrightarrow H$ and a unit $\eta : k \longrightarrow H$ such that $\eta(Id \otimes \eta) = \eta(\eta \otimes Id)$ (associativity) and $Id = \eta(u \otimes Id) = \eta(Id \otimes u)$ (unitary property), where Id is the identity map of H .

Definition 2. (see [20]). H is called a coalgebra if H has a comultiplication $\Delta : H \longrightarrow H \otimes H$ and a counit $\varepsilon : H \longrightarrow k \ni (Id \otimes \Delta)\Delta = (\Delta \otimes Id)\Delta$ (this property is known as coassociativity of Δ) and $Id = (\varepsilon \otimes Id)\Delta = (Id \otimes \varepsilon)\Delta$ (this property is known as counitary property).

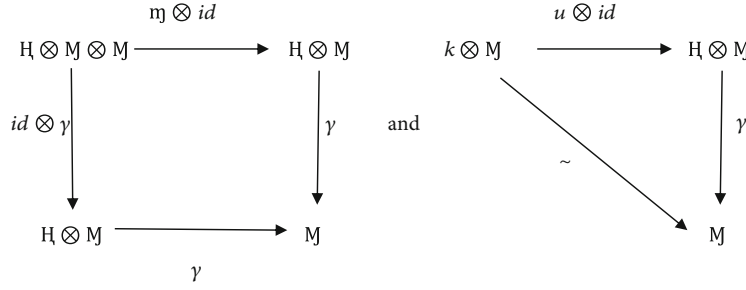


FIGURE 1: Commute.

Definition 3. (see [20]). A bialgebra is quintuple $(H, \mu, \Delta, \eta, \epsilon)$ where H is a vector space, μ is multiplication, Δ is comultiplication, η is unit, and ϵ is counit. An algebra is a k -vector space H endowed with k -linear maps.

$$\mu : H \otimes H \longrightarrow H, \quad \Delta : H \longrightarrow H \otimes H, \quad \epsilon : H \longrightarrow \mathbb{C}, \quad \eta : \mathbb{C} \longrightarrow H.$$

Definition 4. (see [21]). If we have a unique element $\rho \in \text{Hom}_k(H, H)$ such that $id_H * \rho = \rho * id_H = \mu \epsilon$, where $*$ is the convolution in $\text{Hom}_k(H, H)$. Then, H becomes a Hopf algebra with ρ as the antipode of H .

Definition 5. (see [20]). A bialgebra H over a field k is called a weak Hopf algebra if there is an element T in the convolution algebra $\text{Hom}_k(H, H)$ such that $id * T * id = id$ and $T * id * T = T$, where T is a weak antipode of H see.

Definition 6. (see [9]). A WHA H with a weak antipode T is a semilattice graded if $H = \bigoplus_{\lambda \in Y} H_\lambda$, where the graded summands H_λ ; $\lambda \in Y$ are subweak HA. Obviously, these are HAs with antipodes the restrictions $T|_{H_\lambda}$ for each $\lambda \in Y$. Then, there exist homomorphisms $\varphi_{\lambda, \mu} : H_\lambda \longrightarrow H_\mu$ if $\lambda \mu = \mu$ such that $a \in H_\lambda$ and $b \in H_\mu$, the multiplication $a.b$ in H is given by $a.b = \varphi_{\lambda, \lambda \mu}(a) \varphi_{\mu, \lambda \mu}(b)$.

Definition 7. (see [13]). Let (H, η, ν) be a k -algebra. A left A -module is a pair (M, γ) , where M is a k -vector space, and $\gamma : H \otimes M \longrightarrow M$ such that the diagrams in Figures 1(a) and 1(b).

The Cat. of left H -module is denoted by $H^{\mathcal{M}}$.

Definition 8. (see [15]). Let (H, Δ, ϵ) be a k -algebra, a right H -comodule is a k -vector space M with a k -linear map $\rho : M \longrightarrow M \otimes H$ such that the diagrams in Figures 2(a) and 2(b).

The Cat. of right H -comodule is denoted by \mathcal{M}^H .

Definition 9. (see [15]). An H -bimodule is a left and a right module M with structure maps $\mu_l : H \otimes M \longrightarrow M$ and $\mu_r : M \otimes H \longrightarrow M$ that satisfy the conditions in Figure 3.

3. Double Weak Hopf Quiver

Suppose $S = \bigcup_{\mu \in Y} G_\mu$ is a Clifford monoid and Y represents semilattice. We take S as the basis for the underlying vector space of the weak Hopf algebra $H = kS$ which is of course a “SLGWHA”. Let S^* be the dual basis of S and is a basis for the dual vector space $H^* = (kS)^* = \bigoplus_{\mu \in Y} (kG_\mu)^* = \bigoplus_{\mu \in Y} kG_\mu^* = \bigoplus_{\mu \in Y} H_\mu^* = kS^* = (kS)^*$, where G_μ^* is dual basis of the basis G_μ of a vector space $H_\mu^* = kG_\mu^*$, which equips the structure of a Hopf algebra for every $\mu \in Y$. If $x \in S$ and $\bar{y} \in S^*$, then $u = (x, \bar{y})$ becomes an element of $S \times S^*$ which is the basis of the vector space $\bar{H} = kS \otimes (kS)^*$, where \bar{H} is also a weak Hopf algebra.

Definition 10. For all $u, v \in S \times S^*$ such that $u = (x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda^*$ and $v = (s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta^*$, where $x_\mu \in G_\mu$, $s_\gamma \in G_\gamma$; $\bar{y}_\lambda \in G_\lambda^*$, $\bar{t}_\delta \in G_\delta^*$. Then, the mapping $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_\mu \times G_\lambda^* \longrightarrow G_\gamma \times G_\delta^*$. If $\mu \geq \gamma$ or $\lambda \leq \delta$ then the total arrows from u and $\varphi^{\gamma, \delta}_{\mu, \lambda}(u)$ to v are equal, which is equal to $R_{c_\mu, \bar{c}_\lambda}$, if there exists $(c_\mu, c_\lambda^*) \in \mathcal{C}_\mu \times \mathcal{C}_\lambda^*$ such that $v = (c_\mu, c_\lambda^*) \varphi^{(\gamma, \delta)}_{(\mu, \lambda)}(u)$. R is ramification data (RD) of $S \times S^*$, which is the sum of the RD R_{c_γ, c_δ^*} of the group $G_\gamma \times G_\delta^*$; $\gamma, \delta \in Y$.

The ramification data of Clifford monoid $S \times S^* = \bigcup_{\lambda, \mu \in Y} (x_\lambda, \bar{y}_\mu)$, where Y is a semilattice. The RD R of $S \times S^*$ which is sum of the RD R_{c_γ, c_δ^*} of the group $G_\gamma \times G_\delta^*$; $\gamma, \delta \in Y$. Now, $R = \sum_{\lambda, \mu \in Y} R_{\mu, \lambda} = \sum_{\lambda, \mu \in Y} (\sum_{C_\mu \in \mathcal{C}_\mu, C_\lambda^* \in \mathcal{C}_\lambda^*} R_{c_\mu, \bar{c}_\lambda} C_\mu \times C_\lambda^*)$, where $R_{\mu, \lambda} = \sum_{u=x_\mu, \bar{y}_\lambda} \in C_\mu \times C_\lambda^* R_{c_\mu, \bar{c}_\lambda} C_\mu \times C_\lambda^*$ and $C_\mu \times C_\lambda^*$ is the conjugacy class of a subgroup $G_\mu \times G_\lambda^*$.

(i) Then, R is considered to be a positive central element of ring $kS \otimes (kS)^*$. Moreover, \mathcal{C}_μ and \mathcal{C}_λ^* denote the collections of all the conjugacy classes of G_μ and G_λ^* , respectively, for all $\mu, \lambda \in Y$

(ii) If R is ramification data of $S \times S^*$ and \bar{T} is a quiver such that

(a) the set of vertices of $\bar{T}(S \times S^*, R)$ is the set of all elements of $S \times S^*$

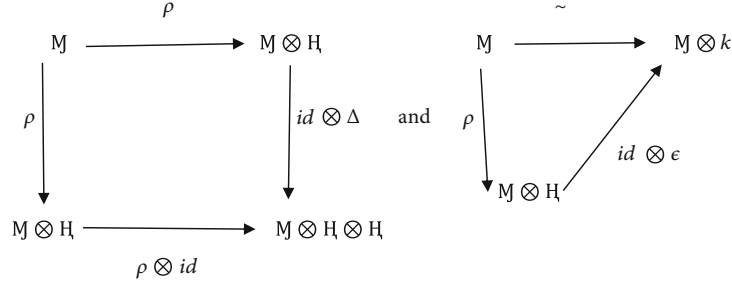


FIGURE 2: Commute.

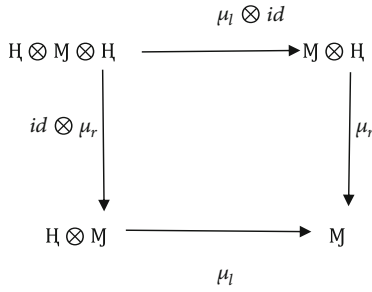


FIGURE 3: Left and right actions commute.

- (b) for all $u, v \in S \times S^*$ such that $u = (x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda^*$ and $v = (s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta^*$ where $x_\mu \in G_\mu, s_\gamma \in G_\gamma; \bar{y}_\lambda \in G_\lambda^*, \bar{t}_\delta \in G_\delta^*$

and for $\mu \not\leq \gamma$ and $\lambda \not\leq \delta$, \nexists any arrow from u to v . If $\mu \geq \gamma$ or $\lambda \leq \delta$ then the total arrows from u to v and $\varphi^{\gamma, \delta}_{\mu, \lambda}(u)$ to v are equal, which are equal to R_{C_γ, C_δ^*} ; if there exists $(C_\mu, C_\lambda^*) \in \mathcal{C}_\mu \times \mathcal{C}_\lambda^*$ such that $v = (c_\mu, c_\lambda^*) \varphi^{\gamma, \delta}_{\mu, \lambda}(u)$, where $\varphi^{\gamma, \delta}_{\mu, \lambda}$ is a mapping given by $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_\mu \times G_\lambda^* \rightarrow G_\gamma \times G_\delta^*$. Then, $\bar{T} = \bar{T}(S \times S^*, R)$ is called a double weak Hopf quiver of R corresponding to the Clifford monoid $S \times S^*$. The maps $\varphi^{\mu, \lambda}_{\mu, \lambda}$ is an identity mapping, as a homomorphism of the group $G_\mu \times G_\lambda^*$. If $\varphi^{\gamma, \delta}_{\mu, \lambda}$ and $\varphi^{\sigma, \rho}_{\gamma, \delta}$ be any two homomorphisms as defined above then $\varphi^{\sigma, \rho}_{\gamma, \delta} \varphi^{\gamma, \delta}_{\mu, \lambda} = \varphi^{\sigma, \rho}_{\mu, \lambda}$ and for any $u, v \in S \times S^*$ we have $u = (x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda^*$ and $v = (s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta^*$, then $u.v = \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v)$; provided that $\gamma \leq \mu$ iff $\mu\gamma = \gamma$ and $\delta \leq \lambda$ iff $\lambda\delta = \delta$ for the partial ordering “ \leq ” in a semilattice Y , for all $\{\mu, \lambda, \gamma, \sigma, \delta\} \in Y$. We denote \bar{T} by $\bar{T}(S \times S^*, R)$ is called a double quiver of a Clifford monoid. We have an algebra $\bar{H} = \bigoplus_{\mu, \lambda \in Y} H_\mu \otimes H_\lambda^* = k S \otimes (kS)^* \cong k\bar{T}(S \times S^*, R)$, where $H_\mu = kG_\mu$ and $H_\lambda^* = (kG_\lambda)^*$. \bar{H} is a weak Hopf algebra if there exist an element $\bar{T} = (T, T^*)$ in $\text{Hom}_k(\bar{H}, \bar{H})$ such that $id * \bar{T} * id = id$ and $\bar{T} * id * \bar{T} = \bar{T}$, where $*$ is the convolution product in $\text{Hom}_k(\bar{H}, \bar{H})$ and a weak antipode of \bar{H} is denoted by \bar{T} , also see [7, 8, 20]. The WHA \bar{H} with a weak antipode \bar{T} is a “SLGWHA” if $\bar{H} = \bigoplus_{\mu, \lambda \in Y} H_\mu \otimes H_\lambda^*$ is a semilattice graded sum, where $H_{\mu, \lambda}; \mu, \lambda \in Y$ are sub-WHA which are HA with antipodes the restriction $\bar{T}|_{H_{\mu, \lambda}}$ for each $\mu, \lambda \in Y$ [9]. There

TABLE 1: Semilattices.

| . | α | β | γ | ρ | σ | δ |
|----------|----------|----------|----------|----------|----------|----------|
| α | α | α | α | α | α | α |
| β | α | β | β | α | β | β |
| γ | α | β | γ | α | β | γ |
| ρ | α | α | α | ρ | ρ | ρ |
| σ | α | β | β | ρ | σ | σ |
| δ | α | β | γ | ρ | σ | δ |

is WHA homomorphism $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_\mu \times G_\lambda^* \rightarrow G_\gamma \times G_\delta^*$, where $\varphi^{\gamma, \delta}_{\mu, \lambda} = (\varphi_{\mu, \lambda}, \varphi^*_{\gamma, \delta})$ and $G_\alpha \times G_\beta$; $\alpha, \beta \in Y$ is subgroup of $S \times S^*$. For two homomorphism $\varphi^{\gamma, \delta}_{\mu, \lambda}, \varphi^{\sigma, \rho}_{\gamma, \delta}$ their composition $\varphi^{\sigma, \rho}_{\gamma, \delta} \varphi^{\gamma, \delta}_{\mu, \lambda} = \varphi^{\sigma, \rho}_{\mu, \lambda}$ from $H_\lambda \otimes H_\mu$ to $H_\sigma \otimes H_\rho$ if $\lambda\mu = \mu$. Such that for all $u = (x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda^*$ and $v = (s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta^*$, where $x_\mu \in G_\mu, s_\gamma \in G_\gamma; \bar{y}_\lambda \in G_\lambda^*, \bar{t}_\delta \in G_\delta^*$, where the multiplication $u.v$ in \bar{H} is given by $u.v = \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v)$, where $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_\mu \times G_\lambda^* \rightarrow G_\gamma \times G_\delta^*$ is the mapping from the subgroup $G_\mu \times G_\lambda^*$ to $G_\gamma \times G_\delta^*$ for all $\mu \geq \gamma$ or $\lambda \leq \delta; \lambda, \mu, \gamma, \delta \in Y$.

$$\begin{aligned} u.v &= \varphi^{\gamma, \delta}_{\mu, \lambda}(x_\mu, \bar{y}_\lambda) \varphi^{\sigma, \rho}_{\gamma, \delta}(s_\gamma, \bar{t}_\delta) = (\varphi_{\mu, \lambda}, \varphi^*_{\gamma, \delta})(x_\mu, \bar{y}_\lambda) (\varphi_{\gamma, \delta}, \varphi^*_{\sigma, \rho})(s_\gamma, \bar{t}_\delta) \\ &= \varphi_{\mu, \lambda}(x_\mu), \varphi^*_{\gamma, \delta}(\bar{y}_\lambda) \varphi_{\gamma, \delta}(s_\gamma), \varphi^*_{\sigma, \rho}(\bar{t}_\delta). \end{aligned} \quad (7)$$

For two mappings $\varphi^{\sigma, \rho}_{\gamma, \delta} \varphi^{\gamma, \delta}_{\mu, \lambda} = \varphi^{\sigma, \rho}_{\mu, \lambda}$.

A Clifford monoid $S \times S^*$ is a regular semigroup $S \times S^*$ with identity. Let $u, v \in S \times S^*$ where $u = (g_\mu, g^*_\delta) \in G_\mu \times G_\delta^*$ and $v = (g_\gamma, g^*_\rho) \in G_\gamma \times G_\rho^*$ then

$$\begin{aligned} u.v &= \varphi^{\gamma, \delta}_{\mu, \lambda}(u) \varphi^{\sigma, \rho}_{\gamma, \delta}(v) = (\varphi_{\mu, \lambda}, \varphi^*_{\gamma, \delta})(g_\mu, g^*_\delta) (\varphi_{\gamma, \delta}, \varphi^*_{\sigma, \rho})(g_\gamma, g^*_\rho) \\ &= (\varphi_{\mu, \lambda}(g_\mu), \varphi^*_{\gamma, \delta}(g^*_\delta)) (\varphi_{\gamma, \delta}(g_\gamma), \varphi^*_{\sigma, \rho}(g^*_\rho)) \\ &= (\varphi_{\mu, \lambda}(g_\mu) \varphi_{\gamma, \delta}(g_\gamma), \varphi^*_{\gamma, \delta}(g^*_\delta) \varphi^*_{\sigma, \rho}(g^*_\rho)) = (\varphi_{\gamma, \lambda}(g_\gamma), \varphi^*_{\gamma, \rho}(g^*_\rho)) \\ &= (g_\lambda, g^*_\gamma) \in G_\lambda \times G_\gamma^* \text{ where } \mu\gamma = \gamma \&\sigma\rho = \rho \text{ or } \gamma \leq \mu \&\rho \leq \sigma. \end{aligned} \quad (8)$$

In the discussion below, we see that if $\bar{H} = kS \otimes (kS)^*$ is a

WHA, then

$$(\overline{H})^* = (kS \otimes (kS)^*)^*, \quad (9)$$

$\overline{H}^* = ((kS)^*)^* \otimes kS^* = kS \otimes kS^* = \overline{H}$ (\cdot : S is a finite semi-group); thus, \overline{H} is self dual of \overline{H} , where S^* is the dual basis of the WHA H and is also a Clifford monoid. The multiplication in $\overline{H} = H \otimes H^*$ is given by $a.b = \varphi^{\gamma, \delta}_{\mu, \lambda}(a)\varphi^{\sigma, \rho}_{\gamma, \delta}(b)$ for $a, b \in \overline{H} = H \otimes H^*$ of the form $a = x \otimes \bar{y}$ & $b = s \otimes \bar{t}$. Then, their multiplication $a.b = xs \otimes \bar{t}\bar{y}$ is in $\overline{H} = H \otimes H^*$. The structure of comultiplication in $\overline{H} = H \otimes H^*$ is defined by

$$\Delta(a) = \Delta(x \otimes \bar{y}) = \sum_{\substack{x' \otimes x'' \in H \\ \bar{y}' \otimes \bar{y}'' \in H^*}} (x' \otimes \bar{y}') \otimes (x'' \otimes \bar{y}'') = \sum_{\substack{x' \otimes x'' \in H \\ \bar{y}' \otimes \bar{y}'' \in H^*}} (x'x'' \otimes \bar{y}'\bar{y}''), \quad (10)$$

for all $a = x \otimes \bar{y} \in \overline{H} = H \otimes H^*$, where $x \in H$ & $\bar{y} \in H^*$. Since the weak antipode T in H is an element of $H \text{om}_{\kappa}(H, H)$ and the weak antipode T^* exists in H^* which is an element of $H \text{om}_{\kappa}(H^*, H^*)$ so that \bar{T} exists in $\overline{H} = H \otimes H^*$ as an element of $H \text{om}_{\kappa}(\overline{H}, \overline{H})$. $id_{\overline{H}} * \bar{T} * id_{\overline{H}} = id_{\overline{H}}$ satisfying $\bar{T} * id_{\overline{H}} * \bar{T} = \bar{T}$. This makes \overline{H} is a weak Hopf algebra. If H is finite dimensional vectors space, so is H^* hence $\overline{H} = H \otimes H^*$ is also a finite dimensional. Thus,

$$\therefore \overline{H}^* = (H \otimes H^*)^* \cong H^{**} \otimes H^* \cong H \otimes H^* = \overline{H}. \quad (11)$$

The idempotents of S^* lie in its center $C(S^*)$. S^* is a regular monoid such that $S^* = \cup_{\lambda \in Y} G^*_{\lambda}$ and $G^*_{\lambda} G^*_{\mu} \subseteq G^*_{\lambda\mu} \forall \lambda, \mu \in Y$, G^*_{λ} for $\lambda \in Y$ and is dual basis of the dual $H^*_{\lambda} = kG^*_{\lambda}$ of the Hopf algebra $H_{\lambda} = kG_{\lambda}$. Further, for any $\lambda, \mu \in Y$ with $\lambda\mu = \mu$, there is a homomorphism $\varphi^{\gamma, \delta}_{\mu, \lambda} : G_{\mu} \times G_{\lambda} \rightarrow G_{\gamma} \times G_{\delta}$ and if $\lambda\mu = \mu$ and $\mu\nu = \nu$, then $\varphi^*_{\lambda, \mu} \varphi^*_{\mu, \nu} = \varphi^*_{\lambda, \nu}$ with $\varphi^{\mu, \lambda}_{\mu, \lambda}$ is an identity mapping of $G_{\mu} \times G_{\lambda}$. The partial ordering “ \leq ” in Y is given by “ $\mu \leq \lambda \iff \lambda\mu = \mu \forall \mu, \lambda \in Y$.”

The immediate observations regarding the WHQ and its algebra are as follows: The vertices of $\overline{\Gamma}(S \times S^*, R)$ are identified by the elements of $S \times S^*$. To find the number of vertices of the double quiver $\overline{\Gamma}(S \times S^*, R)$, first, we consider the following example.

Example 11. Let $Y = \{\alpha, \beta, \gamma, \rho, \sigma, \delta\}$ Be a semilattice as given in Table 1

For a ring R with identity, $R^{2 \times 2}$ denotes the 2×2 full matrix ring over R , $U(R)$ is the group consisting of all units in R . Let Z be the ring of integers. For a prime p , Z_p is a field and $U(Z_p^{2 \times 2})$ is just 2×2 general linear group $GL_2(Z_p)$ over Z_p .

Take $S = \cup_{u \in Y} G_u$ where $Y = \{\alpha, \beta, \gamma, \rho, \sigma, \delta\}$ as given in [20] where $G_{\alpha} = \{e_{\alpha}\}$ and $G_{\delta} = \{e_{\delta}\}$ are the trivial groups, $G_{\beta} = GL_2(Z_2)$, $G_{\gamma} = U(Z_4^{2 \times 2})$, $G_{\rho} = GL_2(Z_3)$, $G_{\sigma} = U(Z_6^{2 \times 2})$. Thus, $kS = \bigoplus_{u \in Y} kG_u$ and $(kS)^* = (\bigoplus_{u \in Y} kG_u)^* = \bigoplus_{v \in Y} (kG_v)^*$. Thus, $\overline{H} = kS \otimes (kS)^* = \bigoplus_{u \in Y} kG_u \otimes \bigoplus_{v \in Y} (kG_v)^* = \bigoplus_{u \in Y} (kG_u \otimes (kG_v)^*)$ which bears the structures of algebra

and coalgebra. Thus, \overline{H} appears as the path coalgebra of the double quiver $\overline{\Gamma}(S \times S^*, R)$.

In the above example, the number of vertices of DWHQ is $\mathfrak{m} = 440 \times 440 = 193600$.

Total number of arrows in the double weak Hopf quiver is 2043168. If $R_{(c_{\gamma}, c_{\delta}^*)}$ denotes the number of arrows originating from $u = (x_{\mu}, \bar{y}_{\lambda}) \in (G_{\mu} \times G_{\lambda}^*)$ to $v = (s_{\gamma}, \bar{t}_{\delta}) \in G_{\gamma} \times G_{\delta}^*$ where $\mu, \lambda, \gamma, \delta \in Y$ such that $\mu \geq \gamma$ or $\lambda \leq \delta$.

The number of vertices of $\overline{\Gamma}(S \times S^*, R)$ is $\mathfrak{m} = 440 \times 440 = 193600$.

The count of arrows in double weak Hopf quiver $\overline{\Gamma}(S \times S^*, R)$ is obtained as 2043168. That can be observed from that Table 2.

If there is an arrow from some element $u = (x_{\mu}, \bar{y}_{\lambda}) \in G_{\mu} \times G_{\lambda}^*$ to some element $v = (s_{\gamma}, \bar{t}_{\delta}) \in G_{\gamma} \times G_{\delta}^*$ then there are arrows from each $u = (x_{\mu}, \bar{y}_{\lambda})$ to $v = (s_{\gamma}, \bar{t}_{\delta})$. The dimension of the vector space corresponding to DWHQ $\overline{\Gamma}(S \times S^*, R)$ is the number of vertices of the DWHQ. The arrows from each idempotent to itself are the existing loops. Therefore, the loops' quantity is the order of the semilattice Y . A Clifford monoid $S \times S^*$ which if finite implies that the DWHQ of the WHA $\overline{H} = kS \otimes (kS)^*$ has no loop if and only if $R = 0$. Then, the quiver is a digraph with $|S \times S^*|$ number of isolated vertices. Otherwise, the DWHQ is a connected digraph.

Let N denote the number of arrows of the quiver $\overline{\Gamma}(S \times S^*, R)$. Let $N_{\mu, \lambda}$ denote the number of all the arrows originating from $u = (x_{\mu}, \bar{y}_{\lambda})$ which is the vertex of the subgroup $G_{\mu} \times G_{\lambda}^*$ and $N^{\gamma, \delta}$ denote the number of all the arrows ending at the vertex represented by the element $v = (s_{\gamma}, \bar{t}_{\delta})$ of the subgroup $G_{\gamma} \times G_{\delta}^*$. Then, we have the following fact.

Lemma 12. *If $\overline{\Gamma}(S \times S^*, R)$ Is a DWHQ with $\mathfrak{m} = |S| = |S^*|$. Then,*

- The total number of vertices in $\overline{\Gamma}(S \times S^*, R)$ is \mathfrak{m}^2
- Total number of arrows in $\overline{\Gamma}(S \times S^*, R)$ is N

Proof. (a) The total number of vertices of the DWHQ is $|\overline{\Gamma}_0| = \dim H = \dim(kS \otimes (kS)^*) = \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}^2$, where $\mathfrak{m} = \dim(kS) = \dim(kS)^*$. \square

Proof. (b) Total no of arrows of DWHQ $\overline{\Gamma}$ originating from

$$N = \sum_{\mu, \lambda \in Y} N_{\mu, \lambda} |G_{\mu} \times G_{\lambda}^*| = R = \sum_{\substack{C_{\gamma} \in \mathcal{C}_{\gamma}, C_{\delta}^* \in \mathcal{C}_{\delta}^* \\ \gamma, \delta \in Y}} R_{C_{\gamma}, \mathcal{C}_{\delta}^*},$$

$$N' = \sum_{\gamma, \delta \in Y} N^{\gamma, \delta} |G_{\gamma} \times G_{\delta}^*| = R = \sum_{\substack{C_{\gamma} \in \mathcal{C}_{\gamma}, C_{\delta}^* \in \mathcal{C}_{\delta}^* \\ \gamma, \delta \in Y}} R_{C_{\gamma}, \mathcal{C}_{\delta}^*}. \quad (12)$$

\square

\square

Hence, $N = \dot{N} =$ Total numbers of arrows of the DWHQ.

Lemma 13. Let $S = \bigcup_{\mu \in Y} G_\mu$ be a Clifford monoid and $\bar{H} = kS \otimes (kS)^*$ be the WHA for the DWHQ \bar{T} . Then, the following arguments are true:

(i) If M is a left \bar{H} -comodule, then $M =$

$$\bigoplus_{u=(x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda} {}^u M$$

(ii) If M is a right \bar{H} -comodule, then $M =$

$$\bigoplus_{v=(s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta} M^v$$

(iii) If M is a \bar{H} -bicomodule, then $M =$

$$\bigoplus_{\substack{u=(x_\mu, \bar{y}_\lambda) \in G_\mu \times G_\lambda \\ v=(s_\gamma, \bar{t}_\delta) \in G_\gamma \times G_\delta}} {}^{u,v} M$$

Proof. For its proof, see Lemma 4.1 in [4]. \square

Lemma 14. A weak Hopf (WH) \bar{H} -bimodule M can be decomposed into WH \bar{H} -subbimodules of M , i.e., $M =$

$$\bigoplus_{\substack{\gamma \leq \mu \\ \lambda \leq \delta \\ \mu, \lambda, \gamma, \delta \in Y}} M_{\mu, \lambda}^{\gamma, \delta}, \text{ where } M_{\mu, \lambda}^{\gamma, \delta} = K - \text{span}({}^v M^u) \text{ with}$$

$$v = (c_\mu, c_\lambda^*) \varphi_{\mu, \lambda}^{\gamma, \delta}(u) \text{ and } {}^v M^u = {}^v M^w \oplus {}^x M^y \oplus \dots \oplus {}^z M^u,$$

where $v = (x_\gamma, \bar{y}_\delta)$, $w = (e_\delta, \bar{y}_\eta)$, $x = (e_\eta, \bar{y}_\sigma)$, $y = (x_\sigma, \bar{y}_\rho)$, \dots , $z = (x_\alpha, \bar{y}_\beta)$ and $u = (x_\mu, \bar{y}_\lambda)$ with $\lambda \geq \beta \geq \dots \geq \rho \geq \sigma \geq \eta \geq \delta$ and $\mu \geq \alpha \geq \dots \geq \sigma \geq \delta \geq \gamma$, where $\lambda, \beta, \dots, \rho, \sigma, \eta, \delta, \mu, \alpha, \dots, \gamma \in Y$.

Proof. The proof clearly follows from [17, Lemma 4.1]. \square

Lemma 15. If M is a WH \bar{H} -bimodule, then

$$\dim {}^v M^u = R_{\mu, \delta} = \begin{cases} \sum R_{c_\mu, c_\delta^*} (C_\mu \times C_\delta^*), & \text{if } \delta > \mu, \\ 1, & \text{if } \delta = \mu, \\ 0, & \text{if } \delta < \mu. \end{cases} \quad (13)$$

Proof. It is obvious from [17, Lemma 4.3]. \square

From the following proposition, we obtain that the set \bar{T} of group-like elements of path coalgebra of a DWHQ \bar{T} is a Clifford monoid.

Proposition 16. If $\bar{T}(S \times S^*, R)$ is a double WHQ for the RD R of $S \times S^*$, then $\bar{T}_0 = G(k\bar{T})$ and $\bar{T}_0 = S \times S^*$. Moreover, $k\bar{T}_0 \simeq k(S \times S^*)$, the algebra of $S \times S^*$ is a sub-WHA of $k\bar{T}$.

Proof. Initially, we prove that the set $G(k\bar{T})$ is equal to \bar{T}_0 . By the multiplication structure that is defined in $k\bar{T}$, for every vertex $i \in \bar{T}_0$, $e_i \in K\bar{T}_0$, we have $\Delta(e_i) = e_i \otimes e_i$ and $\epsilon(e_i) = 1$. Thus, $\bar{T}_0 \subseteq G(k\bar{T})$. Let $u \in G(k\bar{T})$ and take $u = \sum_{l \geq 0} u_l$ with $u_l \in \bar{T}_l$. We prove that $u_l = 0$ for $l > 0$. Suppose on contrary that $n (\neq 0)$ is the greatest l , with $u_l \neq 0$. Thus,

$$\Delta(u) = u_n \otimes u_n + \sum_{\substack{i, j > 0 \\ i \neq j}} u_i \otimes u_j, \quad (14)$$

where $u_n \otimes u_n \in k\Gamma_n \otimes k\Gamma_n^*$ and $u_i \otimes u_j \notin k\Gamma_n \otimes k\Gamma_n^*$ for all $i, j > 0, i \neq j$. \square

Notice that $\Delta(u) = \sum_{l \geq 0} \Delta(u_l) = u_l \otimes s(u_l) + \sum_{l \geq 1} v_l \dots v_{l+1} \otimes v_i \dots v_i + t(u_l) \otimes u_l$ has no term that belongs to $k\Gamma_n \otimes k\Gamma_n^*$, a contradiction. Thus, $u \in \bar{T}_0$ implies that $G(k\bar{T}) \subseteq \bar{T}_0$, and we get $G(k\bar{T}) = \bar{T}_0$. By the definition, the collection of vertices \bar{T}_0 signifies the collections of components of a Clifford monoid $S \times S^*$. Thus, $\bar{T}_0 = S \times S^* = G(k\bar{T})$ and $k\bar{T}_0 \simeq k(S \times S^*) \simeq kS \otimes kS^*$ is a WHA by [8]. The path coalgebra $k\bar{T}$ is pointed; therefore, $k\bar{T}_0$ becomes the sum of its simple subcoalgebras. Thus, $k\bar{T}_0$ is a subcoalgebra of $k\bar{T}$ which is also a WHA. Thus, $k\bar{T}_0$ is sub-WHA of $k\bar{T}$. Here, the path coalgebra $k\Gamma^*$ is dual of the coalgebra $k\Gamma$ of the quiver Γ of the Clifford monoid S . Each summand kG_λ^* of $k\Gamma^* = \bigoplus_{\lambda \in Y} kG_\lambda^*$ is the dual of the coalgebra kG_λ .

There is a necessary and sufficient condition between the SLGWHA $\bar{H} = k\bar{T}$ and the existence of a DWHQ $\bar{T}(S \times S^*, R)$ corresponding to $S \times S^*$ with some ramification data R . We have the following theorem for this argument.

Theorem 17. If $\bar{T}(S \times S^*, R)$ is a DWHQ, then the following arguments are equivalent:

- (i) $k\bar{T}$ equips a SLGWHA structure such that each of its graded summands is itself a graded weak Hopf algebra
- (ii) \bar{T} is DWHQ $\bar{T}(S \times S^*, R)$ of $S \times S^*$ with ramification data R

Proof. (i) Let $\bar{T}(S \times S^*, R)$ be an arbitrary double Weak Hopf quiver of $k\bar{T}$ admitting the structure of SLGWHA.

$\bar{H} = \bigoplus_{\mu, \lambda \in Y} (H_\mu \otimes H_\lambda^*)$, with $\Delta(\bar{H}_{\mu, \gamma, \delta \lambda}) \subseteq \bar{H}_{\mu, \lambda} \otimes \bar{H}_{\gamma, \delta}$ where $\bar{H}_{\alpha, \beta} = H_\alpha \otimes H_\beta^* \forall \alpha, \beta \in Y$, with $(H_\mu \otimes H_\lambda^*)(H_\gamma \otimes H_\delta^*) \subseteq H_{\mu, \gamma} \otimes H_{\delta, \lambda}^*$, $\mu, \lambda, \delta, \gamma \in Y$ such that $\mu \geq \gamma$ and $\lambda \leq \delta$.
 $= H_\gamma \otimes H_\lambda^*$, which is graded.

Thus, $\epsilon(\bar{H}) \setminus G(\bar{H}) = \{0\}$, and $\epsilon(G(\bar{H})) = \{1\}$ and for each $a = x \otimes \bar{y} \in \bar{H}$, where $x \in H = k\Gamma$ and $\bar{y} \in H^* = (k\Gamma)^*$, is also graded, see [4]. \square

Moreover, $G(\overline{H}) = \cup_{\lambda, \mu \in Y} (G_\mu \times G_\lambda^*)$. Hence, $\overline{\Gamma}_0 = G(\overline{H})$ denotes the vertices of the quiver $\overline{\Gamma}$ and the IC $v(k\overline{\Gamma})^u = \{p \in k\overline{\Gamma} \mid \delta_L(p) = v \otimes p, \delta_R(p) = p \otimes u\}$ gives the comodule structure in $\overline{H} = k\overline{\Gamma}$, and in particular $v(k\Gamma_1)^u$ remains constant iff $u, v \in H_\alpha \otimes H_\alpha^* \forall \alpha \in Y$. Thus, $S \times S^* = G(\overline{H}) = G(H) \times G(H^*) = \Gamma \times \Gamma^* = \overline{\Gamma}$. Thus, $\overline{\Gamma}$ is a double weak Hopf quiver.

Conversely, if $\overline{\Gamma}$ is a double weak Hopf quiver of some Clifford monoid $\overline{S} = S \times S^*$ with some ramification data R . i.e., $\overline{S} = \cup_{\lambda, \mu \in Y} G_\mu \times G_\lambda^*$ is a Clifford monoid with respect to ramification data R as given in Definition 1 above. The vector space $k\overline{\Gamma}$ equips the structures of algebra and coalgebra, and there exists $\overline{T} \in \text{Hom}(k\overline{\Gamma}, k\overline{\Gamma}^*)$ as a weak antipode satisfying $id_{k\overline{\Gamma}} * k\overline{T} * id_{k\overline{\Gamma}} = id_{k\overline{\Gamma}} \& \overline{T} * id_{k\overline{\Gamma}} * \overline{T} = \overline{T}$. Thus, $k\overline{\Gamma}$ is a weak Hopf such that $k\overline{\Gamma} = k\Gamma \otimes k\Gamma^* = \sum_{\mu, \lambda \in Y} kG_\mu \otimes kG_\lambda^*$ which is semilattice graded $(kG_\mu \otimes kG_\lambda^*) (kG_\gamma \otimes kG_\delta^*) \subseteq kG_{\mu, \gamma} \otimes kG_{\delta, \lambda}^*$, where each grading summand $kG_\mu \otimes kG_\lambda^*$ is a Hopf algebra $\forall \mu, \lambda \in Y$. Thus, $k\overline{\Gamma}$ appears as the $k\Gamma_0 = k(S \times S^*)$ bimodule and bicomodule.

Moreover $\Delta(k\overline{\Gamma})_{\mu, \lambda, \delta, \gamma} \subseteq (k\overline{\Gamma})_{\mu, \lambda} \otimes (k\overline{\Gamma})_{\gamma, \delta}$ and $G(k\overline{\Gamma}) = \cup_{\lambda, \mu \in Y} (G_\mu \times G_\lambda^*) = \Gamma_0 = S \times S^* = \overline{S}$.

Thus, $k\overline{\Gamma} = \overline{H}$ is a ‘‘semilattice-graded weak Hopf algebra’’.

If we use the product of arrows a and b of a WHQ as

$$a.b = [t(a), b][a, s(b)] + [a, t(b)][s(a), b], \quad (15)$$

and Def. 3.4 in [1], we have the following result for weak Hopf quiver.

Theorem 18. *Let S be a Clifford monoid and k be a field, B be a kS -weak Hopf module and let C be an associated cotensor coalgebra corresponding to the WHA structure by [17, Th. 4.5]. Identify the coalgebra C with the path coalgebra of the WHQ of M by taking a basis of the IC of the bicomodule M . The lengths of the paths α and β are n and m , respectively.*

Then, $\alpha.\beta = \sum_{d \in D_n^{n+m}} (\alpha.\beta)_d$, where $|D_n^{n+m}| = \binom{n+m}{n}$.

Proof. Consider α and β as the paths of length n and m , respectively. Let $d \in D_n^{n+m}$ and $\bar{d} \in D_m^{n+m}$ the complement sequence of d obtained by interchanging 0 and 1 mutually. Now, consider

$$(\alpha.\beta)_d = \left[(d\alpha)_{m+n}, (\bar{d}\beta)_{m+n} \right] \cdots \left[(d\alpha)_1, (\bar{d}\beta)_1 \right] \quad (16)$$

which and belong to the IC of type $(t(\alpha)t(\beta), s(\alpha)s(\beta))$ and lies in the $(m+n)$ -cotensor power of M . \square

In case $d_i = 1$ then $(\bar{d}\beta)_i$ is a vector that acts on the right on $(d\alpha)_i$ (the arrow) and the action results as $[(d\alpha)_i, (\bar{d}\beta)_i]$. If on the other hand $d_i = 0$, then $[(d\alpha)_i, (\bar{d}\beta)_i]$ is the result of left action of $(d\alpha)_i$ (the vertex) on the arrow $(\bar{d}\alpha)_i$.

The source of the first term of the element $(\alpha.\beta)_d$ is the

$$s[(d\alpha)_1, (\bar{d}\beta)_1] = s((d\alpha)_1)s((\bar{d}\beta)_1) = s(\alpha)s(\beta), \quad (17)$$

and sequence of the terms of $(\alpha.\beta)_d$ is a concatenated.

Now, using Th. 3.8 and Lemma 3.10 in [1], we have the required result.

A similar result can be stated for the co-algebra C^* and bicomodule M^* for the dual WHA $H^* = kS^*$ of a dual WHQ Γ^* , which is given as follows, by using Th. 3.9 and [17, Th. 4.5].

Theorem 19. *Let S be the Clifford monoid and S^* be its module as the dual basis of the vector space kS . Let B^* be kS^* weak Hopf module and let C^* be the associated dual cotensor coalgebra corresponding to the DWHQ Γ^* . Identify the dual coalgebra C^* with the path coalgebra of the DWHQ Γ^* of the bicomodule M^* by selecting a basis of IC of M^* . The lengths of the paths $\bar{\alpha}$ and $\bar{\beta}$ are n and m , respectively. Then,*

$$\bar{\beta}.\bar{\alpha} = \sum_{\bar{d} \in \bar{D}_m^{n+m}} (\bar{\beta}.\bar{\alpha})_{\bar{d}} \quad (18)$$

where $\bar{d} \in \bar{D}_m^{n+m}$ (and $\bar{d} \in \bar{D}_n^{n+m} = D_n^{n+m}$)

Proof. Proof is obvious using above Th. 2.9 and [9, Th. 3.8]. \square

By using the coalgebras and bicomodules of Γ and Γ^* , we can obtain the coalgebra \bar{C} , bimodule, and bicomodule \bar{M} of the double weak Hopf quiver $\overline{\Gamma} = \Gamma \times \Gamma^*$ and using the above two theorem, Th. 2.9 and Th. 2.10, we have the result as given bellow.

Theorem 20. *Let S be the Clifford monoid, S^* be its dual and k be a field. Let \bar{M} be a $kS \otimes kS^*$ weak Hopf bimodule and bicomodule and \bar{C} be the associated cotensor coalgebra corresponding to the WHA structure of the coalgebra $k\overline{\Gamma}$. Identify coalgebra \bar{C} with the path coalgebra of the DWHQ $\overline{\Gamma}$ of M by selecting a basis of the IC of \bar{M} . The lengths of the paths $\bar{\alpha}$ and $\bar{\beta}$ are n and m , respectively. Then,*

$$\bar{\alpha}.\bar{\beta} = \sum_{\bar{d} \in \bar{D}_{n,m}^{n+m}} (\bar{\alpha}.\bar{\beta})_{\bar{d}}, \quad (19)$$

where $\bar{\alpha} = (\alpha, \bar{\alpha})$, $\bar{\beta} = (\beta, \bar{\beta})$, $\bar{D}_{n,m}^{n+m} = (D_n^{n+m}, \bar{D}_m^{n+m})$ and $\bar{d} = (d, \bar{d})$.

Proof. Proof is obvious using the proofs of Theorems 18 and 19. \square

4. Conclusion

In this article, the generalized form of a quiver, called the double weak Hopf algebra (DWHQ), was introduced. Further, proposed quivers were used to obtain their path

coalgebras that appeared as the weak Hopf algebra (WHA). Additionally, the study also discussed the module structures of path coalgebra of the double weak quivers and some of their interesting properties. The proposed work contributes in the module theory of a weak Hopf algebra that corresponds to the double weak Hopf quiver.

Data Availability

Data sharing does not applicable to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors do not have any conflicts.

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