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Retraction

Retracted: Roughness in Hypervector Spaces

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] N. Abughazalah, N. Yaqoob, and M. Usman, "Roughness in Hypervector Spaces," *Journal of Function Spaces*, vol. 2022, Article ID 5451971, 9 pages, 2022.

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Research Article

Roughness in Hypervector Spaces

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This paper examines rough sets in hypervector spaces and provides a few examples and results in this regard. We also investigate the congruence relations-based unification of rough set theory in hypervector spaces. We introduce the concepts of lower and upper approximations in hypervector spaces.

1. Introduction

Giuseppe Peano [1], an Italian mathematician, was the first to define vector space as an abstract algebraic structure in 1888. However, the theory did not emerge until 1920. This idea gained attraction in the 1930s and was applied to a wide range of mathematical and scientific disciplines. On a fundamental level, the vector space hypothesis is a unifying and summarizing theory, as it increased interest in science while also driving new disclosures. The most basic example of vector space in the plane is $\mathbb{R}^2$. More studies extended this to Euclidean space ($\mathbb{R}^n$). This area has now developed, and it establishes various regions, such as discrete variable math, coordination space, and functional spaces.

Marty [2] proposed the definition of hypergroups and discussed some of its features in 1934, which is the birth year of hyperstructures. Marty, Krasner, Kuntzmann, Croisot, Dresher, Ore, Eaton, Pall, Campaigne, Griffith, PrenoPitz, Utsumi, Dietzman, Vikrov, and Zappa studied the subject as a general hypothesis and its applications in various areas of mathematics in the 1940s, including geometry, group theory, ring theory, and field theory. In the fields of geometry, graphs, hypergraphs, lattices, fuzzy sets, rough sets, automata, cryptography, artificial intelligence, and many others, this theory has a wide variety of applications, see [3, 4]. In 1982, Pawlak introduced the rough set theory [5]. It is a well-known mathematical method for dealing with imprecise, inconsistent, and incomplete data and knowledge based on illogical relationships. The approximation of sets is one of the primary research concerns that rough sets address, while the algorithm of the analysis or reasoning for connected data is another. Financial and business, science, art, establishing automated computational systems, information and decision systems, and data analysis are just few of the uses. Several writers have investigated roughness in various algebraic structures. Roughness in hemirings was introduced by Ali et al. [6]. In the modules [7] and rings [8], Davvaz looked at roughness. Qurashi and Shabir [9, 10] explored rough subsets in quantales and quantale modules. Shabir and others studied roughness in S-acts [11] and in ordered semigroups [11].

In 2009, Wu et al. [12, 13] had explored a new idea of roughness in vector spaces by using congruence relations. In 1983, Krasner had demonstrated the notion of the hyperring and hyperfield [14]. Later in 1988, Scafati-Tallini gave an idea of hypervector space [15, 16] but used the classical field to define hypervector space. Ameri and Dehghan [17] introduced some results on dimensions of hypervector spaces. In 2010, Roy and Samanta [18] compiled a brief review of hypervector space; they used the hyperfield to define hypervector space. Taghavi and Hosseinzadeh [19–21] added very useful results to the theory of hypervector spaces. Several writers have investigated fuzziness in hypervector spaces, for instance, Ameri [22], Ameri and Dehghan [17, 23, 24], Dehghan [25], and Roy and Samanta [26]. Muhiuddin and Al-Roqi [27] studied the concept of double-framed soft sets in hypervector spaces. Muhiuddin [28] applied intersectional soft sets theory to generalized hypervector spaces, see also [29, 30].

In this paper, we introduced the concept of lower and upper rough subsets in hypervector spaces. In Section 2, we added some very basic definition which will be helpful in our further studies. In Section 3, we provided some results related to congruences in hypervector spaces, and in the last section, we applied roughness to hypervector spaces.

2. Preliminaries and Basic Definitions

In this section, we added some basic definitions and examples on vector spaces, rough sets, hyperfields, and hypervector spaces.

Definition 1 (see [12]). An equivalence relation σ in a vector space V over a field F is called a congruence relation if $\forall s, t, u \in V$ and $\forall a \in F: (s, t) \in \sigma \Longrightarrow (s+u, t+u) \in \sigma$ and $(a * s, a * t) \in \sigma$.

Example 1. Consider the vector space $V = Z_4$ over the field $F = \{0, 1\}$, consider the following relation as

$$\sigma = \{(0,0), (1,1), (2,2), (3,3), (0,2), (2,0), (1,3), (3,1)\}. \tag{1}$$

Then, σ is a congruence relation on V.

Definition 2 (see [2]). Let $M \neq \varnothing$. Then, $\circ : M \times M \longrightarrow \mathscr{P}^*(M)$ is called hyperoperation on M, where $\mathscr{P}^*(M) = P(M) - \varnothing$ represents the set of all non-empty subsets of M. For any $\varnothing \neq P, Q \subseteq M$, we denote

$$P \circ Q = \bigcup_{p \in P, q \in Q} p \circ q. \tag{2}$$

Definition 3 (see [2]). A hypergroupoid $(M, \circ)$ is known as semihypergroup, if $\forall s, t, u \in M$, we have $s \circ (t \circ u) = (s \circ t) \circ u$.

Definition 4 (see [2]). A semihypergroup is called hypergroup if reproductive axiom holds in M,$\forall s \in M$, $M \circ s = s \circ M = M$.

Definition 5 (see [31]). A polygroup is a system $\langle M, \circ, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on M, $\circ : M \times M \longrightarrow \mathscr{P}^*(M)$, and the following axioms hold for all $x, y, z \in M$:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$

(ii) $e \circ x = x = x \circ e$

(iii) $x \in y \circ z \Rightarrow y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$

Example 2. Let $M = \{s, t, u\}$ and the hyperoperation $\circ$ defined in Table 1.

Then, $(M, \circ)$ forms a polygroup.

Definition 6 (see [32]). Let σ be a relation on hypergroup M and $\varnothing \neq P, Q \subseteq M$. Then,

(1) $Q\sigma P$ if $\exists q \in Q$ and $p \in P$ such that $q\sigma p$

(2) $Q\sigma^- P$ if $\forall q \in Q \exists p \in P$ and $\forall p \in P \exists q \in Q$ such that $q\, \sigma p$

(3) $Q\sigma^= P$ if $\forall q \in Q$ and $\forall p \in P$ such that $q\sigma p$

Definition 7 (see [33]). Let σ be an equivalence relation on M . Then, σ is called

(1) left congruence as if $q\sigma p$, then $s \circ q \sigma^- s \circ p, \forall s \in M$

(2) right congruence as if $q\sigma p$, then $q \circ s \sigma^- p \circ s, \forall s \in M$

(3) strongly left congruence as if $q\sigma p$, then $s \circ q \sigma^= s \circ p, \forall s \in M$

(4) strongly right congruence as if $q\sigma p$, then $q \circ s \sigma^= p \circ s$, $\forall s \in M$

Definition 8 (see [5]). The lower approximation of $S \subseteq M$, with respect, to σ is the set

$$\underline{\mathrm{apr}}(S) = \{s \in M : [s]_\sigma \subseteq S\}, \tag{3}$$

and the upper approximation is the set

$$\overline{\mathrm{apr}}(S) = \{s \in M : [s]_\sigma \cap S \neq \varnothing\}. \tag{4}$$

Then, the set S in M is called a rough set if $\overline{\mathrm{apr}}(S) \neq \underline{\mathrm{apr}}(S)$, otherwise definable.

Theorem 9 (see [5]). *Let $S, T \subseteq M$ and σ be an equivalence relation on M. Then, the following holds:*

(1) $\underline{apr}(S) \subseteq S \subseteq \overline{apr}(S)$

(2) $\underline{apr}(\varnothing) = \overline{apr}(\varnothing) = \varnothing$ *and* $\underline{apr}(M) = \overline{apr}(M) = M$

(3) $\overline{apr}(S \cup T) = \overline{apr}(S) \cup \overline{apr}(T)$

(4) If $S \subseteq T$, *then* $\underline{apr}(S) \subseteq \underline{apr}(T)$ *and* $\overline{apr}(S) \subseteq \overline{apr}(T)$

(5) $\overline{apr}(S \cap T) \subseteq \overline{apr}(S) \cap \overline{apr}(T)$

(6) $\underline{apr}(S) \cap \underline{apr}(T) = \underline{apr}(S \cap T)$

(7) $\underline{apr}(S) \cup \underline{apr}(T) \subseteq \underline{apr}(S \cup T)$

(8) $\underline{apr}(-S) = -\overline{apr}(S)$

(9) $\overline{apr}(-S) = -\underline{apr}(S)$

(10) $\underline{apr}\,\underline{apr}(S) = \overline{apr}\underline{apr}(S) = \underline{apr}(S)$

Table 1: A polygroup M.

$\circ$	s	t	u
s	s	t	u
t	t	$\{s,t\}$	u
u	u	u	$\{s,t\}$

(11) $\bar{apr}\,\bar{apr}(S) = \underline{apr}\,\bar{apr}(S) = \underline{apr}(S)$

A Krasner hyperfield is defined as follows.

Definition 10 (see [14]). A nonempty set F under hyperoperation "$\oplus$" and a binary operation "." is called a hyperfield if the following holds:

(1) $(F, \oplus)$ is commutative polygroup

(2) $(F \setminus \{0\}, .)$ is an abelian group

(3) Left an right distributive laws holds with respect to "$\oplus$" and "." in F

Example 3 (see [14]). Let $F = \{m, n\}$. Then, $(F, \oplus, .)$ is a hyperfield under hyperoperation "$\oplus$" and binary operation "." defined in the Cayley Tables 2 and 3.

Definition 11 (see [18]). Consider $V \neq \emptyset$ with a hyperoperation "$\boxplus$" and $(V, \boxplus)$ be a commutative polygroup. Also let ($F, \oplus, .$) be a Krasner hyperfield, then V is a hypervector space over F, if $\exists$ a hyperoperation $* : F \times V \longrightarrow \mathscr{P}^*(V)$, where $\mathscr{P}^*(V) = P(V) \setminus \{\emptyset\}$ and following holds:

(1) $m * (\alpha \boxplus \beta) \subseteq (m * \alpha) \boxplus (m * \beta), \forall \alpha, \beta \in V$ and $m \in F$

(2) $(m \oplus n) * \alpha \subseteq (m * \alpha) \boxplus (n * \alpha), \forall \alpha \in V$ and $m, n \in F$

(3) $(m.n) * \alpha = m * (n * \alpha), \forall \alpha \in V$ and $m, n \in F$

(4) $\alpha = 1_F * \alpha$ and $\tau = 0_F * \alpha$, where $1_F, 0_F \in F$ and $\alpha, \tau \in V$ (τ is zero vector in V).

If equality holds in 1,2, then V is called a good hypervector space.

Example 4. Consider a commutative polygroup $(V, \boxplus)$, where $V = \{\alpha, \beta, \gamma, \delta\}$ and the hyperoperation $\boxplus$ on V is defined in Table 4.

Also consider the same Krasner hyperfield $(F, \oplus, .)$, defined in Example 3. Then, $(V, F, *)$ forms a hypervector space under the hyperoperation $* : F \times V \longrightarrow \mathscr{P}^*(V)$ given as in Table 5.

Definition 12 (see [18]). Let $\emptyset \neq U$ is a subset of a hypervector space V over hyperfield F. Then, U is called a hypersubspace of V over F if U is itself a hypervector space under the same hyperoperations "$\boxplus$" and "$*$". Therefore, a subset U of

Table 2: Hyperoperation "$\oplus$" defined on $F = \{m, n\}$.

$\oplus$	m	n
m	m	n
n	n	$\{m,n\}$

a hypervector space V is a hypersubspace of V if and only if the following properties hold:

(1) $\forall \alpha, \beta \in U$, $\alpha \boxplus \beta \subseteq U$

(2) U has a zero element

(3) each element of U has an inverse with respect to $\boxplus$

(4) $\forall a \in F$ and $\forall \alpha \in U$, $a * \alpha \in U$

Example 5. From Example 4, consider $U = \{\alpha, \beta\} \subseteq V$. Then, $(U, F, *)$ is a hypersubspace of $(V, F, *)$.

3. Congruence in Hypervector Spaces

Here, we introduce some results on congruences in hypervector spaces.

Definition 13. Let σ be an equivalence relation on a hypervector space $(V, F, *)$. Then, σ is congruence in $(V, F, *)$ if

(1) $\forall \alpha, \beta \in V$, if $\alpha \sigma \beta$, then $\forall \gamma \in V$, we have $\alpha \boxplus \gamma \sigma^{-} \beta($$\gamma$ and $\gamma \boxplus \alpha \sigma^{-} \gamma \boxplus \beta$

(2) $\forall m \in F$ and $\alpha \sigma \beta$ such that $m * \alpha \sigma^{-} m * \beta$

From above definition we may also write $[\alpha]_\sigma$ for the representation of $\sigma -$ congruence class containing element $\alpha \in V$.

Definition 14. For $\emptyset \neq S, T \subseteq V$, the linear sum of S and T is given as

$$S \boxplus T = \cup \{\alpha \boxplus \beta : \alpha \in S, \beta \in T\}, \quad (5)$$

and product of S with some $k \in F$

$$k * S = \{k * \alpha : \alpha \in S\}. \quad (6)$$

Theorem 15. *Let σ be a congruence relation on V. Then, $\sigma -$ congruence class $[\tau]_\sigma$, where τ is zero vector in V, is hypersubspace of V, and moreover $[\alpha]_\sigma = \alpha \boxplus [\tau]_\sigma$.*

Proof. Let $\alpha, \beta \in [\tau]_\sigma$. Then, $(\alpha, \tau) \in \sigma$ and $(\beta, \tau) \in \sigma$. Since σ is a congruence relation on hypervector space V, so

$$(\alpha, \boxplus, \beta, \tau \boxplus \beta) \in \sigma^{-} \Rightarrow (\alpha \boxplus \beta, \beta) \in \sigma^{-}, \quad (7)$$

Table 3: Binary operation "." defined on $F = \{m, n\}$.

.	m	n
m	m	m
n	m	n

Table 4: Hyperoperation "$\boxplus$" defined on $V = \{\alpha, \beta, \gamma, \delta\}$.

$\boxplus$	α	β	γ	δ
α	α	β	γ	δ
β	β	$\{\alpha, \beta\}$	γ	δ
γ	γ	γ	$\{\alpha, \beta, \gamma, \delta\}$	$\{\gamma, \delta\}$
δ	δ	δ	$\{\gamma, \delta\}$	$\{\alpha, \beta, \gamma, \delta\}$

and also $(\beta, \tau) \in \sigma^-$ by transitivity of σ,

$$(\alpha \boxplus \beta, \tau) \in \sigma^- \Rightarrow \alpha \boxplus \beta \subseteq \tau]_\sigma. \tag{8}$$

Since τ is a zero element of V and $\tau \in \tau]_\sigma$. So, $[\tau]_\sigma$ has a zero element.

Now, let $\beta \in \tau]_\sigma$ and $(-1_F) \in F$. Then,

$$(-1_F) * (\beta, \tau) \in \sigma^- \Rightarrow ((-1_F) * \beta, (-1_F) * \tau) \in \sigma^- \Rightarrow (-\beta, \tau) \in \sigma^-. \tag{9}$$

This implies that $-\beta \in \tau]_\sigma$.

Also for $\alpha \in \tau]_\sigma$ and $k \in F$, we have

$$(k * \alpha, k * \tau) \in \sigma^- \Rightarrow (k * \alpha, \tau) \in \sigma^- \Rightarrow k * \alpha \in \tau]_\sigma. \tag{10}$$

Hence, $[\tau]_\sigma$ is a hypersubspace of V.

Let $\beta \in \alpha]_\sigma \Rightarrow (\beta, \alpha) \in \sigma$ since σ is a congruence on V. Then, we can write as

$$((-\alpha) \boxplus \beta, (-\alpha) \boxplus \alpha) \in \sigma^- \Rightarrow ((-\alpha) \boxplus \beta, \tau) \in \sigma^-. \tag{11}$$

Therefore, $(-\alpha) \boxplus \beta \subseteq \tau]_\sigma$. This implies that $[\alpha]_\sigma \subseteq \alpha \boxplus \tau]_\sigma$. Now, if $\beta \in \alpha \boxplus \tau]_\sigma$, then $\exists \gamma \in \tau]_\sigma$, such that $\beta \in \alpha \boxplus \gamma$, since $\gamma \in \tau]_\sigma$, so $(\gamma, \tau) \in \sigma$ and

$$((-\alpha) \boxplus \beta, \tau) \in \sigma^- ((-\alpha) \boxplus \beta, (-\alpha) \boxplus \alpha) \in \sigma^-. \tag{12}$$

As σ is a congruence relation so $(\beta, \alpha) \in \sigma^-$. This implies that $\beta \in \alpha]_\sigma$. Therefore, $\alpha \boxplus \tau]_\sigma \subseteq \tau]_\sigma$. Hence, $[\alpha]_\sigma = \alpha \boxplus \tau]_\sigma$. □

Theorem 16. *Let U be a hypersubspace of a hypervector space V, such that $\alpha \boxplus (-\alpha) \subseteq U, \forall \alpha \in V - U$ and $\sigma_U = \{(\alpha, \beta): \alpha, \beta \in V, \alpha \boxplus (-\beta) \subseteq U\}$. Then, σ_U is a congruence on V defined by U and $[\tau]_{\sigma_U} = U$.*

Table 5: A hyperoperation $* : F \times V \longrightarrow \mathscr{P}^*(V)$.

$*$	α	β	γ	δ
m	α	α	α	α
n	α	β	γ	δ

Proof. Since U is hypersubspace of V, then $\tau \in U$ and $\tau \in \alpha \boxplus (-\alpha) \subseteq U \Rightarrow (\alpha, \alpha) \in \sigma_U$. Hence, σ_U is reflexive also if $(\alpha, \beta) \in \sigma_U$, then $\alpha \boxplus (-\beta) \subseteq U$. Since U is itself a hypervector space, so $\beta \boxplus (-\alpha) \subseteq U \Rightarrow (\beta, \alpha) \in \sigma_U$, so σ_U is symmetric. Now if $(\alpha, \beta) \in \sigma_U$ and $(\beta, \gamma) \in \sigma_U \Rightarrow \alpha \boxplus (-\beta) \subseteq U$ and $\beta \boxplus (-\gamma) \subseteq U$. Consider

$$\alpha \boxplus (-\gamma) = (\alpha \boxplus \tau) \boxplus (-\gamma) \subseteq \alpha \boxplus ((-\beta) \boxplus \beta) \boxplus (-\gamma) = (\alpha \boxplus (-\beta)) \boxplus (\beta \boxplus (-\gamma)) \subseteq U \boxplus U = U. \tag{13}$$

Hence, $(\alpha, \gamma) \in \sigma_U$. Therefore, σ_U is transitive. Thus, σ_U is an equivalence relation on V. Further let $(\alpha, \beta) \in \sigma_U \Rightarrow \alpha \boxplus (-\beta) \subseteq U$. Now,

$$(\alpha \boxplus \gamma) \boxplus (-(\beta \boxplus \gamma)) = \alpha \boxplus \gamma \boxplus (-\beta) \boxplus (-\gamma) = (\alpha \boxplus (-\beta)) \boxplus (\gamma \boxplus (-\gamma)) \subseteq U \boxplus U = U. \tag{14}$$

Similarly, we can show that $(\gamma \boxplus \alpha) \boxplus (-(\gamma \boxplus \beta)) \subseteq U$. Now, let $m \in F$ and $(\alpha, \beta) \in \sigma_U \Rightarrow \alpha \boxplus (-\beta) \subseteq U$. As $\alpha, \beta \in V$, so it is clear that $m * \alpha$ and $m * \beta$ will be in V. Thus, by definition of σ_U,

$$(m * \alpha) \boxplus (-(m * \beta)) \subseteq U. \tag{15}$$

Therefore, σ_U is a congruence on V. Now as if $\alpha \in \tau]_{\sigma_U} \Rightarrow (\alpha, \tau) \in \sigma_U \Rightarrow (\alpha \boxplus (-\tau)) \subseteq U \Rightarrow \alpha \in U$. If $\alpha \in U$, then $(\alpha, \tau) \in \sigma_U \Rightarrow \alpha \in \tau]_{\sigma_U}$. Thus, $U = [\tau]_{\sigma_U}$. □

Corollary 17. *Let U be a hypersubspace of a hypervector space V then $[\alpha]_{\sigma_U} = \alpha \boxplus U$.*

Proof. Since we know that $[\tau]_\sigma$ is a hypersubspace of V, then we can say that $U = [\tau]_\sigma$. Also since σ_U is congruence on V and $U = [\tau]_{\sigma_U}$ particularly if we take $\sigma = \sigma_U$, then, we have $U = [\tau]_\sigma$. So we have one to one correspondence between set of hypersubspaces of V and set of all congruence relations on V. Hence, $[\alpha]_{\sigma_U} = \alpha \boxplus U$. □

Theorem 18. *Let U be a hypersubspace of a hypervector space V. Then, $\forall \alpha, \beta \in V$ and $0 \neq k \in F$*

$$\begin{aligned} \sigma_U(\alpha) \boxplus \sigma_U(\beta) &= \sigma_U(\alpha \boxplus \beta), \\ k * \sigma_U(\alpha) &= \sigma_U(k * \alpha). \end{aligned} \tag{16}$$

Proof.

(1) By using Corollary 17, we get

$$\sigma_U(\alpha)\boxplus\sigma_U(\beta) = (\alpha\boxplus U)\boxplus(\beta\boxplus U) = (\alpha\boxplus\beta)\boxplus(U\boxplus U) = (\alpha\boxplus\beta)(U = \sigma_U\boxplus\alpha(\beta). \tag{17}$$

Hence, $\sigma_U(\alpha)\boxplus\sigma_U(\beta) = \sigma_U(\alpha\boxplus\beta)$.

(2) By using Corollary 17, we get

$$k * \sigma_U(\alpha) = k * (\alpha\boxplus U) = (k * \alpha)\boxplus(k * U) \subseteq (k * \alpha)\boxplus U = \sigma_U(k * \alpha). \tag{18}$$

Also,

$$\begin{aligned}\sigma_U(k * \alpha) &= (k * \alpha)\boxplus U = (k * \alpha)\boxplus(1_F * U) = (k * \alpha)\boxplus\left(\left(k.k^{-1}\right) * U\right)\\ &= (k * \alpha)\boxplus\left(k * \left(k^{-1} * U\right)\right) \subseteq (k * \alpha)\boxplus(k * U)\\ &= k * (\alpha\boxplus U) = k * \sigma_U(\alpha).\end{aligned} \tag{19}$$

Hence, $k * \sigma_U(\alpha) = \sigma_U(k * \alpha)$.

Since for any two hypersubspaces U_1 and U_2 of hypervector space $V, U_1\boxplus U_2$ and $U_1 \cap U_2$ are also a hypersubspaces of V which leads to following result. □

Theorem 19. *If U_1 and U_2 are hypersubspaces of a hypervector space V, then*

$$\begin{aligned}\sigma_{U_1\boxplus U_2}(\alpha\boxplus\beta) &= \sigma_{U_1}(\alpha)\boxplus\sigma_{U_2}(\alpha),\\ \sigma_{U_1\cap U_2}(\alpha) &= \sigma_{U_1}(\alpha) \cap \sigma_{U_2}(\alpha).\end{aligned} \tag{20}$$

Proof.

(1) By using Corollary 17, we get

$$\sigma_{U_1\boxplus U_2}(\alpha\boxplus\beta) = (\alpha\boxplus\beta)\boxplus(U_1\boxplus U_2) = (\alpha\boxplus U_1)\boxplus(\beta\boxplus U_2) = \sigma_{U_1}(\alpha)\boxplus\sigma_{U_2}(\beta). \tag{21}$$

Therefore, $\sigma_{U_1\boxplus U_2}(\alpha\boxplus\beta) = \sigma_{U_1}(\alpha)\boxplus\sigma_{U_2}(\alpha)$.

(2) By using Corollary 17, we get

$$\sigma_{U_1\cap U_2}(\alpha) = \alpha\boxplus(U_1 \cap U_2) = (\alpha\boxplus U_1) \cap (\alpha\boxplus U_2) = \sigma_{U_1}(\alpha) \cap \sigma_{U_2}(\alpha). \tag{22}$$

Therefore, $\sigma_{U_1\cap U_2}(\alpha) = \sigma_{U_1}(\alpha) \cap \sigma_{U_2}(\alpha)$. □

4. Rough Subsets in Hypervector Spaces

In this section, we study the properties of lower and upper rough subsets in hypervector spaces.

Definition 20. Let σ be a congruence relation on a hypervector space V. Let $\varnothing \neq X \subseteq V$, then the sets

$$\begin{aligned}\underline{\mathrm{apr}}(X) &= \{\alpha \in V : \sigma(\alpha) \subseteq X\},\\ \bar{\mathrm{apr}}(X) &= \{\alpha \in V : \sigma(\alpha) \cap X \neq \varnothing\},\end{aligned} \tag{23}$$

are called, respectively, lower and upper approximations of set X with respect to σ.

If $\bar{\mathrm{apr}}(X) \neq \underline{\mathrm{apr}}(X)$, then X is called rough set otherwise X is definable.

Definition 21. Let U be a hypersubspace of a hypervector space V. Let $\varnothing \neq X \subseteq V$, then the sets

$$\begin{aligned}\underline{\mathrm{apr}}_U(X) &= \{\alpha \in V : (\alpha\boxplus U) \subseteq X\},\\ \bar{\mathrm{apr}}_U(X) &= \{\alpha \in V : (\alpha\boxplus U) \cap X \neq \varnothing\},\end{aligned} \tag{24}$$

are called repectively, lower and upper approximations of set X with respect to hypersubspace U. If $\underline{\mathrm{apr}}_U(X) \neq \bar{\mathrm{apr}}_U(X)$, then X is a rough set in approximation space (V, U), otherwise definable.

Example 6. Consider a commutative polygroup $(V, \boxplus)$, where $V = \{\alpha, \beta, \gamma, \delta\}$ and the hyperoperation (on V is defined in Table 6.

Also consider the Krasner hyperfield $(F, \oplus, .)$, where $F = \{l, m, n\}$. The hyperoperation "$\oplus$" and "." are, respectively, defined in Tables 7 and 8.

Then, $(V, F, *)$ forms a hypervector space under the hyperoperation $* : F \times V \longrightarrow \mathscr{P}^*(V)$ defines as follows:

$$*(x, y) = \begin{cases}\alpha, & \text{if } x = l,\\ y, & \text{otherwise.}\end{cases} \tag{25}$$

Now, consider $U = \{\alpha, \gamma\}$. Then,

$$\alpha\boxplus U = U, \beta\boxplus U = \{\beta, \delta\}, \gamma\boxplus U = U \text{ and } \delta\boxplus U = \{\beta, \delta\}. \tag{26}$$

Now, consider $S = \{\alpha, \delta\}$, then $\underline{\mathrm{apr}}(S) = \varnothing$ and $\bar{\mathrm{apr}}(S) = \{\alpha, \beta, \gamma, \delta\}$. Since $\underline{\mathrm{apr}}(S) \neq \bar{\mathrm{apr}}(S)$, so S is a rough subset of V.

Theorem 22. *Let U be a hypersubspace of a hypervector space V and S is a non-empty subset of V. Then,*

$$\bar{apr}_U(S) = S\boxplus U. \tag{27}$$

Proof. For $\alpha \in \bar{\mathrm{apr}}_U(S)$, we have $(\alpha\boxplus U) \cap S \neq \varnothing$. Then, $\exists x \in S$ and $\gamma \in U$ such that

$$x \in \alpha\boxplus\gamma \Rightarrow \alpha \in x\boxplus(-\gamma) \subseteq S\boxplus U, \tag{28}$$

thus, $\alpha \in S\boxplus U$. Now, let $\alpha \in S\boxplus U$. Then, $\exists x \in S$ and $\gamma \in U$ for which

$$\alpha \in x\boxplus\gamma \Rightarrow x \in \alpha\boxplus(-\gamma) \subseteq \alpha(U). \tag{29}$$

Table 6: Hyperoperation "⊞" defined on $V = \{\alpha, \beta, \gamma, \delta\}$.

$\boxplus$	α	β	γ	δ
α	α	β	γ	δ
β	β	$\{\alpha,\beta\}$	δ	$\{\gamma,\delta\}$
γ	γ	δ	$\{\alpha,\gamma\}$	$\{\beta,\delta\}$
δ	δ	$\{\gamma,\delta\}$	$\{\beta,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$

Table 7: Hyperoperation "⊕" defined on $F = \{l, m, n\}$.

$\oplus$	l	m	n
l	l	m	n
m	m	$\{l,n\}$	$\{m,n\}$
n	n	$\{m,n\}$	$\{l,m\}$

Thus, $(\alpha \boxplus U) \cap S \neq \varnothing$. This implies that $\alpha \in \overline{\mathrm{apr}}_U(S)$. Hence, $\overline{\mathrm{apr}}_U(S) = S \boxplus U$. □

Theorem 23. *Let V be a hypervector space and U be its hypersubspace. If $\varnothing \neq S, T \subseteq V$. Then, the following holds:*

(1) $\underline{apr}_U(S) \subseteq S \subseteq \overline{apr}_U(S)$;

(2) $\underline{apr}_U(S \cap T) = \underline{apr}_U(S) \cap \underline{apr}_U(T)$;

(3) $\overline{apr}_U(S \cup T) = \overline{apr}_U(S) \cup \overline{apr}_U(T)$;

(4) $\underline{apr}_U(S) \cup \underline{apr}_U(T) \subseteq \underline{apr}_U(S \cup T)$;

(5) $\overline{apr}_U(S \cap T) \subseteq \overline{apr}_U(S) \cap \overline{apr}_U(T)$;

(6) If $S \subseteq T$, then $\underline{apr}_U(S) \subseteq \underline{apr}_U(T)$ and $\overline{apr}_U(S) \subseteq \overline{apr}_U(T)$.

Proof. (1) The proof is obvious.

(2) Let $x \in \underline{\mathrm{apr}}_U(S \cap T) \Longrightarrow x(U \subseteq S \cap T$, so $x \boxplus U \subseteq S$ and $x(U \subseteq T$. Thus, $x \in \underline{\mathrm{apr}}_U(S)$ and $x \in \underline{\mathrm{apr}}_U(T)$. This implies that $x \in \underline{\mathrm{apr}}_U(S) \cap \underline{\mathrm{apr}}_U(T)$. Now let $y \in \underline{\mathrm{apr}}_U(S) \cap \underline{\mathrm{apr}}_U(T)$. Then, $y \in \underline{\mathrm{apr}}_U(S)$ and $y \in \underline{\mathrm{apr}}_U(T)$. So, $y \boxplus U \subseteq S$ and $y \boxplus U \subseteq T$. This implies that $y \boxplus U \subseteq S \cap T$. Thus, $y \in \underline{\mathrm{apr}}_U(S \cap T)$. Hence, $\underline{\mathrm{apr}}_U(S \cap T) = \underline{\mathrm{apr}}_U(S) \cap \underline{\mathrm{apr}}_U(T)$

(3) Consider $x \in \overline{\mathrm{apr}}_U(S \cup T)$, so

$$\begin{aligned} x \in \overline{\mathrm{apr}}_U(S \cup T) &\Leftrightarrow (x \boxplus U) \cap (S \cup T) \neq \varnothing \Leftrightarrow ((x \boxplus U) \cap S) \cup ((x \boxplus U) \cap T) \neq \varnothing \\ &\Leftrightarrow (x \boxplus U) \cap S \neq \varnothing \text{or} (x \boxplus U) \cap T \neq \varnothing \text{g}\, x \in \overline{\mathrm{apr}}_U(S) \text{or} x \in \overline{\mathrm{apr}}_U(T) \\ &\Leftrightarrow x \in \overline{\mathrm{apr}}_U(S) \cup \overline{\mathrm{apr}}_U(T). \end{aligned} \tag{30}$$

Table 8: Binaryoperation "." defined on $F = \{l, m, n\}$.

.	l	m	n
l	l	l	l
m	l	m	n
n	l	n	m

Thus, $\overline{\mathrm{apr}}_U(S \cup T) = \overline{\mathrm{apr}}_U(S) \cup \overline{\mathrm{apr}}_U(T)$. The proofs of (4) and (5) are similar to (1).

(6) If $S \subseteq T$, then we prove that $\underline{\mathrm{apr}}_U(S) \subseteq \underline{\mathrm{apr}}_U(T)$. Let $\alpha \in \underline{\mathrm{apr}}_U(S) \Rightarrow \alpha \boxplus U \subseteq S \subseteq T$, so $\alpha \boxplus U \subseteq T \Rightarrow \alpha \in \underline{\mathrm{apr}}_U(T)$. Similarly, we can prove $\overline{\mathrm{apr}}_U(S) \subseteq \overline{\mathrm{apr}}_U(T)$ □

Theorem 24. *Let U be a hypersubspace of a hypervector space V and S is non-empty subset of V. If $U \subseteq S$, then $U \subseteq \underline{apr}_U(S)$ and $\underline{apr}_U(S) \neq \varnothing$.*

Proof. Let $\alpha \in U$ since U is hypersubspace of V. Then, we have

$$\alpha \boxplus U \subseteq U \subseteq S \Longrightarrow U \subseteq \underline{\mathrm{apr}}_U(S). \tag{31}$$

As clearly $U \subseteq \underline{\mathrm{apr}}_U(S)$ and $U \neq \varnothing \Rightarrow \underline{\mathrm{apr}}_U(S) \neq \varnothing$. □

Theorem 25. *Let U be hypersubspace of a hypervector space V and S, T are non-empty subsets of V then following holds*

(1) $\overline{apr}_U \boxplus S(T) = \overline{apr}_U(S) \boxplus \overline{apr}_U(T)$

(2) $\underline{apr}_U(S) \boxplus \underline{apr}_U(T) \subseteq \underline{apr}_U \boxplus S(T)$

(3) $\overline{apr}_U(k * S) = k * \overline{apr}_U(S)$

(4) $k * \underline{apr}_U(S) \subseteq \underline{apr}_U(k * S)$

Proof.

(1) Using Theorem 22, we get

$$\begin{aligned} \overline{\mathrm{apr}}_U(S \boxplus T) &= (S \boxplus T)(U = (S \boxplus T)((U(U) = (S \boxplus U) \boxplus (T \boxplus U) \\ &= \overline{\mathrm{apr}}_U(S) \boxplus \overline{\mathrm{apr}}_U(T). \end{aligned} \tag{32}$$

Therefore, $\overline{\mathrm{apr}}_U(S \boxplus T) = \overline{\mathrm{apr}}_U(S) \boxplus \overline{\mathrm{apr}}_U(T)$.

(2) Let $x \in \underline{\mathrm{apr}}_U(S) \boxplus \underline{\mathrm{apr}}_U(T)$. Then, $\exists \beta \in \underline{\mathrm{apr}}_U(S)$ and $\gamma \in \underline{\mathrm{apr}}_U(T)$, such that $x \in \beta \boxplus \gamma$, as since $\beta \in \underline{\mathrm{apr}}_U(S$) and $\gamma \in \underline{\mathrm{apr}}_U(T)$, so $\beta \boxplus U \subseteq S$ and $\gamma \boxplus U \subseteq T$. Now consider

$$x⊞U \subseteq (\beta⊞\gamma)(U = (\beta⊞\gamma)⊞(U⊞U) = (\beta⊞U)⊞(\gamma⊞U) \subseteq S⊞T. \tag{33}$$

This implies that $x \in \underline{apr}_U(S⊞T)$. Hence, $\underline{apr}_U(S)⊞\underline{apr}_U(T) \subseteq \underline{apr}_U(S⊞T)$.

(3) Using Theorem 22, we get

$$\begin{aligned} \overline{apr}_U(k * S) &= (k * S)⊞U = (k * S)⊞(1_F * U) = (k * S)⊞\left((k.k^{-1}) * U\right) \\ &= (k * S)⊞\left(k * (k^{-1} * U)\right) \subseteq (k * S)⊞(k * U) = k * (S⊞U) \\ &= k * \overline{apr}_U(S). \end{aligned} \tag{34}$$

Also,

$$k * \overline{apr}_U(S) = k * (S⊞U) = (k * S)⊞(k * U) \subseteq (k * S)⊞U = \overline{apr}_U(k * S). \tag{35}$$

Hence, $\overline{apr}_U(k * S) = k * \overline{apr}_U(S)$.

(4) Let $\alpha \in k * \underline{apr}_U(S)$, then $\exists\, \beta \in \underline{apr}_U(S)$, such that $\alpha \in k * \beta$. Since $\beta \in \underline{apr}_U(S) \Longrightarrow \beta⊞U \subseteq S$. Now consider

$$\begin{aligned} \alpha⊞U \subseteq (k * \beta)⊞U &= (k * \beta)⊞(1_F * U) = (k * \beta)⊞\left((k.k^{-1}) * U\right) \\ &= (k * \beta)⊞(k * \left(k^{-1} * U\right) \\ &= k * \left(\beta⊞(k^{-1} * U)\right) \subseteq k * (\beta⊞U) \subseteq k * S. \end{aligned} \tag{36}$$

This implies that $\alpha \in \underline{apr}_U(k * S)$. Hence, $k * \underline{apr}_U(S) \subseteq \underline{apr}_U(k * S)$. □

Theorem 26. *Let U_1 and U_2 be hypersubspaces of a hypervector space V and S, T be non-empty subsets of V. Then, the following holds:*

(1) $\underline{apr}_{U_1}(S) \cap \underline{apr}_{U_2}(S) \subseteq \underline{apr}_{U_1 \cap U_2}(S)$;

(2) $\overline{apr}_{U_1 \cap U_2}(S) \subseteq \overline{apr}_{U_1}(S) \cap \overline{apr}_{U_2}(S)$;

(3) $\overline{apr}_{U_1⊞U_2}(S(T) = \overline{apr}_{U_1}(S)⊞\overline{apr}_{U_2}(T)$;

(4) $\underline{apr}_{U_1}(S)⊞\underline{apr}_{U_2}(T) \subseteq \underline{apr}_{U_1⊞U_2}(S⊞T)$;

(5) If $U_1 \subseteq U_2$ then $\overline{apr}_{U_1}(S) \subseteq \overline{apr}_{U_2}(S)$ and $\underline{apr}_{U_2}(S) \subseteq \underline{apr}_{U_1}(S)$.

Proof.

(1) Consider $x \in \underline{apr}_{U_1}(S) \cap \underline{apr}_{U_2}(S)$, we have

$$\begin{aligned} & x \in \underline{apr}_{U_1}(S) \cap \underline{apr}_{U_2}(S) \Rightarrow x \in \underline{apr}_{U_1}(S) \text{ and } x \in \underline{apr}_{U_2}(S). \\ & \Rightarrow x⊞U_1 \subseteq S \text{ and } x⊞U_2 \subseteq S \Rightarrow (x⊞U_1) \cap (x⊞U_2) \subseteq S \\ & \Rightarrow x⊞(U_1 \cap U_2) \subseteq S \Rightarrow x \in \underline{apr}_{(U_1 \cap U_2)}(S). \end{aligned} \tag{37}$$

Therefore, $\underline{apr}_{U_1}(S) \cap \underline{apr}_{U_2}(S) \subseteq \underline{apr}_{U_1 \cap U_2}(S)$.

(2) Consider

$$\overline{apr}_{U_1 \cap U_2}(S) = S⊞(U_1 \cap U_2) = (S⊞U_1) \cap (S⊞U_2) = \overline{apr}_{U_1}(S) \cap \overline{apr}_{U_2}(S). \tag{38}$$

Thus, $\overline{apr}_{U_1 \cap U_2}(S) \subseteq \overline{apr}_{U_1}(S) \cap \overline{apr}_{U_2}(S)$.

(3) Consider

$$\begin{aligned} \overline{apr}_{U_1(U_2}(S⊞T) &= (S⊞T)⊞(U_1(U_2) = (S⊞U_1)⊞(T(U_2) \\ &= \overline{apr}_{U_1}(S)⊞\overline{apr}_{U_2}(T). \end{aligned} \tag{39}$$

Thus, $\overline{apr}_{U_1(U_2}(S⊞T) = \overline{apr}_{U_1}(S)⊞\overline{apr}_{U_2}(T)$.

(4) Let $x \in \underline{apr}_{U_1}(S)⊞\underline{apr}_{U_2}(T)$. Then, $\exists \beta \in \underline{apr}_{U_1}(S)$ and $\gamma \in \underline{apr}_{U_2}(T)$, such that $x \in \beta⊞\gamma$. Since $\beta \in \underline{apr}_{U_1}(S)$ and $\gamma \in \underline{apr}_{U_2}(T)$, this implies that $\beta⊞U_1 \subseteq S$ and $\gamma ⊞U_2 \subseteq T$. Consider

$$x⊞(U_1⊞U_2) \subseteq (\beta⊞\gamma)⊞(U_1⊞U_2) = (\beta⊞U_1)⊞(\gamma⊞U_2) \subseteq S⊞T. \tag{40}$$

This implies that $x \in \underline{apr}_{U_1⊞U_2}(S⊞T)$. Hence, $\underline{apr}_{U_1}(S)⊞\underline{apr}_{U_2}(T) \subseteq \underline{apr}_{U_1⊞U_2}(S⊞T)$.

(5) Let $U_1 \subseteq U_2$. Then, $\overline{apr}_{U_1}(S) = S⊞U_1 \subseteq S⊞U_2 = \overline{apr}_{U_2}(S)$. Hence, $\overline{apr}_{U_1}(S) \subseteq \overline{apr}_{U_2}(S)$. Next, consider $\alpha \in \underline{apr}_{U_2}(S)$, then $\alpha⊞U_2 \subseteq S$. Now, $\alpha⊞U_1 \subseteq \alpha⊞U_2 \subseteq S$. This implies that $\alpha \in \underline{apr}_{U_1}(S)$. Hence, $\underline{apr}_{U_2}(S) \subseteq \underline{apr}_{U_1}(S)$.

□

Theorem 27. *For approximation space (V, U) and $\forall\, \emptyset \neq S, T \subseteq V$,*

(1) $\underline{apr}_U(\underline{apr}_U(S)) = \underline{apr}_U(S) = \overline{apr}_U(\underline{apr}_U(S))$

(2) $\underline{apr}_U(\overline{apr}_U(S)) = \overline{apr}_U(S) = \overline{apr}_U(\overline{apr}_U(S))$

(3) $\underline{apr}_U(S) = (\overline{apr}_U(S^c))^c$

(4) $\overline{apr}_U(S) = (\underline{apr}_U(S^c))^c$

(5) $\underline{apr}_U(x⊞U) = \overline{apr}_U(x⊞U) = x⊞U, \forall x \in V$

Proof. 1 to 4 can be proved easily.

(5) Let $y \in \underline{\mathrm{apr}}_U(x\boxplus U)$. Then, $y(U \subseteq x\boxplus U$. This implies that $y \in x\boxplus U$. Thus, $\underline{\mathrm{apr}}_U(x\boxplus U) \subseteq x\boxplus U$. Now let $y \in x\boxplus U$, we have $y\boxplus U \subseteq x\boxplus U$. Thus, $y \in \underline{\mathrm{apr}}_U(x\boxplus U)$. Therefore, $\underline{\mathrm{apr}}_U(x\boxplus U) = x\boxplus U$. Now, using Theorem 22, we get

$$\overline{\mathrm{apr}}_U(x\boxplus U) = (x\boxplus U)\boxplus U = x\boxplus(U\boxplus U) = x\boxplus U. \quad (41)$$

Hence, $\underline{\mathrm{apr}}_U(x\boxplus U) = \overline{\mathrm{apr}}_U(x\boxplus U) = x\boxplus U$. □

Theorem 28. *If S and T are definable sets and $0 \neq k \in F$, then $k * S$,$S \cap T$, $S \cup T$, $S\boxplus T$ are definable sets.*

Proof. Since S, T are definable so $\underline{\mathrm{apr}}_U(S) = \overline{\mathrm{apr}}_U(S)$ and $\underline{\mathrm{apr}}_U(T) = \overline{\mathrm{apr}}_U(T)$. First, we prove that $k * S$ is definable. By using the conditions 3 and 4 of Theorem 25, we get

$$\overline{\mathrm{apr}}_U(k * S) = k * \overline{\mathrm{apr}}_U(S) = k * \underline{\mathrm{apr}}_U(S) \subseteq \underline{\mathrm{apr}}_U(k * S). \quad (42)$$

It is obvious that $\underline{\mathrm{apr}}_U(k * S) \subseteq \overline{\mathrm{apr}}_U(k * S)$. Thus, $\underline{\mathrm{apr}}_U(k * S) = \overline{\mathrm{apr}}_U(k * S)$. Hence, $k * S$ is a definable set. Next, we prove that $S \cap T$ is definable. By using the conditions 2 and 5 of Theorem 23, we get

$$\overline{\mathrm{apr}}_U(S \cap T) \subseteq \overline{\mathrm{apr}}_U(S) \cap \overline{\mathrm{apr}}_U(T) = \underline{\mathrm{apr}}_U(S) \cap \underline{\mathrm{apr}}_U(T) = \underline{\mathrm{apr}}_U(S \cap T). \quad (43)$$

It is obvious that $\underline{\mathrm{apr}}_U(S \cap T) \subseteq \overline{\mathrm{apr}}_U(S \cap T)$. Thus, $\underline{\mathrm{apr}}_U(S \cap T) = \overline{\mathrm{apr}}_U(S \cap T)$. Hence, $S \cap T$ is a definable set. Similarly, by using conditions 3 and 4 of Theorem 23, we can prove that $S \cup T$ is a definable set. Lastly, we prove that $S\boxplus T$ is a definable set. By using the conditions 1 and 2 of Theorem 25, we get

$$\overline{\mathrm{apr}}_U(S\boxplus T) = \overline{\mathrm{apr}}_U(S)\boxplus\overline{\mathrm{apr}}_U(T) = \underline{\mathrm{apr}}_U(S)\boxplus\underline{\mathrm{apr}}_U(T) \subseteq \underline{\mathrm{apr}}_U(S\boxplus T). \quad (44)$$

It is obvious that $\underline{\mathrm{apr}}_U(S\boxplus T) \subseteq \overline{\mathrm{apr}}_U(S\boxplus T)$. Thus, $\underline{\mathrm{apr}}_U(S\boxplus T) = \overline{\mathrm{apr}}_U(S\boxplus T)$. Hence, $S(T)$ is a definable set. □

5. Conclusion

This paper created a bond between rough sets and hypervector space. Based on a congruence, we defined the lower (upper) approximations of a subset of the hypervector space. To begin, we intended the relationship between a congruence relation and hypersubspaces of a hypervector space. Second, some lower and upper approximation characterizations in hypervector spaces are structured. Because of the close relationship between hypervector spaces and Automata (artificial intelligence) and related disciplines, we believe this research will provide a strong tool in approximate reasoning. We believe that the rough hypervector spaces illustrated here will be useful in hyperstructure theory and rough set application.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors of this paper declare that they have no conflict of interest.

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