Research Article

Fuzzy Set Theoretic Approach to Generalized Ideals in BCK/BCI-Algebras


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This paper deals with the study of generalizations of fuzzy subalgebras and fuzzy ideals in BCK/BCI-algebras. In fact, the notions of \((e,e \vee (k^*, q_k))\)-fuzzy subalgebras, \((e,e \vee (k^*, q_k))\)-fuzzy ideals, and \((e,e \vee (k^*, q_k))\)-fuzzy ideals in BCK/BCI-algebras are introduced. Some examples are provided to demonstrate the logic of the concepts used in this paper. Moreover, some characterizations of these notions are discussed. In addition, the concept of \((e,e \vee (k^*, q_k))\)-fuzzy commutative ideals is introduced, and several significant characteristics are discussed. It is shown that for a BCK-algebra \(\mathcal{A}\), every \((e,e \vee (k^*, q_k))\)-commutative ideal of a BCK-algebra is an \((e,e \vee (k^*, q_k))\)-fuzzy ideal, but the converse does not hold in general; a counter example is constructed.

1. Introduction

To deal with possible complexity associated with expectations, state imprecision, and desires, a fuzzy set is a valuable method, as proposed by Zadeh [1]. Fuzzy set theory has since become an active research area in a different field. The idea of quasicoincidence of a fuzzy point with a fuzzy set, as stated in [2], was crucial in generating some different types of fuzzy subgroups, known as \((\alpha, \beta)\)-fuzzy subgroups, as introduced by Bhakat and Das in [3]. The \((e,e \vee)\)-fuzzy subgroup, in particular, is an important and valuable generalization of Rosenfeld’s fuzzy subgroup. The idea of \((\alpha, \beta)\)-fuzzy subalgebras in BCK/BCI-algebras is also interesting and useful generalizations of some well-known ideas of fuzzy subalgebras (see, for example, [4–11]). Several similar ideas based on fuzzy subalgebras and ideals have recently been studied in various algebras (see, for example, [12–18]). Many researchers have also extended fuzzy set theory and related concepts to different fields (see, e.g., [19–24]).

Motivated by a lot of work on ideal theory in BCK/BCI-algebras based on the fuzzy set theory, it is reasonable to introduce a generalized version of the fuzzy ideals of BCK/BCI-algebras. To that aim, in Section 2, some basic definitions and elementary results are presented. Then, in Section 3, we present the concepts \((e,e \vee (k^*, q_k))\)-fuzzy subalgebras, \((e,e \vee (k^*, q_k))\)-fuzzy ideals, and \((e,e \vee (k^*, q_k))\)-fuzzy ideals and associated characteristics are investigated. In Section 4, the concept of \((e,e \vee (k^*, q_k))\)-fuzzy commutative ideals is presented and various properties are investigated, as well as their relationship with \((e,e \vee (k^*, q_k))\)-fuzzy ideals.

2. Preliminaries

An algebra \(\mathcal{A} = (\mathcal{A}; \ast, 0)\) is called a BCI-algebra if \(\forall\ \alpha, \Theta, \zeta \in \mathcal{A}\),

\[
(1) \ ( (\alpha \ast \Theta) \ast (\alpha \ast \zeta)) \ast (\zeta \ast \Theta) = 0
\]
(2) \((\omega \ast (\omega \ast \Theta)) \ast \Theta = 0\)

(3) \(\omega \ast \omega = 0\)

(4) \(\omega \ast \Theta = 0\) and \(\Theta \ast \omega = 0 \Rightarrow \omega = \Theta\)

If \(\mathcal{A}\) satisfies (2), (3), (40), (43), and \(0 \ast \omega = 0\), then \(\mathcal{A}\) is a BCK-algebra.

Any BCK-algebra \(\mathcal{A}\) satisfies the following:

(5) \(\omega \ast 0 = \omega\)

(6) \((\omega \ast \Theta) \ast \psi = (\omega \ast \psi) \ast \Theta\)

Define a partially ordered \(\leq\) on \(\mathcal{A}\) as \(\omega \leq \Theta \Leftrightarrow \omega \ast \Theta = 0\).

A subset \((\Omega, S)\) of \(\mathcal{A}\) is said to be a subalgebra if for all \(\omega, \omega' \in S\) and \(\omega \leq \omega' \in S\) implies \(\omega \in S\).

A mapping \(\mathcal{F} : \mathcal{A} \rightarrow [0, 1]\) is said to be a fuzzy set (briefly, FS) of \(\mathcal{A}\). If \(\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\omega) \land \mathcal{F}(\Theta), \forall \omega, \Theta \in \mathcal{A}\), then \(\mathcal{F}\) is said to be a fuzzy subalgebra. If \(\mathcal{F}(0) \geq \mathcal{F}(\omega)\) and \(\mathcal{F}(\omega) \geq \mathcal{F}(\omega \ast \Theta) \land \mathcal{F}(\Theta), \forall \omega, \Theta \in \mathcal{Z}\), then \(\mathcal{F}\) is said to be a fuzzy ideal.

**Definition 1.** Let \(\omega \in \mathcal{A}\) and \(\epsilon \in [0, 1]\). An ordered fuzzy point (briefly, OFP) \(\omega_\epsilon\) of \(\mathcal{A}\) is defined as

\[
\omega_\epsilon(\Theta) = \begin{cases} 
\epsilon, & \text{if } \Theta \in (\omega], \\
0, & \text{if } \Theta \in (\omega),
\end{cases}
\]

\(\forall \Theta \in \mathcal{A}\).

Clearly, \(\omega_\epsilon\) is a FS of \(\mathcal{A}\). For FS \(\mathcal{F}\) of \(\mathcal{A}\), we represent \(\omega_\epsilon \leq \mathcal{F}\) as \(\omega_\epsilon \in \mathcal{F}\) in the sequel. So, \(\omega_\epsilon \in \mathcal{F}\) if and only if \(\mathcal{F}(\omega) \geq \epsilon\).

**Lemma 2** (see [4]). A FS \(\mathcal{F}\) in \(\mathcal{A}\) is an \((\epsilon, \epsilon \land \nu)\)-fuzzy subalgebra of \(\mathcal{A}\) if and only if

\((\forall \omega, \Theta \in \mathcal{A})(\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land 0.5)\).

**Lemma 3** (see [25]). A FS \(\mathcal{F}\) in \(\mathcal{A}\) is an \((\epsilon, \epsilon \land \nu)\)-fuzzy subalgebra of \(\mathcal{A}\) if and only if

\((\forall \omega, \Theta \in \mathcal{A})(\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\Theta) \land \nu(\Theta) \land \frac{1 - \kappa}{2})\).

**3. \((\epsilon, \epsilon \land \nu)\)-Fuzzy Ideals**

In what follows, \(\mathcal{A}\) denotes a BCK/BCI-algebra unless otherwise specified.

**Definition 4.** Let \(\omega_\epsilon\) be an OFP of \(\mathcal{A}\) and \(\kappa \in (0, 1]\). Then, \(\omega_\epsilon\) is said to be \((\kappa^*, \nu)\)-quasicoincident with a FS \(\mathcal{F}\) of \(\mathcal{A}\), represented as \(\omega_\epsilon(\kappa^*, \nu) \in \mathcal{F}\), if \(\mathcal{F}(\omega) + \epsilon > \kappa^*\).

Let us take, \(0 \leq \kappa < \kappa^* \leq 1\). For an OFP \(\omega_\epsilon\), we define

(1) \(\omega_\epsilon(\kappa^*, \nu) \in \mathcal{F}\) if \(\mathcal{F}(\omega) + \epsilon > \kappa^*\).

(2) \(\omega_\epsilon \in \mathcal{F}\) if \(\omega_\epsilon(\kappa^*, \nu) \in \mathcal{F}\) or \(\omega_\epsilon(\kappa^*, \nu) \notin \mathcal{F}\).

(3) \(\omega_\epsilon \in \mathcal{F}\) if \(\omega_\epsilon(\kappa^*, \nu) \in \mathcal{F}\) does not hold for \(\epsilon \in (\kappa^*, \nu)\).

**Definition 5.** A FS \(\mathcal{F}\) of \(\mathcal{A}\) is said to be an \((\epsilon, \epsilon \land \nu)\)-FSA of \(\mathcal{A}\) if \(\omega_\epsilon \in \mathcal{F}\) and \(\Theta_\delta \in \mathcal{F}\) imply \(\omega_\epsilon \ast \Theta_\delta \in \mathcal{F}\) for \(\forall \epsilon, \delta \in (0, 1]\).

**Example 6.** Consider a BCI-algebra \(\mathcal{A} = \{0, 1, 2\}\) with operation \((\ast)\), which is described in Table 1. Define \(\mathcal{F} : \mathcal{A} \rightarrow [0, 1]\) by the following:

\[
\mathcal{F}(\omega) = \begin{cases} 
0.65, & \text{if } \omega = 0, \\
0.73, & \text{if } \omega = 1, \\
0.32, & \text{if } \omega \in \{2, 3\}.
\end{cases}
\]

Take \(\kappa^* = 0.7\) and \(\kappa = 0.1\). Then, it yields that \(\mathcal{F}\) is an \((\epsilon, \epsilon \land \nu)\)-FSA of \(\mathcal{A}\).

**Theorem 7.** A FS \(\mathcal{F}\) of \(\mathcal{A}\) is an \((\epsilon, \epsilon \land \nu)\)-FSA of \(\mathcal{A}\) if

\[
\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land \frac{\kappa^* - \kappa}{2},
\]

\(\forall \omega, \Theta \in \mathcal{A}\).

**Proof.** \(\Rightarrow\) On the contrary, assume that \(\mathcal{F}(\omega \ast \Theta) > \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land \frac{\kappa^* - \kappa}{2}\), for some \(\omega, \Theta \in \mathcal{A}\). Choose \(\epsilon \in (0, 1]\) such that

\[
\mathcal{F}(\omega \ast \Theta) < \epsilon \leq \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land \frac{\kappa^* - \kappa}{2}.
\]

Then, \(\omega_\epsilon \in \mathcal{F}\) and \(\Theta_\delta \in \mathcal{F}\), but \((\omega_\epsilon \ast \Theta_\delta) \notin \mathcal{F}\), which is not possible. Hence, \(\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land \frac{\kappa^* - \kappa}{2}\).

\((\Leftarrow)\) Assume that \(\mathcal{F}(\omega \ast \Theta) \geq \mathcal{F}(\omega) \land \mathcal{F}(\Theta) \land \frac{\kappa^* - \kappa}{2}\), \(\forall \omega, \Theta \in \mathcal{A}\). Then, \(\mathcal{F}(0) = 0\) and \(\mathcal{F}(\omega) = 0\), \(\forall \omega \in \mathcal{A}\). Then, \(\mathcal{F}(\omega) \geq \epsilon\) and \(\mathcal{F}(\Theta) \geq \delta\).

Choose \(\epsilon \in (0, 1]\) such that \(\mathcal{F}(\omega \ast \Theta) \geq \epsilon \land \delta\) imply that \((\omega_\epsilon \ast \Theta_\delta) \notin \mathcal{F}\). If \(\epsilon \land \delta > (\kappa^* - \kappa)/2\), then \(\mathcal{F}(\omega_\epsilon \ast \Theta_\delta) \geq (\kappa^* - \kappa)/2\). So, \(\mathcal{F}(\omega \ast \Theta) + \epsilon \land \delta > (\kappa^* - \kappa)/2\). Therefore, \(\mathcal{F}(\omega \ast \Theta) \in \mathcal{F}\).

**Definition 8.** A FS \(\mathcal{F}\) of \(\mathcal{A}\) is said to be an \((\epsilon, \epsilon \land \nu)\)-FSA of \(\mathcal{A}\).

(1) \(\omega_\epsilon \in \mathcal{F}\) if \(\omega_\epsilon \in \mathcal{F}\) and \(\Theta_\delta \in \mathcal{F}\) imply \(\omega_\epsilon \ast \Theta_\delta \in \mathcal{F}\) for \(\forall \epsilon, \delta \in (0, 1]\).
Example 9. Take a BCI-algebra $\mathcal{A} = \{0, 1, 2, 3\}$ with operation $(\ast)$ which is described in Table 2. Define $\bar{\mathcal{F}} : \mathcal{A} \rightarrow \{0, 1\}$ by the following:

$$\bar{\mathcal{F}}(\omega) = \begin{cases} 0.9, & \text{if } \omega = 0, \\ 0.3, & \text{if } \omega \in \{1, 2, 3\}. \end{cases}$$

Consider “$\kappa = 0.2$” and “$\kappa = 0.1$.” It is easy to check that $\bar{\mathcal{F}}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$.

Definition 10 (see [5]). A FS $\bar{\mathcal{F}}$ of $\mathcal{A}$ is said to be an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-fuzzy ideal (briefly $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI) of $\mathcal{A}$ if:

1. $\omega \in \bar{\mathcal{F}}$ implies $0 \in \mathcal{F}q\bar{\mathcal{F}}$, and
2. $(\omega \ast \Theta), \Theta \in \mathcal{F} \text{ imply } \omega_{\epsilon, \delta} \in \mathcal{F}q\bar{\mathcal{F}}$

$\forall \omega, \Theta \in \mathcal{A}$ and $\epsilon, \delta \in (0, 1]$.

Proposition 11. In $\mathcal{A}$, every $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI is $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI.

Proof. Suppose that $\mathcal{F}$ is any $(\epsilon, \eta) \mathcal{F}$ of $\mathcal{A}$. Take $\omega \in \mathcal{F}$ for $\omega \in \mathcal{A}$ and $\epsilon \in (0, 1]$. Then, by hypothesis, $0 \in \mathcal{F}q\bar{\mathcal{F}}$. It implies that $\mathcal{F}(0) \geq \epsilon$ or $\mathcal{F}(0) + \epsilon = 1$, and so $\mathcal{F}(0) \geq \epsilon$ or $\mathcal{F}(0) + \epsilon = 1$. Therefore, $0 \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. Next, let $(\omega \ast \epsilon) \in \mathcal{F}$ and $\Theta \delta \in \mathcal{F}$. So, $\omega_{\epsilon, \delta} \in \mathcal{F}q\bar{\mathcal{F}}$ implies $\mathcal{F}(\omega) \geq \epsilon$ $\land \delta = \mathcal{F}(\omega) + \epsilon + \delta \epsilon > 1$. Therefore, $\mathcal{F}(\omega) \geq \epsilon$ and $\mathcal{F}(\omega) + \epsilon + \delta \epsilon > 1$. Thus, $\omega \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. Hence, $\mathcal{F}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$.

Remark 12. In general, the converse of Proposition 11 is not valid. The following example demonstrates this.

Example 13. Take a BCK-algebra $\mathcal{A} = \{0, 1, 2, 3, 4\}$ with operation $(\ast)$ which is described in Table 3.

Define $\bar{\mathcal{F}} : \mathcal{A} \rightarrow \{0, 1\}$ by

$$\bar{\mathcal{F}}(\omega) = \begin{cases} 0.4, & \text{if } \omega = 0, \\ 0.2, & \text{if } \omega \in \{1, 2\}, \\ 0.5, & \text{if } \omega = 3, \\ 0.1, & \text{if } \omega = 4. \end{cases}$$

Take $\kappa^* = 0.2$ and $\kappa = 0.1$. Then, $\mathcal{F}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$ but it is not an $(\epsilon, \eta) \mathcal{F}(\kappa, q_e^*)$-FI of $\mathcal{A}$ as $2_{e=0.6} = (4 \ast 2)_{e=0.6} \in \bar{\mathcal{F}}$ and $2_{e=0.6} \in \mathcal{F}$ but $4_{e, \delta=0.6} \in \mathcal{F}q\bar{\mathcal{F}}$.

Definition 14. A FS $\bar{\mathcal{F}}$ of $\mathcal{A}$ is said to be an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*) \mathcal{F}(\kappa^*, q_e^*)$-fuzzy ideal (briefly $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*) \mathcal{F}(\kappa^*, q_e^*)$-FI) of $\mathcal{A}$ if:

1. $\omega \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ implies $0 \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$, and
2. $(\omega \ast \Theta), \Theta \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ and $\Theta \delta \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ imply $\omega_{\epsilon, \delta} \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$

$\forall \omega, \Theta \in \mathcal{A}$ and $\epsilon, \delta \in (0, 1]$.}

Lemma 15. In $\mathcal{A}$, every $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*) \mathcal{F}(\kappa^*, q_e^*)$-FI is $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI.

Proof. Suppose that $\mathcal{F}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$. Let $\omega \in \mathcal{F}$ for $\omega \in \mathcal{A}$ and $\epsilon \in (0, 1]$. Then, $\mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. So, by hypothesis, $0 \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. Suppose that $(\omega \ast \epsilon) \in \mathcal{F}$ and $\delta \delta \in \mathcal{F}$. Then, $(\omega \ast \epsilon) \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ and $\delta \delta \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. Therefore, by hypothesis, $\omega_{\epsilon, \delta} \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$. Hence, $\mathcal{F}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$.

Remark 16. In general, the converse of Lemma 15 is not valid. This is illustrated by the following.

Example 17. From Example 13, define $\bar{\mathcal{F}} : \mathcal{A} \rightarrow \{0, 1\}$ by

$$\bar{\mathcal{F}}(\omega) = \begin{cases} 0.4, & \text{if } \omega = 0, \\ 0.6, & \text{if } \omega \in \{1, 3\}, \\ 0.1, & \text{if } \omega = 2.4. \end{cases}$$

Take “$\kappa = 0$” and “$\kappa^* = 0.7$.” Then, $\mathcal{F}$ is an $(\epsilon, \eta) \mathcal{F}(\kappa^*, q_e^*)$-FI of $\mathcal{A}$ but it is not an $(\epsilon, \eta) \mathcal{F}(\kappa, q_e^*)$-FI of $\mathcal{A}$ as $2_{e=0.95} = (2 \ast 1)_{e=0.95} \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ and $2_{e=0.5} \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ but $2_{e, \delta=0.5} \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$.

Lemma 18. Let $\mathcal{F}$ be a FS of $\mathcal{A}$. Then, $\mathcal{F}(\kappa) \mathcal{F}$ implies $0 \in \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$.

Therefore, $\mathcal{F}(\kappa) \mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$ and $\mathcal{F}(\kappa^*, q_e^*)\mathcal{F}(\kappa^*, q_e^*)\mathcal{F}$.
Proof. \((\Rightarrow)\) Contrary assume that, for some \(\omega \in \mathcal{A}, \tilde{F}(\omega) < F(\omega) \land ((\kappa^* - \kappa)/2)\). Take \(e \in (0, ((\kappa^* - \kappa)/2)]\) such that
\[
\tilde{F}(\omega) < e \leq \tilde{A}((\kappa^* - \kappa)/2).
\] (10)

Then, \(\omega \in \tilde{F}\), but \(0 \leq \epsilon \in ((\kappa^* - \kappa)/2)\), a contradiction. Hence, \(F(\omega) \geq \tilde{F}(\omega) \land ((\kappa^* - \kappa)/2)\).

\((-\Leftarrow)\) Let \(\omega \in \mathcal{A}\) such that \(\omega \in \tilde{F}\). Then, \(\tilde{F}(\omega) \geq e\). So,
\[
\tilde{F}(\omega) \geq \tilde{F}(\omega) \land ((\kappa^* - \kappa)/2) \geq e \land ((\kappa^* - \kappa)/2).
\] (11)

Now, if \(e \leq ((\kappa^* - \kappa)/2)\), then \(\tilde{F}(\omega) \geq e\). Therefore, \(0 \leq \tilde{F}\). Again, if \(e > ((\kappa^* - \kappa)/2)\), then \(\tilde{F}(\omega) \geq ((\kappa^* - \kappa)/2)\). So, \(\tilde{F}(\omega) + e > ((\kappa^* - \kappa)/2) +(\kappa^* - \kappa)/2 = \kappa^* - \kappa\). This follows that \(0 \leq \omega \in \tilde{F}\). Hence, \(0 \leq \omega \in \mathcal{A}\).

**Lemma 19.** Let \(\tilde{F}\) be FS of \(\mathcal{A}\). Then, \((\omega \ast \Theta) \in \tilde{F}\) and \(\omega \ast \Theta \in \tilde{F}\) imply \(\omega \ast \delta \in \nu((\kappa^*, q_\kappa)) \rightarrow \tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\delta) \land ((\kappa^* - \kappa)/2)\).

Proof. \((\Rightarrow)\) On contrary suppose that \(\tilde{F}(\omega) < \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\delta) \land ((\kappa^* - \kappa)/2)\) for some \(\omega, \Theta \in \mathcal{A}.\) Choose \(e \in (0, ((\kappa^* - \kappa)/2)]\) such that \(\tilde{F}(\omega) < e \leq \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\delta) \land ((\kappa^* - \kappa)/2)\). Then, \((\omega \ast \Theta) \in \tilde{F}\), \(\Theta \in \tilde{F}\), but \(\omega \ast \delta \in \nu((\kappa^*, q_\kappa))\), which is not possible. Thus, we have shown that
\[
\tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\delta) \land ((\kappa^* - \kappa)/2).
\] (12)

\((-\Leftarrow)\) Let \((\omega \ast \Theta) \in \tilde{F}\) and \(\Theta \in \tilde{F}\), \(\forall \epsilon, \delta \in (0, 1]\). Then, \(\tilde{F}(\omega \ast \Theta) \geq e \land \tilde{F}(\Theta) \geq \epsilon\). Thus,
\[
\tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2).
\] (13)

Now, if \(\epsilon \land \delta \leq ((\kappa^* - \kappa)/2)\), then \(\tilde{F}(\omega) \geq \epsilon \land \delta\), then \(\omega \in \tilde{F}\); otherwise, i.e., when \(\epsilon \land \delta > ((\kappa^* - \kappa)/2)\), then \(\tilde{F}(\omega) \geq ((\kappa^* - \kappa)/2)\). So, we have
\[
\tilde{F}(\omega) + e \land \delta \geq \epsilon \land \delta \land ((\kappa^* - \kappa)/2).
\] (14)

This implies that \(\omega \in \nu((\kappa^*, q_\kappa)) \rightarrow \tilde{F}\). Hence, \(\omega \in \nu((\kappa^*, q_\kappa)) \rightarrow \tilde{F}\), as required.

**Theorem 20.** A FS \(\tilde{F}\) of \(\mathcal{A}\) is an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of \(\mathcal{A}\) if and only if
\[
\begin{align*}
(1) & \quad \tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \land (((\kappa^* - \kappa)/2), \forall \omega, \Theta \in \mathcal{A}) \\
(2) & \quad \tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \land \tilde{F}(\Theta), (((\kappa^* - \kappa)/2), \forall \omega, \Theta \in \mathcal{A})
\end{align*}
\]

**Lemma 21.** Let \(\tilde{F}\) be an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of \(\mathcal{A}\) such that \(\omega \leq \Theta\). Then, \(\tilde{F}(\omega) \geq \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2)\).

Proof. Let \(\omega \leq \Theta\) for \(\omega, \Theta \in \mathcal{A}\). Then, \(\omega \ast \Theta = 0\). By hypothesis, we have
\[
\begin{align*}
\tilde{F}(\omega) & \geq \tilde{F}(\omega) \land \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2) \\
& \geq \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2).
\end{align*}
\] (15)

**Lemma 22.** Let \(\tilde{F}\) be an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of \(\mathcal{A}\). Then, for any \(\omega, \Theta, \zeta \in \mathcal{A}\),
\[
\omega \ast \Theta \leq \zeta \Rightarrow \tilde{F}(\omega) \geq \tilde{F}(\Theta) \land \tilde{F}(\zeta) \land ((\kappa^* - \kappa)/2).
\] (16)

Proof. Let \(\omega \ast \Theta \leq \zeta\) for \(\omega, \Theta, \zeta \in \mathcal{A}\). We have
\[
\begin{align*}
\tilde{F}(\omega) & \geq \tilde{F}(\omega) \land \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2) \\
& \geq \tilde{F}(\Theta) \land \tilde{F}(\zeta) \land ((\kappa^* - \kappa)/2) \\
& \geq \tilde{F}(\Theta) \land \tilde{F}(\zeta) \land ((\kappa^* - \kappa)/2).
\end{align*}
\] (17)

**Theorem 23.** Every \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of BCK-algebra \(\mathcal{A}\) is an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FS of \(\mathcal{A}\).

Proof. Let \(\tilde{F}\) be an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of \(\mathcal{A}\) and \(\omega, \Theta \in \mathcal{A}\). As \(\omega \ast \Theta \leq \Theta\) in \(\mathcal{A}\), so by Lemma 21, we have
\[
\tilde{F}(\omega) \geq \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2).
\] (18)

Since \(\tilde{F}\) is an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FI of \(\mathcal{A}\), we have
\[
\begin{align*}
\tilde{F}(\omega) & \geq \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2) \\
& \geq \tilde{F}(\Theta) \land \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2) \\
& \geq \tilde{F}(\Theta) \land \tilde{F}(\Theta) \land ((\kappa^* - \kappa)/2).
\end{align*}
\] (19)

Hence, \(\tilde{F}\) is an \((\epsilon, \nu((\kappa^*, q_\kappa)))\)-FS of \(\mathcal{A}\).

**Remark 24.** In general, the converse of Theorem 23 is not valid.

**Example 25.** Take a BCK-algebra \(\mathcal{A} = \{0, 1, 2, 3\}\) with operation \((\ast)\) which is described in Table 4.
Consider the \((\epsilon, \epsilon V(k^*, q_\delta))\)-FS \(\tilde{\mathcal{F}}\) of \(\mathcal{A}\), where \(\tilde{\mathcal{F}} : \mathcal{A} \to [0, 1]\) is defined by

\[
\tilde{\mathcal{F}}(\omega) = \begin{cases} 
0.3, & \text{if } \omega = 0, \\
0.1, & \text{if } \omega \in (1, 2], \\
0.2, & \text{if } \omega = 3.
\end{cases}
\]

(20)

Take \(\kappa^* = 0.8\) and \(\kappa = 0.1\). Then, \(\tilde{\mathcal{F}}\) is not an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\) as \(0_{\epsilon=0.3} = (1 \ast 3)_{\epsilon=0.3} \in \tilde{\mathcal{F}}\) and \(3_{\delta=0.2} \in \tilde{\mathcal{F}}\) but \(1_{\epsilon, \delta=0.2} \not\in \epsilon V(k^*, q_\delta)\tilde{\mathcal{F}}\).

**Theorem 26.** Let \(\tilde{\mathcal{F}}\) be an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FS of \(\mathcal{A}\). Then, \(\tilde{\mathcal{F}}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\) if and only if \(\Theta * \Theta \leq \zeta\) implies \(\tilde{\mathcal{F}}(\Theta) \geq \tilde{\mathcal{F}}(\Theta) \wedge (\zeta \wedge (k^* - \kappa)/2)\).

**Proof.** Suppose that \(\tilde{\mathcal{F}}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\). Take any \(\omega \in [\tilde{\mathcal{F}}]_\epsilon\). Then, \(\omega \in \epsilon V(k^*, q_\delta)\tilde{\mathcal{F}}\). So, \(\tilde{\mathcal{F}}(\omega) \geq \epsilon \wedge (k^* - \kappa)/2\). Now, by Theorem 20, we have \(\tilde{\mathcal{F}}(0) \geq \tilde{\mathcal{F}}(\omega) \wedge (\zeta \wedge (k^* - \kappa)/2)\).

Hence, \(\tilde{\mathcal{F}}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\).

**Theorem 27.** A FS \(\mathcal{F}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\) if and only if the set \(\mathcal{F} \setminus \Theta\) is an ideal of \(\mathcal{A}\), \(\forall \epsilon \in (0, ((k^* - \kappa)/2)]\).

**Proof.** Suppose that \(\mathcal{F}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\). By Theorem 20, we have

\[
\mathcal{F}(0) \geq \mathcal{F}(\omega) \wedge (k^* - \kappa)/2,
\]

(22)

with \(\omega \in \mathcal{F}\). It follows that \(\mathcal{F}(0) \geq \epsilon \wedge (k^* - \kappa)/2\) implies \(\mathcal{F}(0) \geq \epsilon \wedge (k^* - \kappa)/2\). Therefore, \(0 \in \mathcal{F}\).

Next, suppose that \(\omega * \Theta \in \mathcal{F}\) and \(\Theta \in \mathcal{F}\). Then, \(\mathcal{F}(\omega * \Theta) \geq \epsilon\) and \(\mathcal{F}(\Theta) \geq \epsilon\). Again, by Theorem 20, we have

\[
\mathcal{F}(\Theta) \geq \mathcal{F}(\omega * \Theta) \wedge (\mathcal{F}(\Theta) \wedge (k^* - \kappa)/2) \geq \epsilon \wedge (k^* - \kappa)/2 = \epsilon.
\]

(23)

Therefore, \(\omega \in \mathcal{F}\). Hence, \(\mathcal{F}\) is an ideal of \(\mathcal{A}\).

**Theorem 28.** Let \(\mathcal{F}\) be an FS of \(\mathcal{A}\). Then \(\mathcal{F}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-level subset of \(\mathcal{F}\).

**Definition 28.** Let \(\mathcal{F}\) be an FS of \(\mathcal{A}\). The set

\[
[\mathcal{F}]_{\epsilon} = \{ \omega \in \mathcal{A} | \omega \in \epsilon V(k^*, q_\delta)\mathcal{F} \},
\]

(25)

is said to be an \((\epsilon, \epsilon V(k^*, q_\delta))\)-level subset of \(\mathcal{F}\).

**Theorem 29.** Let \(\mathcal{F}\) be an FS of \(\mathcal{A}\). Then \(\mathcal{F}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\) if and only if the \((\epsilon, \epsilon V(k^*, q_\delta))\)-level subset \([\mathcal{F}]_{\epsilon}\) of \(\mathcal{F}\) is an ideal of \(\mathcal{A}\), \(\forall \epsilon \in (0, 1]\).

**Proof.** Suppose that \(\mathcal{F}\) is an \((\epsilon, \epsilon V(k^*, q_\delta))\)-FI of \(\mathcal{A}\). Take any \(\omega \in [\mathcal{F}]_{\epsilon}\). Then, \(\omega \in \epsilon V(k^*, q_\delta)\mathcal{F}\). So, \(\mathcal{F}(\omega) \geq \epsilon \wedge (k^* - \kappa)/2\). Now, by Theorem 20, we have \(\mathcal{F}(0) \geq \mathcal{F}(\omega) \wedge (k^* - \kappa)/2\). Thus, \(\mathcal{F}(0) \geq \epsilon \wedge (k^* - \kappa)/2\).

Therefore, \(\mathcal{F}(0) \geq \epsilon \wedge (k^* - \kappa)/2\) implies \(0 \in [\mathcal{F}]_{\epsilon}\). Also, if \(\epsilon \leq (k^* - \kappa)/2\), then \(\mathcal{F}(0) \geq \epsilon \wedge (k^* - \kappa)/2\) implies \(0 \in [\mathcal{F}]_{\epsilon}\).

Similarly, \(0 \in [\mathcal{F}]_{\epsilon}\) when \(\mathcal{F}(\omega) \geq \epsilon \wedge (k^* - \kappa)/2\).

Next, take any \(\omega * \Theta \in [\mathcal{F}]_{\epsilon}\) and \(\Theta \in [\mathcal{F}]_{\epsilon}\). Then, \(\omega * \Theta \in \epsilon V(k^*, q_\delta)\mathcal{F}\) and \(\Theta \in \epsilon V(k^*, q_\delta)\mathcal{F}\), i.e., either \(\mathcal{F}(\omega * \Theta) \geq \epsilon\) or \(\mathcal{F}(\omega * \Theta) \geq \epsilon \wedge (k^* - \kappa)\). Again, by Theorem 20, we have

\[
\mathcal{F}(\omega * \Theta) \geq \mathcal{F}(\omega * \Theta) \wedge (\mathcal{F}(\Theta) \wedge (k^* - \kappa)/2) \geq \epsilon \wedge (k^* - \kappa)/2 = \epsilon.
\]

(26)

so, \(\omega * \Theta \in \mathcal{F}\). If \(\epsilon \leq (k^* - \kappa)/2\), then

\[
\mathcal{F}(\omega) \geq \mathcal{F}(\omega * \Theta) \wedge (\mathcal{F}(\Theta) \wedge (k^* - \kappa)/2) \geq \epsilon \wedge (k^* - \kappa)/2 = \epsilon.
\]

(27)

So \(\omega * \Theta \in \mathcal{F}\). Hence, \(\omega * \Theta \in \mathcal{F}\).

Case 2. Let \(\mathcal{F}(\omega * \Theta) \geq \epsilon\) and \(\mathcal{F}(\Theta) \geq \epsilon \wedge (k^* - \kappa)\). If \(\epsilon > (k^* - \kappa)/2\), then

\[
\mathcal{F}(\omega) \geq \mathcal{F}(\omega * \Theta) \wedge (\mathcal{F}(\Theta) \wedge (k^* - \kappa)/2) \geq \epsilon \wedge (k^* - \kappa - \epsilon)/2 = k^* - \kappa - \epsilon,
\]

(28)
i.e., $\tilde{F}(\omega) + \epsilon > \kappa^* - \kappa$ and, thus, $\omega_\epsilon(\kappa^*, q_\kappa, q_\kappa)$. If $\epsilon \leq ((\kappa^* - \kappa)/2)$, then

$$\tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \wedge \tilde{F}(\Theta) \wedge \frac{\kappa^* - \kappa}{2}$$

$$\geq \epsilon \wedge (\kappa^* - \kappa - \epsilon) \wedge \frac{\kappa^* - \kappa}{2} = \epsilon,$$

and so $\omega_\epsilon \in \tilde{F}$. Hence, $\omega_\epsilon \in \vee(\kappa^*, q_\kappa)$. Similarly, for other cases, i.e., when $\tilde{F}(\omega \ast \Theta) + \epsilon > \kappa^* - \kappa$, $\tilde{F}(\Theta) \geq \epsilon$ and $\tilde{F}(\omega \ast \Theta) + \epsilon > \kappa^* - \kappa$, $\tilde{F}(\Theta) + \epsilon > \kappa^* - \kappa$, we have $\omega_\epsilon \in \vee(\kappa^*, q_\kappa, q_\delta)^\kappa$. Therefore, for each case, $\omega_\epsilon \in \vee(\kappa^*, q_\kappa)$, and hence $\omega \in [\tilde{F}]_\epsilon$.

$$(\Leftarrow)$$ Let $[\tilde{F}]_\epsilon$ be an ideal of $\mathcal{A}$, $\forall \epsilon \in (0, 1]$. On contrary, let

$$\tilde{F}(0) < \tilde{F}(\omega) \wedge \frac{\kappa^* - \kappa}{2},$$

with $\omega \in \mathcal{A}$. Then, $\exists \epsilon \in (0, 1]$ such that $\tilde{F}(0) < \epsilon \leq \tilde{F}(\omega) \wedge ((\kappa^* - \kappa)/2)$. It implies that $\omega \in [\tilde{F}]_\epsilon$, but $\theta \in [\tilde{F}]_\epsilon$, which is impossible. Therefore,

$$\tilde{F}(0) \geq \tilde{F}(\omega) \wedge \frac{\kappa^* - \kappa}{2}.$$  

Also, if $\tilde{F}(\omega) < \tilde{F}(\omega \ast \Theta) \wedge \tilde{F}(\Theta) \wedge (\kappa^* - \kappa)/2$ for some $\omega, \Theta \in \mathcal{A}$. Then, $\exists \epsilon \in (0, 1)$ such that

$$\tilde{F}(\omega) < \epsilon \leq \tilde{F}(\omega \ast \Theta) \wedge \tilde{F}(\Theta) \wedge (\kappa^* - \kappa)/2.$$  

It follows that $\omega \ast \Theta \in [\tilde{F}]_\epsilon$ and $\Theta \in [\tilde{F}]_\epsilon$, but $\omega \in [\tilde{F}]_\epsilon$, a contradiction. Therefore, $\tilde{F}(\omega) \geq \tilde{F}(\omega \ast \Theta) \wedge \tilde{F}(\Theta) \wedge (\kappa^* - \kappa)/2$. Hence, $\tilde{F}$ is an $(\epsilon, \vee(\kappa^*, q_\kappa))$-FI of $\mathcal{A}$.

4. $(\epsilon, \vee(\kappa^*, q_\kappa))$-Fuzzy Commutative Ideals

Throughout this section, $\mathcal{A}$ will stand for a BCK-algebra.

**Definition 30.** Let $\mathcal{A}$ be a BCK-algebra. An FS $\tilde{F}$ is said to be an $(\epsilon, \vee(\kappa^*, q_\kappa))$-fuzzy commutative ideal (briefly, $(\epsilon, \vee(\kappa^*, q_\kappa))$-FCI) if

1. $\omega_\epsilon \in \tilde{F}$ implies $\omega_\epsilon \in \vee(\kappa^*, q_\kappa)$, and
2. $((\omega \ast \Theta) \ast \varsigma)_\epsilon \in \tilde{F}$ and $\varsigma_\delta \in \tilde{F}$ imply $(\omega \ast (\Theta \ast \varsigma))_{\epsilon+\delta} \in \vee(\kappa^*, q_\kappa)$

$\forall \omega, \Theta, \varsigma \in \mathcal{A}$ and $\epsilon, \delta \in (0, 1]$.

**Example 31.** Take a BCK-algebra $\mathcal{A} = \{0, 1, 2, 3\}$ with operation $(\ast)$ which is described in Table 5.

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Define $\tilde{F} : \mathcal{A} \rightarrow [0, 1]$ by

$$\tilde{F}(\omega) = \begin{cases} 0.5, & \text{if } \omega = 0, \\ 0.3, & \text{if } \omega \in \{1, 2\}, \\ 0.1, & \text{if } \omega = 3. \end{cases}$$

Taking "$\kappa^* = 0.2$" and "$\kappa = 0.1$". Then, it is easy to get, $\tilde{F}$ is an $(\epsilon, \vee(\kappa^*, q_\kappa))$-FCI of $\mathcal{A}$.

**Theorem 32.** A FS $\tilde{F}$ of a BCK-algebra $\mathcal{A}$ is an $(\epsilon, \vee(\kappa^*, q_\kappa))$-FCI of $\mathcal{A}$ if

1. $\tilde{F}(0) \geq \tilde{F}(\omega) \wedge ((\kappa^* - \kappa)/2)$, and
2. $\tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \geq \tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \wedge \tilde{F}(\Theta) \wedge ((\kappa^* - \kappa)/2)$, $\forall \omega, \Theta, \varsigma \in \mathcal{A}$.

**Proof.** $(\Rightarrow)$ Lemma 18 gives us condition (2). To prove that (3) holds in $\mathcal{A}$, assume that (3) does not hold in $\mathcal{A}$, so we have

$$\tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) < \tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \wedge \tilde{F}(\Theta) \wedge \frac{\kappa^* - \kappa}{2},$$

for some $\omega, \Theta, \rho \in \mathcal{A}$. Choose $\epsilon \in (0, 1)$ such that $\tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) < \tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \wedge \tilde{F}(\Theta) \wedge (\kappa^* - \kappa)/2$. Then, $((\omega \ast (\Theta \ast (\Theta \ast \omega))) < \tilde{F}$ and $\rho \in \tilde{F}$, but $(\omega \ast (\Theta \ast (\Theta \ast \omega))) \in \tilde{F}$, which is not possible. Thus,

$$\tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \geq \tilde{F}(\omega \ast (\Theta \ast (\Theta \ast \omega))) \wedge \tilde{F}(\Theta) \wedge \frac{\kappa^* - \kappa}{2}.$$
\( \varepsilon \wedge \delta \) implies \((\varpi \ast (\Theta \ast (\varpi \ast \omega)))_{\nu, \delta} \in \mathcal{F}\); otherwise, i.e., when \(\varepsilon \wedge \delta > (k^* - \delta)/2\), then \(\mathcal{F}(\varpi \ast (\Theta \ast (\varpi \ast \omega))) \geq ((k^* - \delta)/2)\). So, we have

\[
\mathcal{F}(\varpi \ast (\Theta \ast (\varpi \ast \omega))) \geq \varepsilon \wedge \delta > \frac{k^* - \delta}{2} + \frac{k^* - \delta}{2} = k^* - \delta.
\]

This implies that \((\varpi \ast (\Theta \ast (\varpi \ast \omega)))_{\nu, \delta} \in \mathcal{V}(\kappa, \kappa, \mathcal{F})\). Therefore, \((\varpi \ast (\Theta \ast (\varpi \ast \omega)))_{\nu, \delta} \in \mathcal{V}(\kappa, \kappa, \mathcal{F})\). Hence, \(\mathcal{F}\) is an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\).

**Theorem 33.** Every \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of BCK-algebra \(\mathcal{A}\) is an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\).

**Proof.** Let \(\mathcal{F}\) be an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\) and \(\varpi, \Theta \in \mathcal{A}\). So, we have

\[
\mathcal{F}(\varpi) = \mathcal{F}(\varpi \ast (0 \ast (0 \ast \omega))) \geq \mathcal{F}((\varpi \ast 0) \ast \Theta) \wedge \mathcal{F}(\Theta) \wedge \frac{k^* - \varpi}{2}.
\]

Hence, \(\mathcal{F}\) is an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\).

**Remark 34.** The converse of Theorem 33 is not valid in general.

**Example 35.** Take a BCK-algebra \(\mathcal{A} = \{0, 1, 2, 3, 4\}\) with operation \((\ast)\) which is described in Table 6.

\[
\hat{\mathcal{F}}(\varpi) = \begin{cases} 0.5, & \text{if } \varpi = 0, \\ 0.4, & \text{if } \varpi = 1, \\ 0, & \text{if } \varpi \in \{2, 3, 4\}. \end{cases}
\]

5. **Conclusion**

By taking \(\varepsilon = 0\), we have

\[
\mathcal{F}(\varpi \ast (\Theta \ast (\varpi \ast \omega))) \geq \mathcal{F}((\varpi \ast \Theta) \ast 0) \wedge \mathcal{F}(0) \wedge \frac{k^* - \varpi}{2} = \mathcal{F}(\varpi \ast \Theta) \wedge \frac{k^* - \varpi}{2}.
\]

\(\ast\) Assume that equation (40) holds in \(\mathcal{A}\). Let \(\varpi, \Theta \in \mathcal{A}\). As \(\mathcal{F}\) is an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\), so we have

\[
\mathcal{F}(\varpi \ast (\Theta \ast (\varpi \ast \omega))) \geq \mathcal{F}((\varpi \ast \Theta) \ast \varpi) \wedge \mathcal{F}(\varpi) \wedge \frac{k^* - \varpi}{2}.
\]

By assumption and (43), we have

\[
\mathcal{F}(\varpi \ast (\Theta \ast (\varpi \ast \omega))) \geq \mathcal{F}((\varpi \ast \Theta) \ast \varpi) \wedge \mathcal{F}(\varpi) \wedge \frac{k^* - \varpi}{2}.
\]

Hence, \(\mathcal{F}\) is an \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-FCI of \(\mathcal{A}\).

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 \\
4 & 4 & 4 & 3 & 2 \\
\end{array}
\]

The main purpose of the present paper is to introduce the concepts of \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy subalgebras and \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy ideals in BCK/BCI-algebras. We provided some equivalent conditions and different characterizations of the \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy ideals in terms of level subsets and \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-level subsets of BCK/BCI-algebras. It has been shown that in any BCK/BCI-algebras the \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy ideals are \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy subalgebras but the converse does not hold and an example provided in this aim. Furthermore, \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy commutative ideals in BCK-algebras is introduced and some related properties of \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy ideals and \((\varepsilon, \varepsilon, \mathcal{V}(\kappa, \kappa))\)-fuzzy commutative ideals are considered. We hope that this work will provide a deep impact on the upcoming research in this field and other fuzzy algebraic studies to open up new horizons of interest and innovations. In future study, these notions may be extended to different algebras such as rings, hemirings, LA-semigroups, semihypergroups, semihyperrings, BCK/BCI-algebras, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, and EQ-algebras. Some important issues for future work are (1) to develop strategies for obtaining more valuable
results and (2) to apply these notions and results for studying related notions in other algebraic (fuzzy) structures.

**Data Availability**

No data were used to support the study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the manuscript.

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