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Research Article

Bilateral Harnack Inequalities for Stochastic Differential Equation with Multiplicative Noise

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By constructing a coupling with unbounded time-dependent drift, a lower bound estimate of dimension-free Harnack inequality with power is obtained for a large class of stochastic differential equation with multiplicative noise. The key is an application of the inverse Hölder inequality. Combining this with the well-known upper bound, bilateral dimension-free Harnack inequality with power is established. As a dual inequality, the bilateral shift-Harnack inequalities with power are also investigated for stochastic differential equation with additive noise. Applications to the study of heat kernel inequalities are provided to illustrate the obtained inequalities.

1. Introduction

We consider the following stochastic differential equation (SDE for brief) on $\mathbb{R}^d$:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{1}$$

where $\{B_t\}_{t\geq 0}$ is the d-dimensional Brownian motion on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses, and

$$\begin{aligned} b &: \Omega \times [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \\ \sigma &: \Omega \times [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d, \end{aligned} \tag{2}$$

are progressively measurable and locally bounded in the second variable and are continuous in the third variable. We will assume throughout this paper that, for any initial value $X_0 \in \mathbb{R}^d$, the SDE above has a unique strong solution which is nonexplosive and continuous in time t. To establish the bilateral Harnack inequalities, we will introduce the following assumptions:

Assumption 1. For b, σ, there exists an increasing function K on $[0, \infty)$ so that for all $t \geq 0$, all $y', y \in \mathbb{R}^d$, and

$$2\left\langle b\left(t, y'\right) - b(t, y), y' - y\right\rangle + \left\|\sigma\left(t, y'\right) - \sigma(t, y)\right\|_{\mathrm{HS}}^2 \leq K_t\left|y' - y\right|^2, a.s. \tag{3}$$

Assumption 2. There exists a decreasing function $\lambda : [0,\infty) \longrightarrow (0,\infty)$ s.t. for all $x \in \mathbb{R}^d$, $t \geq 0$,

$$\sigma(t, x)^* \sigma(t, x) \geq \lambda_t^2 I, a.s. \tag{4}$$

Assumption 3. For σ, there exists an increasing positive function δ on $[0, \infty)$ s.t. for all $t \geq 0$, all $y', y \in \mathbb{R}^d$, $\delta_t < \lambda_t$, and

$$\left|\left(\sigma\left(t, y'\right) - \sigma(t, y)\right)\left(y' - y\right)\right| \leq \delta_t\left|y' - y\right|, a.s. \tag{5}$$

Assumption 4. For $n \geq 1$, there exists a constant $c_n > 0$ s.t. for any $|y'|, |y|, t \leq n$,

$$\left\|\sigma\left(t, y'\right) - \sigma(t, y)\right\|_{\mathrm{HS}} + \left|b\left(t, y'\right) - b(t, y)\right| \leq c_n |y' - y|, a.s. \tag{6}$$

It follows from the above assumptions that the existence and uniqueness of the strong solution of SDE (1) are ensured (see Protter [1]). For more results about existence and uniqueness of the solution of differential equations, we can see [2–4].

Let X_t^x be the solution to the SDE (1) for $X_0 = x$. In this paper, we will establish the bilateral Harnack inequalities for the operator P_t; for any $f \in \mathscr{B}_b^+(\mathbb{R}^d)$, here $\mathscr{B}_b^+(\mathbb{R}^d)$ is the class of all bounded nonnegative measurable functions:

$$P_t f(x) \coloneqq Ef(X_t^x), \quad x \in \mathbb{R}^d, t \geq 0, \tag{7}$$

where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. In Wang [5], under the above assumptions (Assumptions 1–4), the author has successfully established the dimension-free Harnack inequalities for a large class of SDEs with multiplicative noise by constructing a coupling.

The dimension-free Harnack inequality was firstly established by Wang [6] for diffusion semigroups on Riemannian manifolds under a curvature condition. As a useful tool in the study of diffusion semigroups, in particular, for the uniform integrability, contractivity properties, and estimates on heat kernels, Wang's remarkable work has cause a lot of studies in the last two decades (see Aida and Zhang [7], Arnaudon et al. [8], and Da Prato [9]).

Wang's Harnack inequality has been extended in a large number of papers. For instance, by using coupling method, the upper bounded of dimension-free Harnack inequality has been established in [10] for a delay SDE with additive noise, in Wang and Yuan [11] for a delay SDE with multiplicative noise, in Wang and Zhang [12] for SDE with non-Lipschitz drift and driven by additive anisotropic cylindrical α-stable process, and in Wang et al. [13] for stochastic Burgers equation. Recently, using coupling by change of measures, the dimension-free Harnack inequality is investigated in Wang [14] for a distribution-dependent stochastic differential equation with regular coefficients and in Huang and Wang [15] for a distribution-dependent SDE with singular coefficients. For more details, we can refer to the book of Wang [16] for a deep analysis about dimension-free Harnack inequalities. All existing literatures focus on the upper bound of the dimension-free Harnack inequality. So far, we have not seen the lower bound estimate of dimension-free Harnack inequality with power in any previous literature. The aim of this paper is to establish the lower bound of the dimension-free Harnack inequality with power for SDE with multiplicative noise.

Theorem 1 in Wang [5] gives an upper bound of the dimension-free Harnack inequality with power for SDE with multiplicative noise. Hence, combining Theorem 1 in Wang [5] with our lower bound of the dimension-free Harnack inequality with power for SDE with multiplicative noise, actually, we have the following bilateral dimension-free Harnack inequality with power for SDE with multiplicative noise.

Theorem 5. *Let*

$$\kappa_1(p) \coloneqq \frac{(1-\sqrt{p})K_T}{4\sqrt{p}\delta_{p,T}\left(\delta_{p,T} - \lambda_T(1-\sqrt{p})\right)\left(1-e^{-K_T T}\right)}, \tag{8}$$

where $\delta_{p,T} \coloneqq \max\{\delta_T, \lambda_T(1-\sqrt{p})/2\}$, *and*

$$\kappa_2(p) \coloneqq \frac{(\sqrt{p}-1)K_T}{4\sqrt{p}\delta_{p,T}\left(\lambda_T(\sqrt{p}-1) - \delta_{p,T}\right)\left(1-e^{-K_T T}\right)}, \tag{9}$$

where $\delta_{p,T} \coloneqq \max\{\delta_T, \lambda_T(\sqrt{p}-1)/2\}$.

For $\mathscr{P}_1 \coloneqq (0, (1-(\delta_T/\lambda_T))^2)$ *and* $\mathscr{P}_2 \coloneqq ((1+(\delta_T/\lambda_T))^2, +\infty)$, *the bilateral dimension-free Harnack inequality with power*

$$\begin{aligned}&\sup_{p\in\mathscr{P}_1}\left(P_T f^p(x)\right)^{1/p} \exp\left[\kappa_1(p)|x-y|^2\right] \\ &\quad \leq P_T f(y) \leq \inf_{p\in\mathscr{P}_2}\left(P_T f^p(x)\right)^{1/p} \exp\left[\kappa_2(p)|x-y|^2\right],\end{aligned} \tag{10}$$

holds for all $T > 0, x, y \in \mathbb{R}^d$ *and* $f \in \mathscr{B}_b^+(\mathbb{R}^d)$.

In this paper, we will give a direct proof about the lower bound estimate of dimension-free Harnack inequality with power for a large class of SDE with multiplicative noise, rather than the reciprocal transformation based on the upper bound of dimension-free Harnack inequality with power. Of course, based on the reciprocal transformation of our lower bound estimate, we can also get the upper bound of dimension-free Harnack inequality with power. Unfortunately, our method does not adapt to establish the lower bound estimate of dimension-free log-Harnack inequality. At the same time, there is no evidence that the lower bound estimate of dimension-free log-Harnack inequality can be easily obtained by reciprocal transformation. This also leaves an open problem of how to establish the lower bound estimate of dimension-free log-Harnack inequality for a large class of SDE with additive noise or multiplicative noise.

Let $p_t(x, y)$ be the density of the operator P_t with respect to a Radon measure μ. It follows from Theorem 5 that the following corollary on bilateral heat kernel inequalities is a direct consequence.

Corollary 6. *Assume that Assumptions 1–4 hold and the operator* P_t *have a strictly positive density* $P_t(x, y)$ *with respect to* μ. *Then, for* $p > (1+\delta_T/\lambda_T)^2$, *the upper bound of* *heat kernel inequality*

$$\int_{\mathbb{R}^d} p_T(x,z)\left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{1/(p-1)}, \quad \mu(dz) \le \exp\left[\frac{p\kappa_2(p)}{p-1}|x-y|^2\right], \tag{11}$$

holds, and for $0<p<(1-\delta_T/\lambda_T)^2$, *the lower bound of heat kernel inequality, and for all* $T>0$,

$$\mathrm{int}_M p_T(x,z)\left(\frac{p_T(x,z)}{p_T(y,z)}\right)^{1/p}\mu(dz) \ge \exp\left[\kappa_1(p)|x-y|^2\right], \quad \forall x,y\in\mathbb{R}^d, \tag{12}$$

holds.

The main aim of this paper is to establish lower bound of dimension-free Harnack inequality with power for SDE with multiplicative noise. As a dual inequality, the shift-Harnack inequality has been developed and applied in Wang [17]. Obviously, it is relatively easy to establish lower bound of shift-Harnack inequality by the method used in this paper.

Next, we establish the bilateral dimension-free Wang's shift-Harnack inequalities with power for P_t introduced in Wang [17]. We now only consider the additive noise for which the SDE (1) reduces to

$$dX_t = b(t,X_t)dt + \sigma(t)dB_t, \quad X_0 = x. \tag{13}$$

Theorem 7. *Let* σ *from* $[0,\infty)$ *to* $\mathbb{R}^d\otimes\mathbb{R}^d$ *and* b *from* $[0,\infty)\times\mathbb{R}^d$ *to* $\mathbb{R}^d$ *be measurable and satisfy the assumptions in Section 1. Furthermore, the function* $\sigma(t)$ *is invertible and satisfies the assumption.*

Assumption 8. There exists a decreasing function λ from $[0,\infty)$ to $[0,\infty)$ s.t. for all $t\ge 0$,

$$\sigma(t)^*\sigma(t) \ge \lambda_t^2 I, a.s. \tag{14}$$

Then,

(1) for any $p\in(0,1)$, $T>0$, $x,v\in\mathbb{R}^d$, and for any $f\in\mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_T f)(x) \ge \left(P_T f^p(v+\cdot)\right)^{1/p}(x)\exp\left[\frac{1}{2(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{15}$$

(2) for any $p>1$, $T>0$, $x,v\in\mathbb{R}^d$, and for any $f\in\mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_T f)(x) \le \left(P_T f^p(v+\cdot)\right)^{1/p}(x)\exp\left[\frac{1}{2(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{16}$$

It is quite remarkable that the lower bound of dimension-free Harnack inequality with power in Theorem 5 is essentially equivalent to the upper bounded of dimension-free Harnack inequality with power through appropriate transformation. The same remark as Theorem 5 is also true for bilateral dimension-free shift-Harnack inequalities with power in Theorem 7. In the future, we want to establish bilateral Harnack inequalities for various models, for example, functional SDEs with additive/multiplication noise, distribution-dependent SDEs, distribution-dependent SDEs with singular coefficients, and SDEs driven by cylindrical α-stable processes. However, the method we use here is not suitable for the lower bounded dimension-free log-Harnack inequality. How to establish the lower bounded dimension-free log-Harnack inequality is a very interesting and meaningful question.

In Section 2, we construct a coupling and prove several auxiliary results which will be needed for the proof of theorem. The dimension-free bilateral Harnack inequality with power and dimension-free bilateral shift-Harnack inequality with power are given in Sections 3–4, respectively. As applications, bilateral heat kernel estimates are derived in Section 5.

2. Auxiliary Results

Let $x,y\in\mathbb{R}^d$, $T>0$, and

$$0<p<\left(1-\frac{\delta_T}{\lambda_T}\right)^2, \tag{17}$$

be fixed such that $x\ne y$. Due to $\delta_T<\lambda_T$, we have

$$\theta_T := \frac{2\delta_T}{\lambda_T(1-\sqrt{p})} \in (0,2). \tag{18}$$

For any $\theta\in(0,2)$, write

$$\kappa_t = \frac{2-\theta}{K_T}\left(1-e^{K_T(t-T)}\right), \quad t\in[0,T]. \tag{19}$$

Then, function κ is smooth and strictly positive on $[0,T)$ such that

$$2 - K_T\kappa_t + \kappa_t' = \theta, t\in[0,T]. \tag{20}$$

We now construct the following coupling:

$$dX_t = \sigma(t,X_t)dB_t + b(t,X_t)dt, \quad X_0 = x,$$
$$dY_t = \sigma(t,Y_t)dB_t + b(t,Y_t)dt + \frac{1}{\kappa_t}\sigma(t,Y_t)\sigma(t,X_t)^{-1}(X_t-Y_t)dt, \quad Y_0 = y. \tag{21}$$

We set $\inf\varnothing = T$ by convention. Let

$$\zeta_n := \inf\{t \in [0, T): |Y_t| \geq n\}, \tag{22}$$

and $\zeta = \lim_{n\longrightarrow\infty} \zeta_n$ is the explosion time of Y_t, and then, the coupling (X_t, Y_t) is a well-defined continuous process for $t < T\wedge\zeta$. For $t < T\wedge\zeta$, let

$$d\tilde{B}_t = dB_t + \frac{1}{\kappa_t}(X_t - Y_t)\sigma(t, X_t)^{-1}dt. \tag{23}$$

If $\zeta = T$ and

$$R_s := \exp\left[-\int_0^s \kappa_t^{-1}\langle (X_t - Y_t)\sigma(t, X_t)^{-1}, dB_t\rangle - \frac{1}{2}\int_0^s \kappa_t^{-2}|\sigma(t, X_t)^{-1}(X_t - Y_t)|^2 dt\right], \tag{24}$$

is a uniformly integrable martingale for $s \in [0, T)$, then by the martingale convergence theorem, $R_T := \lim_{t\uparrow T} R_t$ exists and $\{R_t\}_{t\in[0,T]}$ is a martingale. In this case, by the Girsanov theorem, $\{\tilde{B}_t\}_{t\in[0,T)}$ is a d-dimensional Brownian motion under the probability $R_T\mathbb{P}$. We can rewrite (21) as

$$\begin{aligned} dX_t &= \sigma(t, X_t)d\tilde{B}_t + b(t, X_t)dt - \frac{X_t - Y_t}{\kappa_t}dt, \quad X_0 = x, \\ dY_t &= \sigma(t, Y_t)d\tilde{B}_t + b(t, Y_t)dt, \quad Y_0 = y. \end{aligned} \tag{25}$$

Let

$$\tau_n := \inf_{t\in[0,T)}\{t : |X_t| + |Y_t| \geq n\}. \tag{26}$$

We have $\tau_n \uparrow \zeta$ as $n \longrightarrow \infty$ because of the nonexplosibility of X_t.

By the Lemma 2.1 in Wang [5], we know that for any $s \in [0, T]$, the following limits

$$\begin{aligned} R_{s\wedge\zeta} &:= \lim_{n\longrightarrow\infty} R_{s\wedge\tau_n\wedge(T-1/n)}, \\ R_{T\wedge\zeta} &:= \lim_{s\uparrow T} R_{s\wedge\zeta}, \end{aligned} \tag{27}$$

exist such that $\{R_{s\wedge\zeta}\}_{s\in[0,T)}$ with respect to $\mathbb{P}$ is a uniformly integrable martingale, and

$$\mathbb{Q}(\zeta = T) = 1, \tag{28}$$

so that $\mathbb{Q} = R_T\mathbb{P}$.

Lemma 9. *Assume Assumptions 1–4. Then, for $s \in [0, T)$,*

$$E_{s,\zeta}^{\mathbb{Q}} \int_0^{s\wedge\zeta} \frac{|X_t - Y_t|^2}{\kappa_t^2} dt \leq \frac{1}{\theta\kappa_0}|x - y|^2, \tag{29}$$

where the expectation $\mathbb{E}_{s,\zeta}^{\mathbb{Q}}$ is under the probability measure $\mathbb{Q} = R_{s\wedge\zeta}\mathbb{P}$.

Proof. Fixed $s \in [0, T)$. Using the Itô's formula for $|X_t - Y_t|^2$ and Assumption 1, we have, for $t \leq s\wedge\tau_n$,

$$\begin{aligned} d|X_t - Y_t|^2 &= -\frac{2}{\kappa_t}|X_t - Y_t|^2 dt + 2\langle (b(t, X_t) - b(t, Y_t)), (X_t - Y_t)_t\rangle dt \\ &+ \|\sigma(t, X_t) - \sigma(t, Y_t)\|_{HS}^2 dt \\ &+ 2\langle (X_t - Y_t)(\sigma(t, X_t) - \sigma(t, Y_t)), d\tilde{B}_t\rangle \leq -\frac{2}{\kappa_t}|X_t - Y_t|^2 dt \\ &+ K_T|X_t - Y_t|^2 dt + 2\langle (X_t - Y_t)(\sigma(t, X_t) - \sigma(t, Y_t)), d\tilde{B}_t\rangle. \end{aligned} \tag{30}$$

Combining this with the fact $2 - K_T\kappa_t + \kappa'_t = \theta$, we obtain

$$\begin{aligned} d\frac{|X_t - Y_t|^2}{\kappa_t} &\leq -\frac{2 - K_T\kappa_t + \kappa_{t'}}{\kappa_t}|X_t - Y_t|^2 dt \\ &+ \frac{2}{\kappa_t}\langle (X_t - Y_t)(\sigma(t, X_t) - \sigma(t, Y_t)), d\tilde{B}_t\rangle = -\frac{\theta}{\kappa_t^2}|X_t - Y_t|^2 dt \\ &+ \frac{2}{\kappa_t}\langle (X_t - Y_t)(\sigma(t, X_t) - \sigma(t, Y_t)), d\tilde{B}_t\rangle, \end{aligned} \tag{31}$$

for $t \leq s\wedge\tau_n$. Multiplying this inequality by $(1/\theta)$ and then integrating in $[0, s\wedge\tau_n]$, we get

$$\begin{aligned} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\kappa_t} dt &\leq -\frac{|X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}|^2}{\theta\kappa_{s\wedge\tau_n}} + \frac{1}{\theta\kappa_0}|x - y|^2 \\ &+ \int_0^{s\wedge\tau_n} \frac{2}{\theta\kappa_t}\langle (X_t - Y_t)(\sigma(t, X_t) - \sigma(t, Y_t)), d\tilde{B}_t\rangle. \end{aligned} \tag{32}$$

By the Girsanov theorem, $\{\tilde{B}_t\}_{t\leq s\wedge\tau_n}$ ia the d-dimensional Brownian motion under the probability measure $R_{s\wedge\tau_n}\mathbb{P}$. Taking expectation $\mathbb{E}_{s,n}^{\mathbb{Q}}$ with respect to $R_{s\wedge\tau_n}\mathbb{P}$, hence we obtain, for any $s \in [0, T)$,

$$E_{s,n}^{\mathbb{Q}} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\kappa_t^2} dt \leq \frac{1}{\theta\kappa_0}|y - x|^2, \quad n \geq 1. \tag{33}$$

By the dominated convergence theorem, we have

$$\mathbb{E}_{s,\zeta}^{\mathbb{Q}} \int_0^{s\wedge\zeta} \frac{|X_t - Y_t|^2}{\kappa_t^2} dt \leq \frac{1}{\theta\kappa_0}|x - y|^2, \tag{34}$$

where the expectation $\mathbb{E}_{s,\zeta}^{\mathbb{Q}}$ is under the probability measure $\mathbb{Q} = R_{s\wedge\zeta}\mathbb{P}$; hence, we obtain the desired result. □

Lemma 10. *Assume Assumptions 1–4. Let*

$$r_T = \frac{\lambda_T^2\theta_T^2}{4\delta_T^2 - 4\theta_T\lambda_T\delta_T} < -1. \tag{35}$$

Then,

$$\sup_{s\in[0,T]} ER_{s\wedge\zeta}^{1+r_T} \le \exp\left[\frac{\theta_T K_T(2\delta_T-\lambda_T\theta_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T-\lambda_T\theta_T)(1-e^{-K_T T})}\right]. \tag{36}$$

Proof. By the definition of R_s, we have

$$\begin{aligned} ER_{s\wedge\tau_n}^{1+r_T} &= E_{s,n}^{\mathbb{Q}} R_{s\wedge\tau_n}^{r_T} = E_{s,n}^{\mathbb{Q}} \exp\left[-r_T\int_0^{s\wedge\tau_n}\frac{1}{\kappa_t}\left\langle\sigma(t,X_t)^{-1}(X_t-Y_t), d\tilde{B}_t\right\rangle\right. \\ &\quad \left. + \frac{r_T}{2}\int_0^{s\wedge\tau_n}\frac{\left|\sigma(t,X_t)^{-1}(X_t-Y_t)\right|^2}{\kappa_t^2}dt\right]. \end{aligned} \tag{37}$$

Observing that for any exponential integrable martingale Λ_t with respect to $R_{s\wedge\tau_n}\mathbb{P}$, one has

$$\begin{aligned} \mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\Lambda_t + r_T\langle\Lambda\rangle_t}{2}\right] &= \mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\Lambda_t - r_T^2\beta\langle\Lambda\rangle_t}{2} + \frac{r_T(\beta r_T+1)\langle\Lambda\rangle_t}{2}\right] \\ &\le \left(\mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta\Lambda_t - r_T^2\beta^2\langle\Lambda\rangle_t}{2}\right]\right)^{1/\beta} \\ &\quad \times \left(\mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)}\langle\Lambda\rangle_t\right]\right)^{(\beta-1)/\beta} \\ &= \left(\mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)}\langle\Lambda\rangle_t\right]\right)^{(\beta-1)/\beta}, \end{aligned} \tag{38}$$

where $\beta > 1$. It follows that

$$\begin{aligned} ER_{s\wedge\tau_n}^{1+r_T} &\le \left(E_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)}\int_0^{s\wedge\tau_n}\frac{\left|\sigma(t,X_t)^{-1}(X_t-Y_t)\right|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta} \\ &\le \left(E_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)\lambda_T^2}\int_0^{s\wedge\tau_n}\frac{|(X_t-Y_t)|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta}. \end{aligned} \tag{39}$$

Noting that $r_T < -1$, take $\beta = 1 + \sqrt{1 + r_T^{-1}}$, i.e., which minimizes $(r_T\beta(r_T\beta+1))/(2(\beta-1))$ for $\beta \in (1, \infty)$ (see Figure 1), such that

$$\begin{aligned} \frac{r_T\beta(r_T\beta+1)}{2(\beta-1)\lambda_T^2} &= \frac{\left(r_T-\sqrt{r_T(1+r_T)}\right)\left((r_T+1)-\sqrt{r_T(1+r_T)}\right)}{-2/r_T\sqrt{r_T(1+r_T)}\lambda_T^2} \\ &= \frac{\left(r_T-\sqrt{r_T(1+r_T)}\right)^2}{2\lambda_T^2} = \frac{\theta_T^2}{8\delta_T^2}. \end{aligned} \tag{40}$$

Hence, noting that $(\beta-1)/\beta \in (0,1)$, we can write (39) as

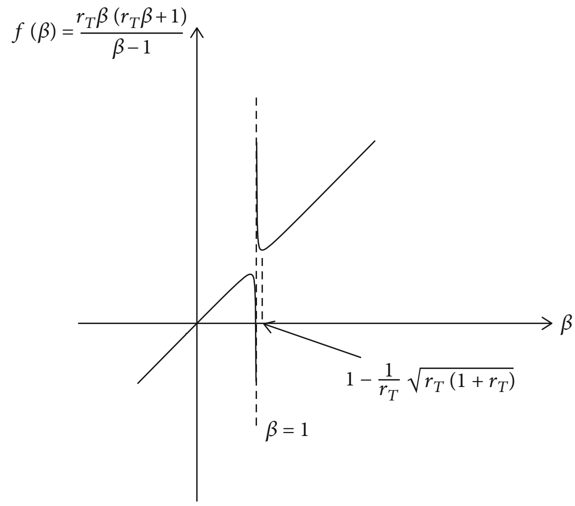

FIGURE 1: Minimize $(r_T\beta(r_T\beta+1))/\beta-1$ when $r_T<-1$.

$$\begin{aligned} ER_{s\wedge\tau_n}^{1+r_T} &\le \left(E_{s,n}^{\mathbb{Q}} \exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)\lambda_T^2}\int_0^{s\wedge\tau_n}\frac{|(X_t-Y_t)|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta} \\ &\le \left(E_{s,n}^{\mathbb{Q}} \exp\left[\frac{\theta_T^2}{8\delta_T^2}\int_0^{s\wedge\tau_n}\frac{|(X_t-Y_t)|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta}. \end{aligned} \tag{41}$$

Furthermore, we see from (40) that

$$r_T - \sqrt{r_T(1+r_T)} = -\frac{\lambda_T\theta_T}{2\delta_T}. \tag{42}$$

Noting that $r_T\beta = r_T - \sqrt{r_T(1+r_T)}$, it follows from the above relation (42) that

$$\begin{aligned} \sqrt{r_T(1+r_T)} &= r_T + \frac{\lambda_T\theta_T}{2\theta_T} = \frac{\lambda_T^2\theta_T^2}{4\delta_T^2 - 4\lambda_T\theta_T\delta_T} + \frac{\lambda_T\theta_T}{2\delta_T} \\ &= \frac{\lambda_T^2\theta_T^2}{2\delta_T(2\delta_T-2\lambda_T\theta_T)} + \frac{\lambda_T\theta_T}{2\delta_T} \\ &= \frac{\lambda_T^2\theta_T^2}{2\delta_T(2\delta_T-2\lambda_T\theta_T)} + \frac{\lambda_T\theta_T(2\delta_T-2\lambda_T\theta_T)}{2\delta_T(2\delta_T-2\lambda_T\theta_T)} \\ &= \frac{\lambda_T^2\theta_T^2 + \lambda_T\theta_T(2\delta_T-2\lambda_T\theta_T)}{2\delta_T(2\delta_T-2\lambda_T\theta_T)} \\ &= \frac{2\lambda_T\theta_T\delta_T - \lambda_T^2\theta_T^2}{2\delta_T(2\delta_T-2\lambda_T\theta_T)}. \end{aligned} \tag{43}$$

Due to the choice of β and the definition of r_T (35) and

(43), we can compute

$$\begin{aligned} \frac{\beta-1}{\beta} &= \frac{-1/r_T\sqrt{r_T(1+r_T)}}{1-1/r_T\sqrt{r_T(1+r_T)}} = \frac{\sqrt{r_T(1+r_T)}}{\sqrt{r_T(1+r_T)}-r_T} \\ &= \frac{\left(2\lambda_T\theta_T\delta_T-\lambda_T^2\theta_T^2\right)/(2\delta_T(2\delta_T-2\lambda_T\theta_T))}{\lambda_T\theta_T/2\delta_T} \\ &= \frac{2\lambda_T\theta_T\delta_T-\lambda_T^2\theta_T^2}{(\lambda_T\theta_T/2\delta_T)\times 2\delta_T(2\delta_T-2\lambda_T\theta_T)} \\ &= \frac{2\lambda_T\theta_T\delta_T-\lambda_T^2\theta_T^2}{\lambda_T\theta_T(2\delta_T-2\lambda_T\theta_T)} = \frac{2\delta_T-\lambda_T\theta_T}{2\delta_T-2\lambda_T\theta_T}. \end{aligned} \tag{44}$$

For any $\alpha > 0$, it follows from (32) that

$$\begin{aligned} &\mathbb{E}_{s,n}^{\mathbb{Q}} \exp\left[\alpha\int_0^{s\wedge\tau_n}\frac{|X_t-Y_t|^2}{\kappa_t^2}dt\right] \\ &\le \exp\left[\frac{\alpha}{\theta_T\kappa_0}|x-y|^2\right]\cdot\mathbb{E}_{s,n}^{\mathbb{Q}}\exp\left[\frac{2\alpha}{\theta_T}\int_0^{s\wedge\tau_n}\frac{1}{\kappa_t}\left\langle (X_t-Y_t)(\sigma(t,X_t)-\sigma(t,Y_t)),d\tilde{B}_t\right\rangle\right] \\ &\le \exp\left[\frac{\alpha K_T}{\theta_T(2-\theta_T)(1-e^{-K_TT})}|x-y|^2\right]\cdot\left(\mathbb{E}_{s,n}^{\mathbb{Q}}\exp\left[\frac{8\alpha^2\delta_T^2}{\theta_T^2}\int_0^{s\wedge\tau_n}\frac{|X_t-Y_t|^2}{\kappa_t^2}dt\right]\right)^{1/2} \end{aligned} \tag{45}$$

Taking $\alpha = \theta_T^2/8\delta_T^2$, we have

$$E_{s,n}^{\mathbb{Q}}\exp\left[\frac{\theta_T^2}{8\delta_T^2}\int_0^{s\wedge\tau_n}\kappa_t^{-2}|X_t-Y_t|^2dt\right] \le \exp\left[\frac{\theta_T K_T|x-y|^2}{4\delta_T^2(2-\theta_T)(1-e^{-K_TT})}\right]. \tag{46}$$

Now, returning to (41), and noticing that $(\beta-1)/\beta \in (0,1)$, then we obtain

$$\begin{aligned} \mathbb{E}R_{s\wedge\tau_n}^{1+r_T} &\le \left(\mathbb{E}_{s,n}^{\mathbb{Q}}\exp\left[\frac{r_T\beta(r_T\beta+1)}{2(\beta-1)}\int_0^{s\wedge\tau_n}\frac{\left|\sigma(t,X_t)^{-1}(X_t-Y_t)\right|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta} \\ &\le \left(\mathbb{E}_{s,n}^{\mathbb{Q}}\exp\left[\frac{\theta_T^2}{8\delta_T^2}\int_0^{s\wedge\tau_n}\frac{|X_t-Y_t|^2}{\kappa_t^2}dt\right]\right)^{(\beta-1)/\beta} \\ &\le \left(\exp\left[\frac{\theta K_T}{4\delta_T^2(2-\theta_T)(1-e^{-K_TT})}|y-x|^2\right]\right)^{(\beta-1)/\beta} \\ &= \exp\left[\frac{\theta K_T(2\delta_T-\lambda_T\theta_T)}{8\delta_T^2(2-\theta_T)(\delta_T-\lambda_T\theta_T)(1-e^{-K_TT})}|y-x|^2\right]. \end{aligned} \tag{47}$$

Therefore, by letting $n \longrightarrow \infty$, the Fatou lemma implies that

$$\sup_{s\in[0,T]} ER_{s\wedge\zeta}^{1+r_T} \le \exp\left[\frac{\theta_T K_T(2\delta_T-\lambda_T\theta_T)}{8\delta_T^2(2-\theta_T)(\delta_T-\lambda_T\theta_T)(1-e^{-K_TT})}|x-y|^2\right]. \tag{48}$$

This proves the assertion. □

3. Proof of Lower Bounded of Harnack Inequality

We will only give the proof of the lower bounded of the dimension-free Harnack inequality with power, since the upper bounded of the dimension-free Harnack inequality with power has been established in Wang [5]. We will first introduce an inverse Hölder inequality, which is the key building stone for this paper to obtain successfully lower bound of the dimension-free Harnack inequality with power. In the literature, we can find the earlier version of inverse Hölder inequality with integral form in Hardy et al. [18], P23-24, and more inverse Hölder inequalities can be found in Nehari [19] and Lin and Bai [20] for the following type and the references there cited.

Lemma 11. *For any $p \in (0,1)$, $q := p/(p-1) < 0$, then*

$$\frac{1}{p}+\frac{1}{q}=1, \tag{49}$$

and for any positive random variables X, Y, the inverse Hölder inequality holds:

$$E[XY] \ge \left(E[X^p]\right)^{1/p}\left(E[Y^q]\right)^{1/q}. \tag{50}$$

For our purpose to establish the dimension-free Harnack inequality with power and for simplicity, we only consider $p \in (0,1)$, although the inverse Hölder inequality also holds for $p < 0$. With the inverse Hölder inequality in hand, we now can start to prove the lower bounded of dimension-free Harnack inequality with power.

Theorem 12. *Let*

$$\kappa_1(p) := \frac{(1-\sqrt{p})K_T}{4\sqrt{p}\delta_{p,T}\left(\delta_{p,T}-\lambda_T(1-\sqrt{p})\right)(1-e^{-K_TT})}, \tag{51}$$

where $\delta_{p,T} := \max\{\delta_T, \lambda_T(1-\sqrt{p})/2\}$. If Assumptions 1–4 hold, then for $0 < p < (1-\delta_T/\lambda_T)^2$, the dimension-free Harnack inequality with power

$$P_Tf(y) \ge \left(P_Tf^p(x)\right)^{1/p}\exp\left[\kappa_1(p)|x-y|^2\right], \tag{52}$$

holds for all $T > 0, x, y \in \mathbb{R}^d$ and $f \in \mathscr{B}_b^+(\mathbb{R}^d)$.

Proof. Since $\{R_{s\wedge\zeta}\}_{s\in[0,T)}$ with respect to $\mathbb{P}$ is a uniformly integrable martingale and $\{\tilde{B}_t\}_{t\in[0,T]}$ is the d-dimensional Brownian motion under new probability measure $\mathbb{Q}$, thus Y_t can be solved up to time T. Let

$$\tau := \inf\{t \in [0,T]: X_t = Y_t\}, \tag{53}$$

and set $\inf\varnothing = \infty$ by convention. We claim $\tau \le T$ and thus, $X_T = Y_T$, $\mathbb{Q}$-a.s. Indeed, if for some $\omega \in \Omega$ such that τ

$(\omega) > T$, by the continuity of the processes, we have

$$\inf_{t\in[0,T]} |X_t(\omega) - Y_t(\omega)|^2 > 0. \tag{54}$$

So, on the set $\{\tau > T\}$, we have

$$\int_0^T \frac{|X_t - Y_t|^2}{\kappa_t^2} dt = \infty. \tag{55}$$

But according to Lemma 9, we obtain

$$E_{\mathbb{Q}} \int_0^T \frac{|X_t - Y_t|^2}{\kappa_t^2} dt < \infty. \tag{56}$$

Hence, we conclude that $\mathbb{Q}(\tau > T) = 0$. Therefore, $X_T = Y_T$, $\mathbb{Q}$-a.s.

Now, since $X_T = Y_T$ and $\{\tilde{B}_t\}_{t\in[0,T]}$ is the d-dimensional Brownian motion under new probability measure $\mathbb{Q}$, we have

$$(P_T f(y))^p = \left(E_{\mathbb{Q}}[f(Y_T)]\right)^p = \left(E\left[R_{T\wedge\zeta} f(X_T)\right]\right)^p. \tag{57}$$

Using the inverse Hölder inequality, for $0 < p < 1$, hence $-1 < p - 1 < 0$, we have

$$\left(E\left[R_{T\wedge\zeta} f(X_T)\right]\right)^p \ge P_T f^p(x) \left(E R_{T\wedge\zeta}^{p/(p-1)}\right)^{p-1}. \tag{58}$$

Hence, we arrive at

$$(P_T f(y))^p \ge P_T f^p(x) \left(E R_{T\wedge\zeta}^{p/(p-1)}\right)^{p-1} = P_T f^p(x) \left(E R_{T\wedge\varsigma}^{1+r_T}\right)^{p-1}. \tag{59}$$

Here, $r_T = 1/(p-1) < -1$.

Let $\theta = \theta_T$. Noting the choice of the θ (18), we have

$$r_T = \frac{1}{p-1} = \frac{\lambda_T^2 \theta^2}{4\delta_T^2 - 4\theta\lambda_T\delta_T}, \tag{60}$$

that is,

$$(p-1)\lambda_T^2\theta^2 = 4\delta_T^2 - 4\theta\lambda_T\delta_T. \tag{61}$$

In fact, in (18), we choose

$$\theta = \frac{2\delta_T}{\lambda_T(1-\sqrt{p})}, \tag{62}$$

which solves the equation $(p-1)\lambda_T^2\theta^2 = 4\delta_T^2 - 4\theta\lambda_T\delta_T$.

Observing that the function x^{p-1} is a decreasing function with respect to x in $(0,\infty)$ if $0 < p < 1$, it follows from Lemma 10 that

$$\begin{aligned} P_T f(y))^p &\ge P_T f^p(x) \left[E R_{T\wedge\varsigma}^{1+r_T}\right]^{p-1} \\ &\ge P_T f^p(x) \exp\left[\frac{(p-1)\theta K_T(2\delta_T - \lambda_T\theta)}{8\delta_T^2(2-\theta)(\delta_T - \lambda_T\theta)(1-e^{-K_T T})}|x-y|^2\right]. \end{aligned} \tag{63}$$

Replacing θ with $2\delta_T/(\lambda_T(1-\sqrt{p}))$ in (63), we can rewrite (63) and then obtain

$$\begin{aligned} (P_T f(y))^p &\ge P_T f^p(x) \exp\left[\frac{2\delta_T\sqrt{p}(1-p)K_T}{8\delta_T^2(\delta_T(\sqrt{p}+1) + \lambda_T(p-1))(1-e^{-K_T T})}|x-y|^2\right] \\ &= P_T f^p(x) \exp\left[\frac{\sqrt{p}(1-\sqrt{p})K_T}{4\delta_T(\delta_T - \lambda_T(1-\sqrt{p}))(1-e^{-K_T T})}|x-y|^2\right]. \end{aligned} \tag{64}$$

This completes the proof of this theorem. □

Remark 13. In this theorem, we give a direct proof about the lower bound estimate of dimension-free Harnack inequality with power for a large class of stochastic differential equation with multiplicative noise, rather than the reciprocal transformation based on the upper bound of dimension-free Harnack inequality with power. Of course, based on the reciprocal transformation of our lower bound estimate, we can also get the upper bound of dimension-free Harnack inequality with power. Unfortunately, our method does not adapt to establish the lower bound estimate of dimension-free log-Harnack inequality. At the same time, there is no evidence that the lower bound estimate of dimension-free log-Harnack inequality can be easily obtained by reciprocal transformation. This also leaves an open problem of how to establish the lower bound estimate of dimension-free log-Harnack inequality for a large class of stochastic differential equation with additive noise or multiplicative noise.

4. Bilateral Shift-Harnack Inequalities

In this section, we establish the bilateral dimension-free Wang's shift-Harnack inequalities with power for P_t, and the upper bounded of this inequality has been introduced in Wang (2014). As seen in the previous section, the study for the multiplicative noise case is very complicated; we hence only consider the additive noise for which the SDE (1) reduces to

$$dX_t = b(t, X_t)dt + \sigma(t)dB_t, X_0 = x. \tag{65}$$

Proposition 14. *Let function σ from $[0,\infty)$ to $\mathbb{R}^d \times \mathbb{R}^d$ and b from $[0,\infty) \times \mathbb{R}^d$ to $\mathbb{R}^d$ be two progressively measurable process and satisfy assumptions in Section 1. Furthermore, the function $\sigma(t)$ is invertible and satisfies the following assumption.*

Assumption 15. There exists a decreasing function λ from $[0,\infty)$ to $[0,\infty)$ s.t. for all $x\in\mathbb{R}^d$, $t\geq 0$,

$$\sigma(t)^*\sigma(t)\geq\lambda_t^2 I, a.s. \tag{66}$$

Then,

(1) for any $p\in(0,1)$, $T>0, x, v\in\mathbb{R}^d$, and any $f\in\mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_Tf)(x)\geq\left(P_Tf^p(v+\cdot)\right)^{1/p}(x)\exp\left[\frac{|v|^2}{2(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{67}$$

(2) for any $p>1$, $T>0, x, v\in\mathbb{R}^d$, and any $f\in\mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_Tf)(x)\leq\left(P_Tf^p(v+\cdot)\right)^{1/p}(x)\exp\left[\frac{|v|^2}{2(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{68}$$

Proof. Let $Y_t=X_t+(t/T)v$, $t\in[0,T]$. Then, we have

$$dY_t=b(t,Y_t)dt+\sigma(t)d\tilde{B}_t, Y_0=x, t\in[0,T], \tag{69}$$

where

$$\begin{aligned}\tilde{B}_t&:=B_t+\int_0^t\eta_s ds,\\ \eta_t&:=\sigma^{-1}(t)\left[\frac{v}{T}+b(t,X_t)-b(t,X_t+tv/T)\right].\end{aligned} \tag{70}$$

Let

$$R_T=e^{-\int_0^T\langle\eta_t,dB_t\rangle-1/2\int_0^T|\eta_s|^2ds}. \tag{71}$$

By the Girsanov theorem, we have

$$(P_Tf)(x)=E[R_Tf(Y_T)]=E[R_Tf(X_T+v)]. \tag{72}$$

Similar to (58), by inverse Hölder inequality, we obtain

$$(P_Tf)(x)\geq\left(P_Tf^p(v+\cdot)\right)^{1/p}(x)\left(E\left[R_T^{p/(p-1)}\right]\right)^{(p-1)/p}. \tag{73}$$

Due to (71), we have

$$\begin{aligned}\mathbb{E}\left[R_T^{p/(p-1)}\right]&\leq\mathbb{E}\left[M_t\exp\left[\frac{p}{2(p-1)^2}\int_0^T|\eta_s|^2ds\right]\right]\\&=\mathbb{E}\left[M_T\exp\left[\frac{p}{2(p-1)^2}\int_0^T\sigma^{-2}(t)\left[\frac{v}{T}+b(t,X_t)-b\left(t,X_t+\frac{tv}{T}\right)\right]^2ds\right]\right]\\&\leq\mathbb{E}\left[M_T\exp\left[\frac{p}{2(p-1)^2\lambda_T^2}\int_0^T\left[\frac{v}{T}+b(t,X_t)-b\left(t,X_t+\frac{tv}{T}\right)\right]^2ds\right]\right].\end{aligned} \tag{74}$$

Here, the last inequality holds according to Assumption 15, and

$$M_t:=\exp\left[-\frac{p}{p-1}\int_0^t\eta_s dB_s-\frac{p^2}{2(p-1)^2}\int_0^t|\eta_s|^2ds\right]. \tag{75}$$

Now, noticing from Assumption 1 that

$$|b(t,x)-b(t,y)|\leq K_T|x-y|, \tag{76}$$

we see that for all $x\in\mathbb{R}$,

$$\left|\frac{v}{T}+b(t,x)-b(t,x+tv/T)\right|\leq\frac{1+K_Tt}{T}|v|. \tag{77}$$

Thus, in view of the above assertion (77) and $\mathbb{E}[M_T]\leq 1$, estimate (74) becomes

$$E\left[R_T^{p/(p-1)}\right]\leq\exp\left[\frac{p|v|^2}{2(p-1)^2\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right]. \tag{78}$$

Using again the fact that the function x^{p-1} is a decreasing function with respect to x in $(0,\infty)$ if $-1<p-1<0$, it follows from (73) that

$$\begin{aligned}(P_Tf)(x)&\geq\left(P_Tf^p(v+\cdot)\right)^{1/p}(x)\left(E\left[R_T^{p/(p-1)}\right]\right)^{(p-1)/p}\\&\geq\left(P_Tf^p(v+\cdot)\right)^{1/p}(x)\exp\left[\frac{|v|^2}{2(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right].\end{aligned} \tag{79}$$

This completes the proof of the first part of this theorem. Noticing that (78) holds for all $p>0$, the second part of this theorem can be obtained. □

In fact, we can get a better conclusion than Proposition 14.

Theorem 16. *Let function σ from $[0,\infty)$ to $\mathbb{R}^d\times\mathbb{R}^d$ and function b from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}^d$ be measurable and satisfy assumptions in Section 1. Furthermore, the function $\sigma(t)$ is invertible and satisfies Assumption 15. Then,*

(1) for any $p\in(0,1)$, $T>0, x, v\in\mathbb{R}^d$, and $f\in\mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_Tf)(x) \geq (P_Tf^p(v+\cdot))^{1/p}(x)\exp\left[\frac{\sqrt{p}|v|^2}{2(1+\sqrt{p})(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{80}$$

(2) for any $p>1$, $T>0$, $x, v \in \mathbb{R}^d$, and $f \in \mathscr{B}_b^+(\mathbb{R}^d)$,

$$(P_Tf)(x) \leq (P_Tf^p(v+\cdot))^{1/p}(x)\exp\left[\frac{\sqrt{p}|v|^2}{2(1+\sqrt{p})(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right] \tag{81}$$

Proof. Similar to (39), we have

$$\begin{aligned}\mathbb{E}\left[R_T^{p/(p-1)}\right] &\leq \left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\frac{p}{2(p-1)^2}\int_0^T|\eta_s|^2ds\right\}\right]\right)^{\sqrt{p}/\sqrt{p}+1}\\ &= \left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\frac{p}{2(p-1)^2}\int_0^T\sigma^{-2}(t)\left[\frac{v}{T}+b(t,X_t)-b\left(t,X_t+\frac{tv}{T}\right)\right]^2ds\right\}\right]\right)^{\sqrt{p}/\sqrt{p}+1}\\ &\leq \left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\frac{p}{2(p-1)^2\lambda_T^2}\int_0^T\left[\frac{v}{T}+b(t,X_t)-b\left(t,X_t+\frac{tv}{T}\right)\right]^2ds\right\}\right]\right)^{\sqrt{p}/\sqrt{p}+1},\end{aligned} \tag{82}$$

Due to (77), we have

$$\mathbb{E}\left[R_T^{p/(p-1)}\right] \leq \exp\left[\frac{p\sqrt{p}|v|^2}{2(1+\sqrt{p})(p-1)^2\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right]. \tag{83}$$

(1) For p that satisfies $0<p<1$, using the fact that the function x^{p-1} is a decreasing function with respect to x in $(0,\infty)$, it follows from (73) based on the inverse Hölder inequality that

$$\begin{aligned}(P_Tf)(x) &\geq (P_Tf^p(v+\cdot))^{1/p}(x)\left(\mathbb{E}\left[R_T^{p/(p-1)}\right]\right)^{(p-1)/p}\\ &\geq (P_Tf^p(v+\cdot))^{1/p}(x)\exp\left[\frac{\sqrt{p}|v|^2}{2(1+\sqrt{p})(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right].\end{aligned} \tag{84}$$

This completes the proof of the lower bounded

(2) For $p>1$, by Hölder inequality, we have

$$(P_Tf)(x) \leq (P_Tf^p(v+\cdot))^{1/p}(x)\left(\mathbb{E}\left[R_T^{p/(p-1)}\right]\right)^{(p-1)/p} \tag{85}$$

Notice that (83) holds for all $p>0$; hence, we get

$$(P_Tf)(x) \leq (P_Tf^p(v+\cdot))^{1/p}(x)\exp\left[\frac{\sqrt{p}|v|^2}{2(1+\sqrt{p})(p-1)\lambda_T^2}\left(\frac{1}{T}+K_T+\frac{K_T^2T}{3}\right)\right]. \tag{86}$$

This completes the proof of this theorem. □

Remark 17. From this simplified proof of the bilateral dimension-free shift-Harnack inequality with power for additive case, we can easily see that the moment estimation (83) is the key to establishing the bilateral dimension-free shift-Harnack inequality with power. We need to specifically point out that the moment estimation (83) holds for all $p>0$. To establish the upper bound, we need to use the moment estimation (83) and Hölder inequality, but to establish the lower bound, we need to use the moment estimation (83) and the inverse Hölder inequality.

5. Proof of the Corollary 6

Let $p_t(x,y)$ be the density of P_t with respect to μ which is a Radon measure. Then, it follows from Corollary 1.2 in Wang [5] that for $p>(1+\delta_T/\lambda_T)^2$, the Harnack inequality with power in Wang [5] is equivalent to the following heat kernel inequalities:

$$\begin{aligned}&\int_{\mathbb{R}} p_T(x,z)\left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{1/(p-1)}\mu(dz)\\ &\leq \exp\left[\frac{K_T\sqrt{p}|x-y|^2}{4\delta_{p,T}(\sqrt{p}+1)\left[(\sqrt{p}-1)\lambda_T-\delta_{p,T}\right](1-e^{-K_TT})}\right].\end{aligned} \tag{87}$$

For all $x, y \in \mathbb{R}^d$, $T>0$, hence, (11) holds. Due to this, then we will complete the proof of the Corollary 6, if we prove the following proposition:

Proposition 18. *Assume that Assumptions 1–4 hold. Let $p_t(x,y)$ be a strictly positive density of the operator P_t with respect to a Radon measure μ. Then, for $0<p<(1-\delta_T/\lambda_T)^2$, the following heat kernel inequality*

$$\begin{aligned}&\int_M p_T(x,z)\left(\frac{p_T(x,z)}{p_T(y,z)}\right)^{1/p}\mu(dz)\\ &\geq \exp\left[\frac{(1-\sqrt{p})K_T}{4\sqrt{p}\delta_T(\delta_T-\lambda_T(1-\sqrt{p}))(1-e^{-K_TT})}|x-y|^2\right],\end{aligned} \tag{88}$$

holds for all $x, y \in \mathbb{R}^d$.

Proof. By Theorem 12, we have

$$(P_Tf(y))^p \geq P_Tf^p(x)\exp\left[\tilde{\kappa}(p)|x-y|^2\right]. \tag{89}$$

Here,

$$\tilde{\kappa}(p) \coloneqq \frac{\sqrt{p}(1-\sqrt{p})K_T}{4\delta_T(\delta_T - \lambda_T(1-\sqrt{p}))(1-e^{-K_T T})}. \tag{90}$$

Let $P_T(x, dy)$ be a transition probability such that

$$P_T f(x) = \int_E f(y) P_T(x, dy), x \in E. \tag{91}$$

Obviously, the above inequality implies that $P_T(x, \cdot)$ and $P_T(y, \cdot)$ are equivalent to each other. Indeed, if $P_T(x, A) = 0$, then applying the above inequality to $f = I_A$, we conclude that $P_T(y, A) = 0$.

Let

$$p_{T,(x,y)}(z) \coloneqq \frac{P_T(x, dz)}{P_T(y, dz)}, \tag{92}$$

be the Radon-Nikodym derivative of $P_T(x, \cdot)$ with respect to $P_T(y, \cdot)$.

Applying (89) to $f(z) \coloneqq \{p_{T,(x,y)}(z)\}^{1/p}$, we have

$$\begin{aligned}
(P_T f(x))^p &\geq P_T f^p(y) \exp\left[\tilde{\kappa}(p)|x-y|^2\right] \\
&= \exp\left[\tilde{\kappa}(p)|x-y|^2\right] \int_E \left\{p_{T,(x,y)}(z)\right\}^{p/p} P(y, dz) \\
&= \exp\left[\tilde{\kappa}(p)|x-y|^2\right] \int_E p_{T,(x,y)}(z) P(y, dz) \\
&= \exp\left[\tilde{\kappa}(p)|x-y|^2\right] \int_E P(x, dz) = \exp\left[\tilde{\kappa}(p)|x-y|^2\right].
\end{aligned} \tag{93}$$

So we obtain

$$(P_T f(x))^p \geq \exp\left[\tilde{\kappa}(p)|x-y|^2\right], \tag{94}$$

that is,

$$P_T f(x) \geq \exp\left[\frac{\tilde{\kappa}(p)}{p}|x-y|^2\right]. \tag{95}$$

Thus,

$$\int_M p_T(x,z)\left(\frac{p_T(x,z)}{p_T(y,z)}\right)^{1/p} \mu(dz) \geq \exp\left[\frac{\tilde{\kappa}(p)}{p}\right]|x-y|^2. \tag{96}$$

Therefore, the desired result in (88) holds. □

Data Availability

All data, models, and code generated or used during the study appear in the submitted article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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