

Research Article

Approximation Properties of a New Type of Gamma Operator Defined with the Help of k -Gamma Function

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Received 13 May 2022; Accepted 27 June 2022; Published 2 August 2022

Academic Editor: Yusuf Gurefe

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With the help of the k -Gamma function, a new form of Gamma operator is given in this article. Voronovskaya type theorem, weighted approximation, rates of convergence, and pointwise estimates have been found for approximation features of the newly described operator. Finally, numerical examples have been provided to demonstrate that the operator is approaching the function.

1. Introduction

One of the most important topics in mathematical analysis is approximation theory. The theory is studied in almost every subject, including engineering and physics. Many mathematicians have made investigation in this area. In 1885 [1], Weierstrass claimed that polynomials can approximate every function in the closed interval $[a, b]$. Besides, theorems about this subject are prepared by Korovkin around 1950 [2]. The Korovkin approximation theorem is one of the well-known theorems in mathematics. Their theorems indicate that a series of positive linear operators can converge to the identity operator under specific condition [2]. As a result using these theorems, some studies on linear and positive linear operators have been added to the literature. For example, King [3] introduced the Bernstein operator to preserve the function $a_2(h) = h^2$ in 2003. Then, King constructed a new set of operators with respect to the test functions $\{1, h, h^2\}$ and obtained their linear combinations. On the other hand, one of these operators is the Gamma operator which is constructed by

Lupas and Müller [4]. The classical Gamma operator in [4] is expressed as follows:

$$K_m(\varphi; y) = \frac{y^{m+1}}{\Gamma(m+1)} \int_0^\infty e^{-yv} v^m \varphi\left(\frac{m}{v}\right) dv, \forall y \in (0, \infty), m \in \mathbb{N}. \quad (1)$$

Then, in the literature, some researchers introduced the generalizations of Gamma and beta functions and also the extensions of Gamma-type operators and their extensions [5–14]. One of the studies of this topic was by Daz and Pariguan [15]; they introduced and researched k -Gamma function when they were assessing Feynman integrals. k -Gamma function has been showed up various effects on mathematics and applications. One of these effects has been working the Schrodinger equation for harmonium and related models in view of important operations in quantum chemistry [16]. The others have used k -Gamma function for combinatorial analysis in statistic.

According to these studies, the k -Gamma function was defined by Daz and Parigiuan as follows:

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt, (k \in \mathbb{R}, \text{Re}(z) > 0). \quad (2)$$

As can be seen from the definition, Γ_k is a one parameter deformation of the classical Gamma function such that $\Gamma_k \rightarrow \Gamma$ as $k \rightarrow 1$. For $k=2$, it reduces to an integral of Gaussian functions [17]. When we get $u = -t^k/k$ in equation (2), we find the expression $\Gamma_k(z) = k^{(z/k)-1} \Gamma(z/k)$, and all properties of the classic Gamma operator can be generalized into k -Gamma function. It also led to a few new conclusions for k -Gamma function. A few of them are given that

$$\begin{aligned} \Gamma_k(k) &= 1, \\ \Gamma_k(z+k) &= z\Gamma_k(z), \\ (z)_{n,k} &= \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}, \end{aligned} \quad (3)$$

in [15]. For more such properties of k -Gamma and related functions, we can refer to the article [11, 15, 17].

The primary goal of this research is to give the k -Gamma operator given by (4) and its approximation properties. For the operator in (4), in Section 2, we will use Korovkin theorem in [18]. Then, in Section 3, we will consider the Voronovskaya type theorem. In Section 4, we will examine the weighted approximation. Later, we will give the rates of convergence with Peetre's \mathcal{K} -functional and Lipschitz class in Section 5. Moreover, in Section 6, we will obtain pointwise estimates, and finally, in Section 7, we will show the numerical examples for the operators in (4).

2. A New Modification of Gamma Operators Defined with the Help of k -Gamma Function

We shall see a new type of Gamma operators defined with the help of the k -Gamma function in this section, and some findings will be presented in the rest of the article. In this paper, we will use the expressions $a_z(h) = h^z$ and $\psi_{y,z} = (h-y)^z$, $y \in (0, \infty)$ as polynomial functions. The modified representation of the classical Gamma operator is shown as follows:

$$K_m^*(\varphi; y) = \frac{y^{m+1+(1/k)}}{\Gamma_k(mk+k+1)} \int_0^\infty e^{-y^v} (vk)^{m+(1/k)} \varphi\left(\frac{m}{v}\right) dv, \forall y \in (0, \infty), k > 0, m \in \mathbb{N}, \quad (4)$$

where for $v > 0$, $\varphi \in C_\gamma(0, \infty) = \{\varphi \in C(0, \infty) : \varphi(u) = O(u^\gamma), \text{ as } u \rightarrow \infty\}$ for $m > \gamma$. Here $C(0, \infty)$ is the set of continuous functions on $(0, \infty)$. This modified operator is clearly positive and linear in this case. Furthermore, the new Gamma operator defined with the help of the k -Gamma function is directly preserved constant, and test functions are provided in case of limit.

We note that for special case of $k = (1/p)$ ($p \in \mathbb{N}$) in (4), we have Schurer variant of Gamma operators in (1).

The following lemma will be presented without proof and used in fundamental theorems for the rest of the paper.

Lemma 1. *Let $y \in (0, \infty)$. The following are the moment values:*

$$\begin{aligned} K_m^*(a_0(h); y) &= a_0(y), \\ K_m^*(a_1(h); y) &= \frac{mk}{mk+1} a_1(y), \\ K_m^*(a_2(h); y) &= \frac{(mk)^2}{(mk+1)(mk-k+1)} a_2(y), \\ K_m^*(a_3(h); y) &= \frac{(mk)^3}{(mk+1)(mk-k+1)(mk-2k+1)} a_3(y), \\ K_m^*(a_4(h); y) &= \frac{(mk)^4}{(mk+1)(mk-k+1)(mk-2k+1)(mk-3k+1)} a_4(y). \end{aligned} \quad (5)$$

By generalizing the moment values, we have the following lemma.

Lemma 2. *Let $y \in (0, \infty)$ and $z \in \mathbb{N}$, $K_m^*(a_0(h); y) = a_0(y)$. Then, the general formula for the following moment values is obtained*

$$K_m^*(a_z(h); y) = \frac{(mk)^z}{\prod_{i=0}^{z-1} (mk-ki+1)} a_z(y), z = 1, 2, \dots \quad (6)$$

Lemma 3. *Let $y \in (0, \infty)$. Using the equations in Lemma 1, the following are obtained:*

$$\begin{aligned} K_m^*(\psi_{y,0}(h); y) &= 1, \\ K_m^*(\psi_{y,1}(h); y) &= \frac{-1}{mk+1} a_1(y), \\ K_m^*(\psi_{y,2}(h); y) &= \frac{mk^2 - k + 1}{(mk+1)(mk-k+1)} a_2(y), \\ K_m^*(\psi_{y,3}(h); y) &= \left(\frac{m^3 k^3}{(mk+1)(mk-k+1)(mk-2k+1)} \right. \\ &\quad \left. - \frac{3m^2 k^2}{(mk+1)(mk-k+1)} + \frac{3mk}{mk+1} - 1 \right) a_3(y), \\ K_m^*(\psi_{y,4}(h); y) &= \left(\frac{m^4 k^4}{(mk+1)(mk-k+1)(mk-2k+1)(mk-3k+1)} \right. \\ &\quad - 4 \frac{m^3 k^3}{(mk+1)(mk-k+1)(mk-2k+1)} \\ &\quad \left. + 6 \frac{m^2 k^2}{(mk+1)(mk-k+1)} - 4 \frac{mk}{mk+1} + 1 \right) a_4(y). \end{aligned} \quad (7)$$

As a result of our research, the Schurer variant of Gamma operators have not been defined or used. Also, if it

is realized that $k \in \mathbb{R}^+$, it is obtained that our operators are a generalization of the Schurer type operators.

Throughout this paper, we use the norm $\|\varphi\| = \sup \{\varphi(y) : y \in (0, \infty)\}$ for $\varphi \in C(0, \infty)$.

Lemma 4. *Let $\varphi \in C_B(0, \infty)$. Then, we get*

$$\|K_m^*(\varphi; y)\| \leq \|\varphi\|. \tag{8}$$

Proof. By using the result of Lemma 1, we have

$$\begin{aligned} \|K_m^*(\varphi)\| &\leq \frac{y^{m+1+(1/k)}}{\Gamma_k(mk+k+1)} \int_0^\infty e^{-yv} (vk)^{m+(1/k)} \left| \varphi\left(\frac{m}{v}\right) \right| dv \\ &\leq \|\varphi\| \frac{y^{m+1+(1/k)}}{\Gamma_k(mk+k+1)} \int_0^\infty e^{-yv} (vk)^{m+(1/k)} dv \\ &= \|\varphi\| K_m^*(a_0(h); y) = \|\varphi\|. \end{aligned} \tag{9}$$

Thus, we obtain the desired result. Because the moments are conserved in the limit state of the Korovkin test functions, K_m^* is an approximation process on any compact $T \subset (0, \infty)$, according to the Korovkin theorem in [18]. \square

Theorem 5. *Let $\varphi \in C(0, \infty) \cap E$, where $E = \{\varphi : \lim_{y \rightarrow \infty} (\varphi/1 + y^2) = k \text{ constant}\}$. Then, consistently in each compact subset of $(0, \infty)$, we have*

$$\lim_{m \rightarrow \infty} K_m^*(\varphi; y) = \varphi(y). \tag{10}$$

Proof. By using Lemma 1, when $z = 0, 1, 2$, we get

$$\lim_{m \rightarrow \infty} K_m^*(a_z(h); y) = a_z(y) \tag{11}$$

for uniformly each compact subset of $(0, \infty)$. Then, using the Korovkin theorem in [18], we give $\lim_{m \rightarrow \infty} K_m^*(\varphi; y) = \varphi(y)$ for uniformly each compact subset of $(0, \infty)$. \square

3. Voronovskaya Type Theorem

By establishing Voronovskaya's theorem below, we will illustrate the asymptotic behavior of $(K_m^*)_{m \geq 1}$ operators in this section.

Theorem 6. *Let $\varphi \in C(0, \infty) \cap E$ such that $\varphi', \varphi'' \in C(0, \infty) \cap E$. The following limit is valid:*

$$\lim_{m \rightarrow \infty} m[K_m^*(\varphi; y) - \varphi(y)] = -\frac{1}{k} y \varphi'(y) + \frac{1}{2} y^2 \varphi''(y). \tag{12}$$

Proof. From the definition of Taylor formula

$$\varphi(h) = \varphi(y) + \varphi'(y)(h-y) + \frac{1}{2} \varphi''(y)(h-y)^2 + \Omega(h, y)(h-y)^2, \tag{13}$$

where

$$\Omega(h, y) = \frac{\varphi'''(\delta) - \varphi'''(y)}{2}, \tag{14}$$

such that δ lying between y and h and

$$\lim_{h \rightarrow y} \Omega(h, y) = 0. \tag{15}$$

When the $(K_m^*)_{m \geq 1}$ operator is applied to (13), we get

$$\begin{aligned} K_m^*(\varphi; y) &= \varphi(y) + \varphi'(y) K_m^*((h-y); y) + \frac{1}{2} \varphi''(y) K_m^*((h-y)^2; y) \\ &\quad + K_m^*(\Omega(h, y)(h-y)^2; y). \end{aligned} \tag{16}$$

To get the formula

$$\begin{aligned} m[K_m^*(\varphi; y) - \varphi(y)] &= \varphi'(y) m K_m^*((h-y); y) \\ &\quad + \frac{1}{2} \varphi''(y) m K_m^*((h-y)^2; y) \\ &\quad + m K_m^*(\Omega(h, y)(h-y)^2; y), \end{aligned} \tag{17}$$

multiply both sides of the last inequality by m . In the limit case, this equation is

$$\begin{aligned} \lim_{m \rightarrow \infty} m[K_m^*(\varphi; y) - \varphi(y)] &= \varphi'(y) \lim_{m \rightarrow \infty} m K_m^*((h-y); y) \\ &\quad + \frac{1}{2} \varphi''(y) \lim_{m \rightarrow \infty} m K_m^*((h-y)^2; y) \\ &\quad + \lim_{m \rightarrow \infty} m K_m^*(\Omega(h, y)(h-y)^2; y). \end{aligned} \tag{18}$$

We know the values

$$\begin{aligned} \lim_{m \rightarrow \infty} m K_m^*((h-y); y) &= \lim_{m \rightarrow \infty} m \left[\frac{-1}{mk+1} \right] y = -\frac{1}{k} y, \\ \lim_{m \rightarrow \infty} m K_m^*((h-y)^2; y) &= \lim_{m \rightarrow \infty} m \left[\frac{mk^2 - k + 1}{(mk+1)(mk-k+1)} \right] y^2 = y^2, \end{aligned} \tag{19}$$

using Lemma 3. So, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} m[K_m^*(\varphi; y) - \varphi(y)] &= -\frac{1}{k} y \varphi'(y) + \frac{1}{2} \varphi''(y) y^2 \\ &\quad + \lim_{m \rightarrow \infty} m K_m^*(\Omega(h, y) \psi_{y,2}(h); y). \end{aligned} \tag{20}$$

We show that the limit to the right of the equation in (20) is equal to zero. It can easily be said from the Cauchy-

Schwarz inequality that

$$mK_m^* \Omega(h, y) \psi_{y,2}(h; y) \leq \sqrt{K_m^*(\Omega^2(h, y); y)} \sqrt{m^2 K_m^*(\psi_{y,4}(h); y)}. \quad (21)$$

Then, using Korovkin theorem, we have

$$\lim_{m \rightarrow \infty} K_m^*(\Omega^2(h; y), y) = \Omega^2(y, y) = 0, \quad (22)$$

since $\Omega^2(y, y) = 0$ and $\Omega(\cdot, y) \in C(0, \infty) \cap E$ and bounded as $h \rightarrow \infty$ and in view of fact that

$$K_m^*(\psi_{y,4}(h); y) = O\left(\frac{1}{m^2}\right), \quad (23)$$

where $K_m^*(\psi_{y,4}(h); y) = (3m^2k^4 + m(18k^4 - 22k^3 + 6k^2) - 6k^3 + 11k^2 - 6k + 1)/((mk + 1)(mk - k + 1)(mk - 2k + 1)(mk - 3k + 1))$. The proof is completed when equations (21) and (22) are written in (13). \square

4. Weighted Approximation

The Korovkin theorem for weighted approximation of the operators in (4) is given in this section. To demonstrate this, we will follow the theorems given by Gadjiev [19].

Consider $\vartheta(y) = 1 + y^2$ as continuous weighted function on \mathbb{R} , with $\lim_{|y| \rightarrow \infty} \vartheta(y) = \infty$, $\vartheta(y) \geq 1$ for all $y \in [0, \infty)$. Let

us have a look at the weighted spaces below. The property $|\varphi(y)| \leq N_\varphi \vartheta(y)$ represents the weighted space of real-valued functions φ on \mathbb{R} . This subspace is denoted by

$$B_\vartheta[0, \infty) = \{\varphi : [0, \infty) \rightarrow [0, \infty) : |\varphi(y)| \leq N_\varphi \vartheta(y), y \in [0, \infty)\}. \quad (24)$$

N_φ is a constant depending on the functions φ .

Since, the weighted subspaces of $B_\vartheta[0, \infty)$ is given by

$$C_\vartheta[0, \infty) = \{\varphi \in B_\vartheta[0, \infty) : \varphi \text{ is continuous on } \mathbb{R}\} = C[0, \infty) \cap B_\vartheta[0, \infty). \quad (25)$$

Eventually, additional subspace for all $\varphi \in C_\vartheta[0, \infty)$ for which $\lim_{|y| \rightarrow \infty} \varphi(y)/\vartheta(y)$ exists finitely defined as

$$C_\vartheta^x[0, \infty) = \left\{ \varphi \in C_\vartheta[0, \infty) : \lim_{|y| \rightarrow \infty} \frac{\varphi(y)}{\vartheta(y)} = \kappa_\varphi \text{ exists and it is finite} \right\}. \quad (26)$$

This κ_φ is a constant dependent on the φ functions. All three mapping spaces above are normed spaces endowed with

$$\|\varphi\|_\vartheta = \sup_{y \in (0, \infty)} \frac{|\varphi(y)|}{\vartheta(y)}. \quad (27)$$

Lemma 7. Let $\varphi \in C_\vartheta(0, \infty)$. Then, for the modified operator $K_m^*(\varphi)$, we have

$$\|K_m^*(\varphi)\|_\vartheta \leq C \|\varphi\|_\vartheta, \quad (28)$$

which imply that the sequence of the modified operators $K_m^*(\varphi)$ is an approximation process from $C_\vartheta(0, \infty)$ to $B_\vartheta(0, \infty)$.

Proof. The desired result of this lemma is easily obtained from properties of the modified Gamma operator and Lemma 1. \square

Gadjiev proposed a weighted approach to linear positive operator sequences for unbounded intervals in [19]. The following theorem is similar to the Gadjiev theorem.

Theorem 8. Let $\varphi \in C_\vartheta^x(0, \infty)$. For the modified Gamma operator, the following equality holds:

$$\lim_{m \rightarrow \infty} \|K_m^*(\varphi; y) - \varphi(y)\|_\vartheta = 0. \quad (29)$$

Proof. It will be enough to show that equivalence is attained for $\lim_{m \rightarrow \infty} \|K_m^*(a_z; y) - a_z\|_\vartheta = 0$, $z = 0, 1, 2$ using the theorem in [19]. For $z = 0$, we have $\|K_m^*(a_0; y) - a_0\|_\vartheta = 0$. Now, let us examine the cases $z = 1, 2$. When the necessary results for these situations are used,

$$\begin{aligned} \|K_m^*(a_1; y) - a_1\|_\vartheta &= \sup_{y \in (0, \infty)} \frac{|K_m^*(a_1; y) - a_1|}{1 + y^2} \\ &= \sup_{y \in (0, \infty)} \frac{|(mk/mk + 1)y - y|}{1 + y^2} \\ &\leq \left| \frac{mk}{mk + 1} - 1 \right| \sup_{y \in (0, \infty)} \frac{y}{1 + y^2} \leq \left| \frac{1}{mk + 1} \right| \end{aligned} \quad (30)$$

is obtained. If we take the limit of this expression, it becomes

$$\lim_{m \rightarrow \infty} \frac{1}{mk + 1} = 0. \quad (31)$$

Then, we have

$$\begin{aligned} \|K_m^*(a_2; y) - a_2\|_\vartheta &= \sup_{y \in (0, \infty)} \frac{|K_m^*(a_2; y) - a_2|}{1 + y^2} \\ &= \sup_{y \in (0, \infty)} \frac{|m^2 k^2 / ((mk + 1)(mk - k + 1))y^2 - y^2|}{1 + y^2} \\ &\leq \left| \frac{m^2 k^2}{(mk + 1)(mk - k + 1)} - 1 \right| \sup_{y \in (0, \infty)} \frac{y^2}{1 + y^2} \\ &\leq \left| \frac{mk^2 - 2mk + k - 1}{(mk + 1)(mk - k + 1)} \right|. \end{aligned} \quad (32)$$

If we take the limit of this expression, it becomes

$$\lim_{m \rightarrow \infty} \frac{mk^2 - 2mk + k - 1}{(mk + 1)(mk - k + 1)} = 0. \tag{33}$$

As a result of the equations obtained above, the evidence is finished. \square

5. The Rates of Convergence

Now, we can concentrate on the rates of convergence the modified Gamma operator in terms of the modulus continuity. We shall now show that $K_m^*(\varphi)$ outperforms the classical operator in terms of error estimation. Let us define the following in light of this goal.

The modulus of continuity of w is denoted by $\omega_{y_0}(\varphi, \delta)$ for interval $(0, y_0]$, $y_0 \geq 0$ and can be described as follows:

$$\omega_{y_0}(\varphi, \delta) = \sup_{|h-y| \leq \delta, y, h \in (0, y_0]} |\varphi(h) - \varphi(y)|. \tag{34}$$

The modulus of continuity $\omega_{y_0}(\varphi, \delta) \rightarrow 0$ is easily understood as $\delta \rightarrow 0$ for the function $\varphi \in C_B(0, \infty)$, where $C_B(0, \infty)$ is defined as space of all continuous and bounded functions on the interval $(0, \infty)$. Now, let us look at the rates of convergence theorem for $(K_m^*)_{m \geq 1}$.

Theorem 9. For $y_0 > 0$ and $\varphi \in C_B(0, \infty)$, let $\omega_{y_0+1}(\varphi, \delta)$ be the modulus of continuity on the finite interval $(0, y_0 + 1] \subset (0, \infty)$. Then, the following inequality exists:

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq 3N_\varphi \left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} \right) y_0^2 (1 + y_0)^2 \\ &\quad + 2\omega_{y_0+1} \left(\varphi, \sqrt{\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)}} y_0^2 \right), \end{aligned} \tag{35}$$

where N_φ is a constant only according as φ .

Proof. Now, let $\varphi \in C_B(0, \infty)$, $0 < y \leq y_0$, and $h > y_0 + 1$. Then, we can conclude that

$$|\varphi(h) - \varphi(y)| \leq |\varphi(h)| + |\varphi(y)| \leq 3N_\varphi (h - y)^2 (1 + y_0)^2 \tag{36}$$

for $h - y > 1$. Then, again let $\varphi \in C_B(0, \infty)$, $0 < y \leq y_0$. So, the following inequality holds

$$|\varphi(h) - \varphi(y)| \leq \omega_{y_0+1}(\varphi, |h - y|) \leq \omega_{y_0+1}(\varphi, \delta) \left(1 + \frac{1}{\delta} |h - y| \right) \tag{37}$$

for $h \leq y_0 + 1$. As a result, from the above inequality, we

deduce that

$$|\varphi(h) - \varphi(y)| \leq 3N_\varphi (h - y)^2 (1 + y_0)^2 + \omega_{y_0+1}(\varphi, \delta) \left(1 + \frac{1}{\delta} |h - y| \right) \tag{38}$$

for $0 < y \leq y_0$ and $0 < h < \infty$. Applying K_m^* and Cauchy-Schwarz inequality to (38), we obtain

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq 3N_\varphi K_m^*((h - y)^2; y) (1 + y_0)^2 \\ &\quad + \omega_{y_0+1}(\varphi, \delta) \left(1 + \frac{1}{\delta} \sqrt{K_m^*((h - y)^2; y)} \right) \\ &\leq 3N_\varphi \left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} \right) (1 + y_0)^2 \\ &\quad + 2\omega_{y_0+1} \left(\varphi, \sqrt{\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)}} y_0^2 \right). \end{aligned} \tag{39}$$

By choosing $\delta = \sqrt{((mk^2 - k + 1)/((mk + 1)(mk - k + 1)))} y_0^2$, we can conclude the proof. \square

Let

$$C_B^2(0, \infty) = \left\{ \varphi \in C_B(0, \infty) : \varphi', \varphi'' \in C_B(0, \infty) \right\}, \tag{40}$$

with the norm

$$\|\varphi\|_{C_B^2(0, \infty)} = \|\varphi\|_{C_B(0, \infty)} + \|\varphi'\|_{C_B(0, \infty)} + \|\varphi''\|_{C_B(0, \infty)} \tag{41}$$

also

$$\|\varphi\|_{C_B(0, \infty)} = \sup_{y \in (0, \infty)} |\varphi(y)| \tag{42}$$

in [20].

Theorem 10. Let K_m^* be the operator defined in (4). Then, for any $\varphi \in C_B^2(0, \infty)$, we have

$$|K_m^*(\varphi; y) - \varphi(y)| \leq \frac{1}{2} \sqrt{\tau} (2 + \sqrt{\tau}) \|\varphi\|_{C_B^2(0, \infty)}, \tag{43}$$

where τ is $K_m^*(\psi_{y,2}; y)$ in Lemma 3.

Proof. Let $\varphi \in C_B^2(0, \infty)$. When referring to the Taylor series, obtain

$$\varphi(h) = \varphi(y) + \varphi'(y)(h - y) + \frac{1}{2} \varphi''(\xi)(h - y)^2, \tag{44}$$

where ξ between y and h , from which it follows:

$$|\varphi(h) - \varphi(y)| \leq N_1 |h - y| + \frac{1}{2} N_2 (h - y)^2, \tag{45}$$

where

$$\begin{aligned} N_1 &= \sup_{y \in (0, \infty)} |\varphi'(y)| = \|\varphi'\|_{C_B(0, \infty)} \leq \|\varphi\|_{C_B^2(0, \infty)}, \\ N_2 &= \sup_{y \in (0, \infty)} |\varphi''(y)| = \|\varphi''\|_{C_B(0, \infty)} \leq \|\varphi\|_{C_B^2(0, \infty)}, \end{aligned} \quad (46)$$

because of (41). Thus, we have

$$|\varphi(h) - \varphi(y)| \leq \left(|h - y| + \frac{1}{2}(h - y)^2 \right) \|\varphi\|_{C_B^2(0, \infty)}. \quad (47)$$

Since

$$|K_m^*(\varphi; y) - \varphi(y)| = |K_m^*(\varphi(h) - \varphi(y); y)| \leq K_m^*(|\varphi(h) - \varphi(y)|; y), \quad (48)$$

and $K_m^*(|h - y|; y) \leq K_m^*((h - y)^2; y)^{1/2} = \sqrt{\tau}$, we get

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq \left(K_m^*(|h - y|; y) + \frac{1}{2} K_m^*((h - y)^2; y) \right) \|\varphi\|_{C_B^2(0, \infty)} \\ &\leq \frac{1}{2} \sqrt{\tau} (2 + \sqrt{\tau}) \|\varphi\|_{C_B^2(0, \infty)}. \end{aligned} \quad (49)$$

The desired result is obtained. \square

The Peetre's \mathcal{K} -functional is expressed by

$$K_2^*(\varphi, \delta) = \inf_{C_B^2(0, \infty)} \left\{ \|\varphi - u\|_{C_B(0, \infty)} + \delta \|u\|_{C_B^2(0, \infty)} : u \in C_B^2(0, \infty) \right\}. \quad (50)$$

The second-order modulus of continuity is defined by

$$\omega_2(\varphi, \delta) = \sup_{0 < \mathbf{u} < \delta y \in (0, \infty)} |\varphi(y + 2\mathbf{u}) - 2\varphi(y + \mathbf{u}) + \varphi(\mathbf{u})| \quad (51)$$

in [20]. The relation ω_2 and K_2^* is as follows:

$$K_2^*(\varphi, \delta) \leq N \left\{ \omega_2(\varphi; \delta) + \min(1, \delta) \|\varphi\|_{C_B(0, \infty)} \right\} \quad (52)$$

in [21].

Theorem 11. Let $K_m^*(\cdot, \cdot)$ be the operator defined in (4). Then, for any $\varphi \in C_B(0, \infty)$, we have

$$|K_m^*(\varphi; y) - \varphi(y)| \leq 2N \left\{ \omega_2\left(\varphi; \sqrt{\Delta_{m,y}}\right) + \min(1, \Delta_{m,y}) \|\varphi\|_{C_B(0, \infty)} \right\}, \quad (53)$$

where N is a positive constant and $\Delta_{m,y} = \sqrt{\tau}(2 + \sqrt{\tau})/2$ and $\tau = K_m^*(\psi_{y,2}; y)$.

Proof. We prove this by using Theorem 10. Let $u \in C_B^2(0, \infty)$. Since

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq |K_m^*(\varphi - u; y)| + |K_m^*(\varphi; y) - u(y)| + |\varphi(y) - u(y)| \\ &\leq 2\|\varphi - u\|_{C_B(0, \infty)} + \frac{1}{2} \sqrt{\tau} (2 + \sqrt{\tau}) \|u\|_{C_B^2(0, \infty)} \\ &\leq 2 \left(\|\varphi - u\|_{C_B(0, \infty)} + \frac{1}{4} \sqrt{\tau} (2 + \sqrt{\tau}) \|u\|_{C_B^2(0, \infty)} \right). \end{aligned} \quad (54)$$

By taking infimum over all $u \in C_B^2(0, \infty)$ on the right side of the last inequality and by using (50), we get

$$|K_m^*(\varphi; y) - \varphi(y)| \leq 2K_2^* \left(\varphi; \frac{\sqrt{\tau}(2 + \sqrt{\tau})}{4} \right). \quad (55)$$

This completes the proof, by using equation (52). \square

6. Pointwise Estimates

Let us look at some pointwise estimates of rates of convergence of $K_m^*(\varphi; y)$. At first, the relationship between the local approximation and the local smoothness of the function is given. In this direction, let us give the following definitions. Let $s \in (0, 1]$ and $\widehat{I} \subset (0, \infty)$. In this case, a function $\varphi \in C_B(0, \infty)$ can be called $\text{Lip}_{N_\varphi}(s)$ on \widehat{I} if the following condition holds:

$$|\varphi(v) - \varphi(y)| \leq N_{\varphi,s} |v - y|^s, \quad v \in (0, \infty) \text{ and } y \in \widehat{I}, \quad (56)$$

where $N_{\varphi,s}$ is a constant that relies on φ and s mentioned above.

Theorem 12. Let $\varphi \in C_B(0, \infty) \cap \text{Lip}_{N_\varphi}(s)$ such that s and \widehat{I} given as above. In the circumstances, we give

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq N_{\varphi,s} \left[\left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} a_2(y) \right)^{s/2} \right. \\ &\quad \left. + 2(d(y, \widehat{I}))^s \right], \quad y \in (0, \infty), \end{aligned} \quad (57)$$

where $N_{\varphi,s}$ given above and $d(y, \widehat{I})$ is the distance between y and \widehat{I} . This distance is described as:

$$d(y, \widehat{I}) = \inf \{ |v - y|, v \in \widehat{I} \} \quad (58)$$

Proof. Let us define the closure of the set \widehat{I} as \widehat{I} . Then, one can argue that at least one point $v_0 \in \widehat{I}$ occurs where

$$d(y, \widehat{I}) = |y - v_0|. \quad (59)$$

Then, due to the monotonicity properties of $(K_m^*)_{m \geq 1}$, we

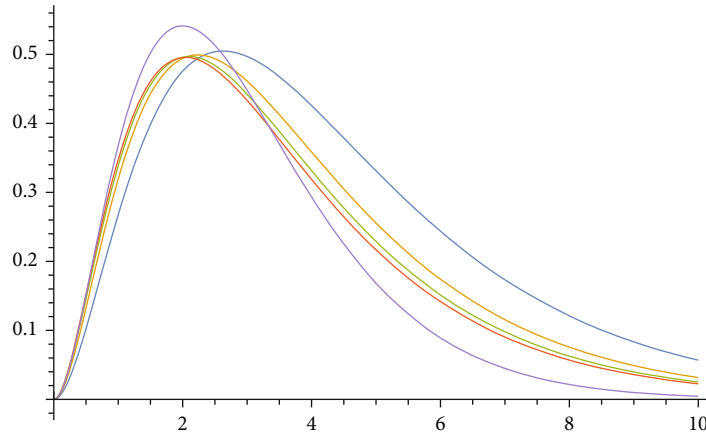


FIGURE 1: Graphics of the K_m^* operator for $k = 1/3, 1, 3, 30$, respectively, and $m = 10$ is fixed (blue for $k = 1/3$, orange for $k = 1$, green for $k = 3$, red for $k = 30$, and purple for $\varphi(y)$).

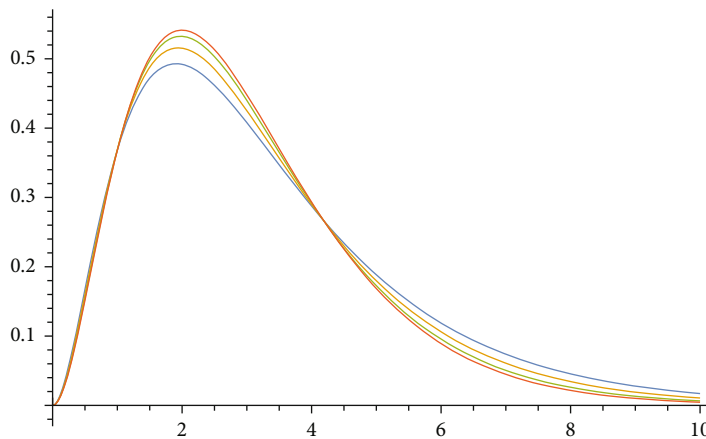


FIGURE 2: Graphics of the K_m^* operator for $m = 10, 20, 60$, respectively, and $k = 3$ is fixed (blue for $m = 10$, orange for $m = 20$, green for $m = 60$, and red for $\varphi(y)$).

deduce that

$$\begin{aligned}
 |K_m^*(\varphi; y) - \varphi(y)| &\leq K_m^*(|\varphi(v) - \varphi(v_0)|; y) + K_m^*(|\varphi(y) - \varphi(v_0)|; y) \\
 &\leq N_{\varphi, s} [K_m^*(|v - v_0|^s; y) + |y - v_0|^s] \\
 &\leq N_{\varphi, s} [K_m^*(|v - y|^s; y) + 2|y - v_0|^s].
 \end{aligned}
 \tag{60}$$

Then, from the definition of Hölder’s inequality, we have

$$\begin{aligned}
 |K_m^*(\varphi; y) - \varphi(y)| &\leq N_{\varphi, s} \left[K_m^*(|v - y|^2; y)^{s/2} + 2(d(y, \hat{\Gamma}))^s \right] \\
 &= N_{\varphi, s} \left[\left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} a_2(y) \right)^{s/2} + 2(d(y, \hat{\Gamma}))^s \right],
 \end{aligned}
 \tag{61}$$

which concludes the theorem. □

Now, let us try to determine the local direct approximation of the new Gamma operator modification. Let us start

with the Lipschitz type maximum function of order s presented in [22] for this goal, that is,

$$\tilde{\omega}_s(\varphi, y) = \sup_{0 < v < \infty, v \neq y} \frac{|\varphi(v) - \varphi(y)|}{|v - y|^s}, \tag{62}$$

where $s \in (0, 1]$ and $y \in (0, \infty)$.

Theorem 13. For $\varphi \in C_B(0, \infty)$ and $\omega_s \in (0, 1]$, the following inequality holds:

$$|K_m^*(\varphi; y) - \varphi(y)| \leq \tilde{\omega}_s(\varphi, y) K_m^* \left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} a_2(y) \right)^{s/2}
 \tag{63}$$

for $y \in (0, \infty)$.

Proof. Thanks to the definition of $\omega_s(\phi, y)$ given above and well-known Hölder inequality, we deduce that

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq K_m^*(|\varphi(v) - \varphi(y)|; y) \leq \tilde{\omega}_s(\varphi, y) K_m^*(|v - y|^s; y) \\ &\leq \tilde{\omega}_s(\varphi, y) K_m^*(|v - y|^2; y)^{s/2} \\ &\leq \tilde{\omega}_s(\varphi, y) K_m^* \left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} a_2(y) \right)^{s/2}. \end{aligned} \quad (64)$$

As a result, the desired outcome is achieved. \square

Now, finally, let us consider the following Lipschitz type space with two parameters, $c, d > 0$, such that

$$\text{Lip}_N^{c,d}(s) = \left(\varphi \in C(0, \infty): |\varphi(v) - \varphi(y)| \leq N \frac{|v - y|^s}{(cy^2 + dy + v)^{s/2}}; y, v \in (0, \infty) \right) \quad (65)$$

introduced in [23] where $s \in (0, 1]$ and N is a positive constant.

Theorem 14. For $\varphi \in \text{Lip}_N^{c,d}(s)$ and $y \in (0, \infty)$, then, we have

$$|K_m^*(\varphi; y) - \varphi(y)| \leq N \left[\frac{[(mk^2 - k + 1)/((mk + 1)(mk - k + 1))] a_2(y)}{cy^2 + dy} \right], \quad (66)$$

where $c, d > 0$.

Proof. The proof is divided into two parts. For the first, we use $s = 1$, which means

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq K_m^*(|\varphi(v) - \varphi(y)|; y), \\ &\leq NK_m^* \left(\frac{|v - y|}{\sqrt{cy^2 + dy + 1}}; y \right), \\ &\leq \frac{N}{\sqrt{cy^2 + dy}} K_m^*(|v - y|; y), \end{aligned} \quad (67)$$

for $\varphi \in \text{Lip}_N^{c,d}(s)$ and $y \in (0, \infty)$. We conclude that

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq \frac{N}{\sqrt{cy^2 + dy}} [K_m^*(|v - y|^2; y)]^{1/2} \\ &\leq N \left[\frac{[(mk^2 - k + 1)/((mk + 1)(mk - k + 1))] a_2(y)}{cy^2 + dy} \right]^{1/2}, \end{aligned} \quad (68)$$

by using the well-known Cauchy-Schwarz inequality, which validates the theory for $s = 1$. Then, let us consider $s \in (0, 1)$.

For $\varphi \in \text{Lip}_N^{c,d}(s)$ and $y \in (0, \infty)$, we obtain that

$$|K_m^*(\varphi; y) - \varphi(y)| \leq \frac{N}{(cy^2 + dy)^{s/2}} K_m^*(|v - y|^s; y). \quad (69)$$

We derive that

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq \frac{N}{(cy^2 + dy)^{s/2}} K_m^*(|v - y|^s; y) \\ &\leq \frac{N}{(cy^2 + dy)^{s/2}} (K_m^*(|v - y|; y))^s \end{aligned} \quad (70)$$

with the help of the well-known Hölder inequality. Finally, we have

$$\begin{aligned} |K_m^*(\varphi; y) - \varphi(y)| &\leq \frac{N}{(cy^2 + dy)^{s/2}} (K_m^*(|v - y|^2; y))^{s/2} \\ &\leq N \left[\frac{[(mk^2 - k + 1)/((mk + 1)(mk - k + 1))] a_2(y)}{cy^2 + dy} \right]^{s/2}, \end{aligned} \quad (71)$$

which completes the proof by applying the well-known Cauchy-Schwarz inequality. \square

For the case of $c = 1$ and $d = 0$, we have the following corollary.

Corollary 15. The local estimate in parametric Lipschitz space is obtained for special fixed parameters $c = 1$ and $d = 0$.

$$|K_m^*(\varphi; y) - \varphi(y)| \leq N \left(\frac{mk^2 - k + 1}{(mk + 1)(mk - k + 1)} \right) \quad (72)$$

for $\varphi \in \text{Lip}_N^{1,0}(s)$ and $y \in (0, \infty)$.

7. Numerical Example

In this section of the article, we provide some numerical examples to verify the rates of convergence of $K_m^*(\varphi; y)$ in two dimensions ($m = 10$ is fixed for Figure 1 and $k = 3$ is fixed for Figure 2). In our first example, we compare the operator $K_m^*(\varphi; y)$ with the classical Gamma operator.

In this example, $K_m^*(\varphi; y)$ and $\varphi(y) = y^2 e^{-y}$ applied for $\varphi: [0, 10] \rightarrow [0, \infty)$.

In Figure 1, it is seen that the operator puts closer to the function as the value of k gets larger ($m = 10$ is fixed). In Figure 2, it is seen that the operator puts closer to the function as the value of m gets larger ($k = 3$ is fixed).

8. Concluding Remarks

We have defined a new form of Gamma operator by considering k -Gamma function. With the operator defined, the conditions of the Korovkin theorem are completed. Later, Voronovskaya type theorem, weighted approximation, the rates of convergence, and pointwise estimates are obtained. Finally, we give numerical example to confirm its approximation.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

I would like to thank the Scientific and Technological Research Council of Turkey (TUBİTAK) for the graduate scholarship that supported the second author.

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