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Research Article

A Study of Spiral-Like Harmonic Functions Associated with Quantum Calculus

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This article introduces new subclasses of harmonic univalent functions associated with q-difference operator. Modified q-multiplier transformation is defined, and certain geometric properties such as the sufficient condition, distortion result, extreme points, and invariance of convex combination of the elements of the subclasses are discussed by employing the newly defined q-operator. Also, various well-known results already proved in the literature are pointed out.

1. Introduction

A function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is known to be real harmonic function in domain $\mathscr{D}$ if θ_{xx} and θ_{yy} are continuous in $\mathscr{D}$ and satisfies

$$\theta_{xx}(x, y) + \theta_{yy}(x, y) = 0. \tag{1}$$

Continuous function $h : \Omega(\subset\mathbb{C}) \longrightarrow \mathbb{C}$ defined by $h(z) = \theta_1(x, y) + i\theta_2(x, y)$ is harmonic if both $\theta_1(x, y)$ and $\theta_2(x, y)$ are real harmonic in Ω. We found that, in any simply connected domain Ω, every harmonic function $h(z)$ can be expressed by $h(z) = h_1(z) + \bar{h}_2(z)$, where h_1 and h_2 are analytic in Ω, and are called, respectively, the analytic and coanalytic parts of h.

The class of complex-valued harmonic functions $h = h_1 + \bar{h}_2$ defined in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$ and normalized by $h_1(0) = h_2(0) = h_1'(0) - 1 = 0$ is denoted by $\mathscr{H}$. The function in the class $\mathscr{H}$ has the following power series representation:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} \mathrm{b}_n z^n}. \tag{2}$$

It is clear that when $h_2(z)$ is identically zero, the class $\mathscr{H}$ coincides with the class $\mathscr{A}$ of normalized analytic functions in $\mathscr{U}$. Due to Lewy [1], a function $h \in \mathscr{H}$ is locally univalent and sense-preserving in $\mathscr{U}$ if and only if

$$\left|h_1'(z)\right| > \left|h_2'(z)\right|, \text{for} z \in \mathscr{U}. \tag{3}$$

We indicate by $\mathscr{S}_{\mathscr{H}}$ the subclass of $\mathscr{H}$ consisting of all sense-preserving univalent harmonic functions h. Firstly, Clunie and Sheil-Small [2] discussed certain geometric properties of the class $\mathscr{S}_{\mathscr{H}}$ and its subclasses. Later on, several authors contributed in the study of subclasses of the class $\mathscr{S}_{\mathscr{H}}$, for example, see [3–9]. The most prominent author Jahangiri [10] investigated various interesting properties of the class $\mathscr{S}^*_{\mathscr{H}}(\varsigma)$ of starlike harmonic functions of order ς, $(0 \le \varsigma < 1)$, defined by

$$\Re\left(\frac{zh'(z)}{h(z)}\right)=\Re\left(\frac{zh_1'(z)-z\overline{h_2'}(z)}{h_1(z)+\overline{h_2}(z)}\right)\geq\varsigma. \tag{4}$$

For the convenience, we present the notion of q-difference operator briefly. Jackson [11] introduced the q-difference operator and is defined by

$$\partial_q h_1(z)=\frac{h_1(z)-h_1(qz)}{(1-q)z}\,;q\neq 1,z\neq 0, \tag{5}$$

for $q\in(0,1)$ and $h_1\in\mathscr{A}$ with $h_1(z)=z+\sum_{n=2}^{\infty}a_nz^n$. We note that $\lim_{q\longrightarrow 1^-}\partial_q h_1(z)=h_1'(z)$, where $h_1'(z)$ is the ordinary derivative of the function. It is clear that

$$\partial_q\left\{\sum_{n=1}^{\infty}a_nz^n\right\}=\sum_{n=1}^{\infty}[n]_qz^{n-1}, \tag{6}$$

where

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots, \tag{7}$$

for $n\in\mathbb{N}=\{1,2,3,..\}$ and $z\in\mathscr{U}$. For some recent investigations involving q-calculus, we may refer the interested reader to [12–17]. Recently, in [18], Shah and Noor introduced the q-analogue of multiplier transformation $I_{q,\tau}^s:\mathscr{A}\longrightarrow\mathscr{A}$ by

$$I_{q,\tau}^s h_1(z)=z+\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_nz^n, \tag{8}$$

where $h_1\in\mathscr{A}$, $s\in\mathbb{R}$ and $\tau>-1$. It is noted that for nonnegative integer s and $\tau=0$, the operator $I_{q,\tau}^s$ coincides with the Salagean q-differential operator defined in [19]. Moreover, if $q\longrightarrow 1^-$ in (8), then the multiplier transformation studied by the Cho and Kim in [20] is deduced. Nowadays, several subclasses of $\mathcal{S}_{\mathscr{H}}$ associated with operators and q-operators were discussed by the prominent researchers, like [21–26]. In motivation of the above said literature, first, we modify the q-multiplier transformation, and then we define certain new subclasses of $\mathcal{S}_{\mathscr{H}}$. For $h=h_1+\overline{h_2}$ given by (2), we define the modified q-multiplier transformation of h as

$$I_{q,\tau}^s h(z)=I_{q,\tau}^s h_1(z)+(-1)^s I_{q,\tau}^s\overline{h_2}(z), \tag{9}$$

where $I_{q,\tau}^s h_1(z)$ is given by (8) and

$$I_{q,\tau}^s h_2(z)=\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s b_nz^n. \tag{10}$$

It is observed that, for $h_2=0$, the modified q-multiplier transformation defined by (9) turns out to be the $q-$multiplier transformation introduced in [18]. For $h=h_1+\overline{h_2}\in\mathcal{S}_{\mathscr{H}}$, we define a new class $\mathscr{H}\mathcal{S}\mathcal{T}_q(\zeta,\varsigma)$ as the following.

Definition 1. Let $h\in\mathcal{S}_{\mathscr{H}}$. Then $h\in\mathscr{H}\mathcal{S}\mathcal{T}_q(\zeta,\varsigma)$ if

$$\Re\left\{1+e^{i\zeta}\left(\frac{z\partial_q h(z)}{h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{11}$$

where $\varsigma\in[0,1)$, $|\zeta|<\pi/2$ and $q\in(0,1)$.

Particularly, for $q\longrightarrow 1^-$, the class $\mathscr{H}\mathcal{S}\mathcal{T}_q(\zeta,\varsigma)$ reduces to the class denoted by $\mathcal{S}_{\mathscr{H}}^*(\zeta,\varsigma)$ of functions $h\in\mathcal{S}_{\mathscr{H}}$ that satisfies

$$\Re\left\{1+e^{i\zeta}\left(\frac{zh'(z)}{h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{12}$$

where $\varsigma\in[0,1)$ and $|\zeta|<\pi/2$. Moreover, if $\zeta=0$, then the class $\mathcal{S}_{\mathscr{H}}^*(\zeta,\varsigma)$ coincides with the class $\mathcal{S}_{\mathscr{H}}^*(\varsigma)$ introduced by Jahangiri [10]. We further define $\mathscr{H}\overline{\mathcal{S}}\mathcal{T}_q(\zeta,\varsigma)=\mathscr{H}\mathcal{S}\mathcal{T}_q(\zeta,\varsigma)\cap\overline{\mathcal{S}_{\mathscr{H}}}$, where $\overline{\mathcal{S}_{\mathscr{H}}}$ denotes the subclass of $\mathcal{S}_{\mathscr{H}}$ consisting of functions of the type $h_s(z)=h_1(z)+\overline{h_{2,s}}(z)$, where

$$h_1(z)=z-\sum_{n=2}^{\infty}|a_n|z^n\text{and}h_{2,s}(z)=(-1)^s\sum_{n=2}^{\infty}|b_n|z^n. \tag{13}$$

Now, by using modified q-multiplier transformation given by (9), we define the following.

Definition 2. Let $h\in\mathcal{S}_{\mathscr{H}}$. Then $h\in\mathscr{H}\mathcal{S}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)$ if

$$\Re\left\{1+e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^s h(z)}-1\right)\right\}\geq\varsigma\cos\zeta, \tag{14}$$

where $s\in\mathbb{R}$, $\tau>-1$, $\varsigma\in[0,1)$, $|\zeta|<\pi/2$, and $q\in(0,1)$.

Also, we define $\mathscr{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\mathcal{S}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)\cap\overline{\mathcal{S}_{\mathscr{H}}}$, where $\overline{\mathcal{S}_{\mathscr{H}}}$ denotes the subclass of $\mathcal{S}_{\mathscr{H}}$ consisting of functions given by (13). It is noted that, for $s=\tau=0$, we have $\mathscr{H}\mathcal{S}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\mathcal{S}\mathcal{T}_q(\zeta,\varsigma)$ and $\mathscr{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta,\varsigma)=\mathscr{H}\overline{\mathcal{S}}\mathcal{T}_q(\zeta,\varsigma)$. In particular, if we take $\zeta=\tau=0$ and $s=m\in\mathbb{N}$ in above definitions, then we have well-known classes $\mathscr{H}_q^m(\zeta,\varsigma)$ and $\overline{\mathscr{H}}_q^m(\zeta,\varsigma)$ introduced by Jahangiri [22].

The next section presents the main investigations such as the sufficient condition, distortion result, extreme points, and invariance of convex combination of the elements of the subclasses defined as above.

2. Main Results

Theorem 3. *Let $h=h_1+\overline{h_2}\in\mathcal{S}_{\mathscr{H}}$ is given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{[n+\tau]_q}{[1+\tau]_q}-\varsigma\cos\zeta\right)|a_n|+\left(\frac{[n+\tau]_q}{[1+\tau]_q}+\varsigma\cos\zeta\right)|b_n|\right\}\right]\times\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s\leq 2(1-\varsigma\cos\zeta), \tag{15}$$

where $s \in \mathbb{R}$, $\tau > -1$, $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, and $q \in (0, 1)$. Then, $h \in \mathcal{HST}_q^{s,\tau}(\zeta, \varsigma)$.

Proof. We need to prove that if the coefficients of the harmonic function $h = h_1 + \overline{h_2} \in \mathcal{S}_{\mathcal{H}}$ given by (2) satisfy the inequality (15), then it also satisfies (14). It is known that $\Re(w) \geq \xi$ if and only if $|1 - \xi + w| \geq |1 + \xi - w|$. So, it suffices to prove that

$$\begin{aligned} &\left|1 - \varsigma \cos\zeta + 1 + e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^{s}h(z)} - 1\right)\right| \\ &\geq \left|1 + \varsigma \cos\zeta - 1 - e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h(z)}{I_{q,\tau}^{s}h(z)} - 1\right)\right|, \end{aligned} \tag{16}$$

or equivalently,

$$\begin{aligned} &\left|(2 - \varsigma \cos\zeta)I_{q,\tau}^{s}h(z) + e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &\quad - \left|(\varsigma \cos\zeta)I_{q,\tau}^{s}h(z) - e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \geq 0. \end{aligned} \tag{17}$$

From the left hand side,

$$\begin{aligned} &\left|(2 - \varsigma \cos\zeta)I_{q,\tau}^{s}h(z) + e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &\quad - \left|(\varsigma \cos\zeta)I_{q,\tau}^{s}h(z) - e^{i\zeta}\left(I_{q,\tau}^{s+1}h(z) - I_{q,\tau}^{s}h(z)\right)\right| \\ &= \left|\begin{matrix} (2 - \varsigma \cos\zeta)\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z)\right\} \\ -e^{i\zeta}\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z) - I_{q,\tau}^{s+1}h_1(z) + (-1)^s I_{q,\tau}^{s+1}\overline{h_2}(z)\right\} \end{matrix}\right| \\ &\quad - \left|\begin{matrix} (\varsigma \cos\zeta)\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z)\right\} \\ +e^{i\zeta}\left\{I_{q,\tau}^{s}h_1(z) + (-1)^s I_{q,\tau}^{s}\overline{h_2}(z) - I_{q,\tau}^{s+1}h_1(z) + (-1)^s I_{q,\tau}^{s+1}\overline{h_2}(z)\right\} \end{matrix}\right| \\ &\geq \left|\begin{matrix} (2 - \varsigma \cos\zeta)z + (2 - \varsigma \cos\zeta) \\ \times\left\{\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \overline{b_n z^n}\right\} \\ -e^{i\zeta}\left\{\begin{matrix} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \overline{b_n z^n} \\ -\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} \overline{b_n z^n} \end{matrix}\right\} \end{matrix}\right| \\ &\quad - \left|\begin{matrix} (\varsigma \cos\zeta)z + (\varsigma \cos\zeta) \\ \times\left\{\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \overline{b_n z^n}\right\} \\ +e^{i\zeta}\left\{\begin{matrix} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \overline{b_n z^n} \\ -\sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} a_n z^n + (-1)^s \sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^{s+1} \overline{b_n z^n} \end{matrix}\right\} \end{matrix}\right| \\ &\geq (2 - \varsigma \cos\zeta)|z| - \sum_{n=2}^{\infty}\left[2 - \varsigma \cos\zeta + \frac{[n+\tau]_q}{[1+\tau]_q} - 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \\ &\quad \cdot |a_n||z|^n - \sum_{n=1}^{\infty}\left[\varsigma \cos\zeta + \frac{[n+\tau]_q}{[1+\tau]_q} - 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |b_n||z|^n \\ &\quad - \sum_{n=2}^{\infty}\left[\frac{[n+\tau]_q}{[1+\tau]_q} - 1 - \varsigma \cos\zeta\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |a_n||z|^n \\ &\quad - \sum_{n=2}^{\infty}\left[\frac{[n+\tau]_q}{[1+\tau]_q} + \varsigma \cos\zeta + 1\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s |b_n||z|^n \geq 2(1 - \varsigma \cos\zeta) \\ &\quad \times\left[1 - \frac{1}{(1 - \varsigma \cos\zeta)}\left\{\begin{matrix} \sum_{n=2}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q} - \varsigma \cos\zeta\right)|a_n| \\ +\sum_{n=1}^{\infty}\left(\frac{[n+\tau]_q}{[1+\tau]_q} + \varsigma \cos\zeta\right)|b_n| \end{matrix}\right\}\right]\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s. \end{aligned} \tag{18}$$

The above expression is nonnegative by (15). Hence, $h \in \mathcal{HST}_q^{s,\tau}(\zeta, \varsigma)$.

The harmonic function is

$$\begin{aligned} h(z) = z &+ \sum_{n=2}^{\infty}\frac{(1 - \varsigma \cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n} t_n z^n \\ &+ \sum_{n=1}^{\infty}\frac{(1 - \varsigma \cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q + [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n} \overline{\upsilon}_n \overline{z}^n, \end{aligned} \tag{19}$$

where $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$ and $\sum_{n=2}^{\infty}|t_n| + \sum_{n=1}^{\infty}|\upsilon_n| = 1$ show that the coefficient bound given by (15) is sharp. For different choices of parameters, we deduce certain results as follows. If $s = \tau = 0$ in Theorem 3, then we have a following new result. □

Corollary 4. *Let a function $h(z) = h_1(z) + \overline{h_2(z)} \in \mathcal{S}_{\mathcal{H}}$ given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{[n]_q - \varsigma \cos\zeta}{1 - \varsigma \cos\zeta}\right)|a_n| + \left(\frac{[n]_q + \varsigma \cos\zeta}{1 - \varsigma \cos\zeta}\right)|b_n|\right\}\right] \leq 2, \tag{20}$$

where $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, and $q \in (0, 1)$. Then, $h \in \mathcal{HST}_q(\zeta, \varsigma)$.

If $q \longrightarrow 1^-$, then Corollary 4 reduces to a new result as follows:

Corollary 5. *Let a function $h = h_1 + \overline{h_2} \in \mathcal{S}_{\mathcal{H}}$ given by (2) and satisfies*

$$\left[\sum_{n=1}^{\infty}\left\{\left(\frac{n - \varsigma \cos\zeta}{1 - \varsigma \cos\zeta}\right)|a_n| + \left(\frac{n + \varsigma \cos\zeta}{1 - \varsigma \cos\zeta}\right)|b_n|\right\}\right] \leq 2, \tag{21}$$

where $\varsigma \in [0, 1)$ and $|\zeta| < \pi/2$. Then, $h \in \mathcal{S}_{\mathcal{H}}^*(\zeta, \varsigma)$.

If we take $\zeta = \tau = 0$ and $s = m \in \mathbb{N}$, then we have well-known result.

Corollary 6 (see [22]). *Let a function $h = h_1 + \overline{h_2} \in \mathcal{S}_{\mathcal{H}}$ given by (2) and satisfies*

$$\left[\sum_{n=2}^{\infty} [n]_q^m \left([n]_q - \varsigma\right)|a_n| + \sum_{n=1}^{\infty} [n]_q^m \left([n]_q + \varsigma\right)|b_n|\right] \leq 1 - \varsigma, \tag{22}$$

where $\varsigma \in [0, 1)$ and $q \in (0, 1)$. Then, $f \in \mathcal{H}_q^m(\zeta, \varsigma)$.

When $\zeta = 0$ in Corollary 5, we get the sufficient condition for f in $\mathcal{S}_{\mathcal{H}}^*(\varsigma)$ proved by Jahangiri [10]. Moreover, for $\zeta = \varsigma = 0$ in Corollary 5, the sufficient condition for function in the class of starlike harmonic univalent mappings is obtained, see [4]. Now, we state and prove the necessary and sufficient conditions for the harmonic functions $h = h_1 + \overline{h_2}$ to be in $\mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$ as follows.

Theorem 7. *Let $h_s = h + \overline{h_{2,s}} \in \mathcal{S}_{\mathcal{H}}$ given by (13). Then, $h_s \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$ if and only if*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{[n+\tau]_q}{[1+\tau]_q} - \varsigma\cos\zeta\right)|a_n| + \left(\frac{[n+\tau]_q}{[1+\tau]_q} + \varsigma\cos\zeta\right)|b_n|\right\} \cdot \left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s \leq 2(1 - \varsigma\cos\zeta), \tag{23}$$

where $s \in \mathbb{R}$, $\tau > -1$, $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, and $q \in (0, 1)$.

Proof. The sufficient condition is obvious from the Theorem 3, because $\mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma) \subset \mathcal{H}\mathcal{S}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$. We need to prove the necessary condition only; that is, if $h_s \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$, then the coefficients of the function $h_s = h + \overline{h_{2,s}}$ satisfy the inequality (23). Let $h_s \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$. Then, by the definition of $\mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$, we have

$$\Re\left\{1 + e^{i\zeta}\left(\frac{I_{q,\tau}^{s+1}h_s(z)}{I_{q,\tau}^{s}h_s(z)} - 1\right) - \varsigma\cos\zeta\right\} \geq 0, \tag{24}$$

where $s \in \mathbb{R}$, $\tau > -1$, $\varsigma \in [0, 1)$, $|\zeta| < \pi/2$, and $q \in (0, 1)$. Equivalently, we can write (24) as

$$\Re\left\{\frac{(1 - \varsigma\cos\zeta)I_{q,\tau}^{s}h_s(z) + e^{i\zeta}\left(I_{q,\tau}^{s+1}h_s(z) - I_{q,\tau}^{s}h_s(z)\right)}{I_{q,\tau}^{s}h_s(z)}\right\} \geq 0. \tag{25}$$

Substituting $h_s = h + \overline{h_{2,s}}$ in (25) and employing (8) along with (13), and also some computation yields

$$\Re\left[\frac{\begin{array}{c}(1-\varsigma\cos\zeta)z - \sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q - \varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n||z|^n \\ -\sum_{n=1}^{\infty}\left(\left([n+\tau]_q/[1+\tau]_q\right) + \varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n||z|^n\end{array}}{z - \sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n||z|^n - \sum_{n=1}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n||z|^n}\right] \geq 0. \tag{26}$$

For all values of z in $\mathcal{U}$ above required condition must hold. Selecting z on the positive real axis where $0 \leq z = r < 1$, we obtain

$$\frac{\begin{array}{c}(1-\varsigma\cos\zeta) - \sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q - \varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n|r^{n-1} \\ -\sum_{n=1}^{\infty}\left(\left([n+\tau]_q/[1+\tau]_q\right) + \varsigma\cos\zeta\right)\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n|r^{n-1}\end{array}}{z - \sum_{n=2}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|a_n|r^{n-1} - \sum_{n=1}^{\infty}\left([n+\tau]_q/[1+\tau]_q\right)^s|b_n|r^{n-1}} \geq 0. \tag{27}$$

The numerator in (27) is negative for r sufficiently close to 1 whenever the inequality (23) does not hold. Hence, there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (27) is negative. This contradicts the required condition for $h_s \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$, and so the proof is complete. □

Next, we want to discuss the distortion bounds for the function $h \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$, which yields a covering result for this class.

Theorem 8. *If $h \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$ and $|z| = r < 1$, then*

$$(1 - |b_1|)r - Tr^2 \leq |h(z)| \leq (1 + |b_1|)r + Tr^2, \tag{28}$$

with

$$T = \frac{[1+\tau]_q^{s+1}}{[2+\tau]_q^{s}}\left[\frac{1 - \varsigma\cos\zeta}{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta} - \frac{1 + \varsigma\cos\zeta}{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta}|b_1|\right]. \tag{29}$$

Proof. Let $h \in \mathcal{H}\overline{\mathcal{S}}\mathcal{T}_q^{s,\tau}(\zeta, \varsigma)$. Taking absolute value of h, we get

$$\begin{aligned}
|h(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^2 \leq (1 + |b_1|)r + \frac{(1 - \varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^{s}} \\
&\quad \times \sum_{n=2}^{\infty}\left\{\left(\frac{\left\{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^{s}}{(1 - \varsigma\cos\zeta)[1+\tau]_q^{s+1}}\right)(|a_n| + |b_n|)\right\}r^2 \\
&\leq (1 + |b_1|)r + \frac{(1 - \varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^{s}} \\
&\quad \cdot \sum_{n=2}^{\infty}\left\{\left(\frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}}{(1 - \varsigma\cos\zeta)[1+\tau]_q}\right)\left(\frac{[n+\tau]_q}{[1+\tau]_q}\right)^s(|a_n| + |b_n|)\right\}r^2 \\
&\leq (1 + |b_1|)r + \frac{(1 - \varsigma\cos\zeta)[1+\tau]_q^{s+1}}{\left\{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}[2+\tau]_q^{s}} \\
&\quad \cdot \left\{1 - \frac{(1 - \varsigma\cos\zeta)}{(1 - \varsigma\cos\zeta)}|b_1|\right\}r^2, \ (\text{by}(2.2)) \leq (1 + |b_1|)r + Tr^2,
\end{aligned} \tag{30}$$

where T is given by (29). Hence, this is the required right hand inequality. Similarly, one can easily prove the required left hand inequality. □

Letting $r \longrightarrow 1$ and by making use of the left hand inequality of the above theorem, we obtain the following.

Corollary 9 (covering result). *If $h \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$, then*

$$\left\{w : |w| < \frac{L - M(1-\varsigma\cos\zeta)}{L} - \frac{L - M(1+\varsigma\cos\zeta)}{L}|b_1|\right\} \cdot \subset f(\mathscr{E}), \tag{31}$$

where $L = \{[2+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\}[2+\tau]_q^s$ *and* $M = [1+\tau]_q^{s+1}$.

In particular, we obtain the covering results for the newly defined classes and well-known classes of harmonic functions by choosing suitable choices of parameters.

Now, our task is to examine $clco\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$, the extreme points of closed convex hulls of $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$.

Theorem 10. *A function $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$ if and only if*

$$h_s(z) = \sum_{n=1}^{\infty}\left(\nu_n h_n(z) + \omega_n g_{s_n}(z)\right), \tag{32}$$

where $h_1(z) = z$,

$$\begin{aligned} h_n(z) &= z - \frac{(1-\varsigma\cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n} z^n ; (n = 2, 3, \cdots), \\ g_{s_n}(z) &= z + (-1)^s \frac{(1-\varsigma\cos\zeta)[1+\tau]_q}{\left\{[n+\tau]_q + [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n} \bar{z}^n ; (n = 1, 2, 3, \cdots), \end{aligned} \tag{33}$$

with $\sum_{n=1}^{\infty}(\nu_n + \Omega_n) = 1$, $\nu_n, \Omega_n \geq 0$, *and* $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$. *Particularly,* $\{h_n\}$ *and* $\{g_{s_n}\}$ *are the extreme points of* $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$.

Proof. We assume function h_s as given by (32)

$$\begin{aligned} h_s(z) = \sum_{n=1}^{\infty}\left(\nu_n h_n(z) + \Omega_n g_{s_n}(z)\right) &= \sum_{n=1}^{\infty}(\nu_n + \Omega_n)z \\ &- \sum_{n=2}^{\infty}\nu_n R_n z^n + (-1)^s \sum_{n=1}^{\infty}\Omega_n R_n^* \bar{z}^n, \end{aligned} \tag{34}$$

where $R_n = (1-\varsigma\cos\zeta)[1+\tau]_q/\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\}\psi_n$ and $R_n^* = (1-\varsigma\cos\zeta)[1+\tau]_q/\{[n+\tau]_q + [1+\tau]_q\varsigma\cos\zeta\}\psi_n$.

Equating (34) with (13), we get

$$|a_n| = \nu_n R_n \text{and} |b_n| = \Omega_n R_n^*. \tag{35}$$

Now,

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[\frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|a_n| \right. \\ &\quad \left. + \frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|b_n|\right] \\ &= 1 - X_1 + \sum_{n=1}^{\infty}(\nu_n + \Omega_n) = 2 - X_1 \leq 2. \end{aligned} \tag{36}$$

Thus, by Theorem 7, $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$. Conversely, let $h_s \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$. We take

$$\nu_n = \frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|a_n| ; (n = 2, 3, \cdots), \tag{37}$$

$$\Omega_n = \frac{\left\{[n+\tau]_q + [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|b_n| ; (n = 1, 2, \cdots), \tag{38}$$

with $\sum_{n=1}^{\infty}(\nu_n + \Omega_n) = 1$. We follow our required result by substituting the values of $|a_n|$ and $|b_n|$ from the above relations in (13).

Finally, we wish to show that the class $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$ is closed under the convex combination of its elements. □

Theorem 11. *The class $\mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$ is closed under the convex combination.*

Proof. Let $h_{i_s} \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta,\varsigma)$, $(i = 1, 2, \cdots)$, with

$$h_{i_s} = z - \sum_{n=2}^{\infty}|a_{i,n}|z^n + (-1)^s\sum_{n=2}^{\infty}|b_{i,n}|\bar{z}^n. \tag{39}$$

Making use of Theorem 7, we have

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|a_{i,n}| \\ &\quad + \sum_{n=1}^{\infty}\frac{\left\{[n+\tau]_q - [1+\tau]_q\varsigma\cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}|b_{i,n}| \leq 1, \end{aligned} \tag{40}$$

with $\psi_n = ([n+\tau]_q/[1+\tau]_q)^s$.

Now,

$$\sum_{i=1}^{\infty} \mathrm{u}_i h_{i_s} = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} \mathrm{u}_i \left|a_{i,n}\right|\right) z^n + (-1)^s \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \mathrm{u}_i \left|b_{i,n}\right|\right) \bar{z}^n. \tag{41}$$

To prove our result, we use (40) and (41)

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}\left(\sum_{i=1}^{\infty} \mathrm{u}_i \left|a_{i,n}\right|\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{\left\{[n+\tau]_q - [1+\tau]_q \varsigma \cos\zeta\right\}\psi_n}{(1-\varsigma\cos\zeta)[1+\tau]_q}\left(\sum_{i=1}^{\infty} \mathrm{u}_i \left|b_{i,n}\right|\right) \\
&\quad \leq \sum_{i=1}^{\infty} \mathrm{u}_i = 1.
\end{aligned} \tag{42}$$

Therefore, $\sum_{i=1}^{\infty} \mathrm{u}_i h_{i_s} \in \mathscr{H}\bar{\mathscr{S}}\mathscr{T}_q^{s,\tau}(\zeta, \varsigma)$. □

3. Conclusions

In this research, we have defined some new subclasses of harmonic univalent functions related to the q-difference operator. Also, we have introduced and studied the modified q-multiplier transformation. Several geometric properties such as sufficient condition, necessary conditions, distortion results, and invariance of classes under convex combination and extreme points are investigated. It is also noted that our investigations deduced various well-known results. In addition, this work can be extend for multivalent functions and (p, q)-calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no conflict of interest regarding the publication of this article.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

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