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Research Article

Controllability of Linear Fractional Systems with Delay in Control

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This paper discusses the controllability of continuous-time linear fractional systems with control delay. The Atangana-Baleanu fractional derivative with the Caputo approach is used. First, the solution expression for a linear fractional system is obtained. Then, the corresponding fractional delay controllability Gramian matrix is defined, and its non-singularity as necessary and sufficient conditions for the controllability is proved. Finally, another equivalent condition based on the matrix rank formed by the coefficients matrices of the original system is provided that is much easier to check.

1. Introduction

Fractional calculus, with its long-memory property, is an excellent tool for modeling systems. In fact, some dynamic processes in many practical systems (biological, electrochemical, viscoelastic, etc.) are fractional [1, 2]. Fractional-order delay differential equations are applicable for establishing a very realistic model of some processes and systems with memory. The achievements of many researchers about the occurrence of delay in practical systems are presented in [3–5].

Controllability, as one of the dynamic properties of fractional systems, plays a major role in modern control theory and engineering. Controllability of linear systems is established in [6]. Balachandran et al. [7] obtained sufficient conditions for the controllability of nonlinear fractional dynamical systems. The necessary and sufficient conditions for global relative controllability of linear fractional systems containing both lumped constant delay in state variables and distributed delays in admissible controls are presented by Klamka [8].

The most well-known fractional derivatives are Riemann-Liouville and Caputo [9, 10]. The other fractional derivatives that look like the Riemann-Liouville and Caputo ones are presented in [11, 12]. The aforementioned operators with singular kernels have difficulties in the management of many physical phenomena. In 2015, Caputo and Fabrizio presented a new definition of fractional differential operator with exponential kernel [13]. Atangana and Baleanu suggested a generalized fractional derivative with a non-singular kernel containing the Mittag-Leffler function, in 2016 [14]. In [15], some controllability criteria of fractional systems involving the Atangana-Baleanu fractional derivative in Caputo sense are provided.

In the current study, we investigate the controllability of linear fractional systems with control delay

$$\begin{aligned} {}^{ABC}D^{\alpha}x(t) &= Ax(t) + Bu(t) + Cu(t-h), t \geq 0, \\ x(0) &= x_0, \\ u(t) &= v(t), -h \leq t \leq 0, \end{aligned} \tag{1}$$

where ${}^{ABC}D^{\alpha}$ is the Atangana-Baleanu fractional derivative using the Caputo approach of order $0 < \alpha < 1$; $x(t) \in \mathbb{R}^n$ is a state vector; $u(t) \in \mathbb{R}^m$ is a control vector; $A \in \mathbb{R}^{n\times n}$, $B, C \in \mathbb{R}^{n\times m}$ are constant matrices; $h > 0$ is the time control delay; and $v(t)$ is the initial control function. The controllability of fractional systems with delay in control is investigated by Wei [16]. In [17], the constrained controllability of linear fractional control systems with multiple delays in control is discussed. The controllability of linear fractional systems with multiple variable delays and distributed delay in admissible control under Caputo derivative are analyzed by Klamka [18].

The main novelties of this paper are that we present the necessary and sufficient conditions for the controllability of linear fractional systems with control delay under the Atangana-Baleanu fractional derivative using the Caputo approach. We show that these conditions are equivalent to the non-singularity of the controllability Gramian matrix and the full-rank property of a suitably defined matrix.

The following are the major contributions of the paper:

(i) The solution expression of continuous-time linear fractional systems with control delay, involving the Atangana-Baleanu derivative using the Caputo approach, is obtained

(ii) The fractional delay controllability Gramian matrix is defined to deal with the controllability problem. We also show that its non-singularity is equivalent to the controllability of described system

(iii) The control $u(t)$, which steers the considered system from any admissible initial state and initial control to any state, is introduced

(iv) Another controllability criterion based on the rank of the matrix K is provided in Theorem 13

The following notations will be used throughout this paper. Let $A \in \mathbb{R}^{n\times n}$. The symbol A^* is used for the transpose of matrix A. Assuming that $I-(1-\alpha)A$ is non-singular, we set $\widehat{A}=(I-(1-\alpha)A)^{-1}$. The column space and the null space of matrix A are denoted as Im (A) and Ker (A), respectively.

The structure of this study is as follows: Section 2 is dedicated to a brief overview of fractional calculus. In Section 3, some controllability criteria of a linear fractional system with control delay are examined. Finally, a brief conclusion is provided in Section 4.

2. Preliminaries

Let $n-1<\alpha\le n$, $\alpha\in\mathbb{R}^+$, $n\in\mathbb{N}$ and g be a suitable function. In what follows, we recall some basic concepts of fractional calculus. For more details, see [9, 14, 19–23].

Definition 1. The Riemann-Liouville fractional integral and derivative of order α of g are given by

$$\begin{aligned} {}^{RL}_{\alpha}I^{\alpha}_{t}g(t)&=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\theta)^{\alpha-1}g(\theta)d\theta,\\ {}^{RL}_{\alpha}D^{\alpha}_{t}g(t)&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t \frac{g(\theta)}{(t-\theta)^{\alpha+1-n}}d\theta. \end{aligned} \tag{2}$$

For $0<\alpha<1$, the Riemann-Liouville fractional derivative of order α of g is

$${}^{RL}_{\alpha}D^{\alpha}_{t}g(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{g(\theta)}{(t-\theta)^{\alpha}}d\theta. \tag{3}$$

Definition 2. The Caputo fractional derivative of order α of g is defined as

$${}^{C}_{\alpha}D^{\alpha}_{t}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{g^{(n)}(\theta)}{(t-\theta)^{\alpha+1-n}}d\theta. \tag{4}$$

When $0<\alpha<1$, the Caputo fractional derivative of order α of g can be written as

$${}^{C}_{\alpha}D^{\alpha}_{t}g(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{g'(\theta)}{(t-\theta)^{\alpha}}d\theta. \tag{5}$$

Lemma 3. *The following equality holds true for Convolution operator in Riemann-Liouville sense:*

$${}^{RL}_{0}D^{\alpha}_{t}\int_0^t \phi(t-\theta)\varphi(\theta)d\theta=\int_0^t \varphi(t-\theta){}^{RL}_{0}D^{\alpha}_{\theta}\phi(\theta)d\theta+\varphi(t)\lim_{\theta\longrightarrow0^+}{}^{RL}_{0}I^{1-\alpha}_{\theta}\phi(\theta). \tag{6}$$

Moreover, if $g(0)=0$, then ${}^{RL}_{0}D^{\alpha}_{t}(g(t))={}^{C}_{0}D^{\alpha}_{t}(g(t))$.

Lemma 4. *Let the function $g(t)$ has the Laplace transform, then the Laplace transform of the Caputo fractional derivative is*

$$L\left({}^{C}_{0}D^{\alpha}_{t}g(t)\right)(s)=s^{\alpha}L(g(t))(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}g^{(k)}(0),\ n-1<\alpha\le n. \tag{7}$$

For $0<\alpha\le1$, the preceding equation becomes

$$L\left({}^{C}_{0}D^{\alpha}_{t}g(t)\right)(s)=s^{\alpha}L(g(t))(s)-s^{\alpha-1}g(0). \tag{8}$$

Definition 5. The well-known Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},\ (\alpha>0,\beta>0). \tag{9}$$

Lemma 6. *Let* Re $(s)>|a|^{1/\alpha}$, *then*

$$L\left(t^{\beta-1}E_{\alpha,\beta}(\pm a t^{\alpha})\right)(s)=\frac{s^{\alpha-\beta}}{s^{\alpha}\mp a}. \tag{10}$$

Lemma 7. *Let $\beta>0$ and $\nu>0$, then*

$$\frac{1}{\Gamma(\nu)}\int_0^t (t-\theta)^{\nu-1}E_{\alpha,\beta}(\lambda\theta^{\alpha})\theta^{\beta-1}d\theta=t^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda t^{\alpha}). \tag{11}$$

For the matrix Mittag-Leffler function, similar equations are provided.

Definition 8. The Atangana-Baleanu fractional derivative using the Caputo approach of $g\in H^1(a,b)$, $b>a$, and $\alpha\in[0,1]$ is given by

$${}^{ABC}_{a}D^{\alpha}_{t}g(t)=\frac{B(\alpha)}{1-\alpha}\int_a^t g'(\theta)E_{\alpha}\left(-\alpha\frac{(t-\theta)^{\alpha}}{1-\alpha}\right)d\theta, \tag{12}$$

where the normalization function $B(\alpha)$ is satisfied $B(0)=B(1)=1$. Throughout this paper, $B(\alpha)=1$.

Lemma 9. *Let the function $g(t)$ has the Laplace transform, then the Laplace transform of the Atangana-Baleanu fractional derivative in Caputo sense is*

$$L\left({}_0^{ABC}D_t^{\alpha} g(t)\right)(s) = \frac{B(\alpha)}{1-\alpha} \frac{s^{\alpha} L(g(t))(s) - s^{\alpha-1} g(0)}{s^{\alpha} + (\alpha/1-\alpha)}. \quad (13)$$

For brevity's sake, ${}^{C}D^{\alpha}$ and ${}^{ABC}D^{\alpha}$ are used instead of ${}_0^{C}D_t^{\alpha}$ and ${}_0^{ABC}D_t^{\alpha}$, respectively.

3. Controllability Problem

In this section, we firstly present the solution expression of system (1). Then, defining the controllability Gramian matrix, the necessary and sufficient conditions for the controllability of this system are established.

Theorem 10. *Let the initial conditions be $x(0) = x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and there exist ${}^{C}D^{\alpha}u(\theta)$ and ${}^{C}D^{\alpha}v(\theta)$. The solution of linear fractional system with control delay (1) is*

$$\begin{aligned} x(t) &= \Psi_{\alpha,1}(\alpha\hat{A}A, t)x(0) + \int_0^{t-h} \left(\Psi_{\alpha,\alpha}(\alpha\hat{A}A, t-\theta)\hat{A}B + \Psi_{\alpha,\alpha}(\alpha\hat{A}A, t-\theta-h)\hat{A}C\right) \\ &\quad \times \left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta + \int_{t-h}^{t} \Psi_{\alpha,\alpha}(\alpha\hat{A}A, t-\theta)\hat{A}B\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta \\ &\quad + \int_{-h}^{0} \Psi_{\alpha,\alpha}(\alpha\hat{A}A, t-\theta-h)\hat{A}C\left(\alpha v(\theta) + (1-\alpha){}^{C}D^{\alpha}v(\theta)\right)d\theta, \end{aligned} \quad (14)$$

where

$$\Psi_{\alpha,\beta}(A,t) = \sum_{j=0}^{\infty} \frac{A^j t^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)} = t^{\beta-1} \sum_{j=0}^{\infty} \frac{A^j t^{\alpha j}}{\Gamma(\alpha j+\beta)} = t^{\beta-1} E_{\alpha,\beta}(At^{\alpha}). \quad (15)$$

Proof. Taking Laplace transform of system (1), we have

$$\frac{s^{\alpha}L(x(t))(s) - s^{\alpha-1}x(0)}{s^{\alpha}(1-\alpha)+\alpha} = AL(x(t))(s) + BL(u(t))(s) + CL(u(t-h))(s), \quad (16)$$

which can be rewritten as

$$\begin{aligned} &(I-(1-\alpha)A)s^{\alpha}L(x(t))(s) - \alpha AL(x(t))(s) \\ &\quad = s^{\alpha-1}x_0 + (1-\alpha)Bs^{\alpha}L(u(t))(s) \\ &\quad + \alpha BL(u(t))(s) + (1-\alpha)Cs^{\alpha}L(u(t-h))(s) \\ &\quad + \alpha CL(u(t-h))(s). \end{aligned} \quad (17)$$

Then, pre-multiplying both sides of (17) by the matrix $\hat{A}$ gives

$$\begin{aligned} \left(s^{\alpha}I - \alpha\hat{A}A\right)L(x(t))(s) &= s^{\alpha-1}\hat{A}x_0 + (1-\alpha)\hat{A}Bs^{\alpha}L(u(t))(s) \\ &\quad + \alpha\hat{A}BL(u(t))(s) + (1-\alpha)\hat{A}Cs^{\alpha}L(u(t-h))(s) \\ &\quad + \alpha\hat{A}CL(u(t-h))(s). \end{aligned} \quad (18)$$

The above equation can be written as

$$\begin{aligned} L(x(t))(s) &= \left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}s^{\alpha-1}\hat{A}x_0 + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Bs^{\alpha}L(u(t))(s) \\ &\quad + \alpha\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}BL(u(t))(s) \\ &\quad + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Cs^{\alpha}L(u(t-h))(s) \\ &\quad + \alpha\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}CL(u(t-h))(s). \end{aligned} \quad (19)$$

By adding and subtracting $(1-\alpha)(s^{\alpha}I - \alpha\hat{A}A)^{-1}\hat{A}Bs^{\alpha-1}u(0)$ and $(1-\alpha)(s^{\alpha}I - \alpha\hat{A}A)^{-1}\hat{A}Cs^{\alpha-1}u(-h)$, we have

$$\begin{aligned} L(x(t))(s) &= \left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}s^{\alpha-1}\hat{A}x_0 \\ &\quad + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}B\left(s^{\alpha}L(u(t))(s) - s^{\alpha-1}u(0)\right) \\ &\quad + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Bs^{\alpha-1}u(0) \\ &\quad + \alpha\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}BL(u(t))(s) \\ &\quad + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}C\left(s^{\alpha}L(u(t-h))(s) - s^{\alpha-1}u(-h)\right) \\ &\quad + (1-\alpha)\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Cs^{\alpha-1}u(-h) \\ &\quad + \alpha\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}CL(u(t-h))(s). \end{aligned} \quad (20)$$

From Lemma 4 and taking the inverse Laplace transform, we obtain

$$\begin{aligned} x(t) &= L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}s^{\alpha-1}\right)\hat{A}x_0 \\ &\quad + (1-\alpha)L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}BL\left({}^{C}D^{\alpha}u(t)\right)\right) \\ &\quad + (1-\alpha)L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Bs^{\alpha-1}\right)u(0) \\ &\quad + \alpha L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}BL(u(t))(s)\right) \\ &\quad + (1-\alpha)L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}CL\left({}^{C}D^{\alpha}u(t-h)\right)\right) \\ &\quad + (1-\alpha)L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}Cs^{\alpha-1}\right)u(-h) \\ &\quad + \alpha L^{-1}\left(\left(s^{\alpha}I - \alpha\hat{A}A\right)^{-1}\hat{A}CL(u(t-h))(s)\right). \end{aligned} \quad (21)$$

Finally, applying the Convolution theorem, Lemma 6 and equation (21), we get

$$\begin{aligned}
x(t) &= E_{\alpha,1}\left(\alpha\widehat{A}At^{\alpha}\right)\widehat{A}(x_0 + (1-\alpha)Bu(0) + (1-\alpha)Cu(-h)) \\
&\quad + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}\left(\alpha\widehat{A}A(t-\theta)^{\alpha}\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta \\
&\quad + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}\left(\alpha\widehat{A}A(t-\theta)^{\alpha}\right)\widehat{A}C\left(\alpha u(\theta-h) + (1-\alpha){}^{C}D^{\alpha}u(\theta-h)\right)d\theta \\
&= E_{\alpha,1}\left(\alpha\widehat{A}At^{\alpha}\right)\widehat{A}(x_0 + (1-\alpha)Bu(0) + (1-\alpha)Cu(-h)) \\
&\quad + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}\left(\alpha\widehat{A}A(t-\theta)^{\alpha}\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta \\
&\quad + \int_{-h}^{t-h} (t-\theta-h)^{\alpha-1}E_{\alpha,\alpha}\left(\alpha\widehat{A}A(t-\theta-h)^{\alpha}\right)\widehat{A}C\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta.
\end{aligned} \tag{22}$$

Since $Ax_0 + Bu(0) + Cu(-h) = 0$, we have $\widehat{A}(x_0 + (1-\alpha)B u(0) + (1-\alpha)Cu(-h)) = x_0$, and the equation (22) result in (14). □

Definition 11. The system (1) is controllable on $[0, t_1]$, if for every admissible initial state x_0, initial control $u(t)$, and x_1, there exists a control $u(t) \in \mathbb{R}^m$ defined on $[0, t_1]$ such that the corresponding solution of (1) satisfies $x(t_1) = x_1$.

Corresponding to system (1), the controllability Gramian matrix is described as

$$\begin{aligned}
W_{t_1} &= \int_0^{t_1-h} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right) \\
&\quad \times \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right)^{*}d\theta \\
&\quad + \int_{t_1-h}^{t_1} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\right)\left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\right)^{*}d\theta.
\end{aligned} \tag{23}$$

Now, we present the main results of this paper in two following theorems.

Theorem 12. *The linear fractional system with control delay (1) is controllable on $[0, t_1]$, if and only if the controllability Gramian matrix W_{t_1} is non-singular.*

Proof. First, to prove the sufficiency, suppose that W_{t_1} is non-singular.

Let

$$u(t) = \begin{cases}
\dfrac{1}{1-\alpha}\displaystyle\int_0^t \Psi_{\alpha,\alpha}(-\gamma, t-\theta)\left(B^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-\theta\right) + C^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-\theta-h\right)\right) \times W_{t_1}^{-1}\left(x_1 - \Psi_{\alpha,1}\left(\alpha\widehat{A}A, t_1\right)x_0\right)d\theta, & 0 \le t \le t_1-h, \\
\dfrac{B^{*}\widehat{A}^{*}}{1-\alpha}\displaystyle\int_0^t \Psi_{\alpha,\alpha}(-\gamma, t-\theta)\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-\theta\right) \times W_{t_1}^{-1}\left(x_1 - \Psi_{\alpha,1}\left(\alpha\widehat{A}A, t_1\right)x_0\right)d\theta, & t_1-h < t \le t_1, \\
0, & -h \le t \le 0,
\end{cases} \tag{24}$$

where $\gamma = \alpha/(1-\alpha)$. It is demonstrated that $u(t)$ has the Caputo derivative and satisfies in

$$\alpha u(t) + (1-\alpha){}^{C}D^{\alpha}u(t) = \begin{cases}
\left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-t\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-t-h\right)\widehat{A}C\right)^{*} \times W_{t_1}^{-1}\left(x_1 - \Psi_{\alpha,1}\left(\alpha\widehat{A}A, t_1\right)x_0\right), & 0 \le t \le t_1-h, \\
B^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-t\right)W_{t_1}^{-1}\left(x_1 - \Psi_{\alpha,1}\left(\alpha\widehat{A}A, t_1\right)x_0\right), & t_1-h < t \le t_1, \\
0, & -h \le t \le 0.
\end{cases} \tag{25}$$

To prove this, first $0 \le t \le t_1 - h$ is considered. Since $u(0) = 0$, then from Lemma 3, ${}^{RL}D^{\alpha}(u(t)) = {}^{C}D^{\alpha}(u(t))$. Taking Caputo fractional derivative of $u(t)$, we have

$$\begin{aligned}
{}^{C}D^{\alpha}(u(t)) &= {}^{C}D^{\alpha}\left(\frac{1}{1-\alpha}\int_0^t B^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}(-\gamma, t-\theta)\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-\theta\right)y d\theta\right) \\
&\quad + {}^{C}D^{\alpha}\left(\frac{1}{1-\alpha}\int_0^t C^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}(-\gamma, t-\theta)\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-\theta-h\right)y d\theta\right) \\
&= \frac{B^{*}\widehat{A}^{*}}{1-\alpha}\int_0^t \Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-(t-\theta)\right){}^{RL}D^{\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta)y d\theta \\
&\quad + \frac{B^{*}\widehat{A}^{*}}{1-\alpha}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-t\right)\lim_{\theta\longrightarrow 0^{+}}{}^{RL}I^{1-\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta)y \\
&\quad + \frac{C^{*}\widehat{A}^{*}}{1-\alpha}\int_0^t \Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-(t-\theta)-h\right){}^{RL}D^{\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta)y d\theta \\
&\quad + \frac{C^{*}\widehat{A}^{*}}{1-\alpha}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_1-t-h\right)\lim_{\theta\longrightarrow 0^{+}}{}^{RL}I^{1-\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta)y,
\end{aligned} \tag{26}$$

where $y = W_{t_1}^{-1}(x_1 - \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0)$. Following Lemma 7, we get

$${}^{RL}I^{1-\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta) = E_{\alpha,1}(-\gamma\theta^{\alpha}), \tag{27}$$

$${}^{RL}D^{\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta) = \frac{d}{d\theta}\left(E_{\alpha,1}(-\gamma\theta^{\alpha})\right). \tag{28}$$

Equation (28) is easily equivalent to

$$\begin{aligned}
{}^{RL}D^{\alpha}\Psi_{\alpha,\alpha}(-\gamma, \theta) &= \frac{d}{d\theta}\left(E_{\alpha,1}(-\gamma\theta^{\alpha})\right) = \frac{d}{d\theta}\sum_{j=0}^{\infty}\frac{(-\gamma\theta^{\alpha})^{j}}{\Gamma(j\alpha+1)} \\
&= -\gamma\theta^{\alpha-1}\sum_{j=0}^{\infty}\frac{(-\gamma\theta^{\alpha})^{j}}{\Gamma(j\alpha+\alpha)} = -\gamma\theta^{\alpha-1}E_{\alpha,\alpha}(-\gamma\theta^{\alpha}).
\end{aligned} \tag{29}$$

The equations (26), (27), and (29) result in

$$
\begin{aligned}
{}^{C}D^{\alpha}(u(t)) &= -\frac{\gamma B^{*}\widehat{A}^{*}}{1-\alpha}\int_{0}^{t}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-(t-\theta)\right)\Psi_{\alpha,\alpha}(-\gamma,\theta)y d\theta + \frac{B^{*}\widehat{A}^{*}}{1-\alpha}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-t\right)y \\
&\quad - \frac{\gamma C^{*}\widehat{A}^{*}}{1-\alpha}\int_{0}^{t}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-(t-\theta)-h\right)\Psi_{\alpha,\alpha}(-\gamma,\theta)y d\theta + \frac{C^{*}\widehat{A}^{*}}{1-\alpha}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-t-h\right)y \\
&= -\frac{\gamma}{1-\alpha}\int_{0}^{t}\Psi_{\alpha,\alpha}(-\gamma,\theta)\left(B^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-(t-\theta)\right) + C^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-(t-\theta)-h\right)\right)y d\theta \\
&\quad + \frac{1}{1-\alpha}\left(B^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-t\right) + C^{*}\widehat{A}^{*}\Psi_{\alpha,\alpha}\left(\alpha A^{*}\widehat{A}^{*}, t_{1}-t-h\right)\right)y.
\end{aligned} \tag{30}
$$

Now, from (24) and (30), we get the result for $0 \le t \le t_1 - h$. Similarly, the desired results for $t_1 - h < t \le t_1$ and $-h \le t \le 0$ are achieved.

From (14) and (25), we obtain

$$
\begin{aligned}
x(t_1) &= \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0 + \int_{0}^{t_1-h}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C) \\
&\quad \times\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta + \int_{t_1-h}^{t_1}\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta, \\
&= \int_{0}^{t_1-h}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C) \times (\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C)^{*} \\
&\quad \times W_{t_1}^{-1}(x_1 - \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0)d\theta + \int_{t_1-h}^{t_1}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B)(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B)^{*} \\
&\quad \times W_{t_1}^{-1}(x_1 - \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0)d\theta, = \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0 + W_{t_1}W_{t_1}^{-1}(x_1 - \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)x_0) = x_1,
\end{aligned} \tag{31}
$$

which demonstrates that system (1) is controllable.

Now, to prove the necessity, suppose that system (1) is controllable. If W_{t_1} is singular, then a vector $y \ne 0$ exists such that

$$
y^{*}W_{t_1}y = 0, \tag{32}
$$

that is

$$
\begin{aligned}
0 &= y^{*}\int_{0}^{t_1-h}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C) \\
&\quad \times (\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C)^{*}d\theta y \\
&\quad + y^{*}\int_{t_1-h}^{t_1}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B)(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B)^{*}d\theta y.
\end{aligned} \tag{33}
$$

This equation is clearly equivalent to

$$
\begin{aligned}
0 &= \int_{0}^{t_1-h}\left\|y^{*}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C)\right\|^{2}d\theta \\
&\quad + \int_{t_1-h}^{t_1}\left\|y^{*}\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B\right\|^{2}d\theta.
\end{aligned} \tag{34}
$$

It follows from the above equation

$$
\begin{cases}
0 = y^{*}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C), & 0 \le \theta \le t_1 - h, \\
0 = y^{*}\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B, & t_1 - h < \theta \le t_1.
\end{cases} \tag{35}
$$

Let $x_0 = (\Psi_{\alpha,1}(\alpha\widehat{A}A, t_1))^{-1}y$. According to the assumption of controllability, a control $u(t)$ exists on $[0, t_1]$ such that $x(t_1) = 0$. Consequently,

$$
\begin{aligned}
0 &= \Psi_{\alpha,1}(\alpha\widehat{A}A, t_1)(\Psi_{\alpha,1}(\alpha\widehat{A}A, t_1))^{-1}y \\
&\quad + \int_{0}^{t_1-h}(\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B + \Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta-h)\widehat{A}C) \\
&\quad \times\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta \\
&\quad + \int_{t_1-h}^{t_1}\Psi_{\alpha,\alpha}(\alpha\widehat{A}A, t_1-\theta)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha){}^{C}D^{\alpha}u(\theta)\right)d\theta.
\end{aligned} \tag{36}
$$

Then,

$$0 = y^* y + \int_0^{t_1-h} y^* \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right) \times \left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta + \int_{t_1-h}^{t_1} y^* \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta. \tag{37}$$

It follows from the equations (35) and (37) that $y^* y = 0$. So, a contradiction is obtained. Therefore, W_{t_1} is non-singular. □

Theorem 13. *The linear fractional system with control delay (1) is controllable on $[0, t_1]$, if and only if the matrix*

$$K = \left[B\, AB \cdots A^{n-1}B\, C\, AC \cdots A^{n-1}C\right], \tag{38}$$

is full-rank.

Proof. Suppose

$$\widehat{K} := \left[\widehat{A}B\left(\widehat{A}A\right)\widehat{A}B \cdots \left(\widehat{A}A\right)^{n-1}\widehat{A}B\,\widehat{A}C\left(\widehat{A}A\right)\widehat{A}C \cdots \left(\widehat{A}A\right)^{n-1}\widehat{A}C\right]. \tag{39}$$

We prove that $\mathrm{Im}\left(\widehat{K}\right) = \mathrm{Im}\left(W_{t_1}\right)$. To accomplish this, consider the set of reachable states of system (1) as

$$R_t = \{\eta \in R^n : \textit{there exists control } u \textit{ such that } x(t) = \eta\}, \tag{40}$$

from initial conditions $x_0 = 0$ and $u(t) = v(t) = 0, -h \le t \le 0$. We prove that for every $t > 0$, $R_t = \mathrm{Im}\left(\widehat{K}\right) = \mathrm{Im}\left(W_{t_1}\right)$. The proof is in three steps.

The first step is to demonstrate that $R_t \subset \mathrm{Im}\left(\widehat{K}\right)$. Let $\eta \in R_t$, then a control $u(t)$ exists such that

$$\eta = \int_0^{t-h} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t-\theta-h\right)\widehat{A}C\right) \times \left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta + \int_{t-h}^{t} \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t-\theta\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta, \tag{41}$$

which is equal to

$$\begin{aligned}\eta &= \int_0^{t} \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t-\theta\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta \\ &\quad + \int_0^{t-h} \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t-\theta-h\right)\widehat{A}C\left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta \\ &= \lim_{L\longrightarrow\infty} \sum_{j=0}^{L} \left(\widehat{A}A\right)^j \left(\widehat{A}B\beta_j + \widehat{A}C\gamma_j\right),\end{aligned} \tag{42}$$

where

$$\beta_j = \int_0^t \frac{\alpha^j (t-\theta)^{(j+1)\alpha-1}}{\Gamma(\alpha j + \alpha)} \left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta,$$
$$\gamma_j = \int_0^{t-h} \frac{\alpha^j (t-\theta-h)^{(j+1)\alpha-1}}{\Gamma(\alpha j + \alpha)} \left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta, \tag{43}$$

for $j = 0, 1, \cdots, L$.

Equation (42) can be written as the following product:

$$\eta = \lim_{L\longrightarrow\infty} \left[\widehat{A}B\left(\widehat{A}A\right)\widehat{A}B \cdots \left(\widehat{A}A\right)^{L}\widehat{A}B\,\widehat{A}C\left(\widehat{A}A\right)\widehat{A}C \cdots \left(\widehat{A}A\right)^{L}\widehat{A}C\right] \begin{bmatrix}\beta_0 \\ \beta_1 \\ \vdots \\ \beta_L \\ \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_L\end{bmatrix}. \tag{44}$$

By Cayley-Hamilton theorem, $\eta \in \mathrm{Im}\left(\widehat{K}\right)$. Therefore, $R_t \subset \mathrm{Im}\left(\widehat{K}\right)$.

The second step is to demonstrate that $\mathrm{Im}\left(\widehat{K}\right) \subset \mathrm{Im}\, W_{t_1}$. We prove equivalently that $\left(\mathrm{Im}\left(W_{t_1}\right)\right)^\perp \subset \left(\mathrm{Im}\left(\widehat{K}\right)\right)^\perp$, where $\left(\mathrm{Im}\left(W_{t_1}\right)\right)^\perp$ and $\left(\mathrm{Im}\left(\widehat{K}\right)\right)^\perp$ are the orthogonal complements of $\mathrm{Im}\left(W_{t_1}\right)$ and $\mathrm{Im}\left(\widehat{K}\right)$, respectively. We demonstrate that if $\eta \in \left(\mathrm{Im}\left(W_{t_1}\right)\right)^\perp = \mathrm{Ker}\left(W_{t_1}^*\right) = \mathrm{Ker}\left(W_{t_1}\right)$, then $\eta \in \left(\mathrm{Im}\left(\widehat{K}\right)\right)^\perp$. Let $\eta \in \mathrm{Ker}\left(W_{t_1}\right)$, then

$$\eta^* W_{t_1} \eta = 0, \tag{45}$$

that is

$$0 = \int_0^{t_1-h} \left\| \eta^* \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right) \right\|^2 d\theta + \int_{t_1-h}^{t_1} \left\| \eta^* \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B \right\|^2 d\theta. \tag{46}$$

The above equation is clearly equivalent to

$$\begin{cases} 0=\eta^*\left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-\theta\right)\widehat{A}B+\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-\theta-h\right)\widehat{A}C\right), & 0\le\theta\le t_1-h,\\ 0=\eta^*\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-\theta\right)\widehat{A}B, & t_1-h<\theta\le t_1. \end{cases} \tag{47}$$

For the second equation of (47) with $y=(t_1-\theta)^\alpha$, we have

$$\eta^*\sum_{j=0}^{\infty}\frac{\alpha^j(\widehat{A}A)^j y^j}{\Gamma(\alpha j+\alpha)}\widehat{A}B=0. \tag{48}$$

Taking derivative j times $(j=0,1,2,\cdots,n-1)$, with respect to y and taking the limit $y\longrightarrow 0^+$, it follows

$$\eta^*(\widehat{A}A)^j\widehat{A}B=0, j=0,1,\cdots,n-1. \tag{49}$$

Then when $0\le\theta\le t_1-h$, from the Cayley-Hamilton theorem, we have

$$\begin{aligned}\eta^*\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-\theta\right)\widehat{A}B&=\sum_{j=0}^{\infty}\frac{\alpha^j(t_1-\theta)^{(j+1)\alpha-1}}{\Gamma(\alpha j+\alpha)}\eta^*(\widehat{A}A)^j\widehat{A}B\\&=\sum_{j=0}^{n-1}\frac{\alpha^j(t_1-\theta)^{(j+1)\alpha-1}}{\Gamma(\alpha j+\alpha)}\eta^*(\widehat{A}A)^j\widehat{A}B=0.\end{aligned} \tag{50}$$

The preceding equation and the first equation of (47) imply that

$$\begin{aligned}0&=\eta^*\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-\theta-h\right)\widehat{A}C\\&=\eta^*\sum_{j=0}^{\infty}\frac{\alpha^j(\widehat{A}A)^j(t_1-\theta-h)^{(j+1)\alpha-1}}{\Gamma(\alpha j+\alpha)}\widehat{A}C\Rightarrow\eta^*\sum_{j=0}^{\infty}\frac{\alpha^j(\widehat{A}A)^j(t_1-\theta-h)^{j\alpha}}{\Gamma(\alpha j+\alpha)}\widehat{A}C=0.\end{aligned} \tag{51}$$

Setting $y=(t_1-\theta-h)^\alpha$, we can write

$$\eta^*\sum_{j=0}^{\infty}\frac{\alpha^j(\widehat{A}A)^j y^j}{\Gamma(\alpha j+\alpha)}\widehat{A}C=0. \tag{52}$$

As mentioned procedure, differentiating j times with respect to y and taking the limit $y\longrightarrow 0^+$ result in

$$\eta^*(\widehat{A}A)^j\widehat{A}C=0, j=0,1,\cdots,n-1. \tag{53}$$

From (49) and (53), we obtain

$$\begin{aligned}&\eta^*\left[\widehat{A}B\,(\widehat{A}A)\widehat{A}B\cdots(\widehat{A}A)^{n-1}\widehat{A}B\,\widehat{A}C\,(\widehat{A}A)\widehat{A}C\cdots(\widehat{A}A)^{n-1}\widehat{A}C\right]\\&=\eta^*\widehat{K}=0\Rightarrow\widehat{K}^*\eta=0.\end{aligned} \tag{54}$$

It follows that $\eta\in\operatorname{Ker}(\widehat{K}^*)=(\operatorname{Im}(\widehat{K}))^\perp$. Therefore, Im $(\widehat{K})\subset\operatorname{Im}(W_{t_1})$.

The third step is to demonstrate that $\operatorname{Im}(W_{t_1})\subset R_t$. Let $x_1\in\operatorname{Im}(W_{t_1})$, then a vector y exists such that $x_1=W_{t_1}y$.

Let

$$u(t)=\begin{cases}\frac{1}{1-\alpha}\int_0^t\Psi_{\alpha,\alpha}(-\beta,t-\theta)\left(B^*\widehat{A}^*\Psi_{\alpha,\alpha}\left(\alpha A^*\widehat{A}^*,t_1-\theta\right)+C^*\widehat{A}^*\Psi_{\alpha,\alpha}\left(\alpha A^*\widehat{A}^*,t_1-\theta-h\right)\right)yd\theta, & 0\le t\le t_1-h,\\ \frac{B^*\widehat{A}^*}{1-\alpha}\int_0^t\Psi_{\alpha,\alpha}(-\beta,t-\theta)\Psi_{\alpha,\alpha}\left(\alpha A^*\widehat{A}^*,t_1-\theta\right)yd\theta, & t_1-h<t\le t_1,\\ 0, & -h\le t\le 0.\end{cases} \tag{55}$$

It is possible to show that $D^\alpha u(t)$ exists and

$$\alpha u(t)+(1-\alpha)^C D^\alpha u(t)=\begin{cases}\left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-t\right)\widehat{A}B+\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A,t_1-t-h\right)\widehat{A}C\right)^*y, & 0\le t\le t_1-h,\\ B^*\widehat{A}^*\Psi_{\alpha,\alpha}\left(\alpha A^*\widehat{A}^*,t_1-t\right)y, & t_1-h<t\le t_1,\\ 0, & -h\le t\le 0.\end{cases} \tag{56}$$

Then, from Theorem 10 with $x(0) = 0$, we obtain

$$\begin{aligned} x(t_1) &= \int_0^{t_1-h} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right) \\ &\quad \times \left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta \\ &\quad + \int_{t_1-h}^{t_1} \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\left(\alpha u(\theta) + (1-\alpha)^C D^\alpha u(\theta)\right) d\theta, \\ &= \int_0^{t_1-h} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right) \\ &\quad \times \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B + \Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta-h\right)\widehat{A}C\right)^* y d\theta \\ &\quad + \int_{t_1-h}^{t_1} \left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\right)\left(\Psi_{\alpha,\alpha}\left(\alpha\widehat{A}A, t_1-\theta\right)\widehat{A}B\right)^* y d\theta, \\ &= W_{t_1} y = x_1. \end{aligned} \tag{57}$$

It follows that $x_1 \in R_t$. Therefore, Im $(W_{t_1}) \subset R_t$.

Taking into account the three preceding steps, Im $(\widehat{K})$ = Im (W_{t_1}) is concluded. Since, the matrix $\widehat{A}$ is commutative with A, we can write

$$\widehat{K} = \begin{bmatrix} K_1 & 0 \\ 0 & K_1 \end{bmatrix} K, \tag{58}$$

where

$$K_1 = \begin{bmatrix} \widehat{A} & 0 & \cdots & 0 \\ 0 & \widehat{A}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{A}^n \end{bmatrix}. \tag{59}$$

So, Im (K) = Im $(\widehat{K})$ = Im (W_{t_1}). Therefore, the matrix K is full-rank if and only if the Gramian matrix W_{t_1} is nonsingular. Now, from Theorem 12, the desired result is achieved. □

4. Conclusion

In this paper, we investigated the controllability problem of linear fractional system with delay in control, involving the Atangana-Baleanu derivative in Caputo sense. The solution expression of such a system has been stated. Then, we introduced the fractional delay controllability Gramian matrix. According to the controllability Gramian matrix, the necessary and sufficient conditions for the controllability of a linear fractional system with control delay have been presented. Moreover, the desired control has been provided. We also established another controllability criterion based on the rank of the matrix presented in Theorem 13.

As future recommendations, the controllability of semilinear or generally nonlinear fractional systems with different types of delays not only in admissible controls but also in the state variables under the Atangana-Baleanu derivative can be considered. Another important issue to handle is the controllability of fractional systems with different orders of derivatives under the Atangana-Baleanu derivative.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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