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Research Article

Chebyshev Wavelet Analysis

Emanuel Guariglia and Rodrigo Capobianco Guido

Institute of Biosciences, Letters and Exact Sciences, São Paulo State University (UNESP), Rua Cristóvão Colombo 2265, 15054-000 São José do Rio Preto, SP, Brazil

Correspondence should be addressed to Emanuel Guariglia; emanuel.guariglia@gmail.com

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This paper deals with Chebyshev wavelets. We analyze their properties computing their Fourier transform. Moreover, we discuss the differential properties of Chebyshev wavelets due to the connection coefficients. Uniform convergence of Chebyshev wavelets and their approximation error allow us to provide rigorous proofs. In particular, we expand the mother wavelet in Taylor series with an application both in fractional calculus and fractal geometry. Finally, we give two examples concerning the main properties proved.

1. Introduction

In the last four decades, wavelet analysis rose to the role of mathematical theory due to the introduction of multiresolution analysis [1]. In the current literature, 1909 is often recognised as the birth of wavelet analysis, when Haar introduced a complete orthonormal system for the space $L^2([0,1])$. Nowadays, wavelet analysis is a mathematical tool widely applied in different fields. Image compression, electromagnetism, and PDE image are just three examples where wavelet methods currently play a meaningful role (see, e.g., [2–5]). In particular, Mallat produced a fast wavelet decomposition and reconstruction algorithm [6]. Over the course of time, the Mallat algorithm became the base for many wavelet applications in pure and applied science. Quite recently, wavelet analysis was also used for several techniques in image fusion, where each algorithm leads to different image decompositions. Fusing two types of information (temporal and spectral), the discrete wavelet transform (DWT) enabled the development of many DWT-based techniques. An application of the wavelet analysis in image fusion is the discrete shapelet transform (DST), which estimates the degree of similarity between the signal under analysis and a prespecified shape. This discrete transform consists of a fractal-based criterion to redefine the original Daubechies' DWTs, leading to a time-frequency-shape joint analysis. Replacing the fractal-based criterion with a correlation-based formulation, the DST can be improved significantly. More specifically, the DST of the second generation simplifies both the study of filter coefficients and the interpretation of the transformed signal [7].

The main advantage of wavelet analysis is their decomposition of mathematical entities (e.g., images and time series) into components at different scales. This property is a consequence of the multiresolution analysis. Approximation in Fourier basis can lead to unpleasant results, as in the case of Gibbs phenomenon. These approximation problems can occur in any other reconstruction, but wavelets. Likewise, fractal geometry allows us to describe irregular sets by the concepts of fractal dimension and lacunarity [8, 9]. As is well known, irregular sets provide a better representation of different natural phenomena than the classical Euclidean models. Thus, fractal-like sets are currently used for many real-world applications (e.g., antenna theory and dynamical systems). Quite recently, considerable attention has been paid to the application of hybrid methods based both on wavelet analysis and fractal geometry in nonlinear modelling. For a fuller and deeper treatment on fractal-wavelet analysis, we refer the reader to the results of Jorgensen [10, 11].

Chebyshev wavelets are generally used for numerical methods in integral equations and PDEs. In particular, Chebyshev wavelets allowed the introduction of these methods due to the operational matrices P and D defined in (12) and (13), respectively. In [12], Hyedari et al. introduced a numerical method based on Chebyshev wavelets for solution of PDEs with boundary conditions of the telegraph type. Biazar and Ebraimi proposed a method based on Chebyshev wavelets for solving nonlinear systems of Volterra integral equations [13]. Similarly, Singh and Saha Ray [14] dealt with the stochastic Itô-Volterra integral equations by Chebyshev wavelets of the second kind. Following the recent trends in nonlinear analysis, Chebyshev wavelets were also used for the numerical solutions of fractional differential equations (see, e.g., [15, 16]). Current literature showed that these methods depends on the different operational matrices in the sense of [17–19]. Moreover, Chebyshev wavelets provided sharp estimates of functions in Hölder spaces of order α [20].

In this paper, we give new results on Chebyshev wavelets. More precisely, we deal with the differentiability of Chebyshev wavelets and the possibility to use their derivatives to reconstruct a function. The differential properties of Chebyshev wavelets, expressed by the connection coefficients (also called refinable integrals), are given by finite series in terms of the Kronecker delta. Moreover, we treat the p-order derivative of Chebyshev wavelets and compute its Fourier transform. In the same spirit, we expand in Taylor series a function by Chebyshev wavelets and connection coefficients. Accordingly, Taylor expansion of the mother wavelet allows us to define the local fractional derivative of Chebyshev wavelets. More precisely, the introduction of local fractional calculus in these wavelet bases enables us to extend the local fractional derivative to nonsmooth continuous functions (e.g., fractal sets or random signals).

The rest of the paper is divided into three sections. In Section 2 we give some remarks on wavelet analysis and, particularly, on Chebyshev wavelets. Section 3 is devoted to differential properties of Chebyshev wavelets by connection coefficients. In Section 4, we deal with the Taylor expansion of Chebyshev wavelets. Finally, Section 5 extends the sought results on Chebyshev wavelets to fractal-like sets by local fractional calculus.

2. Remarks on Wavelet Analysis

This section is to devoted to recall some basic definitions and properties of wavelet analysis, which will be used throughout the paper. From now on, we refer to the set of natural numbers, denoted by $\mathbb{N}$, as the set of strictly positive integer numbers, that is $\mathbb{N} = \{1, 2, 3, \cdots\}$. Thus, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, we will use the notation $x^{\underline{n}}$ to denote the n th falling factorial of x [21].

Definition 1. The nth-order Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x), x \in [-1, 1], n \in \mathbb{N}_0, \quad (1)$$

so that

$$T_n(\cos\theta) = \cos n\theta, \theta \in [0, \pi], n \in \mathbb{N}_0.$$

Thus, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and so on. Definition 1 refers to the trigonometric representation of these polynomials. In literature, Chebyshev polynomials are usually defined as solutions of some Sturm-Liouville differential equations (today called Chebyshev differential equations). In particular, the definition in terms of Sturm-Liouville form leads us to prove the orthogonality of the Chebyshev polynomials with regards to the weight function:

$$w(x) = \frac{1}{\sqrt{1 - x^2}}, x \in [-1, 1],$$

that is,

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1 - x^2}} \mathrm{d}x = \begin{cases} \dfrac{\pi}{2}\, \delta_{m,n}, & m \neq 0, n \neq 0, \\ \pi, & m = n = 0, \end{cases} \quad (2)$$

where $\delta_{m,n}$ is the Kronecker delta. Furthermore, for any $n, m \in \mathbb{N}_0$, Chebyshev polynomials can be written by the following general recurrence relation:

$$T_{n+m}(x) = 2\, T_n(x)\, T_m(x) - T_{|n-m|}(x),$$

which for $m = 1$ gives

$$T_{n+1}(x) = 2\, x\, T_n(x) - T_{n-1}(x), n \geq 1. \quad (3)$$

Chebyshev polynomials of the second, third, and fourth kinds can be defined and handled in much the same way. Moreover, all four Chebyshev polynomials admits a matrix representation (see [13] for more details). For instance, (3) can be written in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2x & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2x & 1 \end{pmatrix} \begin{pmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \end{pmatrix} = \begin{pmatrix} 1 \\ -x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (4)$$

or

$$At = c,$$

where A is the $(n+1) \times (n+1)$ matrix of the coefficients in (3) while t and c are the left-hand side and the right-hand side vectors in (4), respectively.

The properties of Chebyshev polynomials mentioned above lay the foundation for introducing a corresponding wavelet bases, termed Chebyshev wavelets. To this scope and before going ahead, let us recall the definition of wavelet orthonormal basis on $\mathbb{R}$. Wavelets are a family of functions generated by dilation and translation of one single function

ψ (called mother wavelet). In literature, all other functions of this family are usually called daughter wavelets. Thus, a family of continuous wavelets is given by

$$\psi_{a,b}(x) = |a|^{-1/p}\,\psi\left(\frac{x-b}{a}\right), p > 0, a, b \in \mathbb{R}, a \neq 0, \quad (5)$$

where a and b correspond to the scale factor and time shift, respectively. In what follows, therefore, we can assume $p = 2$ in (5) which is the most common value for p. Clearly, (5) for dilation and translation parameters a^{-k} and nba^{-k} gives the following family of discrete wavelets:

$$\psi_n^k(x) = |a|^{k/2}\,\psi\left(|a|^k x - nb\right), \quad a > 1, b > 0, k, n \in \mathbb{Z}, \quad (6)$$

which for $a = 2$ and $b = 1$ yields

$$\psi_n^k(x) = 2^{k/2}\psi\left(2^k x - n\right), \quad k, n \in \mathbb{Z}.$$

The family of functions (6) is a wavelet basis for $L^2(\mathbb{R})$ which becomes orthonormal for $a = 2$ and $b = 1$.

2.1. Chebyshev Wavelets. Multiresolution analysis shows that Chebyshev wavelets can be built as recursive wavelets for piecewise polynomial spaces on $[0, 1]$. For this construction, we refer the reader to [22, 23], in which the problem is widely discussed.

Definition 2. Let $n = 1, 2, \cdots, 2^{k-1}$ and $m = 0, 1, \cdots, M-1$ with $(k, M) \in \mathbb{N}^2$. Chebyshev wavelets are defined as follows:

$$\psi_{n,m}^k(x) = \begin{cases} 2^{k/2}\tilde{T}_m\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (7)$$

where

$$\tilde{T}_m(x) = \frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)T_m(x).$$

Remark 3. In Definition 2, Chebyshev wavelets depends on four parameters, that is, $\psi_{n,m}^k(x) = \psi(k, n, m, x)$. Moreover, $m \in \mathbb{N}_0$; thus,

$$\tilde{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}}T_m(x), & m > 0. \end{cases}$$

Remark 4. In view of (1), Definition 2 implies that Chebyshev wavelets are defined on the real interval $[0, 1)$. Note that the orthogonality of the Chebyshev polynomials on $[-1, 1]$ with regard to w implies the orthogonality of the Chebyshev wavelets on $[0, 1)$ with regard to the weight function $w_k(x) = w(2^{k-1}x - n + 1)$ with n and x as in Definition 2 (see [24] for more details).

A function $f \in L^2(\mathbb{R})$ can be expanded in terms of the wavelet basis $(\psi_n^k)_{k,n\in\mathbb{Z}}$ as follows:

$$f(x) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\beta_n^k\,\psi_n^k(x). \quad (8)$$

The coefficients ψ_n^k, usually termed wavelet coefficients, are given by $\beta_n^k = \langle f(x), \psi_n^k(x)\rangle$ where $\langle .,. \rangle$ denotes the inner product. The series representation in (8) is called a wavelet series. In the case of Chebyshev wavelets, the previous inner product is defined in $L_w^2([0, 1])$, that is,

$$\langle f(x), g(x)\rangle_w \coloneqq \int_0^1 f(x)\bar{g(x)}w(x)dx, f, g \in L_w^2[0, 1].$$

Thus, any function $f \in L_w^2([0, 1])$ can be expanded in terms of Chebyshev wavelets as follows:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta_{n,m}^k\,\psi_{n,m}^k(x), \quad (9)$$

where the wavelet coefficients are given by

$$\beta_{n,m}^k = \left\langle f(x), \psi_{n,m}^k(x)\right\rangle_{w_k}. \quad (10)$$

2.2. Function Approximation and Operational Matrix. Convergence of series (9) on $L_w^2[0, 1]$ implies that f can be approximated as follows:

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta_{n,m}^k\,\psi_{n,m}^k(x) = B^T\mathbf{\Psi}(x), \quad (11)$$

where B and $\mathbf{\Psi}$ are $\tilde{m} = 2^{k-1}M$ column vectors. For simplicity of notation and without loss of generality, we rewrite (11) as

$$f(x) \simeq \sum_{i=1}^{\tilde{m}}\beta_i\,\psi_i(x) = \mathrm{B}^T\mathbf{\Psi}(x),$$

where $\beta_i = \beta_{n,m}^k$ and $\psi_i = \psi_{n,m}^k$. The index i is given by $i = M(n-1) + m + 1$, and thus,

$$\begin{cases} B \coloneqq (\beta_1, \beta_2, \cdots, \beta_{\tilde{m}})^T, \\ \mathbf{\Psi}(x) \coloneqq (\psi_1, \psi_2, \cdots, \psi_{\tilde{m}})^T. \end{cases}$$

Likewise, Chebyshev wavelets allow us to approximate every function of two variables $u = u(x, y)$ defined over $[0, 1) \times [0, 1)$ as follows:

$$u(x,y) \simeq \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{m}} u_{ij}\,\psi_i(x)\psi_i(y) = \mathbf{\Psi}^T(x)U\mathbf{\Psi}(y),$$

where $U = [u_{ij}]$ being

$$u_{ij} = \left\langle \psi_i(x), \left\langle u(x,y), \psi_j(y)\right\rangle_{w_k(y)}\right\rangle_{w_k(x)}, i, j = 1, 2, \cdots, \tilde{m}.$$

We may now integrate the vector $\mathbf{\Psi}(x)$, precisely given by

$$\int_0^x \mathbf{\Psi}(t)\,dt = P\mathbf{\Psi}(x), \quad (12)$$

where P is the $\tilde{m}\times\tilde{m}$ operational matrix of integration. It is worth noticing that, due to introduction of Chebyshev wavelets, the matrix P is sparse (see [17, 18] for more details). Furthermore, P^n allows the n-times integration of $\mathbf{\Psi}(x)$ given by

$$\underbrace{\int_0^x \cdots \int_0^x}_{n\text{–times}} \mathbf{\Psi}(t)dt_1 \cdots \mathrm{d}t_n = P^n\mathbf{\Psi}(x).$$

Similarly, we can differentiate $\mathbf{\Psi}(x)$ as follows:

$$\frac{d\mathbf{\Psi}(x)}{dx} = D\mathbf{\Psi}(x),$$

and so

$$\frac{d^n\mathbf{\Psi}(x)}{dx^n} = D^n\mathbf{\Psi}(x), \quad (13)$$

where D is the $\tilde{m}\times\tilde{m}$ operational matrix of differentiation [12, 25].

Finally, we point out that the product of two Chebyshev wavelets can be approximated as [19] follows:

$$\mathbf{\Psi}(x)\mathbf{\Psi}^T(x)X \simeq \tilde{X}\mathbf{\Psi}(x),$$

where X is a $\tilde{m}$ column vector and $\tilde{X}$ is a $\tilde{m}\times\tilde{m}$ matrix. In literature, $\tilde{X}$ is called the operational matrix of product. In particular, for $X = B$, we get

$$\mathbf{\Psi}(x)\mathbf{\Psi}^T(x)B \simeq \tilde{B}\mathbf{\Psi}(x), \quad (14)$$

where $\tilde{B}$ is a diagonal $\tilde{m}\times\tilde{m}$ matrix. In recent years, approximation (14) has been applied for solving integral equations, PDEs, and boundary valued problems (see, e.g., [17, 18]).

3. Fourier Transform, Differentiability, and Connection Coefficients

In this section, we study the differentiability of Chebyshev wavelets. More precisely, we prove our results in the weighted function space $L^2_w[0,1]$ by the introduction of connection coefficients.

3.1. Differentiability of L^2_w Functions in Chebyshev Wavelet Bases

Theorem 5. *(L^2_w convergence) A function $f \in L^2_w[0,1]$ with bounded second derivative on $[0,1]$, i.e., $|f''(x)| \le A$ for any $x \in [0,1]$, can be expanded as an infinite sum of Chebyshev wavelets and the series converges uniformly to the function f, that is*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \beta^k_{n,m}\psi^k_{n,m}(x), \quad (15)$$

where $\beta^k_{n,m} = \langle f(x), \psi^k_{n,m}(x)\rangle_{w_k}$.

Proof. We only sketch the proof. For a fuller treatment, we refer the reader to [24].

First,

$$\beta^k_{n,m} = \int_0^1 f(x)\psi^k_{n,m}(x)w_k(x)dx = \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} \frac{f(x)\,\tilde{T}_m\left(2^k x - 2n + 1\right)}{\sqrt{1-\left(2^k x - 2n + 1\right)^2}}\,dx. \quad (16)$$

Now, if $m > 1$, the change of variable $2^k x - 2n + 1 = \cos\theta$ in (16) gives

$$\begin{aligned}\beta^k_{n,m} &= \frac{\sqrt{2}}{2^{3k/2}m\sqrt{\pi}}\int_0^\pi f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\sin m\theta\,\sin\theta \mathrm{d}\theta \\ &= \frac{1}{2^{5k/2}m\sqrt{2\pi}}\int_0^\pi f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right)h_m(\theta)\mathrm{d}\theta,\end{aligned} \quad (17)$$

where

$$h_m(\theta) = \sin\theta\left(\frac{\sin\,(m-1)\theta}{m-1} - \frac{\sin\,(m+1)\theta}{m+1}\right).$$

Since $n \le 2^{k-1}$, it follows that

$$\beta^k_{n,m} \le \frac{\sqrt{2\pi}A}{(2n)^{5/2}\left(m^2-1\right)}. \quad (18)$$

Similarly, if $m = 1$, (17) implies that

$$\beta^k_{n,1} \le \frac{\sqrt{2\pi}}{(2n)^{3/2}}\max_{0\le x\le 1}\left|f'(x)\right|. \quad (19)$$

Furthermore, for $m = 0$, the series in (15) converges. In fact, $(\psi^k_{n,0})_{n=1}^{\infty}$ is an orthogonal system, which implies the convergence of $\sum_{n=1}^{\infty}\beta^k_{n,0}\,\psi^k_{n,0}(x)$. It follows that

$$\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\,\psi^k_{n,m}(x)\right| \le \left|\sum_{n=1}^{\infty}\beta^k_{n,0}\,\psi^k_{n,0}(x)\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\beta^k_{n,m}\right| < \infty.$$

Accordingly, the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\beta^k_{n,m}\psi^k_{n,m}(x)$ converges to $f(x)$ uniformly, as desired. □

Theorem 5 leads to computation of the approximation error of the wavelet expansion (30), as stated in the following proposition.

Proposition 6. *(estimation) Under the same hypotheses as in Theorem 5, we have*

$$\sigma_{k,M} \le \frac{\sqrt{\pi}}{2}\left(\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^3}\left(\max_{0\le x\le 1}{}^2\left|f'(x)\right|[\![M=1]\!] + \frac{A^2}{4n^2}\sum_{m=M}^{\infty}\frac{1}{(m^2-1)^2}M\neq 1\right)\right)^{1/2},$$

where $[\![\,]\!]$ *is the Iverson bracket notation and*

$$\sigma_{k,M} := \left(\int_0^1\left(f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta^k_{n,m}\,\psi^k_{n,m}(x)\right)^2 w_k(x)dx\right)^{1/2}.$$

Proof. First, recall that the series in (30) can be approximated with the truncated series in (11). Throughout the proof, we write $\beta_{n,m}$ instead of $\beta^k_{n,m}$ to avoid confusion. Accordingly,

$$\begin{aligned}\sigma^2_{k,M} &= \int_0^1\left(f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\beta_{n,m}\,\psi^k_{n,m}(x)\right)^2 w_k(x)\mathrm{d}x\\ &= \int_0^1\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\left(\psi^k_{n,m}(x)\right)^2 w_k(x)dx\\ &= \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\frac{2^{k+1}}{\pi}\int_{(n-1)/2^{k-1}}^{n/2^{k-1}}\frac{T^2_m\left(2^kx-2n+1\right)}{\sqrt{1-\left(2^kx-2n+1\right)^2}}dx.\end{aligned} \tag{20}$$

Now, the change of variable $2^kx-2n+1=x'$ in (20) and relabeling x' as x gives

$$\sigma^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}\frac{2}{\pi}\int_{-1}^{1}\frac{T^2_m(x)}{\sqrt{1-x^2}}dx.$$

Moreover, (2) implies that

$$\int_{-1}^{1}\frac{T^2_m(x)}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{2},\quad m\ge 1,$$

therefore

$$\sigma^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\beta^2_{n,m}. \tag{21}$$

From Definition 2 we see that $M\in\mathbb{N}$, thus combining (18), (19) with (21) it follows

$$\begin{aligned}\sigma^2_{k,M} &= \sum_{n=2^{k-1}+1}^{\infty}\left(\beta^2_{n,1}[\![M=1]\!] + \sum_{m=M}^{\infty}\beta^2_{n,m}[\![M\neq 1]\!]\right)\\ &\le \frac{\pi}{4}\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^3}\left(\max_{0\le x\le 1}{}^2\left|f'(x)\right|[\![M=1]\!] + \frac{A^2}{4n^2}\sum_{m=M}^{\infty}\frac{1}{(m^2-1)^2}[\![M\neq 1]\!]\right).\end{aligned}$$

The proof is complete. □

Theorem 5 and Proposition 6 show uniform convergence and accuracy estimation of Chebyshev wavelets, which lay the foundation for their wide application to the theory of integral equations (see, e.g., [12–15, 17, 18, 24]).

Our next goal is to rewrite Chebyshev wavelets as a power series. We recall [26] that

$$T_m(x) = \frac{m}{2}\sum_{r=0}^{\lfloor m/2\rfloor}\frac{(-1)^r}{m-r}\binom{m-r}{r}(2x)^{m-2r},\ m>0.$$

The change of variable $r'=m-2r$ in the previous series gives

$$\begin{aligned}T_m(x) &= m\sum_{r=0}^{m}\frac{(-1)^{(m-r)/2}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}(2x)^r\\ &= \sum_{r=0}^{m}\frac{2^r m(-1)^{\omega}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}x^r,\quad m+r\in E,\omega=\frac{(m-r)}{2},\end{aligned} \tag{22}$$

where E denotes the set of even numbers. Clearly, $m+r\in E$ implies that $m-r\in E$, hence $m\pm r\in E$. Moreover, the previous change of variable entails that

$$m-2\lfloor m/2\rfloor = \begin{cases}0, & m\in E,\\ 1, & m\in O,\end{cases}$$

thus, the lower index of summation in (22) is $r=0$. For simplicity of notation, we set

$$\alpha^m_r := \frac{2^r m(-1)^{\omega}}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}},\quad m+r\in E,\omega=\frac{(m-r)}{2}, \tag{23}$$

thus,

$$T_m(x) = \sum_{r=0}^{m} \alpha_r^m x^r, \quad m > 0. \tag{24}$$

It is worth noticing that all contributions of the summation index r in (24) are subject to the condition $m + r \in E$ in (23), i.e., $m \pm r \in E$; thus, half of them vanish. More precisely, we have that the lower index of summation is $r = 0$ for $m \in E\backslash\{0\}$ and $r = 1$ for $m \in E$.

We see from (24) that (7) can be rewritten as follows:

$$\psi_{n,m}^k(x) = \begin{cases} 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n + 1\right)^r\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{25}$$

Since the parameters n and k give, respectively, a dilation and a translation of the wavelet basis (7), the wavelet mother ψ is such that $\psi = \psi(m, x)$. As a consequence, the wavelet mother ψ depends only on the its associated Chebyshev polynomial $T_m(x)$. According to the current symbology in wavelet analysis, we define the wavelet mother ψ as follows:

$$\psi(x) \coloneqq \psi_{1,m}^1(x),$$

therefore,

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\alpha_r^m(x-1)^r, & 0 \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

3.2. Connection Coefficients. The differential operators can be represented in wavelet bases if we compute the wavelet decomposition of the derivatives. Let f be a C^p function with $p > 0$ such that $f \in L_w^2[0, 1]$ with bounded second derivative on $[0, 1]$. The wavelet reconstruction (30) allows us to compute the derivatives of f as follows:

$$\frac{d^p}{dx^p} f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \beta_{n,m}^k \frac{d^p}{dx^p}\psi_{n,m}^k(x).$$

Thus, according to (8), the derivatives of f up to order p are uniquely determined by

$$\frac{d^p}{dx^p}\psi_{n,m}^k(x). \tag{26}$$

On the other hand, the first derivative of Chebyshev wavelets are given by

$$\frac{d}{dx}\psi_{n,m}^k(x) = \begin{cases} m\sqrt{\frac{2}{\pi}}\, 2^{3k/2}\, U_{m-1}\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

We note that the first derivative in (27) depends on $U_{m-1}(x)$, i.e., Chebyshev polynomials of the second kind. More specifically, it can be written as a Chebyshev wavelet of the second kind (see [20]). The computation of the derivatives (26) is more complicated for $p > 1$. In particular, high-order derivatives in (26) cannot be easily derived. Therefore, according to (8), we next turn to the wavelet decomposition of the derivatives (26), that is,

$$\frac{d^p}{dx^p}\psi_{n,m}^k(x) = \sum_{l=1}^{\infty}\sum_{q=0}^{\infty}\gamma_{nlmq}^{(p)kh}\,\psi_{l,q}^h(x), \tag{28}$$

with

$$\gamma_{nlmq}^{(p)kh} = \left\langle \frac{d^p}{dx^p}\psi_{n,m}^k(x), \psi_{l,q}^h(x)\right\rangle_{w_h}. \tag{29}$$

The coefficients (29) are called connection coefficients with an obvious intuitive meaning. Their computation can be obtained in the Fourier domain due to the Parseval–Plancherel identity:

$$\langle f, g\rangle \coloneqq \int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx = \int_{-\infty}^{\infty}\hat{f}(\xi)\,\bar{\hat{g}}(\xi)\,d\xi = \left\langle \hat{f}, \hat{g}\right\rangle, \quad f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

where $\hat{f}$ denotes the Fourier transform of f defined as follows:

$$\hat{f}(\xi) \coloneqq \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\,dx, \quad f \in L^1(\mathbb{R}).$$

Therefore,

$$\widehat{\frac{d^p}{dx^p}\psi_{n,m}^k(x)} = (2\pi i\xi)^p\,\hat{\psi}_{n,m}^k(\xi). \tag{30}$$

Let us now compute the Fourier transform of $T_m(x)$.

Lemma 7. *The Fourier transform of Chebyshev polynomials $T_m(x)$ is given by*

$$\hat{T}_m(\xi) = \sum_{r=0}^{m} c_r^m \delta^{(r)}(\xi),$$

where

$$c_r^m = \frac{i^m}{\pi^r}\frac{m}{m+r}\begin{pmatrix}\frac{m+r}{2} \\ \frac{m-r}{2}\end{pmatrix}, \quad m + r \in E. \tag{31}$$

Proof. The proof falls naturally into two parts ($m = 0$ and $m > 0$). For $m = 0$, it follows immediately that $c_0^0 = 1$and

$$T_0(x) = 1 \overset{\mathscr{F}}{\leftrightarrow} \delta(\xi).$$

Let us now turn to the case $m > 0$. We begin by recalling the differentiation property of Fourier transform

$$x^n f(x) \overset{\mathscr{F}}{\leftrightarrow} \left(\frac{i}{2\pi}\right)^n \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \hat{f}(\xi), n \in \mathbb{N}_0, \quad (32)$$

which holds in the space of tempered distributions on the real line $\mathscr{S}'(\mathbb{R})$. Accordingly, from (32) for $f(x) = \delta(x)$, we get

$$x^n \overset{\mathscr{F}}{\leftrightarrow} \left(\frac{i}{2\pi}\right)^n \delta^{(n)}(\xi), n \in \mathbb{N}_0,$$

where δ is the Dirac delta distribution. Therefore,

$$T_m(\xi) \overset{(24)}{=} \sum_{r=0}^{m} \alpha_r^m \mathscr{F}\{x^r\} = \sum_{r=0}^{m} \alpha_r^m \left(\frac{i}{2\pi}\right)^r \delta^{(r)}(\xi), \quad m > 0.$$

Furthermore, since

$$\alpha_r^m \left(\frac{i}{2\pi}\right)^r = c_r^m, m > 0,$$

the desired result plainly follows. □

On the one hand, in the proof of Lemma 7, we used the fact that $\widehat{T}_0(\xi) = \delta(\xi)$. On the other hand, the principal significance of Lemma 7 is that the Fourier transform of Chebyshev polynomials $T_m(x)$ is nothing but a sum of derivatives of the Dirac delta. Condition (31) on coefficients c_r^m implies that if $m \in E$ the Fourier transform of $T_m(x)$ is the sum of all even order derivatives of the Dirac delta. Likewise, if $m \in O$, the Fourier transform of $T_m(x)$ is the sum of all odd order derivatives of the Dirac delta. We note that for $m \in E$ the Fourier transform $\widehat{T}_m(\xi)$ always contains the Dirac delta $\delta(\xi)$. Moreover, the presence of the power i^m in (31) implies that

$$\widehat{T}_m(\xi) \in \begin{cases} \mathbb{R}, & m \in E, \\ \mathbb{I}, & m \in 0, \end{cases}$$

where $\mathbb{I}$ denotes the set of imaginary numbers. These results are shown in Table 1 for the $m = 1, 2, \cdots, 10$.

Now, we are in a position to compute the Fourier transform in (30) and connection coefficients.

Theorem 8. *Let n and k be defined as in (7). Moreover, let $m, p \in \mathbb{N}$. The following statements hold:*

$$\hat{\psi}_{n,m}^k(\xi) = 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m \delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\sum_{t=0}^{r} c_{r,t}^{m,n}\delta^{(t)}(\xi)\right),$$

with

$$c_{r,t}^{m,n} := \left(2^{(k-1)t+r} i^{m-r+t}/\pi^t\right)(m/m+r)\binom{m+r/2}{m-r/2}\binom{r}{t}(1-2n)^{r-t}, m+r \in E,$$

$$\frac{d^p}{dx^p}\hat{\psi}_{n,m}^k(x) = 2^{k/2}(2\pi i\xi)^p\left(\frac{1}{\sqrt{\pi}}\delta_m\delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\sum_{r=0}^{m}\sum_{t=0}^{r} c_{r,t}^{m,n}\delta^{(t)}(\xi)\right),$$

$$\gamma_{nlmq}^{(p)kh} = \frac{2^{1+p+((k-h)k-h/2)}}{\sqrt{\pi}} m \left(\sum_{\substack{0\le r\le m-p: \\ m-p-r\in E}} d_{m,r}^p\left(\sqrt{2}\delta_q\lambda_{\delta_q} + (1-\delta_q)\lambda_{1-\delta_q}\right) - \frac{\sqrt{2\pi}}{4} d_m^p [\![m-p \in E]\!]\right), l \le n,$$

where

$$d_{m,r}^p := \left(\frac{m+p+r}{2} - 1\right)\underline{p-1}\binom{\frac{m+p-r}{2}-1}{p-1},$$

$$\lambda_{1-\delta_q} := \lambda_{\delta_q}\left(1 + \sum_{\nu=1}^{q} \alpha_\nu^q \frac{(s+\nu+2)(s+\nu+1/2)!}{(s+\nu+1)(s+\nu+2/2)!}\right),$$

with α_ν^q as in (23), $d_m^p = d_{m,0}^p$ and

$$\lambda_{\delta_q} := \sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\frac{((s+1)/2)!}{(s+1)\,(s/2)!}, s \in E.$$

Proof. First, from (25), it follows that

$$\hat{\psi}_{n,m}^k(\xi) = 2^{k/2}\left(\frac{1}{\sqrt{\pi}}\delta_m\delta(\xi) + \sqrt{\frac{2}{\pi}}(1-\delta_m)\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n + 1\right)^r\right\}\right).$$

Combining the proof of Lemma 7 and binomial theorem gives

$$\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n+1\right)^r\right\} = \sum_{r=0}^{m}\sum_{t=0}^{r}\alpha_r^m 2^{kr}\binom{r}{t}\left(\frac{-2n+1}{2^k}\right)^{r-t} \cdot \left(\frac{i}{2\pi}\right)^t \delta^{(t)}(\xi) \overset{(23)}{=} \sum_{r=0}^{m}\sum_{t=0}^{r}\frac{2^{(k-1)t+r} i^{m-r+t}}{\pi^t}\frac{m}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}\binom{r}{t} \cdot (1-2n)^{r-t}\delta^{(t)}(\xi), m+r \in E.$$

With the same notation as in Lemma 7, we set

$$c_{r,t}^{m,n} = \frac{2^{(k-1)t+r} i^{m-r+t}}{\pi^t}\frac{m}{m+r}\binom{\frac{m+r}{2}}{\frac{m-r}{2}}\binom{r}{t}(1-2n)^{r-t}, \quad m+r \in E,$$

hence,

$$\mathscr{F}\left\{\sum_{r=0}^{m}\alpha_r^m\left(2^k x - 2n+1\right)^r\right\} = \sum_{r=0}^{m}\sum_{t=0}^{r} c_{r,t}^{m,n}\delta^{(t)}(\xi).$$

This proves (i). Furthermore, (ii) follows straightforwardly from (i) and (30).

Table 1: Fourier transform of Chebyshev polynomials $T_m(x)$ for $m = 1, 2, \cdots, 10$.

m	$T_m(x)$	$\hat{T}_m(\xi)$
1	x	$i\delta'(\xi)/(2\pi)$
2	$2x^2 - 1$	$-\delta(\xi) - \delta''(\xi)/(2\pi^2)$
3	$4x^3 - 3x$	$\left(3\pi^2\delta'(\xi) + \delta^{(3)}(\xi)\right)/(2\pi^3 i)$
4	$8x^4 - 8x^2 + 1$	$\delta(\xi) + \left(4\pi^2\delta''(\xi) + \delta^{(4)}(\xi)\right)/(2\pi^4)$
5	$16x^5 - 20x^3 + 5x$	$i\left(5\pi^4\delta'(\xi) + 5\pi^2\delta^{(3)}(\xi) + \delta^{(5)}(\xi)\right)/(2\pi^5)$
6	$32x^6 - 48x^4 + 18x^2 - 1$	$-\delta(\xi) - \left(9\pi^4\delta''(\xi) + 6\pi^2\delta^{(4)}(\xi) + \delta^{(6)}(\xi)\right)/(2\pi^6)$
7	$64x^7 - 112x^5 + 56x^3 - 7x$	$\left(7\pi^6\delta'(\xi) + 14\pi^4\delta^{(3)}(\xi) + 14\pi^2\delta^{(5)}(\xi) + \delta^{(7)}(\xi)\right)/(2\pi^7 i)$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	$\delta(\xi) + \left(16\pi^6\delta''(\xi) + 20\pi^4\delta^{(4)}(\xi) + 8\pi^2\delta^{(6)}(\xi) + \delta^{(8)}(\xi)\right)/(2\pi^8)$
9	$259x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$	$i\left(9\pi^8\delta'(\xi) + 30\pi^6\delta^{(3)}(\xi) + 27\pi^4\delta^{(5)}(\xi) + 9\pi^2\delta^{(7)}(\xi) + \delta^{(9)}(\xi)\right)/(2\pi^9)$
10	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$	$-\delta(\xi) - \left(25\pi^8\delta''(\xi) + 50\pi^6\delta^{(4)}(\xi) + 35\pi^4\delta^{(6)}(\xi) + 10\pi^2\delta^{(8)}(\xi) + \delta^{(10)}(\xi)\right)/(2\pi^{10})$

Finally, we can prove (iii). From (29), we have

$$\gamma_{nlmq}^{(p)kh} = \int_0^1 \frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x)\, w_h(x)\, dx. \tag{33}$$

By (7),

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right) &= 2^{k/2}\frac{d^p}{dx^p}\left(\frac{1}{\sqrt{\pi}}\delta_m + \sqrt{\frac{2}{\pi}}(1-\delta_m)T_m\left(2^k x - 2n + 1\right)\right) \\ &= 2^{k/2}\sqrt{\frac{2}{\pi}}(1-\delta_m)\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right), \quad \frac{n-1}{2^{k-1}} \\ &\leq x < \frac{n}{2^{k-1}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x) &= 2^{(h+k)/2}\frac{\sqrt{2}}{\pi}(1-\delta_m)\delta_q\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right) \\ &\quad + 2^{(h+k)/2}\frac{2}{\pi}(1-\delta_m)(1-\delta_q)\frac{d^p}{dx^p} \\ &\quad \cdot\left(T_m\left(2^k x - 2n + 1\right)\right)T_q\left(2^h x - 2l + 1\right) \\ &= 2^{(h+k)/2}\frac{d^p}{dx^p}\left(T_m\left(2^k x - 2n + 1\right)\right) \\ &\quad \cdot\left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}(1-\delta_q)T_q\left(2^h x - 2l + 1\right)\right),\end{aligned}$$

which follows from the hypothesis that $m \geq 1$. We proved in Appendix that

$$\frac{d^p}{dx^p}T_m\left(2^k x - 2n + 1\right) = 2^p m \sum_{\substack{0\leq r\leq m-p: \\ m-p-r\in E}} d_{m,r}^p\, T_r\left(2^k x - 2n + 1\right) - 2^{p-1} m\, d_m^p\; m - p \in E,$$

with

$$d_{m,r}^p := \left(\frac{m+p+r}{2} - 1\right)^{\underline{p-1}}\begin{pmatrix}\frac{m+p-r}{2} - 1 \\ p-1\end{pmatrix},$$

and $d_m^p = d_{m,0}^p$. Thus,

$$\begin{aligned}\frac{d^p}{dx^p}\left(\psi_{n,m}^k(x)\right)\psi_{l,q}^h(x) &= 2^{p+(h+k)/2}\, m\left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}(1-\delta_q)T_q\left(2^h x - 2l + 1\right)\right) \\ &\quad \cdot\left(\sum_{0\leq r\leq m-p:m-p-r\in E} d_{m,r}^p\, T_r\left(2^k x - 2n + 1\right) - \frac{d_m^p}{2}\; m - p \in E\right),\end{aligned}$$

Now, we can proceed to compute $\gamma_{nlmq}^{(p)kh}$. Note that, expanding the integrand in (33), we have

$$\begin{cases}\dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}}, & n = 1, 2, \cdots, 2^{k-1}, \\ \dfrac{l-1}{2^{h-1}} \leq x < \dfrac{l}{2^{h-1}}, & l = 1, 2, \cdots, 2^{h-1}.\end{cases} \tag{34}$$

The assumption $l \leq n$ implies that the integrand in (33) holds for $(34)_2$. As a consequence,

$$\gamma_{nlmq}^{(p)kh} = \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} 2^{p+(h+k)/2}\, m \left(\sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p}\, T_r\left(2^k x - 2n + 1\right) - \frac{d_m^p}{2}\, m - p \in E \right)$$
$$\cdot \left(\frac{\sqrt{2}}{\pi}\delta_q + \frac{2}{\pi}\left(1-\delta_q\right) T_q\left(2^h x - 2l + 1\right)\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx$$
$$= 2^{p+((h+k)/2)}\, m \sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p} \left(\frac{\sqrt{2}}{\pi}\delta_q \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_r\left(2^k x - 2n + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right.$$
$$\left. + \frac{2}{\pi}\left(1-\delta_q\right) \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_r\left(2^k x - 2n + 1\right) T_q\left(2^h x - 2l + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right)$$
$$- 2^{p+((h+k)h+k/2)}\, m \frac{d_m^p}{2}\, m - p \in E \left(\frac{\sqrt{2}}{\pi}\delta_q \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx + \frac{2}{\pi}\left(1-\delta_q\right) \right.$$
$$\left. \cdot \int_{(l-1)/2^{h-1}}^{l/2^{h-1}} T_q\left(2^h x - 2l + 1\right) \frac{1}{\sqrt{1-\left(2^h x - 2l + 1\right)^2}}\, dx \right).$$

For simplicity of notation, we indicate the last four integrals with I_1,I_2, I_3, and I_4, respectively. Thus,

$$\gamma_{nlmq}^{(p)kh} = 2^{p+(h+k)/2}\, m \left(\sum_{0\le r\le m-p:m-p-r\in E} d_{m,r}^{p} \left(\frac{\sqrt{2}}{\pi}\delta_q I_1 + \frac{2}{\pi}\left(1-\delta_q\right) I_2 \right) - \frac{d_m^p}{2} \left(\frac{\sqrt{2}}{\pi}\delta_q I_3 + \frac{2}{\pi}\left(1-\delta_q\right) I_4 \right) m - p \in E \right). \quad (35)$$

The change of variable $x' = 2^h x - 2l + 1$ in I_1, and relabeling x' as x, gives

$$I_1 = \frac{1}{2^h}\int_{-1}^{1} T_r\left(2^{k-h}(x + 2l - 1) - 2n + 1\right) \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$\overset{(24)}{=} \frac{1}{2^h}\int_{-1}^{1} \sum_{t=0}^{r} \alpha_t^r \left(2^{k-h}(x + 2l - 1) - 2n + 1\right)^t \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$= \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s} \int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\,\mathrm{d}x. \quad (36)$$

We note that the last computation follows from the binomial theorem. Moreover,

$$\int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\,\mathrm{d}x = \sqrt{\pi}\,\frac{\left((-1)^s + 1\right)\Gamma((s+1)/2)}{s\Gamma(s/2)}, \ \mathrm{Re}\ s > -1, \quad (37)$$

Since $s \ge 0$ in (35), the result above holds in the computation of I_1. Obviously, (36) vanishes for s odd. Therefore,

$$\int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}}\,\mathrm{d}x = 2\sqrt{\pi}\,\frac{\Gamma((s+1)/2)}{s\Gamma(s/2)} = 2\sqrt{\pi}\,\frac{((s+1)/2)!}{(s+1)\,(s/2)!}, \quad s \in E,$$

and so

$$I_1 = \frac{\sqrt{\pi}}{2^{h-1}}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s} \frac{((s+1)/2)!}{(s+1)\,(s/2)!}, \quad s \in E.$$

Let us now pass to the second integral I_2. We can proceed analogously to the computation of I_1. In fact, (35) implies that

$$I_2 = \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j} \alpha_t^r \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}$$
$$\cdot (2l-1)^{j-s}\int_{-1}^{1} \frac{x^s}{\sqrt{1-x^2}} T_q(x)\,\mathrm{d}x \overset{(24)}{=} \frac{1}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j}\sum_{\nu=0}^{q} \alpha_t^r\alpha_\nu^q \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s}\int_{-1}^{1} \frac{x^{s+\nu}}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Furthermore,

$$\int_{-1}^{1} \frac{x^{s+\nu}}{\sqrt{1-x^2}}\,\mathrm{d}x = \sqrt{\pi}\,\frac{\left((-1)^{s+\nu} + 1\right)\Gamma((s+\nu+1)/2)}{2\,\Gamma((s+\nu+2)/2)}, \quad \mathrm{Re}\ (s+\nu) > -1. \quad (38)$$

As in computation of I_1, the result above holds because $s + \nu \ge 0$. We note that (37) vanishes for $s + \nu$ odd. Thus,

$$\int_{-1}^{1} \frac{x^{s+\nu}}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\sqrt{\pi}\Gamma((s+\nu+1)/2)}{\Gamma((s+\nu+2)/2)} = \sqrt{\pi}\,\frac{(s+\nu+2)((s+\nu+1)/2)!}{(s+\nu+1)((s+\nu+2)/2)!}, \quad s+\nu \in E,$$

and so

$$I_2 = \frac{\sqrt{\pi}}{2^h}\sum_{t=0}^{r}\sum_{j=0}^{t}\sum_{s=0}^{j}\sum_{\nu=0}^{q} \alpha_t^r\,\alpha_\nu^q \binom{t}{j}\binom{j}{s} 2^{(k-h)j}(1-2n)^{t-j}(2l-1)^{j-s} \frac{(s+\nu+2)((s+\nu+1)/2)!}{(s+\nu+1)((s+\nu+2)/2)!}, \quad s+\nu \in E.$$

Similarly, we can compute the integral I_3 and I_4. In fact, the same change of variable as in I_1gives

$$I_3 = \frac{1}{2^h}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2^h},$$
$$I_4 = \frac{1}{2^h}\int_{-1}^{1} T_q(x)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{(2)}{=} \frac{\pi}{2^h}\delta_q.$$

Thus, $(1-\delta_q)I_4 = 0$ in (34). This completes the proof. □

Remark 9. In the proof of Theorem 8, the hypothesis $l \le n$ played a fundamental role in the computation of connection coefficients $\gamma_{nlmq}^{(p)kh}$. Indeed, for $n < l$, the integrand in (33) holds for $(34)_1$. Thus, we get a similar computation as in the proof of Theorem 8 but the integral I_1 and I_2 will be defined on $[(n-1)/2^{k-1}, n/2^{k-1}[$. The details are left to the reader.

Remark 10. Theorem 8 allows us to define the connection coefficients of Chebyshev wavelets. Moreover, we note that the proof of Theorem 5 gives an upper bound on connection coefficients $\gamma_{nlmq}^{(p)kh}$, i.e.,

$$\gamma_{nlmq}^{(p)kh} \le \frac{2^{1+p+(k-h)/2}}{\sqrt{\pi}}\, m \sum_{\substack{0\le r\le m-p:\\ m-p-r\in E}} d_{m,r}^{p}\left(\sqrt{2}\delta_q\lambda_{\delta_q} + \left(1-\delta_q\right)\lambda_{1-\delta_q}\right), \quad l \le n. \quad (39)$$

Theorems 5 and 8 allow us the reconstruction for $L_w^2([0,1])$ functions together with their derivative. Moreover, Theorem 8 gives the Fourier transform for any order derivative of Chebyshev wavelets, as the next example shows.

Example 1. Let us consider Chebyshev wavelets for $k = 2$ re and $m = 1$:

$$\psi_{1,1}^{2}(x) = \begin{cases} 2\sqrt{\dfrac{2}{\pi}}(4x-1), & 0 \le x < \dfrac{1}{2}, \\ 0, & \text{otherwise}, \end{cases} \quad \psi_{2,1}^{2}(x) = \begin{cases} 2\sqrt{\dfrac{2}{\pi}}(4x-3), & \dfrac{1}{2} \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

For the sake of simplicity and without loss of generality, we consider only $\widehat{\psi}_{1,1}^{2}$. Thus, we leave it to the reader to deal with $\widehat{\psi}_{2,1}^{2}$.

From Theorem 8, we have

$$\begin{aligned} \widehat{\psi}_{1,1}^{2}(\xi) &= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{1} \sum_{t=0}^{r} c_{r,t}^{1,1} \delta^{(t)}(\xi) = \sqrt{\frac{2}{\pi}} \left(\left(c_{0,0}^{1,1} + c_{1,0}^{1,1} \right) \delta(\xi) + c_{1,0}^{1,1} \delta'(\xi) \right) \\ &= \sqrt{\frac{2}{\pi}} \left((-1+i)\delta(\xi) + \frac{2i}{\pi} \delta'(\xi) \right), \end{aligned}$$

where we used that $c_{0,0}^{1,1} = i$, $c_{1,0}^{1,1} = -1$, and $c_{1,1}^{1,1} = 2i/\pi$. Finally, we conclude that the sixth-order derivative of $\psi_{1,1}^{2}$ has the following Fourier transform:

$$\begin{aligned} \widehat{\frac{d^6}{dx^6} \psi_{1,1}^{2}(x)} &= (2\pi i \xi)^6 \sqrt{\frac{2}{\pi}} \left((-1+i)\delta(\xi) + \frac{2i}{\pi} \delta'(\xi) \right) \\ &= 64\pi^5 \sqrt{\frac{2}{\pi}} \xi^6 \left(\pi (1-i)\delta(\xi) - 2i\delta'(\xi) \right). \end{aligned}$$

Figures 1 and 2 show the graph of $\psi_{1,1}^{2}$ and $\widehat{\psi}_{1,1}^{2}$. We note that the Fourier transform of $\psi_{1,1}^{2}$ exbibits an impulsive behaviour. In particular, the imaginary part of $\widehat{\psi}_{1,1}^{2}$ depends also on the distributional derivative of the Dirac delta, i.e.,

$$\begin{cases} \operatorname{Re} \widehat{\psi}_{1,1}^{2} = \operatorname{Re} \widehat{\psi}_{1,1}^{2}(\delta), \\ \operatorname{Im} \widehat{\psi}_{1,1}^{2} = \operatorname{Im} \widehat{\psi}_{1,1}^{2}\left(\delta, \delta'\right). \end{cases}$$

We may approximate the Dirac delta as [27] follows:

$$\delta(x) \approx \sqrt{\frac{1}{\varepsilon\pi}} e^{-x^2/\varepsilon}, \quad \varepsilon \ll 1,$$

thus,

$$\delta'(x) = -\frac{2x}{\varepsilon\sqrt{\varepsilon\pi}} e^{-x^2/\varepsilon}, \quad \varepsilon \ll 1. \tag{40}$$

Approximation (39) allows us to draw $\widehat{\psi}_{1,1}^{2}$. In Figure 1, we show the imaginary part of $\widehat{\psi}_{1,1}^{2}$ with $\varepsilon = 0.01$. Precisely, it is worth noticing that the Dirac delta $\delta(\xi)$ in Figure 1 was multiplied by a factor of 51 which takes into account the approximation error introduced by (39).

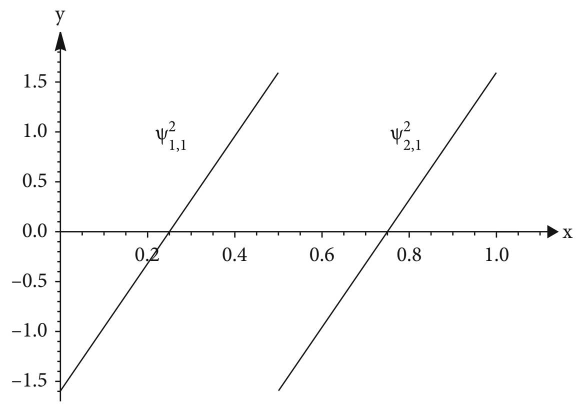

FIGURE 1: Chebyshev wavelets $\psi_{n^*,1}^{2}$ with $n^* = 1, 2$.

4. Taylor Series and Chebyshev Wavelets

In Section 3, we introduced the connection coefficients (29) that allows us to prove the following statement.

Theorem 11. *Let f be a C^q function such that $f \in L_w^2([0,1])$ with bounded second derivative on $[0,1]$. Then, the Taylor series of f in $x = x_0$ is given by*

$$f(x) = f(x_0) + \sum_{t=1}^{\infty} \left(\sum_{n,l=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^{k} \gamma_{nlmq}^{(t)kh} \psi_{l,q}^{h}(x_0) \right) \frac{(x-x_0)^t}{t!}, \tag{41}$$

with $\beta_{n,m}^{k}$ as in (10).

Proof. The wavelet expansion (30) entails that the pth-order derivative of f (with $p \le q$) can be expanded as follows:

$$\begin{aligned} f^{(p)}(x) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \beta_{n,m}^{k} \frac{d^p}{dx^p} \psi_{n,m}^{k}(x) \overset{(28)}{=} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \beta_{n,m}^{k} \sum_{l=1}^{\infty} \sum_{q=0}^{\infty} \gamma_{nlmq}^{(p)kh} \psi_{l,q}^{h}(x) \\ &= \sum_{n,l=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^{k} \gamma_{nlmq}^{(p)kh} \psi_{l,q}^{h}(x), \end{aligned}$$

as desired.

It is worthy noticing that a suitable choice of the initial point x_0 allows us to simplify the Taylor expansion (40). In particular, for $x_0 = 0$, we have that

$$f(x) \overset{(41)}{=} f(0) + \sum_{t=1}^{\infty} \left(\sum_{n,l=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^{k} \gamma_{nlmq}^{(t)kh} \psi_{l,q}^{h}(0) \right) \frac{x^t}{t!}.$$

Moreover, Definition 2 gives that $x_0 = 0 \Longrightarrow l = h = 1$; thus,

$$\psi_{l,q}^{h}(0) = \psi_{1,q}^{1}(0) = 2^{1/2} \tilde{T}_q(-1) = \sqrt{2} \left(\frac{1}{\sqrt{\pi}} \delta_q + \sqrt{\frac{2}{\pi}} (-1)^q \left(1 - \delta_q\right) \right),$$

$$\gamma_{nlmq}^{(t)kh} = \gamma_{n1mq}^{(t)k1}.$$

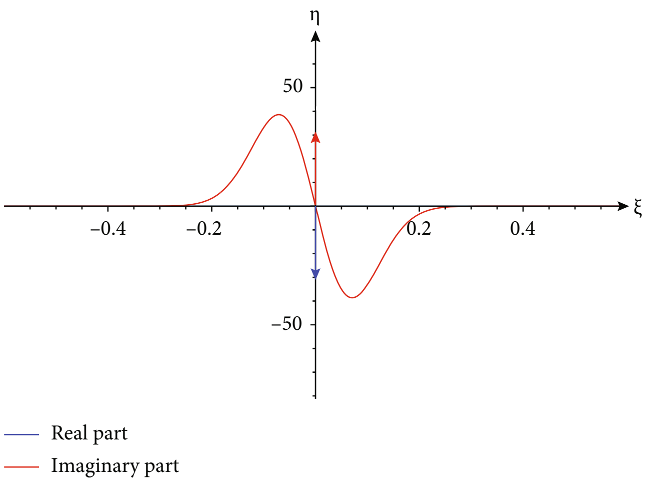

Figure 2: The Fourier transform of $\psi^2_{1,1}$. We drew the imaginary part of $\widehat{\psi}^2_{1,1}$ by (101) with $\varepsilon = 0.01$.

Therefore,

$$f(x) = f(0) + \sqrt{\frac{2}{\pi}} \sum_{t=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta^k_{n,m} \gamma^{(t)k1}_{n1mq} \left(\delta_q + \sqrt{2} (-1)^q (1 - \delta_q) \right) \right) \frac{x^t}{t!}. \tag{42}$$

Explicitly, this means that any function $f \in L^2_w([0,1])$ can be expressed as a power series when the wavelet coefficients $\beta^k_{n,m}$ are finite. Accordingly, the Taylor series for the wavelet mother $\psi(x) = \psi^1_{1,m}(x)$ is given by

$$\psi(x) = \sqrt{\frac{2}{\pi}} \delta_m + \frac{2}{\sqrt{\pi}} (1 - \delta_m)(-1)^m + \sqrt{\frac{2}{\pi}} \sum_{t=1}^{\infty} \left(\sum_{m,q=0}^{\infty} \beta^1_{1,m} \gamma^{(t)11}_{11mq} \left(\delta_q + \sqrt{2} (-1)^q (1 - \delta_q) \right) \right) \frac{x^t}{t!}, \tag{43}$$

since $\psi(0) = \sqrt{2/\pi}\, \delta_m + (2/\sqrt{\pi})(1 - \delta_m)(-1)^m$. We note that in the last computation the sum on the index n reduces to one term because for the wavelet mother $n = 1$.

Now, we are in a position to estimate the approximation error in (28) for a fixed scale of approximation. It is worth pointing out that the approximation depends on the upper bound in the sums. □

Theorem 12 (Approximation error of wavelet derivatives). *Let $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ be such that $p \le q + 1$. Under the same hypotheses as in Definition 2, the approximation error in (28)*

$$\varepsilon_{k,M} := \left(\int_0^1 \left(\frac{d^p}{dx^p} \psi^h_{l,q}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \gamma^{(p)hk}_{lnqm} \psi^k_{n,m}(x) \right)^2 w_k(x) dx \right)^{1/2},$$

is bounded by

$$\varepsilon_{k,M} \le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}} q(q-p+1) \left(\sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} d^2_{m,q,p,r^*} \right)^{1/2}, \quad n \le l,$$

where is the Iverson bracket notation, $d_{q,p}$ stands for d^p_q defined as in Theorem 8 and

$$d_{m,q,p,r^*} := \max_{\substack{0 \le r \le q-p: \\ q-p-r \in E}} d^p_{q,r} \cdot \left(\sqrt{2} \delta_m \lambda_{\delta_m} + (1 - \delta_m) \lambda_{1-\delta_m} \right),$$

with $d^p_{q,r}$ as in Theorem 8.

Proof. The proof can be handled in a similar way as in the proof of Proposition 6. In order to avoid confusion, we rewrite (28) as follows:

$$\frac{d^p}{dx^p} \psi^h_{l,q}(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \gamma^{(p)hk}_{lnqm} \psi^k_{n,m}(x),$$

with

$$\gamma^{(p)hk}_{lnqm} = \left\langle \frac{d^p}{dx^p} \psi^h_{l,q}(x), \psi^h_{n,m}(x) \right\rangle_{w_h}.$$

Moreover, throughout the proof, we denote $\gamma^{(p)hk}_{lnqm}$ briefly by γ_{lnqm}. Therefore,

$$\begin{aligned}
\varepsilon^2_{k,M} &= \int_0^1 \left(\frac{d^p}{dx^p} \psi^k_{n,m}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \gamma_{lnqm} \psi^k_{n,m}(x) \right)^2 w_k(x) dx \\
&= \int_0^1 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \gamma^2_{lnqm} \left(\psi^k_{n,m}(x) \right)^2 w_k(x) dx \\
&= \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \gamma^2_{lnqm} \frac{2^{k+1}}{\pi} \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} \frac{T^2_m (2^k x - 2n + 1)}{\sqrt{1 - (2^k x - 2n + 1)^2}} dx.
\end{aligned}$$

Now, the proof of Proposition 6 gives

$$\varepsilon^2_{k,M} = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \gamma^2_{lnqm}.$$

We note that the coefficients $d^p_{q,r}$ defined as in Theorem 8 are nonnegative. Thus, (38) implies that

$$\begin{aligned}
\gamma_{lnqm} &\le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}} q \sum_{0 \le r \le q-p: q-p-r \in E} d^p_{q,r} \left(\sqrt{2} \delta_m \lambda_{\delta_m} + (1 - \delta_m) \lambda_{1-\delta_m} \right) \\
&\le \frac{2^{1+p+(h-k)/2}}{\sqrt{\pi}} q(q-p+1)\, d_{m,q,p,r^*}, \quad n \le l.
\end{aligned}$$

Accordingly,

$$\gamma_{lnqm}^2 \le \frac{2^{2+2p+h-k}}{\pi} q^2 (q-p+1)^2 \, d_{m,q,p,r^*}^2, \quad n \le l.$$

We conclude that

$$\varepsilon_{k,M}^2 \le \frac{4}{\pi} \, 2^{2p+h-k} q^2 (q-p+1)^2 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} d_{m,q,p,r^*}^2, \quad n \le l,$$

which gives the desired estimate. □

Remark 13. We note that the term d_{m,q,p,r^*}^2 depends on m and n by the definition of $d_{p,r}^p$ in Theorem 8. Moreover, the hypothesis $p \le q+1$ assures us that the derivative

$$\frac{d^p}{dx^p} \psi_{l,q}^h(x),$$

does not identically vanish on its domain, i.e., we exclude trivial cases in the thesis of Theorem 12.

5. Fractional Calculus of Chebyshev Wavelets

This section is devoted to the local fractional calculus over Chebyshev wavelets. Our approach follows new trends in fractional calculus. Indeed, many problems in fractional calculus are nowadays dealt with hybrid methods (e.g., [28]). In particular, we show how the local fractional calculus extends the fractional derivative to nonsmooth continuous functions (e.g., fractal functions). In fact, the introduction of a local fractional derivative on Chebyshev wavelets by connection coefficients allows us to compute the local fractional derivative of non-smooth functions.

5.1. Chebyshev Wavelets on Fractal Sets of Dimension ν. According to the recent results on fractional calculus of wavelet bases (see, e.g., [29]), we can define the Chebyshev wavelets on a fractal set of dimension ν (with $0 < \nu \le 1$) as follows:

$$\psi_\nu(x) \overset{(43)}{=} \sqrt{\frac{2}{\pi}} \delta_m + \frac{2}{\sqrt{\pi}} (1-\delta_m)(-1)^m + \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{m,q=0}^{\infty} \beta_{1,m}^1 \gamma_{11mq}^{(t)11} \left(\delta_q + \sqrt{2} (-1)^q (1-\delta_q) \right) \right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}. \quad (44)$$

Thus, if we set

$$\gamma_m^p := \begin{cases} \sqrt{\frac{2}{\pi}} \delta_m + \frac{2}{\sqrt{\pi}} (1-\delta_m)(-1)^m, & p = 0, \\ \sqrt{\frac{2}{\pi}} \sum_{m,q=0}^{\infty} \beta_{1,m}^1 \gamma_{11mq}^{(t)11} \left(\delta_q + \sqrt{2} (-1)^q (1-\delta_q) \right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}, & p > 0, \end{cases}$$

we have that

$$\psi_\nu(x^\nu) = \sum_{p=0}^{\infty} \frac{\gamma_m^p}{\Gamma(\nu p+1)} x^{\nu p}. \quad (45)$$

We note that power series expansion (44) for $\nu = 1$ reduces to Chebyshev wavelets on a regular domain:

$$\psi_1(x) = \psi(x) = \sum_{p=0}^{\infty} \frac{\gamma_m^p}{p!} x^p.$$

In addition to this, series expansion (44) for $\nu < 1$ gives Chebyshev wavelets in a continuous nonregular domain, which are functions of the variable x^ν. We point out that the variable x^ν can be defined in nonregular domains, such as the Cantor set shown (Figure 3). For more details, we refer the reader to [29].

In the same spirit as (42), any function $f \in L_w^2[0,1]$ can be expanded in Taylor series on a fractal set of dimension ν as follows:

$$f(x) = f(0) + \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{m,q=0}^{\infty} \beta_{n,m}^k \gamma_{n1mq}^{(t)k1} \left(\delta_q + \sqrt{2} (-1)^q (1-\delta_q) \right) \right) \frac{x^{\nu p}}{\Gamma(\nu p+1)}. \quad (46)$$

Obviously, Taylor expansion (45) generalizes (41) on a fractal set of dimension ν. In order to clarify the generalization of power series (44) for nonregular domain, we consider one of the simplest case of nonregular set: the Cantor set. For further considerations on the Cantor set, we refer the reader to [30, 31]. It suffices for our purposes to recall that the Cantor function (or Devil's staircase) f is defined as follows:

$$f(x) \overset{u}{=} \lim_{n \longrightarrow \infty} f_n(x),$$

where $=^u$ denotes uniform convergence, $f_n[0,1] \longrightarrow \mathbb{R}$ such that $f_0(x) = x$ and

$$f_{n+1}(x) = \frac{1}{2} \begin{cases} f_n(3x) & 0 \le x \le \frac{1}{3}, \\ 1, & \frac{1}{3} \le x \le \frac{2}{3}, \\ 1 + f_n(3x-2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Of course, the Cantor function maps the Cantor set C onto $[0,1]$. The family of functions $(f_n)_{n \in \mathbb{N}}$ is a set of polygonal approximations of the Cantor function f. We note that the previous definition of $(f_n)_{n \in \mathbb{N}}$ is equivalent to the following condition:

$$|f_{n+1}(x) - f_n(x)| \le 2^{-n-1}, \quad x \in [0,1], \quad n \in \mathbb{N}.$$

Moreover, Chalice [32] proved that the Cantor function f is the unique monotonic, real-valued function on $[0,1]$

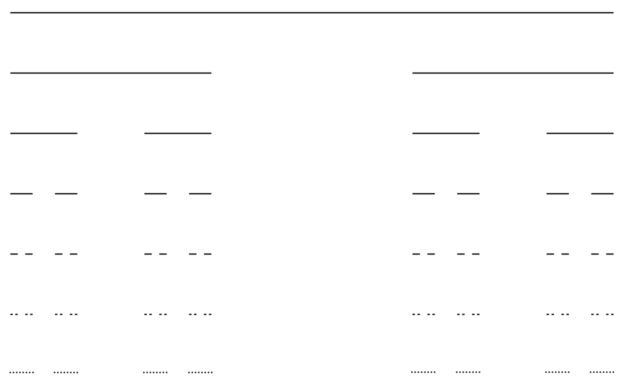

Figure 3: The middle third Cantor set.

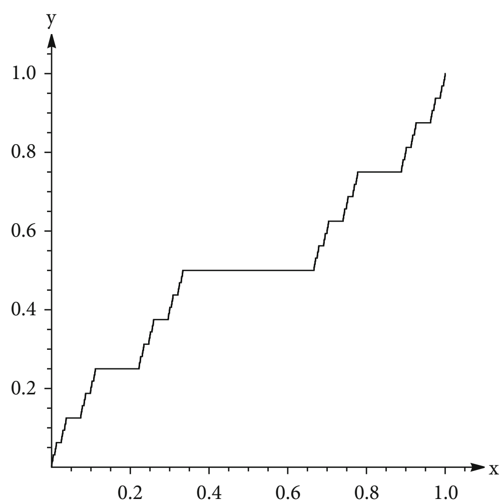

Figure 4: The Cantor function.

such that [label = ()]

$$f(0)=0,$$
$$f\left(\frac{x}{3}\right)=\frac{f(x)}{2},$$
$$f(1-x)=1-f(x).$$

This characterization gives us an easier way to draw the Cantor function f (Figure 4). For more details on the Cantor function, we refer the reader to [33]. We note that the method of Chalice can be used to draw the graphics of the functions $\psi_\nu(x^\nu)$ both on the compact interval $[0,1]$ and on the Cantor set.

5.2. Local fractional Calculus. We recall that if the real function $f(x)$ is such that

$$|f(x)-f(x_0)|<\varepsilon^\nu, \tag{47}$$

with $|x-x_0|<\delta$ being $\varepsilon,\delta>0$, then $f(x)$ is called fractional continuous at $x=x_0$. As in the classical case, we denote this property by

$$\lim_{x\longrightarrow x_0} f(x)=f(x_0).$$

In the same spirit, the function $f(x)$ is called local fractional continuous on the interval $]a,b[$ and denoted by $f\in C_\nu(a,b)$, if it satisfies the condition (46) for $x\in]a,b[$. Now, if $f\in C_\nu(a,b)$, the local fractional derivative of the real function $f(x)$ of order ν at $x=x_0$ is given [34] by

$$f^{(\nu)}(x_0)=\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}f(x)\bigg|_{x=x_0}=\Gamma(1+\nu)\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{(x-x_0)^\nu},\quad 0<\nu\le 1. \tag{48}$$

The function f is called ν-differentiable (or differentiable of order ν) at $x=x_0$ if the limit in (47) exists and is finite. It is worth noticing that the local fractional derivative (47) does not satisfy the generalized Leibniz rule [21]. Nevertheless, if f and g are both ν-differentiable in the interval $]a,b[$ in the sense of (47), we have that

$$(fg)^{(\nu)}=f^{(\nu)}g+fg^{(\nu)}. \tag{49}$$

In fact, (47) implies that

$$\begin{aligned}\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}(f(x)\,g(x))&=\Gamma(1+\nu)\lim_{h\longrightarrow 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h^\alpha}\\&=\Gamma(1+\nu)\lim_{h\longrightarrow 0}\left(\frac{f(x+h)-f(x)}{h^\alpha}\,g(x)+f(x+h)\,\frac{g(x+h)-g(x)}{h^\alpha}\right)\\&=f^{(\nu)}(x)g(x)+f(x)g^{(\nu)}(x).\end{aligned}$$

Thus, the ν-derivative of the product of two functions is the fractional equivalent of the product rule for integer derivatives.

We note that the definition (47) works for almost all the rules of fractional calculus (except, of course, the generalized Leibniz rule). For instance, we have [35] that

$$\frac{d^\nu}{dx^\nu}\sin_\nu(x^\nu)=\cos_\nu(x^\nu),\qquad \frac{d^\nu}{dx^\nu}\cos_\nu(x^\nu)=-\sin_\nu(x^\nu).$$

Moreover,

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^{\nu k}=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k-1)\nu)}x^{(k-1)\nu}, \tag{50}$$

and thus, for $k=1$, it follows that

$$d^\nu x^\nu=\Gamma(1+\nu)\,dx^\nu\Rightarrow(dx)^\nu=\Gamma(1+\nu)\,dx^\nu. \tag{51}$$

In recent years, the differential operator (50) was used to define integral transformations on Cantor sets (see, e.g., [36, 37]). The results of Section 4 allow us to obtain a series expansion of the local fractional derivative (47) as stated in the following theorem.

Theorem 14. *Let $f\in L^2_w[0,1]$ such that the wavelet coefficients (10) exist and are finite. Then, the local fractional*

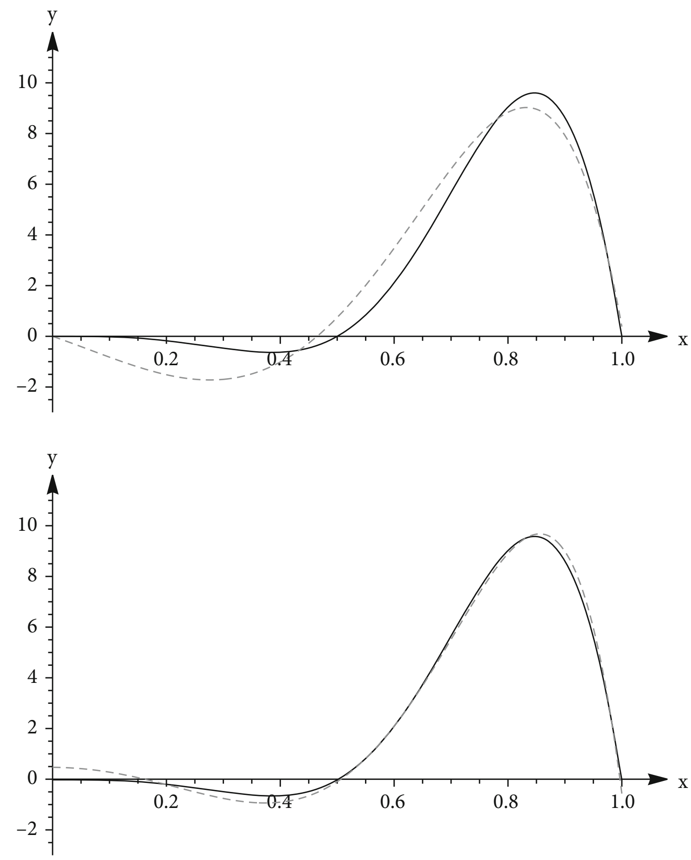

FIGURE 5: Comparison between the function $g(x) = 3x^2 e^{2x} \sin(-2\pi x)$ and its approximation (dashed line) by Chebyshev wavelets with $k^* = 6 \wedge M^* = 21$ (top) and $k^* = 9 \wedge M^* = 31$ (bottom).

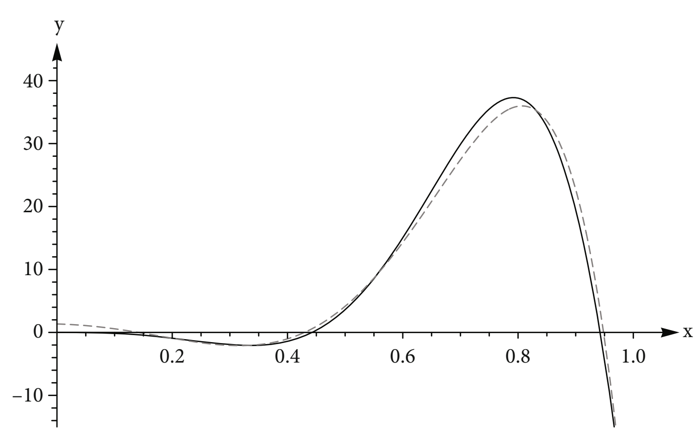

FIGURE 6: Comparison between the local fractional derivative of $g(x) = 3x^2 e^{2x} \sin(-2\pi x)$ for $\nu = 1/2$ and its approximation (dashed line) by Chebyshev wavelets with $k^* = 9 \wedge M^* = 31$.

derivative of f in the neighborhood of x = 0 is given by

$$\frac{d^{\nu}}{dx^{\nu}}f(x)=\sum_{p=1}^{\infty}\frac{A_p(\beta)}{\Gamma(1+(p-1)\nu)}x^{(p-1)\nu},$$

where

$$A_p(\beta):=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\sum_{m,q=0}^{\infty}\beta_{n,m}^{k}\,\gamma_{n1mq}^{(p)k1}\left(\delta_q+\sqrt{2}(-1)^q(1-\delta_q)\right).$$

Proof. The proof follows directly by (45) and (49). In fact, we have

$$\begin{aligned}\frac{d^{\nu}}{dx^{\nu}}f(x)&=\sqrt{\frac{2}{\pi}}\sum_{p=1}^{\infty}\left(\sum_{n=1}^{\infty}\sum_{m,q=0}^{\infty}\beta_{n,m}^{k}\,\gamma_{n1mq}^{(t)k1}\left(\delta_q+\sqrt{2}(-1)^q(1-\delta_q)\right)\right)\\&\quad\cdot\frac{1}{\Gamma(\nu p+1)}\frac{d^{\nu}}{dx^{\nu}}x^{\nu p}\\&\overset{(50)}{=}\sqrt{\frac{2}{\pi}}\sum_{p=1}^{\infty}\left(\sum_{n=1}^{\infty}\sum_{m,q=0}^{\infty}\beta_{n,m}^{k}\,\gamma_{n1mq}^{(t)k1}\left(\delta_q+\sqrt{2}(-1)^q(1-\delta_q)\right)\right)\\&\quad\cdot\frac{1}{\Gamma(1+(p-1)\nu)}x^{(p-1)\nu}.\end{aligned}$$

□

Example 2. Let us consider the function $g:x\in\mathbb{R}\longrightarrow 3x^2 e^{2x}\sin(-2\pi x)\in\mathbb{R}$. This example shows the efficiency of the results proved in this paper. First, we note that $g\in L_w^2[0,1]$. In fact,

$$\int_0^1\frac{3x^2e^{2x}\sin(-2\pi x)}{\sqrt{1-x^2}}dx\approx 5.0275.$$

Moreover, g'' is bounded on $[0,1]$. Accordingly, we can expand the function g in Chebyshev wavelets. Now, (11) implies that g can be approximated in terms of Chebyshev wavelets as follows:

$$3x^2e^{2x}\sin(-2\pi x)\simeq\sum_{n=1}^{2^{k^*-1}}\sum_{m=0}^{M^*-1}\beta_{n,m}^{k}\,\psi_{n,m}^{k}(x),\qquad(52)$$

where increasing the values of both k^* and M^* provides a better approximation in (51). Obviously, wavelet coefficients $\beta_{n,m}^{k}$ are computed by (10). The approximation error of g by Chebyshev wavelets is clearly shown in Figure 5.

On the one hand, Theorem 14 implies that

$$\frac{d^{\nu}}{dx^{\nu}}3x^2e^{2x}\sin(-2\pi x)\simeq\sum_{p=1}^{\infty}\frac{A_p^T(\beta)}{\Gamma(1+(p-1)\nu)}x^{(p-1)\nu},\qquad(53)$$

where the coefficients $A_p^T(\beta)$ is the truncated version of $A_p(\beta)$, i.e.,

$$A_p^T(\beta)=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{2^{k^*-1}}\sum_{m=0}^{M^*-1}\sum_{q=0}^{Q^*-1}\beta_{n,m}^{k}\,\gamma_{n1mq}^{(p)k1}\left(\delta_q+\sqrt{2}(-1)^q(1-\delta_q)\right).$$

On the other hand, (48) gives

$$\begin{aligned}\frac{d^{\nu}}{dx^{\nu}}g(x)&=-3\frac{d^{\nu}}{dx^{\nu}}\left(x^2e^{2x}\sin 2\pi x\right)\\&=-3(2\pi)^{\nu}x^2e^{2x}\left(\left(\frac{\Gamma(3)}{\Gamma(3-\nu)}\frac{1}{(2\pi x)^{\nu}}+\frac{1}{\pi^{\nu}}\right)\sin 2\pi x+\sin\left(2\pi x+\frac{\pi}{2}\nu\right)\right),\end{aligned}\qquad(54)$$

where we used the fact [35] that

$$\frac{d^{\nu}}{dx^{\nu}}e^{ax}=a^{\nu}e^{ax},\quad\frac{d^{\nu}}{dx^{\nu}}\sin ax=a^{\nu}\sin\left(ax+\frac{\pi}{2}\nu\right),\quad a\in\mathbb{R},$$
$$\frac{d^{\nu}}{dx^{\nu}}x^n=\frac{\Gamma(n+1)}{\Gamma(n+1-\nu)}x^{n-\nu},\quad n\geq 0.$$

Comparison between (52) and (53) shows the efficiency of this method even with rough wavelet approximations (Figure 6).

Appendix

Proposition A.1. *Let $p\in\mathbb{N}$ and $m\in\mathbb{N}_0$. Let E be the set of even numbers. Then,*

$$\frac{d^p}{dx^p}T_m(x)=2^p m\sum_{\substack{0\leq r\leq m-p:\\ m-p-r\in E}}d_{m,r}^{p}\,T_r(x)-2^{p-1}m\,d_m^p\,[\![m-p\in E]\!],$$

where

$$d_{m,r}^{p}:=\left(\frac{m+p+r}{2}-1\right)^{\underline{p-1}}\binom{\frac{m+p-r}{2}-1}{p-1},$$

with $d_m^p=d_{m,0}^p$.

Proof. We begin by recalling the following expansion:

$$\frac{d^p}{dx^p}T_m(x)=2^p\sum_{0\leq j\leq(m-p/2)}m(m-1-j)^{\underline{p-1}}\binom{p+j-1}{p-1}T_{m-p-2j}(x)-[\![m-p\in E]\!]2^{p-1}m\left(\frac{m+p}{2}-1\right)^{\underline{p-1}}\binom{\frac{m+p}{2}-1}{p-1},$$

which is due to Prodinger [38]. By the change of variable $r = m - p - 2j$ in the previous sum, we get

$$\frac{d^p}{dx^p} T_m(x) = 2^p m \sum_{\substack{0 \le r \le m-p : \\ m-p-r \in E}} \left(\frac{m+p+r}{2} - 1\right)^{\underline{p-1}} \begin{pmatrix} \frac{m+p-r}{2} - 1 \\ p-1 \end{pmatrix} T_r(x) - [\![m-p \in E]\!] 2^{p-1} m \left(\frac{m+p}{2} - 1\right)^{\underline{p-1}} \begin{pmatrix} \frac{m+p}{2} - 1 \\ p-1 \end{pmatrix},$$

which completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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