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Research Article

Dynamical Behavior of Stochastic Markov Switching Hepatitis B Epidemic Model with Saturated Incidence Rate

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The article researches a stochastic hepatitis B epidemic model with saturated incidence rate, which is perturbed by both white noise and colored noise. Firstly, we obtain a significant criterion R_0^S which relies on environmental noises. By means of Lyapunov function approach, we show that there is a stationary distribution if $R_0^S > 1$. Its condition implies that when white noise is small, in the stochastic model, there exists a stochastic positive equilibrium state without changing the basic properties of its corresponding deterministic model. Secondly, we derive sufficient criteria for extinction of the disease. Finally, we propose a definition of the solution to an impulsive stochastic functional differential equation with Markovian switching (ISFDM).

1. Introduction

Hepatitis B virus is a severe infectious disease that has emerged as one of the greatest threats to human health in the 21st century. An estimated 350 million people worldwide have been infected with hepatitis B virus [1]. The mathematical model to describe hepatitis B virus transmission and its dynamics has been extensively explored, which provides some effective suggestions for further study on the progression and its control [2–5]. Recently, Khan et al. [6] investigated a hepatitis B epidemic model with saturated incidence rate:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \dfrac{\alpha SI}{1+\gamma I} - (\mu_0 + \nu)S, \\ \dfrac{dI}{dt} = \dfrac{\alpha SI}{1+\gamma I} - (\mu_0 + \mu_1 + \beta)I, \\ \dfrac{dR}{dt} = \beta I + \nu S - \mu_0 R, \end{cases} \tag{1}$$

with $S(0) > 0, I(0) > 0$, and $R(0) > 0$. In model (1), the birth rate is denoted by Λ. The transmission rate of hepatitis B is given by α, while μ_0 and μ_1, respectively, demonstrated the natural and disease-induced death rates. Recovery rate is denoted by β, while the vaccination and saturation rates are ν and γ, respectively. According to the theory in [6], model (1) always has the disease-free equilibrium $E^0 = (S^0, 0, R^0)$, where the components are defined as $S^0 = \Lambda/(\mu_0 + \nu)$, and $R^0 = \Lambda\nu/(\mu_0(\mu_0 + \nu))$. If $R_0 < 1$, E^0 is globally asymptotically stable. If $R_0 > 1$, E^0 is unstable and there exists an endemic equilibrium $E^* = (S^*, I^*, R^*)$ which is globally asymptotically stable, where $R_0 = \alpha\Lambda/((\mu_0 + \nu)(\mu_0 + \mu_1 + \beta))$.

In fact, epidemic models are inherently subject to a continuous spectrum of disturbances [7–11]. Many authors demonstrated that the white noise and colored noise have a great destabilizing influence on the epidemic transmission. Moreover, considering the effect of environment noise on the epidemic model has become a popular trend in controlling the spread of disease [12–16]. In this respect, some researches on stochastic hepatitis B virus models have been reported [17–19]. Particularly, in the epidemic model, the disease transmission rate α represents an extremely important coefficient [16, 20]. In this paper, by taking into account the effect of continuous-time Markov chain on the transmission rate α, we consider a stochastic analogue of the deterministic model (1):

$$\begin{cases} dS = \left(\Lambda - \dfrac{\alpha(\xi(t))SI}{1+\gamma I} - (\mu_0+\nu)S\right)dt + \sigma_1(\xi(t))SdB_1(t), \\ dI = \left(\dfrac{\alpha(\xi(t))SI}{1+\gamma I} - (\mu_0+\mu_1+\beta)I\right)dt + \sigma_2(\xi(t))IdB_2(t), \\ dR = (\beta I + \nu S - \mu_0 R)dt + \sigma_3(\xi(t))RdB_3(t), \end{cases} \tag{2}$$

where $B_i(t)$ are independent standard Brownian motions and σ_i^2 stand for the intensities of $B_i(t), i = 1, 2, 3$. $\xi(t), t \geq 0$, is a right-continuous Markov chain on the complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with values in a finite space $\mathscr{M} = \{1, 2, \cdots, N\}$ (see [21, 22]).

It is widely known that the stability of biomathematical model has always been a hot issue in recent years [23–26]. Compared with their corresponding deterministic cases, lots of stochastic models have no traditional positive equilibrium state. Consequently, the research of ergodic stationary distribution of s stochastic biomathematical model has been a research highlight. In addition, model (2) incorporates white noise as well as colored noise possessing important practical significance [27]. The main aim of this article is to prove the existence of stationary distribution for model (2). Above all, to guarantee existence and uniqueness of globally positive solution for model (2), we establish the following conclusion. Since the proof is standard, we omit it here.

Lemma 1. *For any initial value $(S(0), I(0), R(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathscr{M}$, there exists a unique positive solution $(S(t), I(t), R(t), \xi(t)) \in \mathbb{R}_+^3 \times \mathscr{M}$ of model (2) on $t \geq 0$ almost surely (a.s.).*

2. Existence of a Unique and Ergodic Stationary Distribution

Theorem 2. *If $R_0^S > 1$, where*

$$R_0^S = \frac{\sum_{k\in\mathscr{M}} \pi_k \alpha(k)\Lambda}{\left(\mu_0 + \nu + \sum_{k\in\mathscr{M}} \pi_k \left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathscr{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)}, \tag{3}$$

then for any initial value $(S(0), I(0), R(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathscr{M}$, model (2) has a unique stationary distribution which is ergodic.

Proof. In order to prove Theorem 2, we need to validate that the feasibility of (A1), (A2), and (A3) in Lemma 7 in the appendix holds. We have assumed (A1) holds in Section 1. To verify (A3), we need to find a nonnegative C^2-function $V(S, I, R, k)$ and a compact set $D_\varepsilon \in \mathbb{R}_+^4$ such that $LV \leq -1$ for all $(S, I, R, k) \in (\mathbb{R}_+^3 \setminus D_\varepsilon) \times \mathscr{M}$. Construct a C^2-function

$$\begin{aligned} V(S,I,R) &= M(-c_1 \ln S - c_2 \ln I) + \rho(k) + (S+I+R)^{\rho+1} \\ &\quad - \ln S - \ln I - \ln R = MV_1 + V_2 + V_3 + V_4 + V_5, \end{aligned} \tag{4}$$

where $V_1 = -c_1 \ln S - c_2 \ln I + \rho(k)$, $V_2 = (S+I+R)^{\rho+1}$, $V_3 = -\ln S$, $V_4 = -\ln I$, $V_5 = -\ln R$, and $0 < \rho < 2\mu_0 / \max_{i=1,2,3}\{\breve{\sigma}_i^2\}$, where $\breve{\sigma}_i = \max_{k\in\mathscr{M}}\{\sigma_i(k)\}$, and constants M, c_1, c_2, compact set D_ε and function $\rho(k)$ will be determined later. Employing Itô's formula [28–34], we can get

$$\begin{aligned} \mathscr{L}V_1 &= -\frac{c_1\Lambda}{S} + \frac{c_1\alpha(k)I}{1+\gamma I} + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) \\ &\quad - \frac{c_2\alpha(k)S}{1+\gamma I} + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &= +\sum_{l\in\mathscr{M}}\zeta_{kl}\rho(l) - \frac{c_1\Lambda}{S} - \frac{c_2\alpha(k)S}{1+\gamma I} - (1+\gamma I) + \frac{c_1\alpha(k)I}{1+\gamma I} \\ &\quad + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &\quad + (1+\gamma I) + \sum_{l\in\mathscr{M}}\zeta_{kl}\rho(l) \leq -3\sqrt[3]{c_1c_2\alpha(k)\Lambda} + 1 \\ &\quad + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &\quad + \gamma I + \frac{c_1\alpha(k)I}{1+\gamma I} + \sum_{l\in\mathscr{M}}\zeta_{kl}\rho(l). \end{aligned} \tag{5}$$

Choose $M_1(k) = -3\sqrt[3]{c_1c_2\alpha(k)\Lambda} + 1 + c_1(\mu_0+\nu+(1/2)\sigma_1^2(k)) + c_2(\mu_0+\mu_1+\beta+(1/2)\sigma_2^2(k))$; on the basis of the irreducibility of generator matrix Γ, one can find that for $\Theta = (\Theta(1), \Theta(2), \cdots, \Theta(N))$, there exists $\rho = (\rho(1), \rho(2), \cdots, \rho(N))^T$ satisfying the following Poisson system $\Gamma\rho = \left(\sum_{k=1}^N \pi_k \Theta(k)\right)\overleftarrow{1} - \Theta$. Let c_1 and c_2 satisfy

$$\begin{aligned} c_1\left(\mu_0+\nu+\sum_{k\in\mathscr{M}}\pi_k\frac{\sigma_1^2(k)}{2}\right) &= c_2\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathscr{M}}\pi_k\frac{\sigma_2^2(k)}{2}\right) \\ &= \frac{\sum_{k\in\mathscr{M}}\pi_k\alpha(k)\Lambda}{\left(\mu_0+\nu+\sum_{k\in\mathscr{M}}\pi_k\left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathscr{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)}. \end{aligned} \tag{6}$$

Then,

$$\begin{aligned} \mathscr{L}V_1 &\leq -\frac{\sum_{k\in\mathscr{M}}\pi_k\alpha(k)\Lambda}{\left(\mu_0+\nu+\sum_{k\in\mathscr{M}}\pi_k\left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathscr{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)} \\ &\quad + 1 + \gamma I + \frac{c_1\alpha(k)I}{1+\gamma I} \leq -\left(R_0^S - 1\right) + \gamma I + c_1\breve{\alpha}I = -\lambda + \varphi(I), \end{aligned} \tag{7}$$

where

$$\lambda = R_0^S - 1,$$
$$\varphi(I) = \gamma I + c_1 \breve{\alpha} I, \tag{8}$$

and set $\breve{\alpha} = \max_{k\in\mathscr{M}}\{\alpha(k)\}$. Applying Itô's formula, one can obtain

$$\begin{aligned}\mathscr{L}V_2 &= (\rho+1)(S+I+R)^{\rho}(\Lambda - \mu_0 S - (\mu_0+\mu_1)I - \mu_0 R) \\ &\quad + \frac{1}{2}\rho(\rho+1)(S+I+R)^{\rho-1}(\sigma_1^2(k)S^2 + \sigma_2^2(k)I^2 \\ &\quad + \sigma_3^2(k)R^2) \le (\rho+1)(S+I+R)^{\rho}(\Lambda - \mu_0(S+I+R)) \\ &\quad + \max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\frac{\rho}{2}(\rho+1)(S+I+R)^{\rho+1} = \Lambda(\rho+1)(S+I+R)^{\rho} \\ &\quad - (\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{\rho+1} \\ &\le B - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{\rho+1} \\ &\le B - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}),\end{aligned} \tag{9}$$

where

$$\begin{aligned}B = \sup_{(S,I,R)\in\mathbb{R}_+^3}&\left\{\Lambda(\rho+1)(S+I+R)^{\rho} - \frac{1}{2}(\rho+1)\right. \\ &\left.\cdot\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{(\rho+1)}\right\} < \infty.\end{aligned} \tag{10}$$

Denote

$$C = \sup_{(S,I,R)\in\mathbb{R}_+^3}\left\{\theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1} + I^{\rho+1} + R^{\rho+1})\right\}, \tag{11}$$

where $\theta = B + (\mu_0 + \nu + (1/2)\breve{\sigma}_1{}^2) + (\mu_0 + \mu_1 + \beta + (1/2)\breve{\sigma}_2{}^2) + (\mu_0 + (1/2)\breve{\sigma}_3{}^2)$. By using Itô's formula, we also have

$$\begin{aligned}\mathscr{L}V_3 &= -\frac{\Lambda}{S} + \frac{\alpha(k)I}{1+\gamma I} + \mu_0 + \nu + \frac{1}{2}\sigma_1^2(k), \\ \mathscr{L}V_4 &= -\frac{\alpha(k)S}{1+\gamma I} + \mu_0 + \mu_1 + \beta + \frac{1}{2}\sigma_2^2(k), \\ \mathscr{L}V_5 &= -\beta\frac{I}{R} - \nu\frac{S}{R} + \mu_0 + \frac{1}{2}\sigma_3^2(k).\end{aligned} \tag{12}$$

Hence, by (7), (9), and (12), we get

$$\begin{aligned}\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\ &\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}),\end{aligned} \tag{13}$$

where $\widehat{\alpha} = \min_{k\in\mathscr{M}}\{\alpha(k)\}$. Here, we choose that the positive constant M satisfies the following inequality:

$$-M\lambda + C \le -2. \tag{14}$$

For arbitrary $\varepsilon > 0$, define the following bounded closed set:

$$D_\varepsilon = \left\{\varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I \le \frac{1}{\varepsilon}, \varepsilon^2 \le R \le \frac{1}{\varepsilon^2}\right\}, \tag{15}$$

where ε satisfies the following conditions:

$$\begin{aligned}&-\frac{\Lambda}{\varepsilon} + K \le -1, \\ &-M\lambda + M\varphi(\varepsilon) + \breve{\alpha}\varepsilon + C \le -1, \\ &-\frac{\beta}{\varepsilon} + K \le -1, \\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + D \le -1, \\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + E \le -1, \\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + F \le -1,\end{aligned} \tag{16}$$

where

$$\begin{aligned}K &= \sup_{(S,I,R)\in\mathbb{R}_+^3}\{M\varphi(I) + \breve{\alpha}I + C\}, \\ D &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(I^{\rho+1} + R^{\rho+1})\right\}, \\ E &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1} + R^{\rho+1})\right\}, \\ F &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1} + I^{\rho+1})\right\}.\end{aligned} \tag{17}$$

Furthermore,

$$\mathbb{R}_+^3 \setminus D_\varepsilon = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6, \tag{18}$$

where

$$
\begin{aligned}
D_1 &= \left\{(S, I, R) \in \mathbb{R}_+^3, 0 < S < \varepsilon\right\},\\
D_2 &= \left\{(S, I, R) \in \mathbb{R}_+^3, 0 < I < \varepsilon\right\},\\
D_3 &= \left\{(S, I, R) \in \mathbb{R}_+^3, 0 < R < \varepsilon^2, S > \varepsilon, I > \varepsilon\right\},\\
D_4 &= \left\{(S, I, R) \in \mathbb{R}_+^3, S > \frac{1}{\varepsilon}\right\},\\
D_5 &= \left\{(S, I, R) \in \mathbb{R}_+^3, I > \frac{1}{\varepsilon}\right\},\\
D_6 &= \left\{(S, I, R) \in \mathbb{R}_+^3, R > \frac{1}{\varepsilon}\right\}.
\end{aligned}
\tag{19}
$$

Case 1. If $(S, I, R) \in D_1$, we derive that

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -\frac{\Lambda}{\varepsilon} + K \le -1.
\end{aligned}
\tag{20}
$$

Case 2. If $(S, I, R) \in D_2$, we have

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -M\lambda + M\varphi(\varepsilon) + \breve{\alpha}\varepsilon + C \le -1.
\end{aligned}
\tag{21}
$$

Case 3. If $(S, I, R) \in D_3$, we compute

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -\frac{\beta}{\varepsilon} + K \le -1.
\end{aligned}
\tag{22}
$$

Case 4. If $(S, I, R) \in D_4$, we derive

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + D \le -1.
\end{aligned}
\tag{23}
$$

Case 5. If $(S, I, R) \in D_5$, we conclude

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + E \le -1.
\end{aligned}
\tag{24}
$$

Case 6. If $(S, I, R) \in D_6$, we have

$$
\begin{aligned}
\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha} I - \frac{\Lambda}{S} - \frac{\widehat{\alpha} S}{1+\gamma I} - \beta \frac{I}{R} - \nu \frac{S}{R} + \theta\\
&\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1} + I^{\rho+1} + R^{\rho+1}\right)\\
&\le -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\breve{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + F \le -1.
\end{aligned}
\tag{25}
$$

Then, we can obtain that for a sufficiently small ε, $LV < -1$ for any $(S, I, R) \in \mathbb{R}_+^3 \setminus D_\varepsilon$. Therefore, we can verify (A3) in Lemma 7 of the appendix. On the other hand, the diffusion matrix $D(x, k) = \operatorname{diag}\left\{\sigma_1^2(k)S^2, \sigma_2^2(k)I^2, \sigma_3^2(k)R^2\right\}$ of model (2) is positive definite, which implies that condition (A2) in Lemma 7 holds. This completes the proof.□ □

Now, consider the corresponding model (2) without Markov switching:

$$
\begin{cases}
dS = \left(\Lambda - \dfrac{\alpha S I}{1+\gamma I} - (\mu_0 + \nu)S\right)dt + \sigma_1 S dB_1(t),\\
dI = \left(\dfrac{\alpha S I}{1+\gamma I} - (\mu_0 + \mu_1 + \beta)I\right)dt + \sigma_2 I dB_2(t),\\
dR = (\beta I + \nu S - \mu_0 R)dt + \sigma_3 R dB_3(t).
\end{cases}
\tag{26}
$$

Define a parameter

$$
\widehat{R}_0 = \frac{\alpha \int_0^\infty x\pi(x)dx}{\mu_0 + \mu_1 + \beta + \left(\sigma_2^2/2\right)},
\tag{27}
$$

where

$$
\begin{aligned}
\pi(x) &= Q x^{-2-(2(\mu_0+\nu))/\sigma_1^2}\sigma_1^{-2+(2(\mu_0+\nu))/\sigma_1^2} e\\
&\quad -\left(2/\sigma_1^2\right)\left(\left(\Lambda/x\right)+\left(\mu_0+\nu\right)\right), \quad x \in (0, +\infty).
\end{aligned}
\tag{28}
$$

Similar to Theorem 3.1 in [35], it is easy to obtain the following result.

Theorem 3. *Let $(S(t), I(t), R(t))$ be the solution of model (26). If $\widehat{R}_0 < 1$, for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3$,*

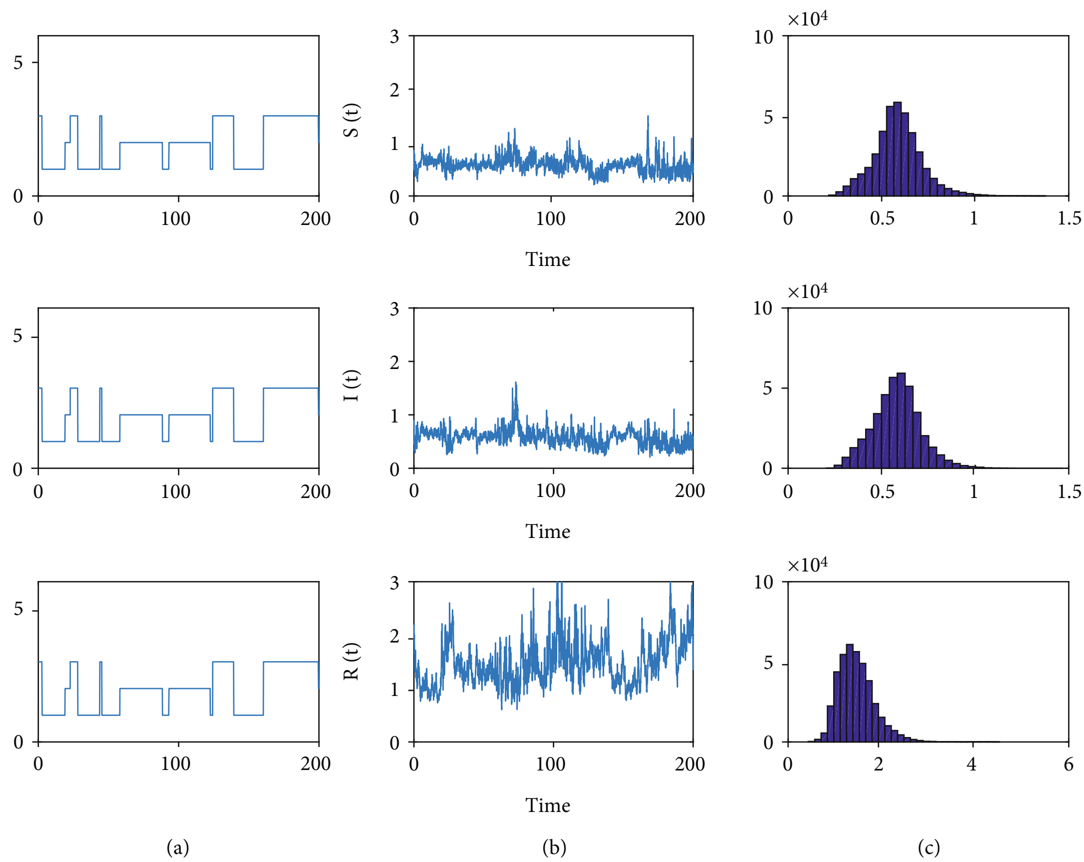

Figure 1: $S(t)$, $I(t)$, and $R(t)$ have ergodic property. The pictures in (a) are Markovian chain. The pictures in (c) are the density functions of model (2) for $k \in \mathcal{M} = \{1, 2, 3\}$. The initial value $S(0) = 0.8$, $I(0) = 0.7$, and $R(0) = 1.1$. Step size $\Delta t = 0.001$.

then the solution $(S(t), I(t), R(t))$ of model (26) satisfies

$$\lim_{t \longrightarrow +\infty} I(t) = 0\, a.s., \tag{29}$$

and the distribution of $S(t)$ converges weakly to the measure which has the density

$$\pi(x) = Qx^{-2-\left((2(\mu_0+\nu))/\sigma_1^2\right)} \sigma_1^{-2+\left((2(\mu_0+\nu))/\sigma_1^2\right)} e^{-\left(2/\sigma_1^2\right)\left((\Lambda/x)+(\mu_0+\nu)\right)}, \quad x \in (0,+\infty), \tag{30}$$

where Q is a constant such that $\int_0^\infty \pi(x)dx = 1$.

Remark 4. In Theorem 2, we derive $R_0^S = R_0$ when $\alpha(k) \equiv \alpha$ and $\sigma_i(k) \equiv 0$. This conclusion accords with practice.

3. Numerical Examples

In this section, we will test our theory conclusion by Milstein's higher order method in [36].

Example 1. Let the generator of the Markov chain ζ_{ij} be

$$\Gamma = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}, \tag{31}$$

in which ζ_{ij} is a right-continuous Markov chain taking value in $\mathcal{M} = \{1, 2, 3\}$. By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \pi_3) = (2/7, 3/7, 2/7)$. Choose parameters $\Lambda = 0.232$, $\gamma = 0.9$, $\mu_0 = 0.000232$, $\nu = 0.02$, $\mu_1 = 0.0000547$, $\beta = 0.12$, $\alpha(1) = 0.0013$, $\alpha(2) = 0.00129$, $\alpha(3) = 0.00132$, $\sigma_1(1) = 0.01$, $\sigma_2(1) = 0.02$, $\sigma_3(1) = 0.06$, $\sigma_1(2) = 0.011$, $\sigma_2(2) = 0.022$, $\sigma_3(2) = 0.055$, $\sigma_1(3) = 0.009$, $\sigma_2(3) = 0.019$, and $\sigma_3(3) = 0.063$. Then, $R_0^S = 1.2226 > 1$. In view of Theorem 2, there is a stationary distribution of model (2), and it is ergodic. Phase portrait of $(S(t), I(t), R(t))$ and histograms of $(S(t), I(t), R(t))$ are plotted in Figure 1.

Example 2. Select parameters $\Lambda = 0.232$, $\gamma = 0.9$, $\mu_0 = 0.000232$, $\nu = 0.02$, $\mu_1 = 0.0000547$, $\beta = 0.12$, $\alpha = 0.04$, $\sigma_1 = 0.1$, $\sigma_2 = 0.08$, and $\sigma_3 = 0.05$. By calculation, $R_0 = \alpha\Lambda/((\mu_0$

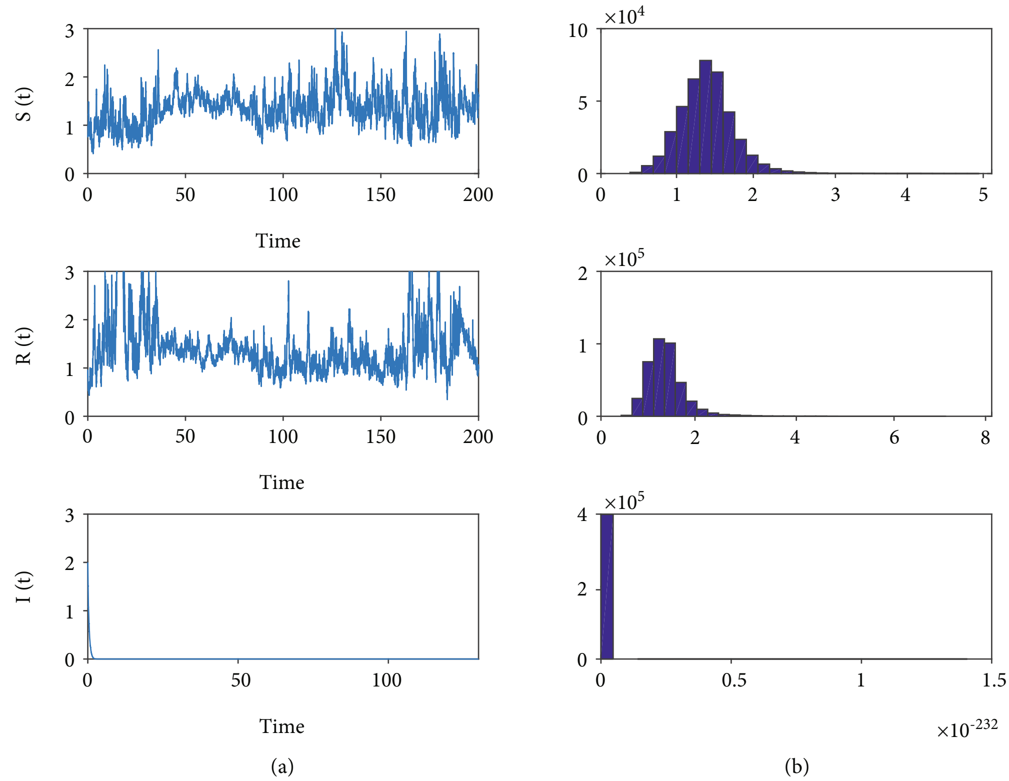

Figure 2: The left column reflects the simulation of number variations of $S(t), R(t)$ and $I(t)$ in model (26) with the initial value $S(0) = 0.8, R(0) = 1.1$, and $I(0) = 0.7$ and the noise intensities given in Example 2. The right column reveals the relevant histogram of density functions of the classes $S(t), R(t)$, and $I(t)$. Step size $\Delta t = 0.001$.

$+\nu)(\mu_0 + \mu_1 + \beta)) = 3.87 > 1$, $\int_0^\infty x\pi(x)dx = 1.16$, and $\widehat{R}_0 = 0.377 < 1$. It means that there exists a unique endemic equilibrium of determined model (1), which is globally asymptotically stable. Instead, in view of Theorem 3, we have $\lim_{t\longrightarrow+\infty} I(t) = 0$ a.s. and the distribution of $S(t)$ in model (26) converges weakly to the measure $\pi(x)$ (see Figure 2).

4. Concluding Remarks

The paper successfully investigates extinction and stationary distribution of a stochastic Markov switching hepatitis B epidemic model with saturated incidence rate. Besides the effect of Markovian switching on the deterministic SIRS epidemic models [37–39], pulse vaccination strategy (PVS) has been adopted to control the outbreaks and fastly tackle the spread of disease by wide areas [40]. In order to help future research, we propose the following definition related to SIR model by taking into account Markovian switching, impulse, and infinite delay.

Definition 5. Considering the following impulsive stochastic functional differential equation with Markovian switching(ISFDM),

$$\begin{cases} dY(t) = F_1\left(t, \zeta(t), Y(t), \int_{-\infty}^{0} Y(t+\theta)d\mu_1(\theta)\right)dt + F_2\left(t, \zeta(t), Y(t), \int_{-\infty}^{0} Y(t+\theta)d\mu_2(\theta)\right)dB(t), \\ t \neq t_k, \quad k \in N, \\ Y(t_k^+) - Y(t_k) = H_k Y(t_k), \quad k \in N, \end{cases} \tag{32}$$

where $Y(t+\theta), -\infty<\theta\leq 0$, represents C_g-value stochastic process, $C_g = \{\psi \in C((-\infty,0]\,;\mathbb{R}^d) \colon \|\psi\|_{c_g} = \sup_{-\infty<s\leq 0} e^{qs}\,|\,\psi(s)| < +\infty\}$, $g(s) = e^{-qs}, q > 0, |\psi(s)| = \sqrt{\psi_1^2(s) + \psi_2^2(s) + \cdots + \psi_d^2(s)}$, and $(\psi_1(s), \psi_2(s), \cdots, \psi_d(s)) \in \mathbb{R}^d$. $H_k > -1$, $\zeta(t)$ denotes the regime switching [41, 42]. For $i = 1, 2$, $\mu_i(\theta)$ is a measure on $(-\infty, 0]$, $0 < t_1 < t_2 < \cdots$, $\lim_{k\longrightarrow+\infty} t_k = +\infty$. The initial condition $Y_0 \in C_g$ and $\zeta(0) = 0$, where $Y_0 = \vartheta = \{\vartheta(\theta) \colon -\infty<\theta\leq 0\}$

is an $\mathscr{F}_0$-measurable C_g-valued random variable such that $\vartheta \in \mathscr{M}^2((-\infty, 0]; \mathbb{R}^d)$ which is the family of all $\mathscr{F}_0$-measurable, $\mathbb{R}^d$-valued processes $\psi(t), t\in(-\infty,0]$ such that $\mathbb{E}\int_{-\infty}^0 |\psi(t)|^2 dt<+\infty$. An $\mathbb{R}^d$-value stochastic process $Y(t)$ defined on $\mathbb{R}$ is called a solution of Equation (32) with initial condition above when $Y(t)$ satisfies the following criterion:

(i) $Y(t)$ is $\mathscr{F}_t$-adapted and continuous on $(0, t_1)$ and $(t_k, t_{k+1}), k \in N$; $F_1(t, \zeta(t), Y(t), \int_{-\infty}^0 Y(t+\theta)d\mu_1(\theta)) \in \mathscr{L}^1(\bar{\mathbb{R}}_+; \mathbb{R}^d)$ and $F_2(t, \zeta(t), Y(t), \int_{-\infty}^0 Y(t+\theta)d\mu_2(\theta)) \in \mathscr{L}^2(\bar{\mathbb{R}}_+; \mathbb{R}^{d\times m})$. Here, the interpretations of $\mathscr{L}^1(\bar{\mathbb{R}}_+; \mathbb{R}^d)$ and $\mathscr{L}^2(\bar{\mathbb{R}}_+; \mathbb{R}^{d\times m})$ can be found in [43]. $B(t)$ stands for a m-dimension standard Brownian motion

(ii) For each $t_k, k \in N$, $Y(t_k^+) = \lim_{t\longrightarrow t_k^+} Y(t)$ and $Y(t_k) = Y(t_k^-) = \lim_{t\longrightarrow t_k^-} Y(t)$ a.s.

(iii) $Y(t)$ satisfies the equivalent integral equation of (32) for almost every $t \in [0,\infty) \setminus t_k$ and satisfies the impulsive criterion at each $t = t_k, k \in N$ with probability one

Remark 6. Liu and Wang [44] give a new definition of a solution of an impulsive stochastic differential equation (ISDE). We propose Definition 5, which generalizes the definition of a solution of ISDE to ISFDM, because time memory and Markovian switching are very important in the fields of infectious disease, biological engineering, chemical engineering, etc.

Appendix

Let $(X(t), \xi(t))$ be the diffusion process described by the following equation [(31)]:

$$dX(t) = b(X(t), \xi(t))dt + \sigma(X(t), \xi(t))dB(t), X(0) = x_0, r(0) = \gamma, \tag{A1}$$

where $b(\cdot, \cdot): \mathbb{R} \times \mathscr{M} \longrightarrow \mathbb{R}^n$, $\sigma(\cdot, \cdot): \mathbb{R} \times \mathscr{M} \longrightarrow \mathbb{R}^{n\times n}$, and $D(x, k) = \sigma(x, k)\sigma^T(x, k) = (d_{ij}(x, k))$. For each $k \in \mathscr{M}$, let $V(\cdot, k)$ be any twice continuously differentiable function; the operator $\mathscr{L}$ can be defined by

$$\mathscr{L}V(x, k) = \sum_{i=1}^n b_i(x, k)\frac{\partial V(x, k)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n d(x, k)\frac{\partial^2 V(x, k)}{\partial x_i \partial x_j} + \sum_{l=1}^N \vartheta_{kl} V(x, l). \tag{A2}$$

According to theorems in [27], it follows the following lemma which provides a criterion for the ergodic stationary distribution of the solution $(X(t), \xi(t))$ to model (A1).

Lemma 7 ([22]). *If the following conditions are satisfied:*

(A1) $\vartheta_{ij} > 0$ for any $i \neq j$.

(A2) For each $k \in \mathscr{M}, D(x, k) = (d_{ij}(x, k))$ is symmetric and satisfies $\lambda|\varpi|^2 \le \langle D(x, k)\varpi, \varpi\rangle \le \lambda^{-1}|\varpi|^2$ for all $\varpi \in \mathbb{R}^n$, with some constant $\lambda \in (0, 1]$ for all $x \in \mathbb{R}^n$.

(A3) There exists a nonempty open set $\mathscr{D}$ with compact closure, satisfying that, for each $k \in \mathscr{M}$, there is a nonnegative function $V(\cdot, k): \mathscr{D}^c \longrightarrow \mathbb{R}$ such that $V(x, k)$ is twice continuously differential and that for some $\alpha > 0$, $\mathscr{L}V(x, k) \le -\alpha$, $(x, k \in \mathscr{D}^c \times \mathscr{M})$, then $(x(t), \xi(t))$ of system (A1) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution.

Data Availability

No data were used in this study.

Conflicts of Interest

The author declares that there are no competing interests.

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