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Research Article

Diverse Precise Traveling Wave Solutions Possessing Beta Derivative of the Fractional Differential Equations Arising in Mathematical Physics

Imran Siddique,[1] **Arshad M. Mirza**,[2] **Kausar Shahzadi**,[1] **M. Ali Akbar**,[3] **and Fahd Jarad**[4,5]

[1]*Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan*
[2]*Department of Physics, University of Management and Technology, Lahore 54770, Pakistan*
[3]*Department of Applied Mathematics, University of Rajshahi, Bangladesh*
[4]*Department of Mathematics, Cankaya University, Etimesgut, Ankara 06790, Turkey*
[5]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*

Correspondence should be addressed to M. Ali Akbar; ali_math74@yahoo.com and Fahd Jarad; fahd@cankaya.edu.tr

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In this paper, we obtain the novel exact traveling wave solutions in the form of trigonometric, hyperbolic and exponential functions for the nonlinear time fractional generalized reaction Duffing model and density dependent fractional diffusion-reaction equation in the sense of beta-derivative by using three fertile methods, namely, Generalized tanh (GT) method, Generalized Bernoulli (GB) sub-ODE method, and Riccati-Bernoulli (RB) sub-ODE method. The derived solutions to the aforementioned equations are validated through symbolic soft computations. To promote the vital propagated features; some investigated solutions are exhibited in the form of 2D and 3D graphics by passing on the specific values to the parameters under the confine conditions. The accomplished solutions show that the presented methods are not only powerful mathematical tools for generating more solutions of nonlinear time fractional partial differential equations but also can be applied to nonlinear space-time fractional partial differential equations.

1. Introduction

Soliton theory has much importance because many equations of mathematical physics have the solution of soliton type. Waves are generated when some disturbance occurs in the phenomena. Soliton interaction takes place when two or more soliton come close to each other. Solitons exhibit particle-like properties because the energy is—at any instant—confined to a limited region of space. The most important technical application of the soliton is that these are used in the optical fibers to carry the digital information. In electromagnetic soliton studies, the transverse electromagnetic wave travels between two strips of super conducting metal.

Fractional calculus has captured the interest of several scholars during the past two centuries. Multiple nonlinear aspects, biological processes, fluid mechanics, chemical processes, etc., are modelled using them. Fractional order partial differential equations (PDEs) serve as the generalization of PDEs in the traditional integer-order. The literature contains several dentitions of fractional derivatives, such as the Hadamard derivative (1892) [1], the Weyl derivative [2], Riesz derivative [3], He's fractional derivative [4], Local derivative [5], Riemann-Liouville [6, 7], Abel-Riemann derivative [8], Caputo [9], Caputo-Fabrizio [10], Atangana-Baleanu derivative in the context of Caputo [11], the conformable fractional derivative [12, 13], and the new truncated M-fractional derivative [14]. Atangana et al. in [15] have recently created the new beta-derivative which satisfies a lot of characteristics that have been considered as limitations for the fractional derivatives. This derivative has some appealing consequences in diverse areas including fluid mechanics, optical physics, chaos theory, biological models, disease analysis, circuit analysis, and others.

Nonlinear fractional differential equations (NLFDEs) occur more frequently in engineering applications and different research areas [16–20]. Then, many real-life problems can be modeled by ordinary or partial differential equations involving the derivatives of fractional order. In order to better understand and apply these physical phenomena in practical scientific research, it is important to find their exact solutions. Finding exact solutions of most of the NLFDEs is not easy, so searching and constructing exact solutions of NLFDEs is a continuing investigation. Recently, many powerful methods for obtaining exact solutions of nonlinear partial differential equations (NLPDEs) have been presented, such as exponential rational function method [21], $\exp_a$ function, and the hyperbolic function methods [22]. (G'/G)-expansion method [23, 24], $(G'/G, 1/G)$-expansion method [25, 26], Sardar-subequation method [27], new subequation method [28], Riccati equation method [29], homotopy perturbation method [30], extended direct algebraic method [31], Kudryashov method [32], Exp-function method [33], the modified extended exp-function method [34], F-expansion method [35], the Backlund transformation method [36], the extended tanh-method [37], Jacobi elliptic function expansion methods [38], extended sinh-Gordon equation expansion method [39], and different other methods [40–43].

The core aim of this work is to establish the exact traveling wave solutions of the fractional generalized reaction doffing model arising in mathematical biology [44, 45] and the density dependent fractional diffusion-reaction equation with the beta-derivative based on three different methods, the Generalized tanh (GT) method [46], Generalized Bernoulli (GB) sub-ODE method [47], and Riccati-Bernoulli (RB) sub-ODE method [48]. These methods are the most direct and effective algebraic methods used for obtaining the exact traveling wave solutions of nonlinear partial differential equations. In [49], Jafari et al. applied the fractional subequation method to construct exact solutions of the fractional generalized reaction Duffing model and in [50], Eslami et al. applied the first integral method to obtain the exact solutions of fractional generalized reaction Duffing model and the exact solutions of fractional diffusion-reaction equation. Uddin et al. [44] obtained the close form solutions of the fractional generalized reaction Duffing model and the density dependent fractional diffusion reaction equation by using the $(G'/G, 1/G)$-expansion method. In [51] Xia et al. applied hyperbolic function to obtain new explicit and exact travelling wave solutions for a class of nonlinear evolution equations. Sonmezoglu [52] applied extended Jacobi elliptic function expansion to construct the exact solutions of these models.

This paper is organized as follows: In Section 2, we present beta derivative and its properties. The descriptions of strategies are given in Section 3. In Sections 4 and 5, we present a mathematical analysis of the models and its solutions via proposed methods. In Section 6, the graphical comparisons of our obtained exact traveling wave solutions are represented in both 2D and 3D plots for various values of parameters. At the end, conclusions are announced in Section 7.

2. Beta Derivative and Its Properties

Definition. The beta-derivative is defined as [15, 53]

$$ {}_{0}^{A}D_{x}^{\alpha}(f(x)) = \lim_{\varepsilon\longrightarrow 0}\frac{f(x+\epsilon(x+(1/\Gamma(\alpha))))-f(x)}{\epsilon}, 0<\alpha\leq 1. \tag{1}$$

Properties of Beta Derivative. Beta derivative has the following properties:

(1)

$$ {}_{0}^{A}D_{x}^{\alpha}[af(x)+bg(x)] = a{}_{o}^{A}D_{x}^{\alpha}f(x)+b{}_{o}^{A}D_{x}^{\alpha}g(x) \tag{2}$$

(2) ${}_{0}^{A}D_{x}^{\alpha}(c)=0$, for any constant c

(3)

$$ {}_{0}^{A}D_{x}^{\alpha}[f(x).g(x)] = g(x){}_{o}^{A}D_{x}^{\alpha}f(x)+f(x){}_{o}^{A}D_{x}^{\alpha}g(x){}_{0}^{A} \tag{3}$$

(4)

$$ {}_{0}^{A}D_{x}^{\alpha}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x){}_{0}^{a}D_{x}^{\alpha}f(x)-f(x){}_{0}^{a}D_{x}^{\alpha}g(x)}{g^{2}(x)} \tag{4}$$

Considering $\epsilon=(x+(1/\Gamma(\alpha)))^{\alpha-1}h$, $h\longrightarrow 0$ when $\epsilon\longrightarrow 0$, therefore we have

$$ {}_{0}^{A}D_{x}^{\alpha}f(x) = \left(x+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\frac{df(x)}{d(x)}, \tag{5}$$

with $\xi=(l/\alpha)(x+(1/\Gamma(\alpha)))^{\alpha}$, where l is a constant.

(5)

$$ {}_{0}^{A}D_{x}^{\alpha}\left[\frac{f(\xi)}{g(x)}\right] = l\frac{df(\xi)}{d(\xi)}. \tag{6}$$

The proofs of the above beta properties were simply presented in [11].

3. Description of Strategies

3.1. Riccati-Bernoulli (RB) Sub-ODE Method. In this section, we represent the basic steps of the RB sub-ODE method [48]. Let us consider the nonlinear partial differential equation of the following form:

$$ F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \cdots) = 0, \tag{7}$$

where $u = u(x, t)$ is an unknown function and F is a polynomial depending on $u(x, t)$ and its various partial derivatives.

Step 1. By wave transformation

$$u(x, t) = u(\xi), \xi = sx + nt + d. \tag{8}$$

The wave variable permits us to reduce Equation (8) into a nonlinear ordinary differential equation for $u = u(\xi)$:

$$H\left(u, u', u'', \cdots\right) = 0, \tag{9}$$

where H is a polynomial of $u(\xi)$ and its total derivative with respect to ξ.

Step 2. Assume that the solution of Equation (9) can be expressed as:

$$u' = a_1 u^{2-m} + b_1 u + c_1 u^m, \tag{10}$$

where a_1, b_1, c_1 and m are constant to be determined later.

Equation (10) has the solution as follows:

Case 1. When $m = 1$, the solution of Equation (10) is

$$u(\xi) = Ce^{(a_1+b_1+c_1)\xi}. \tag{11}$$

Case 2. When $m \neq 1, b_1 = 0, c_1 = 0$, the solution of Equation (10) is

$$u(\xi) = ((a_1(m-1))((\xi - c_1)))^{(1/(m-1))}. \tag{12}$$

Case 3. When $m \neq 1, b_1 \neq 0, c_1 = 0$, then solution of Equation (10)

$$u(\xi) = \left(-\frac{a_1}{b_1} + Ce^{b_1(m-1)\xi}\right)^{(1/(m-1))}. \tag{13}$$

Case 4. When $m \neq 1, a_1 = 0, b_1^2 - 4a_1c_1 < 0$, thus the solution of Equation (10)

$$\begin{aligned} u(\xi) &= \left(-\frac{b_1}{2a_1} + \frac{\sqrt{4a_1c_1 - b_1^2}}{2a_1} \cdot \tan\left(\frac{(1-m)\sqrt{4a_1c_1 - b_1^2}}{2}(\xi + C)\right)\right)^{(1/(m-1))}, \\ u(\xi) &= \left(-\frac{b_1}{2a_1} + \frac{\sqrt{4a_1c_1 - b_1^2}}{2a_1} \cdot \cot\left(\frac{(1-m)\sqrt{4a_1c_1 - b_1^2}}{2}(\xi + C)\right)\right)^{(1/(m-1))}. \end{aligned} \tag{14}$$

Case 5. When $m \neq 1, a_1 \neq 0, b_1^2 - 4a_1c_1 > 0$, the solutions of Equation (10) are

$$u(\xi) = \left(-\frac{b_1}{2a_1} + \frac{\sqrt{b_1^2 - 4a_1c_1}}{2a_1} \cdot \cot\left(\frac{(1-m)\sqrt{b_1^2 - 4a_1c_1}}{2}(\xi + C)\right)\right)^{(1/(m-1))}, \tag{15}$$

$$u(\xi) = \left(-\frac{b_1}{2a_1} + \frac{\sqrt{b_1^2 - 4a_1c_1}}{2a_1} \cdot \tan\left(\frac{(1-m)\sqrt{b_1^2 - 4a_1c_1}}{2}(\xi + C)\right)\right)^{(1/(1-m))}. \tag{16}$$

Case 6. When $m \neq 1, a_1 \neq 0, b_1^2 - 4a_1c_1 = 0$, the solution of Equation (10) is

$$u(\xi) = \left(\frac{1}{a_1(m-1)(\xi + C)} - \frac{b_1}{2a_1}\right)^{(1/(1-m))}, \tag{17}$$

where C is an arbitrary constant.

4. Mathematical Analyses of the Models and Its Solutions

4.1. For Fractional Generalized Reaction Duffing Model. Here, we consider the fractional generalized reaction Duffing model in the forms in [45].

$$\begin{aligned} &\frac{\partial^{2\alpha} u(x, t)}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} u(x, t)}{\partial x^{2\alpha}} + qu(x, t) \\ &\quad + ru^2(x, t) + su^3(x, t) = 0, \quad t > 0, 0 < \alpha \leq 1, \end{aligned} \tag{18}$$

where p, q, r and s are all constants.

If we take $r = 0$, Equation (18) reduces to the following nonlinear wave equation:

$$\begin{aligned} &\frac{\partial^{2\alpha} u(x, t)}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} u(x, t)}{\partial x^{2\alpha}} \\ &\quad + qu(x, t) + su^3(x, t) = 0, \quad t > 0, 0 < \alpha \leq 1. \end{aligned} \tag{19}$$

Let us assume the transformation:

$$u(x, t) = u(\xi), \xi = \frac{k}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{c}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \tag{20}$$

where k and c are constants.

By using Equation (20) into Equation (19), we get the following ODE:

$$c^2u'' + pk^2u'' + qu + su^3 = 0. \tag{21}$$

In the following sections, the proposed methods are applied to extract the required solutions:

4.2. Solutions with GT Method [46]. Considering the homogenous balancing between the terms u'' and u^3 in Equation (21), we get $N = 1$. For $N = 1$, we write the solution of Equation (9) in the following form [46]:

$$u(\xi) = a_0 + a_1 \varphi(\xi), \tag{22}$$

where a_0 and a_1 are unknown parameters.

Substituting Equation (22) into Equation (21) and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0 and a_1 as follows:

$$\varphi^3 : 2c^2 a_1 + 2k^2 p a_1 + s a_1^3 = 0, \tag{23}$$

$$\varphi^2 : 3 s a_0 {a_1}^2 = 0, \tag{24}$$

$$\varphi^1 : 2c^2 C a_1 + 2Ck^2 p a_1 + q a_1 + 3 s a_0^2 a_1 = 0, \tag{25}$$

$$\varphi^0 : q a_0 + s a_0^3 = 0. \tag{26}$$

Solving the system of algebraic equations in (23) with the help of software MATHEMATICA, we obtain the following solutions:

$$a_0 = 0, a_1 = \pm \frac{\sqrt{q}}{\sqrt{C}\sqrt{s}}, c = \pm \frac{\sqrt{-2Ck^2 p - q}}{\sqrt{2}\sqrt{C}}. \tag{27}$$

Case 1. For C <0,

$$u_1(x, t) = \pm \frac{\sqrt{q}}{\sqrt{s}} \left(-i \tanh \sqrt{-C}\xi \right), \tag{28}$$

$$u_2(x, t) = \pm \frac{\sqrt{q}}{\sqrt{s}} \left(-i \coth \sqrt{-C}\xi \right). \tag{29}$$

Case 2. For C >0.

$$u_3(x, t) = \pm \frac{\sqrt{q}}{\sqrt{s}} \left(\tan \left(\sqrt{C}\xi \right) \right). \tag{30}$$

$$u_4(x, t) = \mp \frac{\sqrt{q}}{\sqrt{s}} \left(\cot \left(\sqrt{C}\xi \right) \right). \tag{31}$$

4.3. Solutions with GB Sub-ODE Method [47]. Consider the homogenous balancing in Equation (21), we get $N = 1$. For $N = 1$, we write the solution of Equation (9) in the following form:

$$u(\xi) = a_0 + a_1 \varphi(\xi), \tag{32}$$

where a_0 and a_1 are unknown parameters.

Substituting Equation (32) into Equation (21) and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0 and a_1 as follows:

$$\varphi^3 : 2c^2 \mu^2 a_1 + 2k^2 p \mu^2 a_1 + s {a_1}^3 = 0, \tag{33}$$

$$\varphi^2 : -3c^2 \lambda \mu a_1 - 3k^2 p \lambda \mu a_1 + 3 s a_0 a_1^2 = 0, \tag{34}$$

$$\varphi^1 : q a_1 + c^2 \lambda^2 a_1 + k^2 p \lambda^2 a_1 + 3 s a_0^2 a_1 = 0, \tag{35}$$

$$\varphi^0 : q a_0 + s a_0^3 = 0. \tag{36}$$

Solving the system of algebraic equations in (33) with the help of software MATHEMATICA, we obtain the following solutions:

$$a_0 = \pm i \frac{\sqrt{q}}{\sqrt{s}}, a_1 = \pm \frac{2i\sqrt{q}\mu}{\sqrt{s}\lambda}, c_1 = \pm \frac{\sqrt{2q - k^2 p \lambda^2}}{\lambda}. \tag{37}$$

Case 1.

$$u_1(x, t) = \pm i \frac{\sqrt{q}}{\sqrt{s}} \mp \frac{i\sqrt{q}}{\sqrt{s}} \left(\tanh \left(\frac{\lambda}{2} \xi \right) - 1 \right). \tag{38}$$

Case 2.

$$u_2(x, t) = \pm i \frac{\sqrt{q}}{\sqrt{s}} \mp \frac{i\sqrt{q}}{\sqrt{s}} \left(\coth \left(\frac{\lambda}{2} \xi \right) - 1 \right). \tag{39}$$

4.4. Solutions with RB Sub-ODE Method. Considering the homogenous balancing in Equation (21), we get $N = 1$. For $N = 1$, Equation (9) has the solution:

$$u' = a_1 u^{2-m} + b_1 u + c_1 u^m, \tag{40}$$

where a_1, b_1, c_1, and m are constant to be determined later.

Setting $m = 0$ and each coefficient polynomial to zero gives a set of algebraic equations for a_1, b_1, and c_1 as follows:

$$u^3 : s + 2c^2 a_1^2 + 2k^2 p a_1^2 = 0, \tag{41}$$

$$u^2 : 3c^2 a_1 b_1 + 3k^2 p a_1 b_1 = 0, \tag{42}$$

$$u^1 : q + c^2 {b_1}^2 + k^2 p {b_1}^2 + 2c^2 a_1 c_1 + 2k^2 p a_1 c_1 = 0, \tag{43}$$

$$u^0 : c^2 b_1 c_1 + k^2 p b_1 c_1 = 0. \tag{44}$$

Solving the system of algebraic equations in (41) with the help of software MATHEMATICA, we obtain the following solutions:

$$\begin{aligned} a_1 &= -\frac{\sqrt{s}}{\sqrt{2}\sqrt{-c^2 - k^2 p}}, \\ b_1 &= 0, \\ c_1 &= -\frac{q}{\sqrt{2}\sqrt{-c^2 - k^2 p}\sqrt{s}}. \end{aligned} \tag{45}$$

Case 1. When $m = 1$, we have

$$u(\xi) = C e^{\left(\left(\left(-\sqrt{s} \right) / \left(\sqrt{2}\sqrt{-c^2 - k^2 p} \right) \right) - \left(q / \left(\sqrt{2}\sqrt{-c^2 - k^2 p}\sqrt{s} \right) \right) \right) \xi}. \tag{46}$$

Case 2. When $m \neq 1, a_1 \neq 0$, and $b_1^2 - 4a_1c_1 < 0$, we have

$$u(\xi) = \left(\frac{\sqrt{q}}{\sqrt{s}} \tan \frac{\sqrt{q}}{\sqrt{2}\sqrt{c^2 - k^2 p}} (\xi + C) \right), \quad (47)$$

$$u(\xi) = \left(-\frac{\sqrt{q}}{\sqrt{s}} \cot \frac{\sqrt{q}}{\sqrt{2}\sqrt{c^2 - k^2 p}} (\xi + C) \right). \quad (48)$$

5. Density Dependent Fractional Diffusion Reaction Equation

Density dependent fractional diffusion reaction equation which is widely used in mathematical biology in the form [44, 45]

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + ku(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \\ &= D\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + au(x,t) - bu^2(x,t), \quad t > 0, 0 < \alpha \leq 1, \end{aligned} \quad (49)$$

Let us assume the transformation:

$$u(x,t) = u(\xi), \xi = \frac{p}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^\alpha - \frac{c}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha. \quad (50)$$

Here p and c are constants. By using Equation (50) into Equation (49), we get the following ODE:

$$Dp^2u'' - cu' - kpuu' + au - bu^2 = 0. \quad (51)$$

5.1. Solutions with GT Method [46]. By applying homogenous balancing technique between the terms u'' and uu' into Equation (51), we get $N = 1$. For $N = 1$, we write the solution of Equation (9) in the following form [46]:

$$u(\xi) = a_0 + a_1\varphi(\xi), \quad (52)$$

where a_0 and a_1 are unknown parameters.

Substituting Equation (52) into Equation (51) and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0 and a_1 as follows:

$$\begin{aligned} \varphi^3 &: 2Dp^2a_1 - kpa_1^2 = 0, \\ \varphi^2 &: ca_1 - kpa_0a_1 - ba_1{}^2 = 0, \\ \varphi^1 &: aa_1 + 2CDp^2a_1 - 2ba_0a_1 - Ckpa_1^2 = 0, \\ \varphi^0 &: aa_0 - ba_0^2 + cCa_1 - Ckpa_0a_1 = 0. \end{aligned} \quad (53)$$

By using the software MATHEMATICA, we obtain the following solutions:

$$a_0 = \frac{a}{2b}, a_1 = \pm\frac{ia}{2b\sqrt{C}}, c = \pm\frac{ia}{2\sqrt{C}}, p = 0. \quad (54)$$

Case 1. For C<0,

$$u_1(x,t) = \frac{a}{2b} \pm \frac{a}{2b}\left(\tanh\sqrt{-C}\xi\right), \quad (55)$$

$$u_2(x,t) = \frac{a}{2b} \pm \frac{a}{2b}\left(\coth\sqrt{-C}\xi\right). \quad (56)$$

Case 2. For C>0,

$$u_3(x,t) = \frac{a}{2b} \pm \frac{ia}{2b}\left(\tan\left(\sqrt{C}\xi\right)\right), \quad (57)$$

$$u_4(x,t) = \frac{a}{2b} \mp \frac{ia}{2b}\left(\cot\left(\sqrt{C}\xi\right)\right). \quad (58)$$

5.2. Solutions with GB Sub-ODE Method. By applying homogenous balancing technique between the terms into Equation (51), we get $N = 1$. For $N = 1$, we write the solution of Equation (9) in the following form [47]:

$$u(\xi) = a_0 + a_1\varphi(\xi), \quad (59)$$

where a_0 and a_1 are unknown parameters.

Substituting Equation (59) into Equation (51) and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0 and a_1 as follows:

$$\begin{aligned} \varphi^3 &: 2Dp^2\mu^2a_1 - kp\mu a_1^2 = 0, \\ \varphi^2 &: c\mu a_1 - 3Dp^2\lambda\mu a_1 - kp\mu a_0a_1 - ba_1{}^2 + kp\lambda a_1{}^2 = 0, \\ \varphi^1 &: aa_1 - c\lambda a_1 + Dp^2\lambda^2a_1 - 2ba_0a_1 + kp\lambda a_0a_1 = 0, \\ \varphi^0 &: aa_0 - ba_0^2 = 0. \end{aligned} \quad (60)$$

By using the software MATHEMATICA, we obtain the following solutions:

$$a_0 = \frac{a}{b}, a_1 = -\frac{a\mu}{b\lambda}, c = -\frac{4ab^2D - a^2k^2}{4b^2D\lambda}, p = -\frac{ak}{2bD\lambda}. \quad (61)$$

Case 1.

$$u_1(x,t) = \frac{a}{b} + \frac{a}{2b}\left(\tanh\left(\frac{\lambda}{2}\xi\right) - 1\right). \quad (62)$$

Case 2.

$$u_2(x,t) = \frac{a}{b} + \frac{a}{2b}\left(\coth\left(\frac{\lambda}{2}\xi\right) - 1\right). \quad (63)$$

5.3. Solutions with RB Sub-ODE Method. By applying homogenous balancing technique, the terms u'' and uu' into Equation (54) we get $N = 1$.

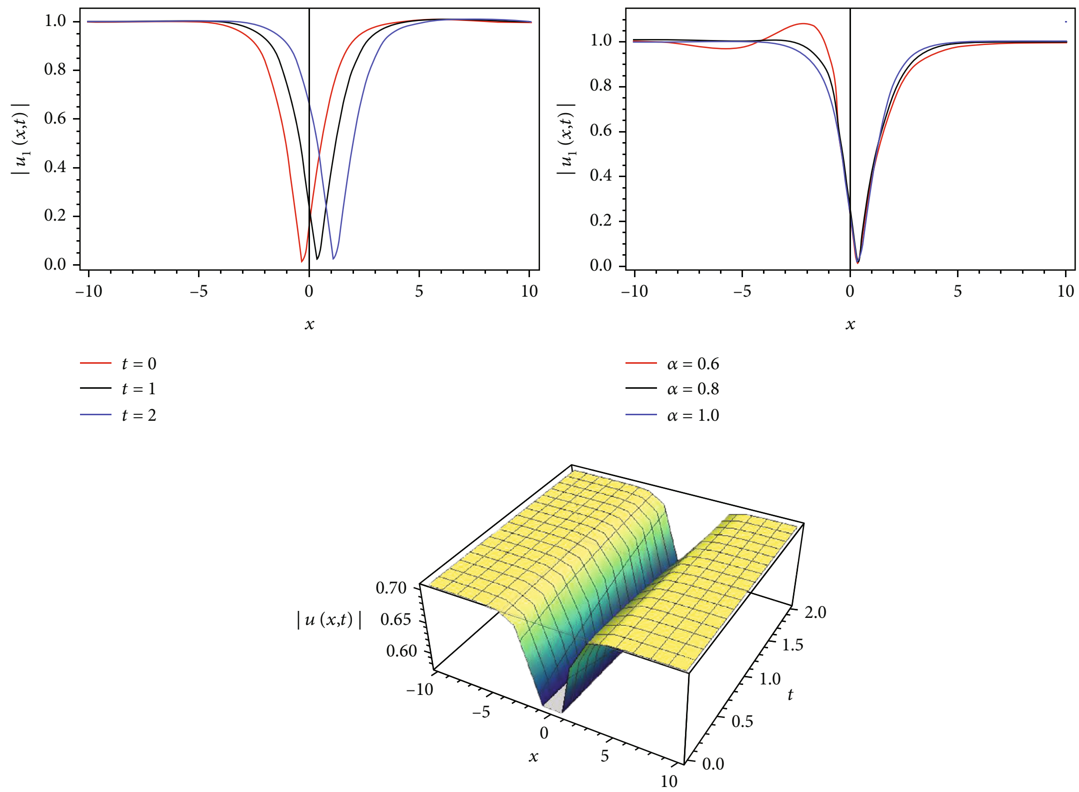

FIGURE 1: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (28) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = -1, c = 0.5\}$.

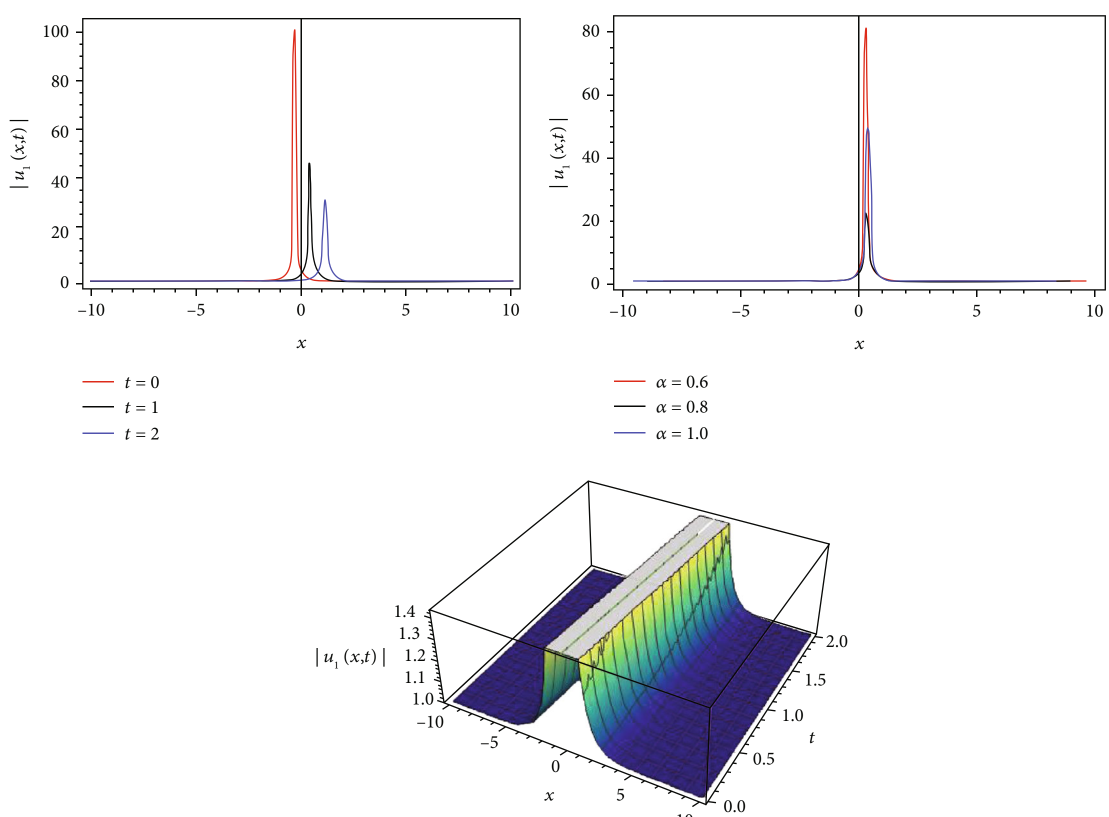

FIGURE 2: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (29) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = -1, c = 0.5.\}$.

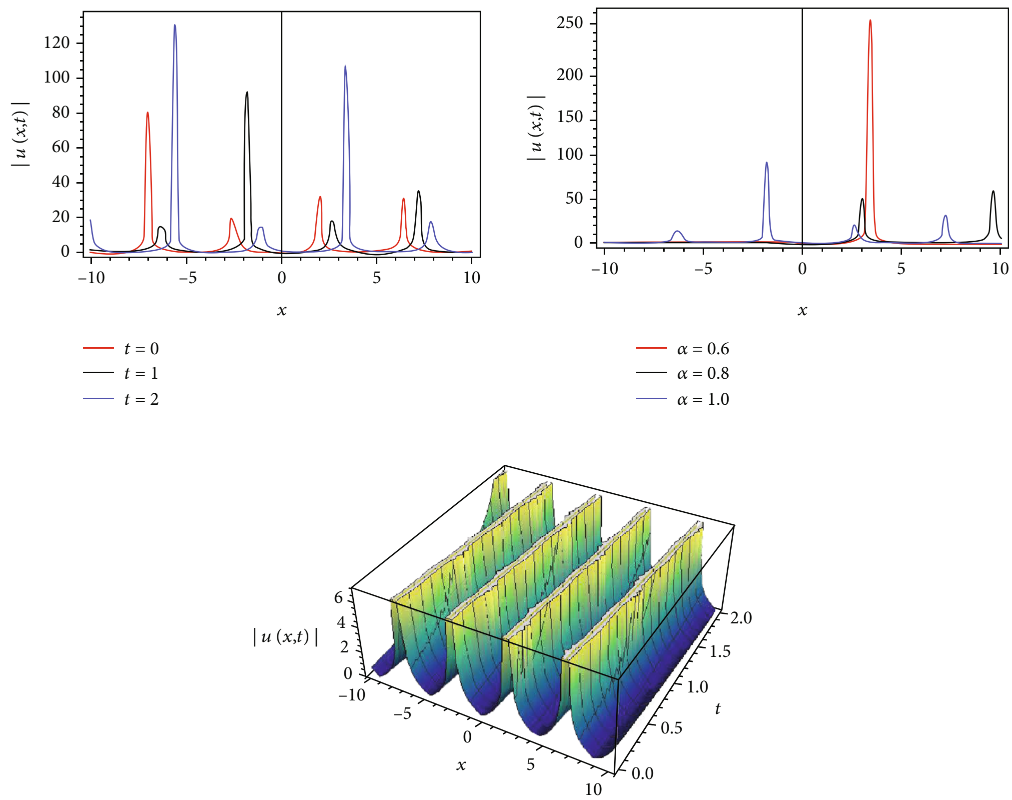

Figure 3: 2D and 3D graphics of Case 2 for trigonometric traveling wave solution (30) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = 1, c = 0.5\}$.

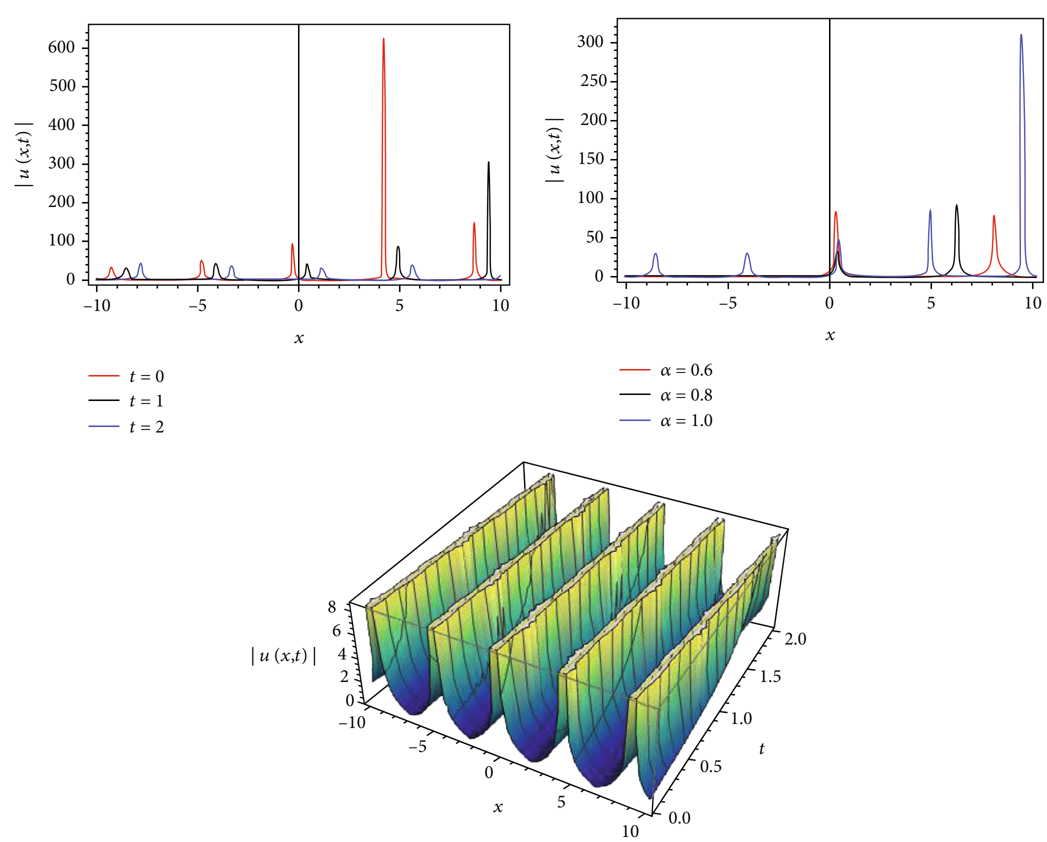

Figure 4: 2D and 3D graphics of Case 2 for trigonometric traveling wave solution (31) at $\{\alpha = 0.6, \mathrm{p} = 0.7, q = 1, s = 1, \mathrm{C} = -1, c = 0.5\}$.

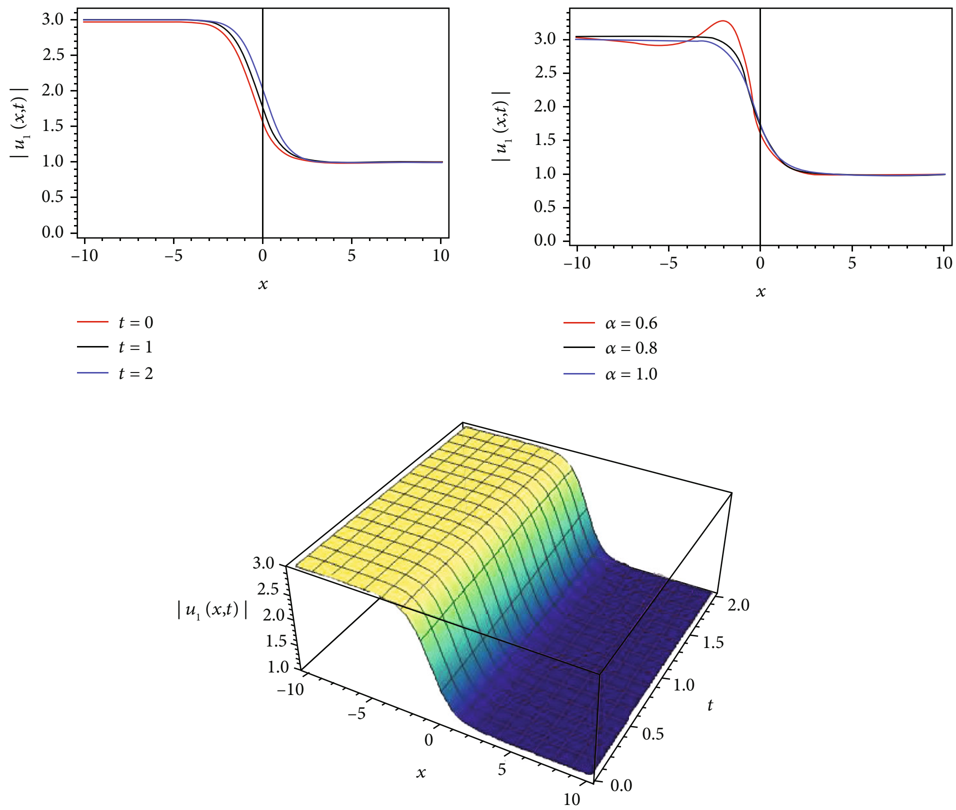

FIGURE 5: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (38) at $\{\alpha = 0.6, \mathrm{k} = 1.5, q = 1, s = 1, \mathrm{c} = 0.5, \lambda = 1\}$.

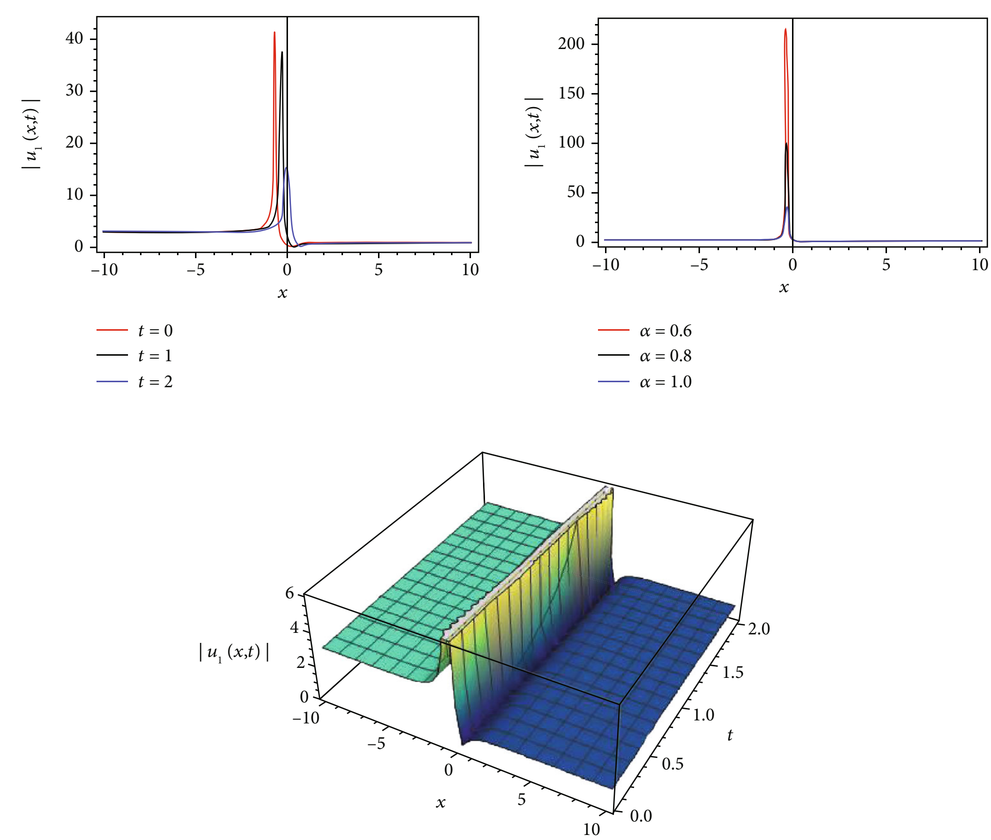

FIGURE 6: 2D and 3D graphics of Case 2 for hyperbolic traveling wave solution (39) at $\{\alpha = 0.6, \mathrm{k} = 1.5, q = 1, s = 1, \mathrm{c} = 0.5, \lambda = 1\}$.

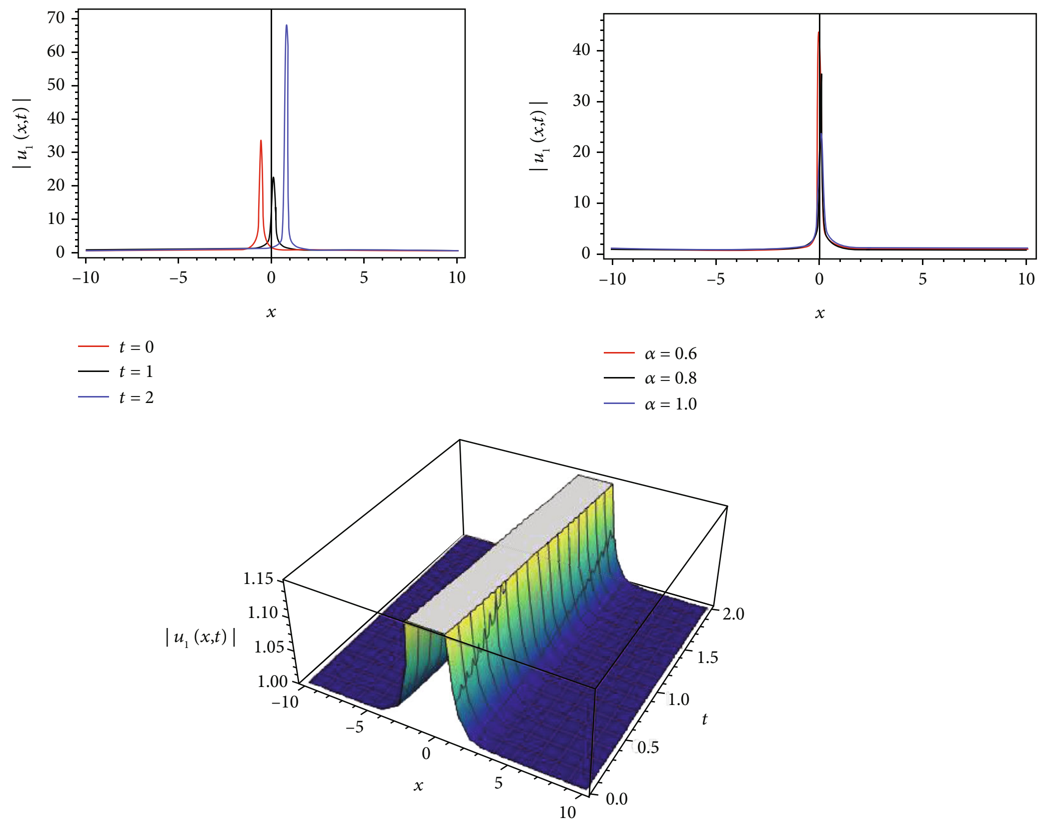

FIGURE 7: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (47) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = -1, c = 0.5\}$

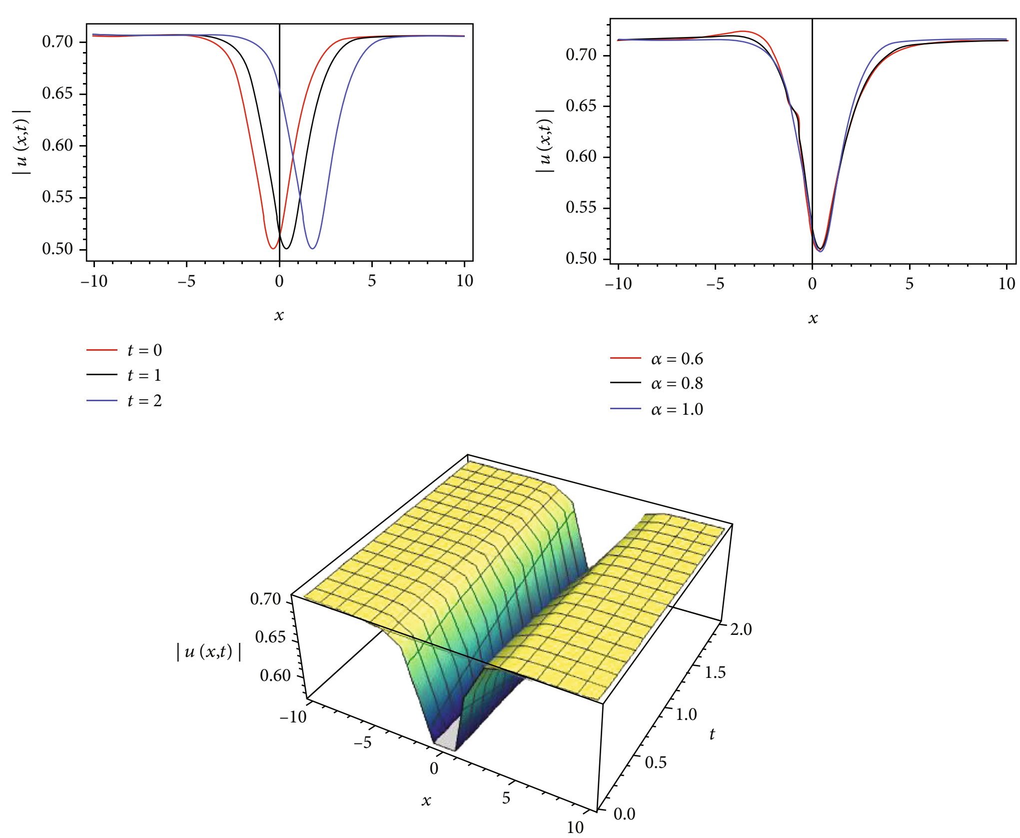

FIGURE 8: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (55) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = -1, c = 0.5\}$

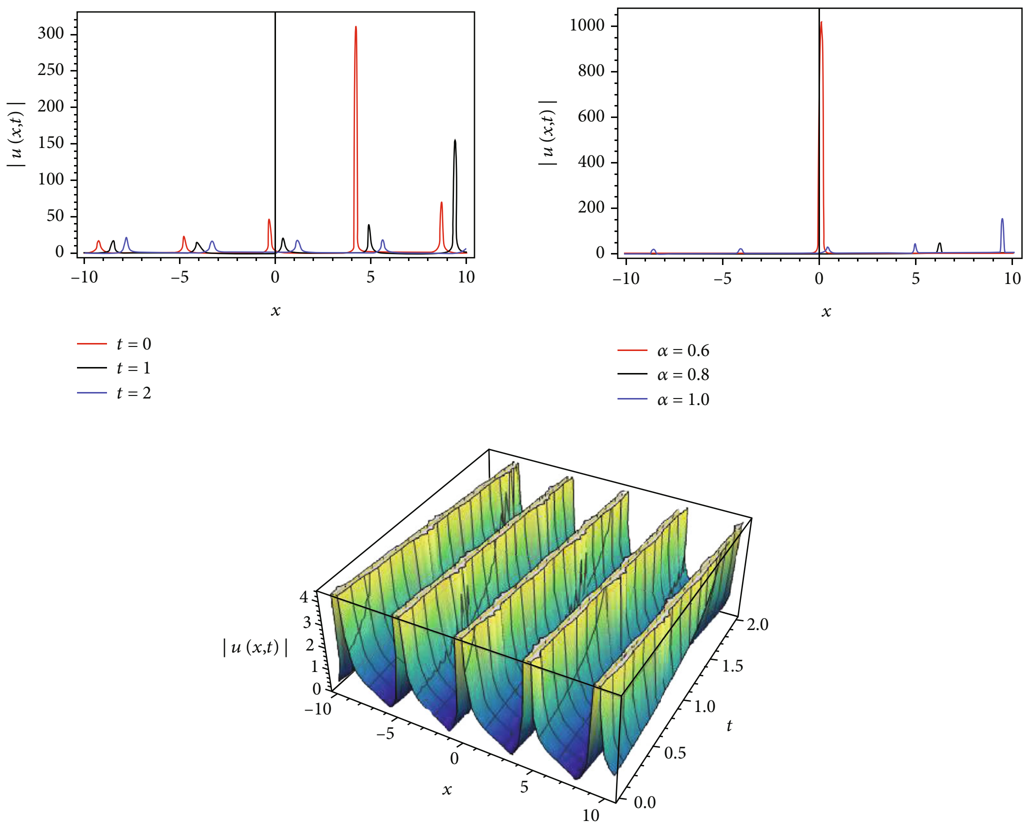

FIGURE 9: 2D and 3D graphics of Case 2 for trigonometric traveling wave solution (57) at $\{\alpha = 0.6, \mathrm{k} = 0.7, q = 1, s = 1, \mathrm{C} = 1, c = 0.5\}$.

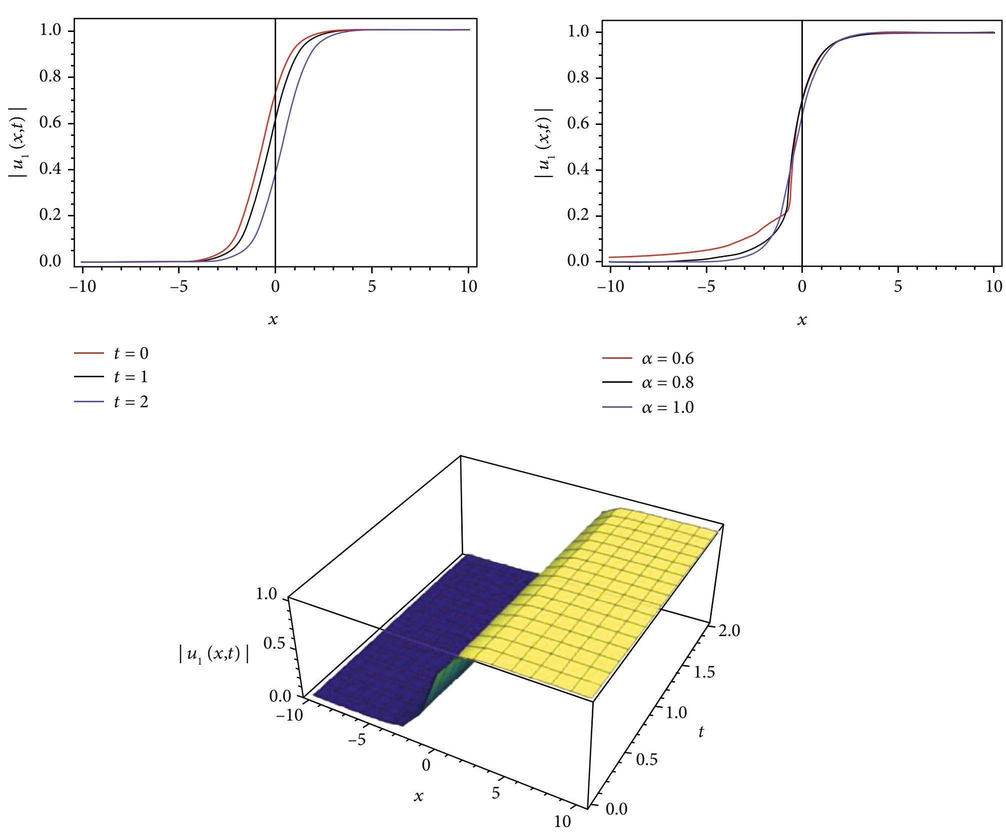

FIGURE 10: 2D and 3D graphics of Case 1 for hyperbolic traveling wave solution (62) at $\{\alpha = 0.6, \mathrm{k} = 0.7, a = 1, b = 1, \mathrm{c} = 0.5, \lambda = 1\}$.

For $N = 1$, Equation (9) has the solution given as:

$$u' = a_1 u^{2-m} + b_1 u + c_1 u^m, \tag{64}$$

where a_1, b_1, c_1 and m are constant to be determined later.

Substituting Equation (64) into Equation (51), setting $m = 0$ and each coefficient polynomial to zero gives a set of algebraic equations for a_1, b_1, and c_1 as follows:

$$u^4 : -kpa_1 + 2Dp^2 a_1^2 = 0, \tag{65}$$

$$u^3 : -b + ca_1 - kpb_1 + 3Dp^2 a_1 b_1 = 0, \tag{66}$$

$$u^2 : a + cb_1 + Dp^2 {b_1}^2 - kpc_1 + 2Dp^2 a_1 c_1 = 0, \tag{67}$$

$$u^1 : cc_1 + Dp^2 b_1 c_1 = 0. \tag{68}$$

By using the software MATHEMATICA, we obtain the following solutions:

$$a_1 = \frac{k}{2Dp}, b_1 = -\frac{ak}{2bDp}, c_1 = 0, c = \frac{\left(4b^2 D + ak^2\right)p}{2bk}. \tag{69}$$

Case 1. When $m = 1$, we have

$$u(\xi) = Ce^{((k/(2Dp))-(ak/(2bDp)))\xi}. \tag{70}$$

Case 2. When $m \neq 1$, $b_1 \neq 0$, and $c_1 = 0$, we have

$$u(\xi) = \left(-\frac{b}{a} + Ce^{((ak/(2bDp))\xi)}\right)^{-1}. \tag{71}$$

The above obtained solutions to the fractional generalized reaction Duffing model and density dependent fractional diffusion reaction equation are compared with those available in the earlier study and claimed to be recorded in the literature for the first time [25, 45].

6. Results and Discussions

To show the dynamics and behavior of our obtained solutions, various exact traveling wave solutions in Equations (28), (29), (30), (31), (38), (39), (48), (55), (57), and (62) are graphically represented and compared in both 3D and 2D plots in Figures 1–10 for various parameters' values. A 3D plot highlights the amount of variation over a while or compares multiple wave items. The 2D line plots are used to represent very high and low frequency and amplitude. The plots are constructed with unique values of $\alpha \in (0, 1]$ for different values of free parameters. The plots denote many natures, such as the trigonometric, hyperbolic and solitary wave solutions, and other forms of the solution generated by the correct physical description by choosing different free parameters. We can observe from the plotted graphs in Figures 1–10 that the wave's frequency and amplitude change with the change of fractional and time parameters.

7. Conclusions

In this article, three methods GT, GB sub-ODE, and RB sub-ODE have been applied to construct a variety of novel exact traveling wave solutions in the form of exponential, hyperbolic, and trigonometric functions of the generalized reaction Duffing model and density dependent fractional diffusion reaction equation arising in Mathematical biology. We have also depicted some of the obtained solutions graphically (3D surface graphs and 2D line plots) and concluded that the results we obtained are accurate, efficient, and versatile in mathematical physics. It is worth to noticing that compared to previous works [25, 26, 44, 45]; the results obtained in this paper are presented for the first time. Lastly, it can be concluded that our offered methods are more effective, reliable, and powerful, which give bounteous consistent solutions to NLPFDEs arise in different fields of nonlinear sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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