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Research Article

Simpson Type Conformable Fractional Inequalities

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In this study, a new Simpson type conformable fractional integral equality for convex functions is established. Based on this identity, some results related to Simpson-like type inequalities are obtained. Also, some estimation results are given for the special cases of the derivative of a function used in our results, and some applications are presented for special means such as the arithmetic, geometric, and logarithmic means.

1. Introduction

Inequalities are extremely useful in mathematics, especially when we deal with quantities that we do not know exactly what they equate too. Often, one can solve a mathematical problem, by estimating an answer, rather than writing down exactly what it is. For more information in this regard, one can see [1–8].

Fractional calculus has been a fascinating area for many researchers in the past and present eras. In the recent two decades, the use of fractional calculus in both pure and applied disciplines of science and engineering has increased significantly. The inequalities involving fractional integrals have become a noticeable approach in recent decades and have acted as a powerful tool for numerous investigations. In recent years, various types of new fractional integral inequalities including Hermite-Hadamard type inequalities have been established via convexity, which provides quite helpful and valid applications in areas such as probability theory, functional inequalities, interpolation spaces, Sobolev spaces, and information theory (see the papers [9, 10]).

The concept of convexity is not a new one even it occurs in some other form in Archimedes' treatment of orbit length. Convex geometry is now a mathematical field in its own right, and significant results have been made in various modern studies such as real analysis, functional analysis, and linear algebra by employing the concept of convexity (see [11–14]). In the last few decades, the subject of convex analysis has got rapid development in view of its geometry and its role in the optimization.

The modern theory of inequalities is another attractive area for researchers in which the notion of convexity plays a major role in improving the estimation bounds of various types of integral inequalities. The following inequality which is known as Simpson's inequality has been studied by several authors (see the papers [15–20]).

Theorem 1. *Let $\psi : [\gamma, \delta] \longrightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (γ, δ) and $\|\psi^{(4)}\|_\infty = \sup |\psi^{(4)}(\varepsilon)| < \infty$. Then,*

$$\left| \int_\gamma^\delta \psi(\varepsilon) d\varepsilon - \frac{\delta-\gamma}{3} \left[\frac{\psi(\gamma)+\psi(\delta)}{2} + 2\psi\left(\frac{\gamma+\delta}{2}\right) \right] \right| \leq \frac{1}{2880} \left\| \psi^{(4)} \right\|_\infty . (\delta-\gamma)^4. \tag{1}$$

The definition below is given in [10, 11].

Definition 2. A function $\psi : [\gamma, \delta] \longrightarrow \mathbb{R}$ is said to be convex on $[\gamma, \delta]$ if the inequality

$$\psi(wa + (1-w)b) \leq w\psi(a) + (1-w)\psi(b) \tag{2}$$

holds for all $a, b \in [\gamma, \delta]$ and $w \in [0, 1]$. If $(-\psi)$ is convex, ψ is concave.

We obtain a new Simpson type identity in this study and use it to derive some results about Simpson-like type inequalities through using conformable fractional integral with some applications.

2. Preliminaries

In this section, we give some definitions and basic results which are useful in obtaining the main results.

Definition 3 (see [21]). Let $\gamma, \delta \in \mathbb{R}$ with $\gamma < \delta$ and $\psi \in L[\gamma, \delta]$.The left and the right Riemann- Liouville fractional integrals $J^{\tau}_{\gamma+}\psi$ and $J^{\tau}_{\delta-}\psi$ of order $\tau > 0$ are defined by

$$\begin{aligned} J^{\tau}_{\gamma+}\psi(\varepsilon) &= \frac{1}{\Gamma(\tau)}\int_{\gamma}^{\varepsilon}(\varepsilon - w)^{\tau-1}\psi(w)dw, \varepsilon > \gamma, \\ J^{\tau}_{\delta-}\psi(\varepsilon) &= \frac{1}{\Gamma(\tau)}\int_{\varepsilon}^{\delta}(w - \varepsilon)^{\tau-1}\psi(w)dw, \varepsilon < \delta, \end{aligned} \tag{3}$$

respectively, where $\Gamma(\tau)$ is the Gamma function defined by $\Gamma(\tau) = \int_0^{\infty} e^{-w} t^{\tau-1} dw$.

In [22], the definition of conformable fractional integrals has been presented as follows.

Definition 4. Let $\tau \in (m, m+1]$, $m = 0, 1, 2, \cdots$, $\beta = \tau - m$, γ, $\delta \in \mathbb{R}$ with $\gamma < \delta$, and $\psi \in L[\gamma, \delta]$. The left and the right conformable fractional integrals $I^{\gamma}_{\tau}\psi$ and $\delta I_{\tau\psi}$ of order $\tau > 0$ are defined by

$$\begin{aligned} I^{\gamma}_{\tau}\psi(\varepsilon) &= \frac{1}{m!}\int_{\gamma}^{\varepsilon}(\varepsilon - w)^{m}(w - \gamma)^{\beta-1}\psi(w)dw, \varepsilon > \gamma, \\ {}^{\delta}I_{\tau}\psi(\varepsilon) &= \frac{1}{m!}\int_{\varepsilon}^{\delta}(w - \varepsilon)^{m}(\delta - w)^{\beta-1}\psi(w)dw, \varepsilon < \delta, \end{aligned} \tag{4}$$

respectively.

Moreover, the papers in [3, 9, 13] contain additional information on conformable fractional integrals. The following are the definitions of beta and incomplete beta functions, as well as the relationship between the gamma and beta functions, as stated in [21].

$$\begin{aligned} \beta(x, y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x, y > 0, \\ \beta_w(x, y) &= \int_0^w t^{x-1}(1-t)^{y-1}dt, x, y > 0, 0 < w \le 1. \end{aligned} \tag{5}$$

3. Main Results

To obtain the main results, first, we need to prove the following lemma:

Lemma 5. *Let $\psi : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I°, $\gamma, \delta \in I^{\circ}$ and $\gamma < \delta$. If $\psi' \in L[\gamma, \delta]$, then*

$$\begin{aligned} &\frac{1}{6}\left[\psi(\gamma) + 4\psi\left(\frac{\gamma+\delta}{2}\right) + \psi(\delta)\right] - \frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)} \\ &\cdot\left[I^{\gamma}_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right) + {}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right] = \frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)} \\ &\cdot\left\{\begin{array}{l} \int_0^1\left(\frac{1}{3}\beta(m+1, \tau-m) - \frac{1}{2}\beta_w(m+1, \tau-m)\right) \times \psi'\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right)dw \\ -\int_0^1\left(\frac{1}{3}\beta(m+1, \tau-m) - \frac{1}{2}\beta_w(m+1, \tau-m)\right) \times \psi'\left(\frac{1-w}{2}\gamma + \frac{1+w}{2}\delta\right)dw \end{array}\right\}. \end{aligned} \tag{6}$$

Proof. We start by considering the following computations which follows from change of variables and using the definition of the conformable fractional integrals.

$$\begin{aligned} I_1 &= \frac{1}{m!}\int_0^1\left(\frac{1}{3}\beta(m+1, \tau-m) - \frac{1}{2}\beta_w(m+1, \tau-m)\right) \\ &\quad\cdot\psi'\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right)dw = \frac{2}{\delta-\gamma}\frac{1}{3}\frac{\Gamma(\tau-m)}{\Gamma(\tau+1)} \\ &\quad\cdot\left(\psi\left(\frac{\gamma+\delta}{2}\right) - \psi(\gamma)\right) - \frac{1}{2.m!}\frac{2}{\delta-\gamma} \\ &\quad\cdot\left[\begin{array}{l} -\left(\int_0^w \varepsilon^n(1-\varepsilon)^{\tau-m-1}d\varepsilon\right)\psi\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right)\Big|_0^1 \\ +\int_0^1 w^n(1-w)^{\tau-m-1}\psi\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right)dw \end{array}\right] \\ &= \frac{2}{\delta-\gamma}\frac{\Gamma(\tau-m)}{\Gamma(\tau+1)}\left(\frac{1}{6}\psi(\gamma) + \frac{1}{3}\psi\left(\frac{\gamma+\delta}{2}\right)\right) \\ &\quad - \frac{1}{2}\left(\frac{2}{\delta-\gamma}\right)^{\tau+1} I^{\gamma}_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right), \end{aligned} \tag{7}$$

and similarly,

$$\begin{aligned} I_2 &= \frac{1}{m!}\int_0^1\left(\frac{1}{3}\beta(m+1, \tau-m) - \frac{1}{2}\beta_w(m+1, \tau-m)\right) \\ &\quad\cdot\psi'\left(\frac{1-w}{2}\gamma + \frac{1+w}{2}\delta\right)dw = \frac{2}{\delta-\gamma}\frac{\Gamma(\tau-m)}{\Gamma(\tau+1)} \\ &\quad\cdot\left(-\frac{1}{6}\psi(\delta) - \frac{1}{3}\psi\left(\frac{\gamma+\delta}{2}\right)\right) + \frac{1}{2}\left(\frac{2}{\delta-\gamma}\right)^{\tau+1} {}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right). \end{aligned} \tag{8}$$

Multiplying $I_1 - I_2$ by $\delta - \gamma/2\Gamma(\tau+1)/\Gamma(\tau-m)$, the proof is completed. □

Remark 6. If we take $\tau = m+1$ in Lemma 5, we have the equality Lemma 5 in [17].

If we take $\tau = 1$ after $\tau = m+1$ in Lemma 5, we have the equality Lemma 1 in [15].

Theorem 7. *Let $\psi : I \subset (0,\infty) \longrightarrow \mathbb{R}$, be a differentiable function on I°, $\gamma, \delta \in I^{\circ}$ and $\gamma < \delta$. If $\psi' \in L[\gamma, \delta]$ and $|\psi'|$ is a convex function on $[\gamma, \delta]$, then*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\
&\quad\cdot\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\Bigg|\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau-m)}{\Gamma(\tau+1)}\\
&\quad\cdot Z_1(\tau,m)\left(\left|\psi'(\gamma)\right|+\left|\psi'(\delta)\right|\right),
\end{aligned}
\tag{9}
$$

where

$$
Z_1(\tau,m)=\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|dw,
\tag{10}
$$

with $m=0,1,2,\cdots$ and $\tau\in(m,m+1]$.

Proof. From Lemma 5 and $|\psi'|$ is convex, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\
&\quad\cdot\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\Bigg|\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\\
&\quad\cdot\left\{\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\times\begin{pmatrix}\frac{1+w}{2}\left|\psi'(\gamma)\right|+\frac{1-w}{2}\left|\psi'(\delta)\right|\\+\frac{1-w}{2}\left|\psi'(\gamma)\right|+\frac{1+w}{2}\left|\psi'(\delta)\right|\end{pmatrix}dw\right\}\\
&=\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\times\left(\left|\psi'(\gamma)\right|+\left|\psi'(\delta)\right|\right)dw\\
&=\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}Z_1(\tau,m)\left(\left|\psi'(\gamma)\right|+\left|\psi'(\delta)\right|\right).
\end{aligned}
\tag{11}
$$

This completes the proof. □

Remark 8. If we take $\tau=m+1$, and after that if we take $\tau=1$ in Theorem 7, we obtain the inequality Corollary 1 in [15].

Theorem 9. *Let $\psi: I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I°, $\gamma,\delta\in I^{\circ}$ and $\gamma<\delta$. If $\psi'\in L[\gamma,\delta]$ and $|\psi'|^q$ is a convex function on $[\gamma,\delta]$ for $q>1$ and $1/p+1/q=1$, then*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\
&\quad\cdot\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\Bigg|\leq\frac{\delta-\gamma}{m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\\
&\quad\cdot\left(\frac{1}{2}\right)^{2/q+1}\left(Z_2(\tau,m)\right)^{1/p}\begin{bmatrix}\left(3\left|\psi'(\gamma)\right|^q+\left|\psi'(\delta)\right|^q\right)^{1/q}\\+\left(\left|\psi'(\gamma)\right|^q+3\left|\psi'(\delta)\right|^q\right)^{1/q}\end{bmatrix},
\end{aligned}
\tag{12}
$$

where

$$
Z_2(\tau,m)=\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|^p dw,
\tag{13}
$$

$m=0,1,2,\cdots$ and $\tau\in(m,m+1]$.

Proof. From Lemma 5 and using the Hölder's integral inequality and the convexity of $|\psi'|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\
&\quad\cdot\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\Bigg|\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\\
&\quad\cdot\left\{\left(\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|^p dt\right)^{1/p}\right.\\
&\quad\times\left[\left(\left|\psi'(\gamma)\right|^q\int_0^1\frac{1+w}{2}dw+\left|\psi'(\delta)\right|^q\int_0^1\frac{1-w}{2}dw\right)^{1/q}\right.\\
&\quad\left.\left.+\left(\left|\psi'(\gamma)\right|^q\int_0^1\frac{1-w}{2}dw+\left|\psi'(\delta)\right|^q\int_0^1\frac{1+w}{2}dw\right)^{1/q}\right]\right\}\\
&\leq\frac{\delta-\gamma}{m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left(\frac{1}{2}\right)^{2/q+1}\left(Z_2(\tau,m)\right)^{1/p}\\
&\quad\cdot\begin{bmatrix}\left(3\left|\psi'(\gamma)\right|^q+\left|\psi'(\delta)\right|^q\right)^{1/q}\\+\left(\left|\psi'(\gamma)\right|^q+3\left|\psi'(\delta)\right|^q\right)^{1/q}\end{bmatrix}.
\end{aligned}
\tag{14}
$$

This completes the proof. □

Remark 10. If we take $\tau=m+1$, and after that if we take $\tau=1$ in Theorem 9, we obtain the inequality Corollary 4 in [15].

Theorem 11. *Let $\psi: I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I°, $\gamma,\delta\in I^{\circ}$ and $\gamma<\delta$. If $\psi'\in L[\gamma,\delta]$ and $|\psi'|^q$ is a convex function on $[\gamma,\delta]$ for $q\geq1$, then*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\
&\quad\cdot\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\Bigg|\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left(Z_1(\tau,m)\right)^{1-1/q}\\
&\quad\cdot\begin{bmatrix}\left(Z_3(\tau,m)\left|\psi'(\gamma)\right|^q+Z_4(\tau,m)\left|\psi'(\delta)\right|^q\right)^{1/q}\\+\left(Z_4(\tau,m)\left|\psi'(\gamma)\right|^q+Z_3(\tau,m)\left|\psi'(\delta)\right|^q\right)^{1/q}\end{bmatrix},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
Z_3(\tau,m)&=\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1+w}{2}dw,\\
Z_4(\tau,m)&=\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1-w}{2}dw,
\end{aligned}
\tag{16}
$$

and $Z_1(\tau,m)$ is defined as in the Theorem 7 with $m=0,1,2,\cdots$ and $\tau\in(m,m+1]$.

Proof. From Lemma 5 and using the power mean inequality, we have that the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\right.\\&\cdot\left.\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\\&\cdot\left\{\left(\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|dw\right)^{1-1/q}\right.\\&\times\left[\left(\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\times\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)\right|^q dw\right)^{1/q}\right.\\&\left.\left.+\left(\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\times\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)\right|^q dw\right)^{1/q}\right]\right\}.\end{aligned}\tag{17}$$

By the convexity of $|\psi'|^q$,

$$\begin{aligned}&\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)\right|^q dw\\&\leq|\psi'(\gamma)|^q\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1+w}{2}dw\\&\quad+|\psi'(\delta)|^q\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1-w}{2}dw,\end{aligned}$$

$$\begin{aligned}&\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)\right|^q dt\\&\leq|\psi'(\gamma)|^q\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1-w}{2}dw\\&\quad+|\psi'(\delta)|^q\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|\frac{1+w}{2}dw.\end{aligned}\tag{18}$$

Using the last two inequalities, we obtain the inequality (15). □

Remark 12. If we take $\tau=m+1$, and after that if we take $\tau=1$ in Theorem 11, we obtain the inequality Theorem 8 in [15].

Theorem 13. *Let $\psi: I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I°, $\gamma,\delta\in I^\circ$ and $\gamma<\delta$. If $\psi'\in L([\gamma,\delta])$ and $|\psi'|^q$ is a convex function on $[\gamma,\delta]$ for $q>1$ and $1/p+1/q=1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|\\&\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left(\frac{1}{2}\right)^{1/q}(Z_2(\tau,m))^{1/p}\\&\times\left[\left(|\psi'(\gamma)|^q+\left|\psi'\left(\frac{\gamma+\delta}{2}\right)\right|^q\right)^{1/q}+\left(\left|\psi'\left(\frac{\gamma+\delta}{2}\right)\right|^q+|\psi'(\delta)|^q\right)^{1/q}\right],\end{aligned}\tag{19}$$

where $Z_2(\tau,m)$ is defined as in Theorem 9 with $m=0,1,2,\cdots$ and $\tau\in(m,m+1]$.

Proof. From Lemma 5 and using the Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|\\&\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left\{\left(\int_0^1\left|\frac{1}{3}\beta(m+1,\tau-m)-\frac{1}{2}\beta_w(m+1,\tau-m)\right|^p dw\right)^{1/p}\right.\\&\times\left[\left(\int_0^1\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)\right|^q dw\right)^{1/q}\right.\\&\left.\left.+\left(\int_0^1\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)\right|^q dw\right)^{1/q}\right]\right\}.\end{aligned}\tag{20}$$

Since $|\psi'|^q$ is convex, we have

$$\begin{aligned}&\int_0^1\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)\right|^q dw\leq\frac{|\psi'(\gamma)|^q+|\psi'(\gamma+\delta/2)|^q}{2},\\&\int_0^1\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)\right|^q dw\leq\frac{|\psi'(\gamma+\delta/2)|^q+|\psi'(\delta)|^q}{2}.\end{aligned}\tag{21}$$

So, we complete the proof. □

Remark 14. If we take $\tau=m+1$, and after that if we take $\tau=1$ in Theorem 13, we obtain the inequality Corollary 4 in [15].

4. Estimation Results

If the function ψ' is bounded, then we have the next result.

Theorem 15. *Let $\psi:[\gamma,\delta]\longrightarrow\mathbb{R}$ be differentiable and continuous mapping on (γ,δ) and let $\psi'\in L[\gamma,\delta]$. Assume that there exist constants $k<K$ such that $-\infty<k\leq\psi'\leq K<+\infty$. Then,*

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|\\&-\Gamma(\tau+1)\Gamma(\tau-m)\delta-\gamma 2m!(K+k)\int_0^1 h(w)\,dw\leq\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{2m!}(K-k)Z_1(\tau,m),\end{aligned}\tag{22}$$

where

$$h(w)=\frac{1}{3}\int_0^1\beta(m+1,\tau-m)dw-\frac{1}{2}\int_0^1\beta_w(m+1,\tau-m)dw.\tag{23}$$

Proof. From Lemma 5, we have that

$$\begin{aligned}&\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\\&=\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left\{\int_0^1 h(w)\left[\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)-\frac{K+k}{2}\right]dw+\int_0^1 h(w)\right.\\&\left.\times\left[\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)-\frac{K+k}{2}\right]dw\right\}+\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{2m!}(K+k)\int_0^1 h(w)dw.\end{aligned}\tag{24}$$

So,

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|$$
$$-\Gamma(\tau+1)\Gamma(\tau-m)\delta-\gamma 2m!(K+k)\int_0^1 h(w)\,dw$$
$$\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left\{\int_0^1|h(w)|\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)-\frac{K+k}{2}\right|dw\right.$$
$$\left.+\int_0^1|h(w)|\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)-\frac{K+k}{2}\right|dw\right\}. \tag{25}$$

Since ψ' satisfies $-\infty<k\leq\psi'\leq K<+\infty$, we have that

$$k-\frac{K+k}{2}\leq\psi'-\frac{K+k}{2}\leq K-\frac{K+k}{2}, \tag{26}$$

which implies that

$$\left|\psi'-\frac{K+k}{2}\right|\leq\frac{K-k}{2}. \tag{27}$$

Hence,

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|$$
$$-\Gamma(\tau+1)\Gamma(\tau-m)\delta-\gamma 2m!(K+k)\int_0^1 h(w)\,dw$$
$$\leq\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{2m!}(K-k)\int_0^1|h(w)|dw\leq\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{2m!}(K-k)Z_1(\tau,m), \tag{28}$$

where $\int_0^1|h(w)|dw=Z_1(\tau,m)$ and $Z_1(\tau,m)$ is defined as in Lemma 5, which completes the proof. □

Remark 16. If we take $\tau=m+1$, and after that if we take $\tau=1$ in Theorem 15, then we obtain

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\psi(\varepsilon)\,d\varepsilon-\frac{K+k}{2}(\delta-\gamma)\int_0^1 h(w)\,dw\right|$$
$$\leq\frac{5}{72}(K-k)(\delta-\gamma). \tag{29}$$

Our next aim is an estimation-type result considering the Simpson-like type conformable fractional integral inequality when ψ' satisfies a Lipschitz condition.

Theorem 17. *Let $\psi:[\gamma,\delta]\longrightarrow\mathbb{R}$ be differentiable and continuous mapping on (a,b) and let $\psi'\in L[a,b]$. Assume that ψ' satisfies the Lipschitz condition for some $L>0$. Then,*

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|$$
$$-\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{m!}\psi'\left(\frac{\gamma+\delta}{2}\right)\int_0^1 h(w)\,dw\leq L\frac{(\delta-\gamma)^2}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}Z_5(\tau,m), \tag{30}$$

where

$$Z_5(\tau,m)=\int_0^1|wh(w)|dw, \tag{31}$$

and $h(w)$ is defined as in Theorem 15.

Proof. From Lemma 5, we have that

$$\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]$$
$$=\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left\{\int_0^1 h(w)\left[\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right]dw\right.$$
$$\left.+\int_0^1 h(w)\left[\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right]dw\right\}$$
$$+\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{m!}\psi'\left(\frac{\gamma+\delta}{2}\right)\int_0^1 h(w)dw. \tag{32}$$

So

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|$$
$$-\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{m!}\psi'\left(\frac{\gamma+\delta}{2}\right)\int_0^1 h(w)\,dw$$
$$\leq\frac{\delta-\gamma}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left\{\int_0^1|h(w)|\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right|dw\right.$$
$$\left.+\int_0^1|h(w)|\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right|dw\right\}. \tag{33}$$

Since ψ' satisfies Lipschitz conditions for some $L>0$, we have that

$$\left|\psi'\left(\frac{1+w}{2}\gamma+\frac{1-w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right|\leq L\left|\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta-\frac{\gamma+\delta}{2}\right|\leq\frac{\delta-\gamma}{2}L|w|,$$
$$\left|\psi'\left(\frac{1-w}{2}\gamma+\frac{1+w}{2}\delta\right)-\psi'\left(\frac{\gamma+\delta}{2}\right)\right|\leq L\left|\frac{1-w}{2}\delta+\frac{1+w}{2}\gamma-\frac{\gamma+\delta}{2}\right|\leq\frac{\delta-\gamma}{2}L|w|. \tag{34}$$

Hence,

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{2^{\tau-1}}{(\delta-\gamma)^{\tau}}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\left[I_{\tau}^{\gamma}\psi\left(\frac{\gamma+\delta}{2}\right)+{}^{\delta}I_{\tau}\psi\left(\frac{\gamma+\delta}{2}\right)\right]\right|$$
$$-\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\frac{\delta-\gamma}{m!}\psi'\left(\frac{\gamma+\delta}{2}\right)\int_0^1 h(w)\,dw\leq L\frac{(\delta-\gamma)^2}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}\int_0^1|wh(w)|dw$$
$$\leq L\frac{(\delta-\gamma)^2}{2.m!}\frac{\Gamma(\tau+1)}{\Gamma(\tau-m)}Z_5(\tau,m), \tag{35}$$

where $\int_0^1|wh(w)|dw=Z_5(\tau,m)$. This ends the proof. □

Remark 18. If we take $\tau = m + 1$, and after that if we take $\tau = 1$ in Theorem 17, we obtain

$$\left|\frac{1}{6}\left[\psi(\gamma)+4\psi\left(\frac{\gamma+\delta}{2}\right)+\psi(\delta)\right]-\frac{1}{\delta-\gamma}\int_\gamma^\delta \psi(\varepsilon)d\varepsilon-(\delta-\gamma)\psi'\left(\frac{\gamma+\delta}{2}\right)\int_0^1 h(w)dw\right| \leq L\frac{2(\delta-\gamma)^2}{81}, \tag{36}$$

with $\int_0^1 |1/3 - w/2| w dw = 4/81$.

5. Applications

5.1. Special Means. For $0 \leq \gamma < \delta$, we consider the following special means:

Theorem 19.

(i) *The arithmetic mean:* $A(\gamma, \delta) = \gamma + \delta/2$

(ii) *The geometric mean:* $G(\gamma, \delta) = \sqrt{\gamma\delta}$

(iii) *The logarithmic mean:* $L(\gamma, \delta) = \delta - \gamma/\ln \delta - \ln \gamma, \gamma \delta \neq 0$

The logarithmic mean: $L_s(\gamma, \delta) = (\delta^{s+1} - \gamma^{s+1}/(s+1)(\gamma-\delta))^{1/s}, s \in \mathbb{Z} \setminus \{0, 1\}$

Next, using the main results obtained in Section 2, we give some applications to special means of real numbers.

Proposition 20. *Let* $0 < \gamma < \delta, s \in \mathbb{N}$. *Then,*

$$\left|\frac{1}{3}A(\gamma^s, \delta^s) + \frac{2}{3}A^s(\gamma, \delta) - L_s^s(\gamma, \delta)\right| \leq \frac{5(\delta-\gamma)}{72} s\left[\gamma^{s-1} + \delta^{s-1}\right]. \tag{37}$$

Proof. The proof is obvious from Remark 8 when applied $\psi(\varepsilon) = \varepsilon^s$. □

Remark 21. If we take $s = 1$ in (37), we obtain the inequality Corollary 7 in [15].

Proposition 22. *Let* $0 < \gamma < \delta, s \in \mathbb{N}$. *Then,*

$$\begin{aligned}&\left|\frac{1}{3}A(\gamma^s, \delta^s) + \frac{2}{3}A^s(\gamma, \delta) - L_s^s(\gamma, \delta)\right| \\ &\quad \leq \frac{\delta-\gamma}{2}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{1/p}\left(\frac{1}{4}\right)^{1/q} \\ &\qquad \left[\left(\left(s\gamma^{s-1}\right)^q + 3\left(s\delta^{s-1}\right)^q\right)^{1/q} + \left(3\left(s\gamma^{s-1}\right)^q + \left(s\delta^{s-1}\right)^q\right)^{1/q}\right].\end{aligned} \tag{38}$$

Proof. The proof is obvious from Remark 10 applied $\psi(\varepsilon) = \varepsilon^s$. □

Remark 23. If we take $s = 1$ in (38), we obtain

$$|A(\gamma, \delta) - L(\gamma, \delta)| \leq (\delta - \gamma)\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{1/p}. \tag{39}$$

Proposition 24. *Let* $0 < \gamma < \delta$, $s \in \mathbb{N}$, *and* $0 < q < 1$. *Then,*

$$\begin{aligned}&\left|\frac{1}{3}A(\gamma^s, \delta^s) + \frac{2}{3}A^s(\gamma, \delta) - L_s^s(\gamma, \delta)\right| \\ &\quad \leq \frac{\delta-\gamma}{2}\left(\frac{5}{36}\right)^{1-1/q}\left(\frac{1}{648}\right)^{1/q} \\ &\qquad \times\left[\left(61\left(s\gamma^{s-1}\right)^q + 29\left(s\delta^{s-1}\right)^q\right)^{1/q} + \left(29\left(s\gamma^{s-1}\right)^q + 61\left(s\delta^{s-1}\right)^q\right)^{1/q}\right].\end{aligned} \tag{40}$$

Proof. The proof is obvious from Remark 12 applied $\psi(\varepsilon) = \varepsilon^s$. □

Remark 25. If we take $s = 1$ in (40), we obtain the inequality Corollary 7 in [15].

Proposition 26. *Let* $0 < \gamma < \delta, s \in \mathbb{N}$. *Then,*

$$\begin{aligned}&\left|\frac{1}{3}A(\gamma^s, \delta^s) + \frac{2}{3}A^s(\gamma, \delta) - L_s^s(\gamma, \delta)\right| \\ &\quad \leq \frac{\delta-\gamma}{12}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{1/p}\left(\frac{1}{4}\right)^{1/q} \\ &\qquad \times\left[\left(\left(s\gamma^{s-1}\right)^q + 3\left(s\left(\frac{\gamma+\delta}{2}\right)^{s-1}\right)^q\right)^{1/q} + \left(3\left(s\left(\frac{\gamma+\delta}{2}\right)^{s-1}\right)^q + \left(s\delta^{s-1}\right)^q\right)^{1/q}\right].\end{aligned} \tag{41}$$

Remark 27. If we take $s = 1$ in (41), we obtain

$$|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{\delta-\gamma}{6}\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{1/p}. \tag{42}$$

Proposition 28. *Let* $0 < \gamma < \delta$. *Then,*

$$\left|\frac{1}{3}A(\alpha, \beta) + \frac{2}{3}G(\alpha, \beta) - L(\alpha, \beta)\right| \leq (\ln \beta - \ln \alpha)\frac{5}{36}A(\alpha, \beta). \tag{43}$$

Proof. The proof is obvious from Remark 8, applied $f(\varepsilon) = e^\varepsilon, \varepsilon > 0$ and $\alpha = e^\gamma, \beta = e^\delta$. □

6. Conclusion

In this paper, using a new identity of Simpson-like type for conformable fractional integral, we obtained some new Simpson type conformable fractional integral inequalities. We also used inequalities such as the Hölder inequality and the power mean inequality to obtain these integral inequalities. Furthermore, we examined some interesting applications. So, this paper is a detailed examination of the Simpson-like type conformable fractional integral inequalities.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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