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Research Article

Applicability of Mönch's Fixed Point Theorem on Existence of a Solution to a System of Mixed Sequential Fractional Differential Equation

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In this paper, we study the existence and uniqueness of the solution for a coupled system of mixed fractional differential equations. The main results are established with the aid of "Mönch's fixed point theorem." In addition, an applied example that supports the theoretical results reached through this study is included.

1. Introduction

Fractional calculus has an extended history, going all the way back to Leibniz's 17th-century explanation of the derivative order in 1965. Mathematicians use fractional calculus to study how derivatives and integrals of noninteger order work and how they change over time. Subsequently, the subject attracted the interest of numerous famous mathematicians, including Fourier, Laplace, Abel, Liouville, Riemann, and Letnikov. For current and wide-ranging analyses of fractional derivatives and their applications, we recommend the monographs [1, 2], and the recently mentioned papers [3, 4].

Many problems in various scientific branches can be successfully studied using partial differential equations, such as theoretical physics, biology, viscosity, electrochemistry, and other physical processes see [5–9]. For example, but not limited, the authors in [10] employed the fractional derivative of the ψ-Caputo type in modeling the logistic population equation, through which they were able to show that the model with the fractional derivative led to a better approximation of the variables than the classical model. In addition, the authors in [11] employed the fractional derivative of the ψ-Caputo type and used the kernel Rayleigh, to improve the model again in modeling the logistic population equation.

The obvious difference between the ordinary differential equation and the fractional differential equation is that the latter is an equation that contains fractional derivatives and also comes in a relationship so that the definition of the fractional derivative is an integral equation on the other side of this equation. Fractional derivatives have drawn the attention of researchers in various fields of research. One of the main goals of solving these equations is to investigate whether these derivatives will help in the future in improving the accuracy of predicting the values of variables in various mathematical models in all sciences, whether in scientific or human aspects.

Before starting this research for solutions to these problems, which are recently considered in the applied sciences, verifying the issue of the existence and uniqueness of such equations is an indispensable thing. To study these conditions, most of the researchers use the most important fixed point theorems in the Banach space, such as the Banach contraction principle and Leray- Schauder theorem see [12–18].

In 2016, Aljoudi et al. [19] published a study investigating the existence results for the following boundary value problem (sequential Hadamard type).

$$\begin{cases} \left({}^{H}D_1^{\nu_1} + \eta^{H}D_1^{\nu_1-1}\right)\xi(\omega) = \varphi_1\left(\omega, \xi(\omega), \zeta(\omega), {}^{H}D_1^{r_1}\zeta(\omega)\right), \\ \left({}^{H}D_1^{\nu_2} + \eta^{H}D_1^{\nu_2-1}\right)\zeta(\omega) = \varphi_2\left(\omega, \xi(\omega), {}^{H}D_1^{r_2}\xi(\omega), \zeta(\omega)\right), \\ \xi(1) = 0,\ \xi(e) = {}^{H}I^{\theta_1}\zeta(\varepsilon_1), \\ \zeta(1) = 0,\ \zeta(e) = {}^{H}I^{\theta_2}\xi(\varepsilon_2), \end{cases} \quad (1)$$

where ${}^{H}D_1^{(.)}, \nu_1, \nu_2 \in (1, 2],\ r_1, r_2 \in (0, 1)$ is the Hadamard fractional derivative, and ${}^{H}I^{\theta_1}$ is the Hadamard fractional integral with order $\theta_1, \theta_2 > 0,\ \varphi_1, \varphi_2 \in C([1, e] \times \mathbb{R}^3, \mathbb{R}), \varepsilon_1, \varepsilon_2 \in [1, e]$.

In 2017, Ahmad and Ntouyas [20] published a study investigating the existence results for the following initial value problem

$$\begin{cases} {}^{CH}D_1^{\nu_1}\left({}^{CH}D_1^{\nu_2}x(t) - f_1(t, x_t)\right) = f_2(t, x_t), & t \in [1, b], \\ x(t) = \varphi(t), & t \in [1-\tau, 1], \\ {}^{CH}D_1^{\nu_2}x(t) = \mu \in \mathbb{R}, \end{cases} \quad (2)$$

where ${}^{CH}D_1^{\nu_i}, \nu_i \in (0, 1), i = 1, 2$ is the Hadamard fractional derivative, $f_i \in [1, b] \times C([-\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R}, \varphi \in C([1-\tau, 1], \mathbb{R}), x_t \in C([-\tau, 0], \mathbb{R})$, where $x_t(\gamma) = x(t+\gamma), \gamma \in [-\tau, 0]$.

Many researchers went deeper in their research beyond the issue of verifying the issue of the existence of a solution to such equations and studied the issue of the stability of these solutions, it can be seen in [21, 22]. Furthermore, many specialists in this field have taken an interest in hybrid partial differential equations see [23–26].

Newly, interest in fractional calculus has increased from a purely mathematical theory and from an applied point of view in various sciences. Focusing on the theory, there are many experts in this field who have studied the existence of solutions for many types' fractional differential equations (FDEs) using the most famous fixed-point theories such as Banach's principle and nonlinear Leary-Schauder alternative. While a few of them tried other theories to examine the existence of solutions to these problems, Derbazi and Baitiche [27] publish one of these scientific papers.

The aim of this paper is to investigate the existence of solutions for the following nonlinear sequential fractional differential equation subject to the Dirichlet boundary conditions.

$$\begin{cases} {}^{C}D^{\alpha_1}\left({}^{CH}D^{\beta_1}\psi(t)\right) = \varsigma(t, \psi(t), \varphi(t)), \\ {}^{C}D^{\alpha_2}\left({}^{CH}D^{\beta_2}\varphi(t)\right) = \xi(t, \psi(t), \varphi(t)), \\ \psi(a) = \psi(T) = 0, \quad \varphi(a) = \varphi(T) = 0, \end{cases} \quad (3)$$

where ${}^{C}D^{\alpha_i}, {}^{CH}D^{\beta_i}$ are the *Caputo* and *Caputo-Hadamard* fractional derivatives of order $0 < \alpha_i, \beta_i \leq 1, i = 1, 2.$ $, a \leq t \leq T$.

In this work, we will try to follow the researchers and specialists in this field, by working to prove the existence of a solution to the problem presented above. In which the work will be presented in this format: Section 2 contains some basic results for fractional calculus. Section 3 shows an important result for the establishment of our main findings, and after that, we present our main findings. In Section 4, an applied example is obtained illustrating what has been obtained in the theoretical aspect of this manuscript. In Section 5, a conclusion and future work section is introduced.

2. Preliminaries

This part is dedicated to presenting some definitions, postulates, and theorems related to the fixed point concept of solutions of differential equations, which will be used to verify the existence of a solution to the system of equations given by Equation (3).

Definition 1 (see [7]). The Hadamard fractional integral of order ν for a continuous function φ is defined as

$${}^{H}I^{\nu}\varphi(\omega) = \frac{1}{\Gamma(\nu)}\int_a^{\omega}\left(\ln\frac{\omega}{\tau}\right)^{q-1}\frac{1}{\tau}\varphi(\tau)d\tau, \nu > 0. \quad (4)$$

Definition 2 (see [7]). The Hadamard fractional derivative of order $\nu > 0$ for a continuous function $\varphi : [a, \infty) \longrightarrow \mathbb{R}$ is defined as

$${}^{H}D^{\nu}\varphi(\omega) = \delta^n\left({}^{H}I^{\nu}\varphi\right)(\omega), \quad (5)$$

$n - 1 < \nu < n, n = [\nu] + 1$, where $\delta = \omega(d/d\omega)$, $[\nu]$ denotes the integer part of the real number ν.

Definition 3 (see [5]). The Caputo-Hadamard fractional derivative of order ν for at least $n-$ times differentiable function $\varphi : [a, \infty) \longrightarrow \mathbb{R}$ is defined as

$${}^{CH}D^{\nu}\varphi(\omega) = \frac{1}{\Gamma(n-\nu)}\int_a^{\omega}\left(\ln\frac{\omega}{\tau}\right)^{n-\nu-1}\delta^n\frac{g(\tau)}{\tau}d\tau. \quad (6)$$

Lemma 4 (see [20]). *Let* $u \in C_\delta^n([a, T], \mathbb{R})$, *where* $C_\delta^n[a, T] = \{u : [a, T] \longrightarrow \mathbb{R} : \delta^{(n-1)}u \in C[a, T]\}$, *then* ${}^{H}I^{\nu}({}^{H}D^{\nu}u)(\omega) = u(\omega) - \sum_{k=1}^{n}c_k(\ln(\omega/a))^{\nu-k}$, *and*

$${}^{H}I^{\nu}\left({}^{CH}D^{\nu}u\right)(\omega) = u(\omega) - \sum_{k=0}^{n-1}c_k\left(\ln\frac{\omega}{a}\right)^k. \quad (7)$$

Denote the Banach space of all continuous functions z from $[a, T]$ into Q by $C([a, T], Q)$,

accompanied by the norm: $\|z\|_\infty = \sup_{a \leq t \leq T}\{z(t)\}$.

Definition 5 (see [28]). The Kuratowski measure of noncompactness $k(\cdot)$.

Defined on bounded set U of Banach space Q is

$$k(U) := \inf \{r > 0 : U = \cup_{i=1}^{m} U_i \text{ and } \operatorname{diam}(U_i) \le r \text{ for } 1 \le i \le m\}. \tag{8}$$

Lemma 6 (see [28]). *Given the Banach space Q with U, V are two bounded proper subsets of Q, then the following properties hold true*

(I) *If $U \subset V$, then $k(U) \le k(V)$;*

(II) *$k(U) = k(\bar{U}) = k(\overline{conv}\, U)$;*

(III) *U is relativly compact $\rightleftarrows k(U) = 0$;*

(IV) *$k(\delta U) = |\delta| k(U), \delta \in \mathbb{R}$;*

(V) *$k(U \cup V) = \max \{k(U), k(V)\}$;*

(VI) *$k(U + V) \le k(U) + k(V), U + V = \{x | x = u + v, u \in U, v \in V\}$;*

(VII) *$k(U + y) = k(U), \forall y \in Q$.*

Lemma 7 (see [29]). *Given an equicontinuous and bounded set $S \subset C([a, T], Q)$, then the function.*

$\omega \mapsto k(S(\omega))$ is continuous on $[a, T]$, $k_C(S) = \max_{\omega \in [a,T]} k(S(\omega))$, and

$$k\left(\int_a^T x(\tau) d\tau\right) \le \left(\int_a^T k(x(\tau)) d\tau\right), S(\tau) = \{x(\tau) : x \in S\}. \tag{9}$$

Definition 8 (see [3]). Given the function $\psi : [a, T] \times Q \longrightarrow Q, \psi$ satisfy the Carathéodory conditions, if the following conditions applies:

(I) $\psi(\omega, z)$ is measurable in ω for $z \in Q$;

(II) $\psi(\omega, z)$ is continuous in $z \in Q$ for $\omega \in [a, T]$.

Theorem 9 (Mönch's fixed point theorem [4]). *Given a bounded, closed, and convex subset $\Omega \subset Q$, such that $0 \in \Omega$, let also T be a continuous mapping of Ω into itself.*

If $S = \overline{conv}\, T(S)$, or $S = T(S) \cup \{0\}$, then $k(S) = 0$, satisfied $\forall S \subset \Omega$, then T has a fixed point.

3. Existence Results

Let $B = \{(\psi(t), \varphi(t)) | (\psi, \varphi) \in C([a, T], \mathbb{R}) \times C([a, T], \mathbb{R})\}$. Obviously, the defined set B is a Banach space with $\|(\psi, \varphi)\|_B = \|\psi\|_\infty + \|\varphi\|_\infty$.

The measurable functions $(\psi, \varphi) \in C([a, T], \mathbb{R}) \times C([a, T], \mathbb{R})$ are said to be solutions of problem Equation (3) if they satisfy problem (3) associated with the given boundary conditions, our next lemma will introduce the solutions of Equation (3), which indeed needed to investigate the existence results.

Lemma 10. *If $p, q \in C([a, T], \mathbb{R})$, then the solution of*

$$\begin{cases} {}^{C}D^{\alpha_1}\left({}^{CH}D^{\beta_1}\psi(t)\right) = p(t), \\ {}^{C}D^{\alpha_2}\left({}^{CH}D^{\beta_2}\varphi(t)\right) = q(t), \\ \psi(a) = \psi(T) = 0, \varphi(a) = \varphi(T) = 0. \end{cases} \tag{10}$$

With $0 < \alpha_i, \beta_i \le 1, i = 1, 2. a \le t \le T$, is given by

$$\begin{aligned} \psi(t) = {} & \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)} \int_a^t \int_a^r \left(\ln \frac{t}{r}\right)^{\beta_1 - 1} (r - x)^{\alpha_1 - 1} p(x) dx \frac{dr}{r} \\ & - \left(\frac{\ln (t/a)}{n(T/a)}\right)^{\beta_1} \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)} \int_a^T \int_a^r \\ & \cdot \left(\ln \frac{T}{r}\right)^{\beta_1 - 1} (r - x)^{\alpha_1 - 1} p(x) dx \frac{dr}{r}, \end{aligned} \tag{11}$$

$$\begin{aligned} \varphi(t) = {} & \frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)} \int_a^t \int_a^r \left(\ln \frac{t}{r}\right)^{\beta_2 - 1} (r - x)^{\alpha_2 - 1} q(x) dx \frac{dr}{r} \\ & - \left(\frac{\ln (t/a)}{n(T/a)}\right)^{\beta_2} \frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)} \int_a^T \int_a^r \\ & \cdot \left(\ln \frac{T}{r}\right)^{\beta_2 - 1} (r - x)^{\alpha_2 - 1} q(x) dx \frac{dr}{r}. \end{aligned} \tag{12}$$

Proof. Apply ${}^{RL}I^{\alpha_i}, i = 1, 2$ to Equation (10), respectively, implies

$${}^{CH}D^{\beta_1}\psi(t) = {}^{RL}I^{\alpha_1} p(t) + c_0, \quad c_0 \in \mathbb{R}, \tag{13}$$

$${}^{CH}D^{\beta_2}\varphi(t) = {}^{RL}I^{\alpha_2} q(t) + d_0, \quad d_0 \in \mathbb{R}. \tag{14}$$

Now, apply ${}^{H}I^{\beta_i}, i = 1, 2$ to Equation (13) and Equation (14), respectively, implies

$$\psi(t) = {}^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1} p\right)(t) + c_0 \frac{(\ln (t/a))^{\beta_1}}{\Gamma(\beta_1 + 1)} + c_1, c_0, c_1 \in \mathbb{R}, \tag{15}$$

$$\varphi(t) = {}^{H}I^{\beta_2}\left({}^{RL}I^{\alpha_2} q\right)(t) + d_0 \frac{(\ln (t/a))^{\beta_2}}{\Gamma(\beta_2 + 1)} + d_1, d_0, d_1 \in \mathbb{R}. \tag{16}$$

Using the conditions $\psi(a) = 0, \varphi(a) = 0$ in Equation (15) and Equation (16), respectively, yields c_1, d_1 are both zeros. Again the conditions $\psi(T) = 0, \varphi(T) = 0$ in Equation (15) and Equation (16), respectively, give

$$c_0 = -\frac{\Gamma(\beta_1+1)}{(\ln(T/a))^{\beta_1}}\, {}^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}p\right)(T),$$
$$d_0 = -\frac{\Gamma(\beta_2+1)}{(\ln(T/a))^{\beta_2}}\, {}^{H}I^{\beta_2}\left({}^{RL}I^{\alpha_2}q\right)(T). \tag{17}$$

Back substituting $c_i, d_i, i=1,2$ obtained above in equations Equation (15) and Equation (16), we get $\psi(t) = {}^{H}I^{\beta_1}({}^{RL}I^{\alpha_1}p)(t) - (\ln(t/a)/\ln(T/a))^{\beta_1}\, {}^{H}I^{\beta_1}({}^{RL}I^{\alpha_1}p)(T)$, and $\varphi(t) = {}^{H}I^{\beta_2}({}^{RL}I^{\alpha_2}q)(t) - (\ln(t/a)/\ln(T/a))^{\beta_2}\, {}^{H}I^{\beta_2}({}^{RL}I^{\alpha_2}q)(T)$. The proof is completed. □

To begin formulating theoretical results regarding the problem of having a solution to the system of fractional differential equations given by Equation (3). We will force the following conditions to be hold true.

(C1). Assume the functions $\varsigma, \xi : [a,T]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy Carathéodory conditions.

(C2). $\exists l_\varsigma, l_\xi \in L^\infty([a,T],\mathbb{R}_+)$, and there exist a nondecreasing continuous function $\vartheta_\varsigma, \vartheta_\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, such that , $\forall t\in[a,T], \forall(\psi,\varphi)\in B$, we have

$$\|\varsigma(t,\psi,\varphi)\|_\infty \le l_\varsigma(t)\vartheta_\varsigma(\|\psi\|_\infty + \|\varphi\|_\infty),$$
$$\|\xi(t,\psi,\varphi)\|_\infty \le l_\xi(t)\vartheta_\xi(\|\psi\|_\infty + \|\varphi\|_\infty). \tag{18}$$

(C3). Let $S\subset B\times B$, be a bounded set, and $\forall t\in[a,T]$, then

$$\kappa(\varsigma(t,S)) \le l_\varsigma(t)\kappa(S),$$
$$\kappa(\xi(t,S)) \le l_\xi(t)\kappa(S). \tag{19}$$

Also, one can use the fact that $(r-a)^{\alpha_1} \le (T-a)^{\alpha_1}$, to deduce that

$$\Xi_1 = \sup_{a\le t\le T}\left\{\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r} + \left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r}\right\}$$
$$\le \frac{2(T-a)^{\alpha_1}(\ln(T/a))^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)},$$
$$\Xi_2 = \sup_{a\le t\le T}\left\{\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}dx\frac{dr}{r} + \left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_2}\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}dx\frac{dr}{r}\right\}$$
$$\le \frac{2(T-a)^{\alpha_2}(\ln(T/a))^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}. \tag{20}$$

Theorem 11. *Assume that the conditions (C1), (C2), and (C3) are satisfied. If* $\max\{\Xi_1\bar{l}_\varsigma, \Xi_2\bar{l}_\xi\} < 1$, *then there exist at least one solution for the boundary value problem Equation (3) on* $[a,T]$.

Proof. Beginning with introducing the following continuous operator $\Upsilon : B \longrightarrow B$, as $\Upsilon = (\Upsilon_1(\psi,\varphi)(t), \Upsilon_2(\psi,\varphi)(t))$, where

$$\Upsilon_1(\psi,\varphi)(t) = \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|dx\frac{dr}{r} - \left(\frac{\ln(t/a)}{\text{n}(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r \cdot\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|dx\frac{dr}{r},$$
$$\Upsilon_2(\psi,\varphi)(t) = \frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}\xi(x,\psi(x),\varphi(x))dx\frac{dr}{r} - \left(\frac{\ln(t/a)}{\text{n}(T/a)}\right)^{\beta_2}\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^T\int_a^r \cdot\left(\ln\frac{T}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}\xi(x,\psi(x),\varphi(x))dx\frac{dr}{r}. \tag{21}$$

According to the conditions (C1) and (C2), the operator Υ is well defined. Then, the following operator equation can be an equivalent equation to the fractional equations given by Equation (11) and Equation (12)

$$(\psi,\varphi) = \Upsilon(\psi,\varphi). \tag{22}$$

Subsequently, proving the existence of the solution to Equation (22) is equivalent to proving the existence of a solution to Equation (3).

Let $\Theta_\varepsilon = \{(\psi,\varphi)\in B : \|(\psi,\varphi)\| \le \varepsilon, \varepsilon>0\}$ be a closed bounded convex ball in B with $\varepsilon \ge \bar{l}_\varsigma\Xi_1\vartheta_\varsigma(\varepsilon) + \bar{l}_\xi\Xi_2\vartheta_\xi(\varepsilon)$, where $\bar{l}_\varsigma = \sup_{a\le t\le T} l_\varsigma(t)$,.

For the possibility of applying Mönch's fixed point theorem, we will proceed in the proof in the form of four steps, and thus, we achieve the desired goal by proving the existence of a solution to the equation given in Equation (3).

Firstly, we show that $\Upsilon\Theta_\varepsilon \subset \Theta_\varepsilon$, for this, we let $t\in[a,T]$, and for any $(\psi,\varphi)\in\Theta_\varepsilon$, we have

$$\|\Upsilon_1(\psi,\varphi)\|_\infty \le \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r \cdot\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r} + \left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r \cdot\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}. \tag{23}$$

Based on (C2), $\forall t\in[a,T]$, observe that

$$\|\varsigma(t,\psi(t),\varphi(t))\|_\infty \le l_\varsigma(t)\vartheta_\varsigma(\|\psi(t)\|_\infty + \|\varphi(t)\|_\infty) \le \bar{l}_\varsigma\vartheta_\varsigma(\varepsilon), \tag{24}$$

then

$$\begin{aligned}\|\Upsilon_1(\psi,\varphi)\|_\infty &\le \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_1-1}\\ &\quad\cdot(r-x)^{\alpha_1-1}l_\varsigma(t)\vartheta_\varsigma\left(\|\psi(t)\|_\infty+\|\varphi(t)\|_\infty\right)dx\frac{dr}{r}\\ &\quad+\left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\\ &\quad\cdot\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}l_\varsigma(t)\vartheta_\varsigma\left(\|\psi(t)\|_\infty+\|\varphi(t)\|_\infty\right)dx\frac{dr}{r},\\ &\le \bar{l}_\varsigma\vartheta_\varsigma\left(\|\psi(t)\|_\infty+\|\varphi(t)\|_\infty\right)\sup_{a\le t\le T}\\ &\quad\cdot\left\{\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r}\right.\\ &\quad\left.+\left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r}\right\}\\ &\le \bar{l}_\varsigma\Xi_1\vartheta_\varsigma(\varepsilon).\end{aligned}\tag{25}$$

Similarly,

$$\begin{aligned}\|\Upsilon_2(\psi,\varphi)\|_\infty &\le \frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_2-1}\\ &\quad\cdot(r-x)^{\alpha_2-1}l_\xi(t)\vartheta_\xi\left(\|\psi(t)\|_\infty+\|\varphi(t)\|_\infty\right)dx\frac{dr}{r}\\ &\quad+\left(\frac{\ln(t/a)}{\mathrm{n}(T/a)}\right)^{\beta_2}\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^T\int_a^r\\ &\quad\cdot\left(\ln\frac{T}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}l_\xi(t)\vartheta_\xi\left(\|\psi(t)\|_\infty+\|\varphi(t)\|_\infty\right)dx\frac{dr}{r}\\ &\le \bar{l}_\xi\Xi_2\vartheta_\xi(\varepsilon).\end{aligned}\tag{26}$$

Equation (25) and Equation (26) imply that

$$\|\Upsilon(\psi,\varphi)\|_B=\|\Upsilon_1(\psi,\varphi)\|_\infty+\|\Upsilon_2(\psi,\varphi)\|_\infty\le\bar{l}_\varsigma\Xi_1\vartheta_\varsigma(\varepsilon)+\le\bar{l}_\xi\Xi_2\vartheta_\xi(\varepsilon)\le\varepsilon.\tag{27}$$

This proves that $\Upsilon\Theta_\varepsilon\subset\Theta_\varepsilon$.

Secondly, we need to show the continuity for Υ to see this, we take the sequence $\{u_n=(\psi_n,\varphi_n)\}\in\Theta_\varepsilon$, such that $u_n\longrightarrow u=(\psi,\varphi)$ as $n\longrightarrow\infty$..

Owing to the Carathéodory continuity of ς, it is obvious that

$$\varsigma((\cdot),\psi_n(\cdot),\varphi_n(\cdot))\longrightarrow\varsigma((\cdot),\psi(\cdot),\varphi(\cdot))\text{as } n\longrightarrow\infty.\tag{28}$$

Keeping in mind was given in (C2), one can deduce that

$$\begin{aligned}&\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma((r),\psi_n(r),\varphi_n(r))-\varsigma((r),\psi(r),\varphi(r))\|_\infty\\ &\quad\le\bar{l}_\varsigma\vartheta_\varsigma(\varepsilon)\left(\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\right).\end{aligned}\tag{29}$$

Together with the Lebesgue dominated convergence theorem and the fact that the function $r\mapsto\bar{l}_\varsigma\vartheta_\varsigma(\varepsilon)((\ln(t/r))^{\beta_1-1}(r-x)^{\alpha_1-1})$ is the Lebesgue integrable on $[a,T]$, we have

$$\begin{aligned}&\left(\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^t\int_a^r\left(\ln\frac{t}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma((r),\psi_n(r),\varphi_n(r))-\varsigma((r),\psi(r),\varphi(r))\|_\infty dx\frac{dr}{r}\right.\\ &\quad+\left(\frac{\ln(t/a)}{\ln(T/a)}\right)^{\beta_1}\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma((r),\psi_n(r),\varphi_n(r))\\ &\quad\left.-\varsigma((r),\psi(r),\varphi(r))\|_\infty dx\frac{dr}{r}\right)\longrightarrow 0\text{as } n\longrightarrow\infty.\end{aligned}\tag{30}$$

Yields to $\|\Upsilon_1(\psi_n,\varphi_n)(t)-\Upsilon_1(\psi,\varphi)(t)\|_\infty\longrightarrow 0$ as $n\longrightarrow\infty.\forall t\in[a,T]$, we get

$$\|\Upsilon_1(\psi_n,\varphi_n)-\Upsilon_1(\psi,\varphi)\|_\infty\longrightarrow 0\text{ as } n\longrightarrow\infty,\tag{31}$$

that is the operator Υ_1 is continuous.

In a like manner, we have

$$\|\Upsilon_2(\psi_n,\varphi_n)-\Upsilon_2(\psi,\varphi)\|_\infty\longrightarrow 0\text{ as } n\longrightarrow\infty,\tag{32}$$

Combining (31) and (32), we obtain

$$\|\Upsilon(\psi_n,\varphi_n)-\Upsilon(\psi,\varphi)\|_\infty\longrightarrow 0\text{ as } n\longrightarrow\infty.\tag{33}$$

From equation (33), we conclude that the operator Υ is continuous.

Third, to verify the equicontinuity for the operator Υ, let $t_1,t_2\in[a,T]$, $(t_1<t_2)$, and for any $(\psi,\varphi)\in\Theta_\varepsilon$, then

$$\begin{aligned}&\|\Upsilon_1(\psi,\varphi)(t_2)-\Upsilon_1(\psi,\varphi)(t_1)\|_\infty\\ &\le\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_1-1}\right]\\ &\quad\times(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}+\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_{t_1}^{t_1}\int_a^r\\ &\quad\times\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}\\ &\quad+\left(\frac{(\ln(t_2/a))^{\beta_1}-(\ln(t_2/a))^{\beta_1}}{\mathrm{l}(\mathrm{n}(T/a))^{\beta_1}\Gamma(\alpha_1)\Gamma(\beta_1)}\right)\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}\\ &\quad\times(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r},\\ &\le\left(\bar{l}_\varsigma(t)\vartheta_\varsigma(\varepsilon)\right)\times\left(\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_1-1}\right]\right.\\ &\quad\times(r-x)^{\alpha_1-1}dx\frac{dr}{r}+\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_{t_1}^{t_1}\int_a^r\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r}\\ &\quad\left.+\left[\frac{(\ln(t_2/a))^{\beta_1}-(\ln(t_1/a))^{\beta_1}}{(\ln(T/a))^{\beta_1}\Gamma(\alpha_1)\Gamma(\beta_1)}\right]\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}dx\frac{dr}{r}\right)\\ &\longrightarrow 0\; as\; t_1\longrightarrow t_2.\end{aligned}\tag{34}$$

Similarly, we get

$$\begin{aligned}&\|\Upsilon_2(\psi,\varphi)(t_2)-\Upsilon_2(\psi,\varphi)(t_1)\|_\infty\\&\le(\bar{l}_\xi(t)\vartheta_\xi(\varepsilon))\times\left(\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_2-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_2-1}\right](r-x)^{\alpha_2-1}dx\frac{dr}{r}\right.\\&+\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_{t_1}^{t_1}\int_a^r\left(\ln\frac{t_2}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}dx\frac{dr}{r}\\&\left.+\left[\frac{(\ln(t_2/a))^{\beta_2}-(\ln(t_2/a))^{\beta_2}}{\mathrm{l}(\mathrm{n}(T/a))^{\beta_2}\Gamma(\alpha_2)\Gamma(\beta_2)}\right]\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_2-1}(r-x)^{\alpha_2-1}dx\frac{dr}{r}\right)\longrightarrow 0\,as\,t_1\longrightarrow t_2.\end{aligned}\tag{35}$$

Note that the R.H.S's of the above inequalities of Equation (34) and Equation (35) are free of $(\psi,\varphi)\in\Theta_\varepsilon$, which implies that Υ is equicontinuous and bounded.

Fourth and finally, we need to satisfy Mönch's hypothesis, so we let$U=U_1\cap U_2$,.

where $U_1, U_2\subseteq\Theta_\varepsilon$. Moreover, U_1, U_2 are assumed to be bounded and equicontinuous, such that

$$U_1\subset\overline{conv}(\Upsilon_1(U_1)\cup\{0\}),\text{and } U_2\subset\overline{conv}(\Upsilon_2(U_2)\cup\{0\}).\tag{36}$$

Thus, the functions $\mathfrak{J}_1(t)=\kappa(U_1(t)),\mathfrak{J}_2(t)=\kappa(U_2(t))$ are continuous on $[a,T]$.

Based on lemma Equation(10), lemma Equation (11), and (C3), we get

$$\begin{aligned}&\mathfrak{J}_1(t)=\kappa(U_1(t))\le\kappa(\overline{conv}(\Upsilon_1(U_1)(t)\cup\{0\}))\le\kappa(\Upsilon_1(U_1)(t))\\&\le\kappa\left\{\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_1-1}\right]\right.\\&\cdot(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}+\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_{t_1}^{t_1}\int_a^r\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}\\&\cdot(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}+\left(\frac{(\ln(t_2/a))^{\beta_1}-(\ln(t_2/a))^{\beta_1}}{\ln(T/a)}\right)\\&\left.\cdot\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\|\varsigma(x,\psi(x),\varphi(x))\|_\infty dx\frac{dr}{r}:(\psi,\varphi)\in U_1\right\}\\&\le\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_1-1}\right](r-x)^{\alpha_1-1}\kappa(\varsigma(x,U_1(x)))dx\frac{dr}{r}\\&+\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_{t_1}^{t_1}\int_a^r\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}\kappa(\varsigma(x,U_1(x)))dx\frac{dr}{r}\\&+\left(\frac{(\ln(t_2/a))^{\beta_1}-(\ln(t_2/a))^{\beta_1}}{\ln(T/a)}\right)\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\left(\ln\frac{T}{r}\right)^{\beta_1-1}\\&\cdot(r-x)^{\alpha_1-1}\kappa(\varsigma(x,U_1(x)))dx\frac{dr}{r}\le\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^{t_1}\int_a^r\left[\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}-\left(\ln\frac{t_1}{r}\right)^{\beta_1-1}\right]\\&\cdot(r-x)^{\alpha_1-1}l_\varsigma(x)\kappa(U_1(x))dx\frac{dr}{r}+\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_{t_1}^{t_1}\int_a^r\left(\ln\frac{t_2}{r}\right)^{\beta_1-1}\\&\cdot(r-x)^{\alpha_1-1}l_\varsigma(x)\kappa(U_1(x))dx\frac{dr}{r}+\left(\frac{(\ln(t_2/a))^{\beta_1}-(\ln(t_2/a))^{\beta_1}}{\ln(T/a)}\right)\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^T\int_a^r\\&\cdot\left(\ln\frac{T}{r}\right)^{\beta_1-1}(r-x)^{\alpha_1-1}l_\varsigma(x)\kappa(U_1(x))dx\frac{dr}{r}\le\Xi_1\bar{l}_\varsigma\|\mathfrak{J}_1\|_\infty.\end{aligned}\tag{37}$$

That is

$\|\mathfrak{J}_1\|\le\Xi_1\bar{l}_\varsigma\|\mathfrak{J}_1\|$, but it is assumed that max $\{\Xi_1\bar{l}_\varsigma,\Xi_2\bar{l}_\xi\}<1$, which implies that $\|\mathfrak{J}_1\|_\infty=0$, i.e.,

$$\mathfrak{J}_1(t)=0,\forall t\in[a,T].\tag{38}$$

In a like manner, we have $\mathfrak{J}_2(t)=0,\forall t\in[a,T]$. So $\kappa(U(t))\le\kappa(U_1(t))=0$ and.

$\kappa(U(t))\le\kappa(U_2(t))=0$, which implies that $U(t)$ is relatively compact in $B\times B$. Now, Arzela-Ascoli is applicable, which means that U is relatively compact in $\Theta_{\varepsilon\cdot,}$ and therefore, using theorem 9, we deduce that the operator Υ has a fixed point (ψ,φ) (solution of the problem Equation (3)) on Θ_ε. And that ends the proof. □

4. Example

In this section, we provide an applied example that supports the theoretical results reached through this study.

Define $\psi_0=\{\psi=(\psi_1,\psi_2,\cdots,\psi_n,\cdots):\lim_{n\longrightarrow\infty}\psi_n=0\}$, it is obvious that z_0 is a Banach space with$\|\psi\|_\infty=\sup_{n\ge1}|\psi_n|$. For this, we consider the following boundary value problem:

$$\begin{cases}{}^{C}D^{0.5}\left({}^{CH}D^{0.75}\psi(t)\right)=\varsigma(t,\psi(t),\varphi(t)), & t\in[1,3],\\{}^{C}D^{0.6}\left({}^{CH}D^{0.9}\varphi(t)\right)=\xi(t,\psi(t),\varphi(t)), & t\in[1,3],\\\psi(1)=\psi(3)=(0,0,0,\cdots,0,\cdots), & \varphi(1)=\varphi(3)=(0,0,0,\cdots,0,\cdots).\end{cases}\tag{39}$$

Here, $\alpha_1=0.5$, $\beta_1=0.75$, $\alpha_2=0.6$, $\beta_2=0.9$, $a=1$, and $T=3$.

Now, let us take for example

$$\begin{aligned}\varsigma(t,\psi(t),\varphi(t))&=\left\{\frac{1}{\ln t+10}\left(\frac{1}{4^n}+\ln(1+|\psi_n|+|\varphi_n|)\right)\right\},n\ge1,\\\xi(t,\psi(t),\varphi(t))&=\left\{\frac{t}{10}\left(\frac{1}{n^4}+\tan^{-1}(1+|\psi_n|+|\varphi_n|)\right)\right\},n\ge1.\end{aligned}\tag{40}$$

$\forall t\in[1,3]$, with$\{\psi_n\}_{n\ge1},\{\varphi_n\}_{n\ge1}\in\psi_0$, assumption (C1) of theorem 11 is satisfied. Furthermore,

$$\begin{aligned}\|\varsigma(t,\psi,\varphi)\|_\infty&\le\left\|\frac{1}{\ln t+10}\left(\frac{1}{4^n}+\ln(1+|\psi_n|+|\varphi_n|)\right)\right\|_\infty\\&\le\frac{1}{\ln t+10}(\|\psi\|+1)=l_\varsigma(t)\vartheta_\varsigma(\|\psi\|).\end{aligned}\tag{41}$$

Similarly,

$$\begin{aligned}\|\xi(t,\psi,\varphi)\|_\infty&\le\left\|\frac{t}{10}\left(\frac{1}{n^4}+\tan^{-1}(1+|\psi_n|+|\varphi_n|)\right)\right\|_\infty\\&\le\frac{t}{10}(\|\psi\|+1)=l_\xi(t)\vartheta_\xi(\|\psi\|).\end{aligned}\tag{42}$$

That is (C2) of theorem 11 is satisfied as well.

Next, if we consider the bounded subset $S \subset \psi_0 \times \psi_0$, we obtain

$$\kappa(\varsigma(t,S)) \le l_\varsigma(t)\kappa(S),\\ \kappa(\xi(t,S)) \le l_\xi(t)\kappa(S), \tag{43}$$

where in our case, we have $l_\varsigma(t) = 1/\ln t + 9, l_\xi(t) = t/10$; the latter two inequalities show that the condition (C2) of the theorem 11 is satisfied.

Finally, we calculate

$$\bar{l}_\varsigma = \frac{1}{10}, \Xi_1 \le \frac{2(T-a)^{\alpha_1}(\ln (T/a))^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)} = 3.6411,\\ \bar{l}_\xi = \frac{3}{10}, \Xi_2 \le \frac{2(T-a)^{\alpha_2}(\ln (T/a))^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\beta_2+1)} = 0.9376. \tag{44}$$

Then, $\max\{\Xi_1\bar{l}_\varsigma, \Xi_2\bar{l}_\xi\} = \max\{0.3611, 0.28128\} = 0.3611 < 1$. So all conditions of theorem 11 satisfied, that is the problem Equation (39) has at least one solution$(\psi, \varphi) \in C([1,3],\psi_0) \times C([1,3],\psi_0)$.

5. Conclusion

In the current paper, we studied the existence and uniqueness of solution for a coupled system of a mixed fractional differential equations. The main results are established by the aid "Mönch's fixed point theorem." In addition, an applied example that supports the theoretical results reached through this study is included. For future work, more investigations can be performed for such a system by applying another type of fractional derivatives to verify the existence and uniqueness issue, stability via Ulam-Hyeres technique is also possible to be verified.

Data Availability

No data sets were used in this study.

Conflicts of Interest

The author declares that he has no conflict of interest.

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References

[1] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Elsevier, 1998.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations, Vol. 204*, Elsevier, 2006.

[3] E. Zeidler, *Nonlinear Functional Analysis and Its Applications: Part 2 B: Nonlinear Monotone Operators*, Springer, 1990.

[4] H. Mönch, "Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 985–999, 1980.

[5] J. Wang, W. Wei, and M. Feckan, "Nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators," *Miskolc Mathematical Notes*, vol. 13, no. 1, pp. 127–147, 2012.

[6] J. R. Wang, Y. Zhou, and M. Fečkan, "On the nonlocal Cauchy problem for semilinear fractional order evolution equations," *Journal of Mathematics*, vol. 12, no. 6, 922 pages, 2014.

[7] M. Awadalla, Y. Y. N. Yannick, K. A. Asbeh, Y. Y. N. Yannick, and K. A. Asbeh, "Modeling the dependence of barometric pressure with altitude using Caputo and Caputo–Fabrizio fractional derivatives," *Journal of Mathematics*, vol. 2020, Article ID 2417681, 9 pages, 2020.

[8] M. K. A. Kaabar, M. Shabibi, J. Alzabut et al., "Investigation of the fractional strongly singular thermostat model via fixed point techniques," *Mathematics*, vol. 9, no. 18, p. 2298, 2021.

[9] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, and S. Rezapour, "Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[10] M. Awadalla, Y. Y. Yameni Noupoue, and K. A. Asbeh, "Psi-Caputo Logistic Population Growth Model," *Journal of Mathematics*, vol. 2021, Article ID 8634280, 9 pages, 2021.

[11] M. Awadalla, Y. Y. Y. Noupoue, and K. Abuasbeh, "Population growth modeling via Rayleigh-Caputo fractional derivative," *Journal of Statistics Applications & Probability*, vol. 10, no. 1, pp. 11–16, 2021.

[12] C. Bai, "Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative," *Journal of Mathematical Analysis and Applications*, vol. 384, no. 2, pp. 211–231, 2011.

[13] B. Ahmad and J. J. Nieto, "Sequential fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3046–3052, 2012.

[14] B. Ahmad and J. J. Nieto, "Boundary value problems for a class of sequential integrodifferential equations of fractional order," *Journal of Function Spaces and Applications*, vol. 2013, article 149659, pp. 1–8, 2013.

[15] M. H. Aqlan, A. Alsaedi, B. Ahmad, and J. J. Nieto, "Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions," *Open Mathematics*, vol. 14, no. 1, pp. 723–735, 2016.

[16] M. Klimek, "Sequential fractional differential equations with Hadamard derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 12, pp. 4689–4697, 2011.

[17] H. Ye and R. Huang, "Initial value problem for nonlinear fractional differential equations with sequential fractional derivative," *Advances in Difference Equations*, vol. 2015, no. 1, 2015.

[18] N. I. Mahmudov and M. Awadalla, "On sequential fractional differential equations with nonlocal integral boundary

conditions," *Journal of Computational Analysis and Applications*, vol. 628, 2019.

[19] S. Aljoudi, B. Ahmad, J. J. Nieto, and A. Alsaedi, "A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions," *Chaos, Solitons & Fractals*, vol. 91, pp. 39–46, 2016.

[20] B. Ahmad and S. Ntouyas, "Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations," *Electronic Journal of Differential Equations*, vol. 2017, no. 36, pp. 1–11, 2017.

[21] R. Agarwal, D. O'Regan, and S. Hristova, "Stability of Caputo fractional differential equations by Lyapunov functions," *Applications of Mathematics*, vol. 60, no. 6, pp. 653–676, 2015.

[22] X. Wang, D. Luo, and Q. Zhu, "Ulam-Hyers stability of Caputo type fuzzy fractional differential equations with time-delays," *Chaos, Solitons & Fractals*, vol. 156, article 111822, 2022.

[23] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Theory of fractional hybrid differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1312–1324, 2011.

[24] S. Sun, Y. Zhao, Z. Han, and Y. Li, "The existence of solutions for boundary value problem of fractional hybrid differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4961–4967, 2012.

[25] B. Ahmad and S. K. Ntouyas, "An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions," *Abstract and Applied Analysis*, vol. 2014, Article ID 705809, 7 pages, 2014.

[26] B. C. Dhage and S. K. Ntouyas, "Existence results for boundary value problems for fractional hybrid differential inclusions," *Topological Methods in Nonlinear Analysis*, vol. 44, no. 1, pp. 229–238, 2014.

[27] C. Derbazi and Z. Baitiche, "Coupled systems of ψ-Caputo differential equations with initial conditions in Banach spaces," *Mediterranean Journal of Mathematics*, vol. 17, article 169, 2020.

[28] M. Benchohra, J. Henderson, and D. Seba, "Measure of noncompactness and fractional differential equations in Banach spaces," *Communications in Applied Analysis*, vol. 12, no. 4, p. 419, 2008.

[29] D. J. Guo, V. Lakshmikantham, and X. Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.