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Research Article

An Approach of Integral Equations in Complex-Valued *b*-Metric Space Using Commuting Self-Maps

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This paper is aimed at establishing some unique common fixed point theorems in complex-valued b-metric space under the rational type contraction conditions for three self-mappings in which the one self-map is continuous. A continuous self-map is commutable with the other two self-mappings. Our results are verified by some suitable examples. Ultimately, our results have been utilized to prove the existing solution to the two Urysohn integral type equations. This application illustrates how complex-valued b-metric space can be used in other types of integral operators.

1. Introduction

In 1922, Banach [1] proved a fixed point theorem (FP-theorem), which is stated as the following: "a single-valued contractive type mapping on a complete metric space has a unique fixed point." After the publication of the Banach FP-theorem, many researchers have contributed their ideas to the theory of FP. Chandok [2, 3], Jungck and Rhoades [4], and Al-Shami and Abo-Tabl [5, 6] proved different contractive types of FPs and a-fixed soft point results in the context of metric spaces.

Bakhtin [7] introduced the idea of b-metric space, while Czerwik [8] proved some fixed point results for nonlinear set-valued contractive type mappings in b-metric spaces. Suzuki in [9] proved basic inequality and some FP-theorems. Jain and Kaur [10] presented a new class of functions to define new contractive maps and established FP-results for these maps. They also extended some results in the framework of b-metric-like spaces. They presented examples and established the application of their main results. They also presented some open problems. Petrusel et al. [11] considered coupled FP-problems for single-valued operators satisfying contraction in said space. They discussed uniqueness, data dependence, and shadowing-property of coupled FP-problem and also established an application for main results. Ameer et al. [12], Boriceanu [13, 14], Bota et al. [15], Czerwik et al. [16, 17], Hussain and Shah [18], Karapinar et al. [19], and Samreen et al. [20] established different contractive type FP and common FP (CFP) results in the context of b-metric spaces.

In 2011, the concept of complex-valued metric space was given by Azam et al. [21], and they proved some CFP-theorems for self-mappings. The notion of said space was proposed by Rouzkard and Imdad [22] which generalizes the results of Azam et al. [21] and established some CFP-results. Abbas et al. [23] presented some generalized CFP-results by using cocyclic mappings in complex-valued metric space. They provided examples to indicate the authenticity of his expressions. Sarwar and Zada [24] used the ideas of $(E.A)$ and (CLR) properties and proved

FP-results for six self-mappings. They showed the existence of their results by establishing some examples. Abbas et al. [25], Nashine et al. [26], Mohanta and Maitra [27], Sintunavarat and Kumam [28], and Verma and Pathak [29] proved some results in the context of complex-valued metric space.

In 2013, Rao et al. [30] introduced the notion of complex-valued b-metric space. Mukheimer et al. [31] established CFP-results on said space by extending and generalizing the results of [30, 31]. In [32], Chantakun et al. extend the work of Dubey et al. [33] by introducing sufficient conditions to prove some CFP-results in complex-valued b-metric space. Yadav et al. [34] used compatible and weakly compatible maps to find CFP-results. They proved the validity of the results by providing some examples and establishing an application. Berrah et al. [35], Hasana [36], Mehmood et al. [37], and Mukheimer [38] established some FP and CFP theorems in complex-valued b-metric spaces.

In this paper, we provide some extended and effective CFP-results for commuting three self-maps on complex-valued b-metric spaces. To verify the validity of our work, we present some illustrative examples in the main section. Further, our results have been utilized to prove the existing solution to the two Urysohn integral type equations. This application is also illustrative of how complex-valued b-metric space can be used in other integral type operators. This paper is organized as follows: In Section 2, we present the preliminary concepts. In Section 3, we establish some extended and modified CFP-results for commuting self-maps in complex-valued b-metric space under the generalized rational type conditions. We also provide authentic examples to indicate the effectiveness of these results. In Section 4, we present an application of the two UITEs to support our main work. Finally, in Section 5, we discuss the conclusion.

2. Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers and $z_i, z_{ii} \in \mathbb{C}$. Define $\leq$ as $z_i \leq z_{ii}$, iff $R_e(z_i) \leq R_e(z_{ii})$ and $I_m(z_i) \leq I_m(z_{ii})$, where R_e denotes the real part and I_m denotes the imaginary part of a complex number. Accordingly, $z_i \leq z_{ii}$, if any one of the following conditions holds:

$(C_1) R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$

$(C_2) R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$

$(C_3) R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$

$(C_4) R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$

Know that $z_i \lneq z_{ii}$ if $z_i \neq z_{ii}$ and one of (C_2), (C_3), and (C_4) is satisfied.

Remark 1 (see [31]). We can easily check the following:

(i) If $a_1, a_2 \in \mathbb{R}$ and $a_1 \leq a_2 \Rightarrow a_1 y \leq a_2 y$, $\forall y \in \mathbb{C}$

(ii) $0 \leq z_i \lneq z_{ii} \Rightarrow |z_i| < |z_{ii}|$

(iii) $z_i \leq z_{ii}$ and $z_{ii} < z_{iii} \Rightarrow z_i < z_{iii}$

Definition 2 (see [8]). Let Ω be a nonempty set and $b \geq 1$ a given real number. A mapping $\eth : \Omega \times \Omega \longrightarrow [0,\infty)$ *is* called a b-metric on Ω if the following conditions are satisfied:

$(b_m 1) \eth(\rho_1, \rho_2) = 0$ if and only if $\rho_1 = \rho_2$

$(b_m 2) \eth(\rho_1, \rho_2) = \eth(\rho_2, \rho_1)$

$(b_m 3) \eth(\rho_1, \rho_2) \leq b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$,

for all $\rho_1, \rho_2, \rho_3 \in \Omega$. Then, $(\Omega, \eth)$ is called a b-metric space.

Definition 3 (see [30]). Let Ω be a nonempty set and $b \geq 1$ a given real number. A mapping $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ *is* called a complex-valued b-metric on Ω if the following conditions are satisfied:

$(Cb_m 1) \eth(\rho_1, \rho_2) \geq 0$ and $\eth(\rho_1, \rho_2) = 0$ if and only if $\rho_1 = \rho_2$

$(Cb_m 2) \eth(\rho_1, \rho_2) = \eth(\rho_2, \rho_1)$

$(Cb_m 3) \eth(\rho_1, \rho_2) \leq b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$,

for all $\rho_1, \rho_2, \rho_3 \in \Omega$. Then, $(\Omega, \eth)$ is called a complex-valued b-metric space.

Example 4. Let $\Omega = R^+$. Define the mapping $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ *by* $\eth(\rho_1, \rho_2) = 7/17|\rho_1 - \rho_2|^2 + i7/17|\rho_1 - \rho_2|^2$, for all $\rho_1, \rho_2 \in \Omega$.

Then, $(\Omega, \eth)$ is a complex-valued b-metric space with $b = 2$.

Definition 5 (see [30, 31]). Let $(\Omega, \eth)$ be a complex-valued b-metric space and $\{\rho_n\}$ a sequence in Ω and $\rho \in \Omega$. Then,

(1) $\{\rho_n\}$ is said to converge to ρ if for every $0 < c^* \in \mathbb{C}$ there exists $N^* \in \mathbb{N}$ such that $\eth(\rho_n, \rho) < c^*$, $\forall n > N^*$. We denote this by $\lim_{n \longrightarrow \infty} \rho_n = \rho$ or $\{\rho_n\} \longrightarrow \rho$ as $n \longrightarrow \infty$

(2) if for every $0 < c^* \in \mathbb{C}$ there exists $N^* \in \mathbb{N}$ such that $\eth(\rho_n, \rho_{n+m}) < c^*$ for all $n > N^*$, $m \in \mathbb{N}$, then $\{\rho_n\}$ is called a Cauchy sequence

(3) if every Cauchy sequence is convergent, then $(\Omega, \eth)$ is called a complete complex-valued b-metric space

Lemma 6 (see [30, 31]). *Let $(\Omega, \eth)$ be a complex-valued b-metric space and let $\{\rho_n\}$ be a sequence in Ω. Then, $\{\rho_n\}$ converges to ρ iff $|\eth(\rho_n, \rho)| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 7 (see [30, 31]). *Let $(\Omega, \eth)$ be a complex-valued b-metric space and let $\{\rho_n\}$ be a sequence in Ω. Then, $\{\rho_n\}$ is a Cauchy sequence iff $|\eth(\rho_n, \rho_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Definition 8 (see [39]). Let $(\Omega, \eth)$ be a complex-valued b-metric space. The self-mappings f_1 and f_2 are said to be commuting if $f_1 f_2 \rho = f_2 f_1 \rho$ for all $\rho \in \Omega$.

3. Main Result

In this section, we prove some CFP theorems in complex-valued g-metric space under the generalized rational type contraction conditions for three self-mappings in which one is continuous. We present some examples for the validation of our work.

Theorem 9. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\begin{aligned}\eth(f_1\rho_1, f_2\rho_2) &\le \kappa_1 \eth(f\rho_1, f\rho_2) \\ &+ \kappa_2 \frac{\eth(f\rho_1, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_2, f_2\rho_2)}{1+\eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_2, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}\end{array}\right\},\end{aligned} \tag{1}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$ and $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\begin{aligned}\Gamma_{2n} &= f\rho_{2n+1} = f_1\rho_{2n}, \\ \Gamma_{2n+1} &= f\rho_{2n+2} = f_2\rho_{2n+1}, \\ &\forall n \ge 0.\end{aligned} \tag{2}$$

□

Now, by using (1),

$$\begin{aligned}\eth(\Gamma_{2n}, \Gamma_{2n+1}) &= \eth(f_1\rho_{2n}, f_2\rho_{2n+1}) \le \kappa_1\eth(f\rho_{2n}, f\rho_{2n+1}) \\ &+ \kappa_2 \frac{\eth(f\rho_{2n}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_{2n}, f_1\rho_{2n}), \eth(f\rho_{2n+1}, f_2\rho_{2n+1}), \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(f\rho_{2n}, f\rho_{2n+1})}, \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n+1})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}, \\ \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}\end{array}\right\} \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n}) + \kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n}, \Gamma_{2n+1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n}, \Gamma_{2n+1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}\end{array}\right\}.\end{aligned} \tag{3}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n}, \Gamma_{2n+1})| &\le \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})| \\ &+ \kappa_2\frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|)} \\ &+ \kappa_3 \min\left\{\begin{array}{c}|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, |\eth(\Gamma_{2n}, \Gamma_{2n+1})|, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n+1})|}{|1+\eth(\Gamma_{2n-1}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}\end{array}\right\}.\end{aligned} \tag{4}$$

After simplification, we get that

$$|\eth(\Gamma_{2n}, \Gamma_{2n+1})| \le \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})|. \tag{5}$$

Again, by using (1) and (2),

$$\begin{aligned}\eth(\Gamma_{2n-1}, \Gamma_{2n}) &= \eth(f_2\rho_{2n-1}, f_1\rho_{2n}) = \eth(f_1\rho_{2n}, f_2\rho_{2n-1}) \le \kappa_1\eth(f\rho_{2n}, f\rho_{2n-1}) \\ &+ \kappa_2\frac{\eth(f\rho_{2n}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_{2n}, f_1\rho_{2n}), \eth(f\rho_{2n-1}, f_2\rho_{2n-1}), \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n-1}, f_2\rho_{2n-1})}{1+\eth(f\rho_{2n}, f\rho_{2n-1})}, \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n-1})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}, \\ \dfrac{\eth(f\rho_{2n-1}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}\end{array}\right\} \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n-2}) + \kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n-2}, \Gamma_{2n-1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n-1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n-2})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n-1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-2}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}\end{array}\right\}.\end{aligned} \tag{6}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n-1}, \Gamma_{2n})| &\le \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n-2})| \\ &+ \kappa_2\frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|)} \\ &+ \kappa_3 \min\left\{\begin{array}{c}|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, |\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|}{|1+\eth(\Gamma_{2n-1}, \Gamma_{2n-2})|}, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}\end{array}\right\}.\end{aligned} \tag{7}$$

After simplification, we get that

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})|\leq\kappa_1|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|. \tag{8}$$

Now, from (8) and (5) and by induction, we have that

$$\begin{aligned}|\eth(\Gamma_{2n},\Gamma_{2n+1})|&\leq\kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n})|\\&\leq\kappa_1^2|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|\leq\cdots\leq\kappa_1^{2n}|\eth(\Gamma_0,\Gamma_1)|.\end{aligned} \tag{9}$$

So, for $m,n\in\mathbb{N}$ with $m>n$,

$$\begin{aligned}|\eth(\Gamma_n,\Gamma_m)|&\leq b|\eth(\Gamma_n,\Gamma_{n+1})|+b|\eth(\Gamma_{n+1},\Gamma_m)|\leq b|\eth(\Gamma_n,\Gamma_{n+1})|\\&\quad+b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})|+\cdots+b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)|\\&\leq b\kappa_1^n|\eth(\Gamma_0,\Gamma_1)|+b^2\kappa_1^{n+1}|\eth(\Gamma_0,\Gamma_1)|+\cdots\\&\quad+b^{m-n}\kappa_1^{m-1}|\eth(\Gamma_0,\Gamma_1)|\\&\leq\left[b\kappa_1^n+b^2\kappa_1^{n+1}+\cdots+b^{m-n}\kappa_1^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)|\\&=\left[b\kappa_1^n+b^2\kappa_1^{n+1}+\cdots+b^{m-n}\kappa_1^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)|\\&=b\kappa_1^n\left[1+b\kappa_1+b^2\kappa_1^2\cdots+b^{m-(n+1)}\kappa_1^{m-(n+1)}\right]\\&\quad\cdot|\eth(\Gamma_0,\Gamma_1)|=b\kappa_1^n\sum_{t=0}^{m-(n+1)}b^t\kappa_1^t|\eth(\Gamma_0,\Gamma_1)|\\&\leq b\kappa_1^n\sum_{t=0}^{\infty}b^t\kappa_1^t|\eth(\Gamma_0,\Gamma_1)|\\&=\frac{b\kappa_1^n}{1-b\kappa_1}|\eth(\Gamma_0,\Gamma_1)|\longrightarrow0,\text{ as }n\longrightarrow\infty.\end{aligned} \tag{10}$$

Therefore, the sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s\in\Omega$ such that $\Gamma_n\longrightarrow s$, as $n\longrightarrow\infty$, or $\lim_{n\longrightarrow\infty}\Gamma_n=s$, and from (2), we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}f\rho_{2n+1}&=s,\\\lim_{n\longrightarrow\infty}f_1\rho_{2n}&=s,\\\lim_{n\longrightarrow\infty}f_2\rho_{2n+1}&=s.\end{aligned} \tag{11}$$

As f is continuous, so

$$\begin{aligned}\lim_{n\longrightarrow\infty}f(f\rho_{2n+1})&=fs,\\\lim_{n\longrightarrow\infty}f(f_1\rho_{2n})&=fs,\\\lim_{n\longrightarrow\infty}f(f_2\rho_{2n+1})&=fs.\end{aligned} \tag{12}$$

Since, (f,f_1) and (f,f_2) are commutable pairs, therefore, from (12), we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty}f_1(f\rho_{2n})&=fs,\\\lim_{n\longrightarrow\infty}f_2(f\rho_{2n+1})&=fs.\end{aligned} \tag{13}$$

Now, we have to show that $fs=s$, so by putting $\rho_1=f\rho_{2n}$ and $\rho_2=\rho_{2n+1}$, in (1),

$$\begin{aligned}\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})&\leq\kappa_1\eth(f(f\rho_{2n}),f\rho_{2n+1})\\&\quad+\kappa_2\frac{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n})))}\\&\quad+\kappa_3\min\left\{\begin{array}{c}\eth(f(f\rho_{2n}),f_1(f\rho_{2n})),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}),f\rho_{2n+1})},\\\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f(f\rho_{2n}),f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))},\\\dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}\end{array}\right\}.\end{aligned} \tag{14}$$

This implies that

$$\begin{aligned}|\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})|&\leq\kappa_1|\eth(f(f\rho_{2n}),f\rho_{2n+1})|\\&\quad+\kappa_2\frac{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|)}\\&\quad+\kappa_3\min\left\{\begin{array}{c}|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}),f\rho_{2n+1})|},\\\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|},\\\dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}\end{array}\right\}.\end{aligned} \tag{15}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (11), (12), and (13), we get that

$$\begin{aligned}|\eth(fs,s)|&\leq\kappa_1|\eth(fs,s)|+\kappa_2\frac{|\eth(fs,s)|\cdot|\eth(s,fs)|}{1/2(|\eth(fs,s)|+|\eth(s,fs)|)}\\&\quad+\kappa_3\min\left\{\begin{array}{c}|\eth(fs,fs)|,|\eth(s,s)|,\\\dfrac{|\eth(fs,fs)|\cdot|\eth(s,s)|}{|1+\eth(fs,s)|},\\\dfrac{|\eth(fs,fs)|\cdot|\eth(fs,s)|}{|\eth(fs,s)|+|\eth(s,fs)|},\\\dfrac{|\eth(s,s)|\cdot|\eth(s,fs)|}{|\eth(fs,s)|+|\eth(s,fs)|}\end{array}\right\}.\end{aligned} \tag{16}$$

After simplification, we get that

$$|\eth(fs,s)|\leq(\kappa_1+\kappa_2)|\eth(fs,s)|\Rightarrow(1-\kappa_1-\kappa_2)|\eth(fs,s)|\leq0. \tag{17}$$

Since $(1-\kappa_1-\kappa_2)\neq 0 \Rightarrow |\eth(fs,s)|=0$, hence, we get that

$$fs=s. \tag{18}$$

Next, we have to show that $f_1s=s$, by the view of (1),

$$\begin{aligned}\eth(f_1s,f\rho_{2n+2}) &= \eth(f_1s,f_2\rho_{2n+1}) \le \kappa_1\eth(fs,f\rho_{2n+1}) \\ &+ \kappa_2\frac{\eth(fs,f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{1/2(\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s))} \\ &+ \kappa_3\min\left\{\begin{array}{c}\eth(fs,f_1s),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\ \dfrac{\eth(fs,f_1s)\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(fs,f\rho_{2n+1})},\\ \dfrac{\eth(fs,f_1s)\cdot\eth(fs,f_2\rho_{2n+1})}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)},\\ \dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)}\end{array}\right\}.\end{aligned} \tag{19}$$

This implies that

$$\begin{aligned}|\eth(f_1s,f\rho_{2n+2})| &\le \kappa_1|\eth(fs,f\rho_{2n+1})| \\ &+ \kappa_2\frac{|\eth(fs,f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{1/2(|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|)} \\ &+ \kappa_3\min\left\{\begin{array}{c}|\eth(fs,f_1s)|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(fs,f\rho_{2n+1})|},\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(fs,f_2\rho_{2n+1})|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|}\end{array}\right\}.\end{aligned} \tag{20}$$

Now, again applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (11) and (18), we have that

$$\begin{aligned}|\eth(f_1s,s)| &\le \kappa_1|\eth(s,s)| + \kappa_2\frac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{1/2(|\eth(s,s)|+|\eth(s,f_1s)|)} \\ &+ \kappa_3\min\left\{\begin{array}{c}|\eth(s,f_1s)|,|\eth(s,s)|,\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|\eth(s,s)|+|\eth(s,f_1s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{|\eth(s,s)|+|\eth(s,f_1s)|}\end{array}\right\}.\end{aligned} \tag{21}$$

This implies that $|\eth(f_1s,s)|\le 0$. Hence,

$$f_1s=s. \tag{22}$$

Now, we have to show that $f_2s=s$, by using (1),

$$\begin{aligned}\eth(f\rho_{2n+1},f_2s) &= \eth(f_1\rho_{2n},f_2s) \le \kappa_1\eth(f\rho_{2n},fs) \\ &+ \kappa_2\frac{\eth(f\rho_{2n},f_2s)\cdot\eth(fs,f_1\rho_{2n})}{1/2(\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n}))} \\ &+ \kappa_3\min\left\{\begin{array}{c}\eth(f\rho_{2n},f_1\rho_{2n}),\eth(fs,f_2s),\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(fs,f_2s)}{1+\eth(f\rho_{2n},fs)},\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(f\rho_{2n},f_2s)}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})},\\ \dfrac{\eth(fs,f_2s)\cdot\eth(fs,f_1\rho_{2n})}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})}\end{array}\right\}.\end{aligned} \tag{23}$$

This implies that

$$\begin{aligned}|\eth(f\rho_{2n+1},f_2s)| &\le \kappa_1|\eth(f\rho_{2n},fs)| \\ &+ \kappa_2\frac{|\eth(f\rho_{2n},f_2s)|\cdot|\eth(fs,f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|)} \\ &+ \kappa_3\min\left\{\begin{array}{c}|\eth(f\rho_{2n},f_1\rho_{2n})|,|\eth(fs,f_2s)|,\\ \dfrac{|\eth(f\rho_{2n},f_1\rho_{2n})|\cdot|\eth(fs,f_2s)|}{|1+\eth(f\rho_{2n},fs)|},\\ \dfrac{|\eth(f\rho_{2n},f_1\rho_{2n})|\cdot|\eth(f\rho_{2n},f_2s)|}{|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|},\\ \dfrac{|\eth(fs,f_2s)|\cdot|\eth(fs,f_1\rho_{2n})|}{|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|}\end{array}\right\}.\end{aligned} \tag{24}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (11) and (18), we get

$$\begin{aligned}|\eth(s,f_2s)| &\le \kappa_1|\eth(s,s)| + \kappa_2\frac{|\eth(s,f_2s)|\cdot|\eth(s,s)|}{1/2(|\eth(s,f_2s)|+|\eth(s,s)|)} \\ &+ \kappa_3\min\left\{\begin{array}{c}|\eth(s,s)|,|\eth(s,f_2s)|,\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_2s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_2s)|}{|\eth(s,f_2s)|+|\eth(s,s)|},\\ \dfrac{|\eth(s,f_2s)|\cdot|\eth(s,s)|}{|\eth(s,f_2s)|+|\eth(s,s)|}\end{array}\right\}.\end{aligned} \tag{25}$$

This implies that $|\eth(s,f_2s)|\le 0$. Hence,

$$f_2s=s. \tag{26}$$

Thus, from (18), (22), and (26), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs=f_1s=f_2s=s. \tag{27}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{28}$$

Then, from (1), we have that

$$\eth(s, s^*) = \eth(f_1 s, f_2 s^*) \leq \kappa_1 \eth(fs, fs^*) + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} + \kappa_3 \min \left\{ \begin{array}{c} \eth(fs, f_1 s), \eth(fs^*, f_2 s^*), \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)}, \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)}, \\ \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right\} = (\kappa_1 + \kappa_2) \eth(s, s^*). \tag{29}$$

This implies that $|\eth(s, s^*)| \leq (\kappa_1 + \kappa_2)|\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2)|\eth(s, s^*)| \leq 0$. Since $(1 - \kappa_1 - \kappa_2) \neq 0 \Rightarrow |\eth(s, s^*)| = 0 \Rightarrow s = s^*$. Hence, prove that f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_3 = 0$ in Theorem 9, we get the following corollary.

Corollary 10. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1 \rho_1, f_2 \rho_2) \leq \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_2 \frac{\eth(f \rho_1, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{1/2(\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1))}, \tag{30}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$ and $b \geq 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_2 = 0$ in Theorem 9, we can get the following corollary.

Corollary 11. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying:*

$$\eth(f_1 \rho_1, f_2 \rho_2) \leq \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_3 \min \left\{ \begin{array}{c} \eth(f \rho_1, f_1 \rho_1), \eth(f \rho_2, f_2 \rho_2), \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_2, f_2 \rho_2)}{1 + \eth(f \rho_1, f \rho_2)}, \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_1, f_2 \rho_2)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)}, \\ \dfrac{\eth(f \rho_2, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)} \end{array} \right\}, \tag{31}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_3 \in [0, 1)$ and $b \geq 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Theorem 12. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1 \rho_1, f_2 \rho_2) \leq \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_2 \frac{\eth(f \rho_1, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{1/2(\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1))} + \kappa_3 \max \left\{ \begin{array}{c} \eth(f \rho_1, f_1 \rho_1), \eth(f \rho_2, f_2 \rho_2), \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_2, f_2 \rho_2)}{1 + \eth(f \rho_1, f \rho_2)}, \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_1, f_2 \rho_2)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)}, \\ \dfrac{\eth(f \rho_2, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)} \end{array} \right\}, \tag{32}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $b \geq 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\begin{aligned} \Gamma_{2n} &= f \rho_{2n+1} = f_1 \rho_{2n}, \\ \Gamma_{2n+1} &= f \rho_{2n+2} = f_2 \rho_{2n+1}, \\ &\forall n \geq 0. \end{aligned} \tag{33}$$

□

Now, by using (32),

$$\begin{aligned} \eth(\Gamma_{2n}, \Gamma_{2n+1}) &= \eth(f_1 \rho_{2n}, f_2 \rho_{2n+1}) \leq \kappa_1 \eth(f \rho_{2n}, f \rho_{2n+1}) + \kappa_2 \frac{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) \cdot \eth(f \rho_{2n+1}, f_1 \rho_{2n})}{1/2(\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n}))} \\ &\quad + \kappa_3 \max \left\{ \begin{array}{c} \eth(f \rho_{2n}, f_1 \rho_{2n}), \eth(f \rho_{2n+1}, f_2 \rho_{2n+1}), \\ \dfrac{\eth(f \rho_{2n}, f_1 \rho_{2n}) \cdot \eth(f \rho_{2n+1}, f_2 \rho_{2n+1})}{1 + \eth(f \rho_{2n}, f \rho_{2n+1})}, \\ \dfrac{\eth(f \rho_{2n}, f_1 \rho_{2n}) \cdot \eth(f \rho_{2n}, f_2 \rho_{2n+1})}{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n})}, \\ \dfrac{\eth(f \rho_{2n+1}, f_2 \rho_{2n+1}) \cdot \eth(f \rho_{2n+1}, f_1 \rho_{2n})}{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n})} \end{array} \right\} \\ &= \kappa_1 \eth(\Gamma_{2n-1}, \Gamma_{2n}) + \kappa_2 \frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n}))} \\ &\quad + \kappa_3 \max \left\{ \begin{array}{c} \eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n}, \Gamma_{2n+1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n+1})}{1 + \eth(\Gamma_{2n-1}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})} \end{array} \right\}. \end{aligned} \tag{34}$$

This implies that,

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le \kappa_1 |\eth(\Gamma_{2n-1},\Gamma_{2n})| + \kappa_2 \frac{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})| \cdot |\eth(\Gamma_{2n},\Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1},\Gamma_{2n+1})| + |\eth(\Gamma_{2n},\Gamma_{2n})|)} + \kappa_3 \max \left\{ \begin{array}{c} |\eth(\Gamma_{2n-1},\Gamma_{2n})|, |\eth(\Gamma_{2n},\Gamma_{2n+1})|, \\ \dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})| \cdot |\eth(\Gamma_{2n},\Gamma_{2n+1})|}{|1+\eth(\Gamma_{2n-1},\Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})| \cdot |\eth(\Gamma_{2n-1},\Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})| + |\eth(\Gamma_{2n},\Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n},\Gamma_{2n+1})| \cdot |\eth(\Gamma_{2n},\Gamma_{2n})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})| + |\eth(\Gamma_{2n},\Gamma_{2n})|} \end{array} \right\}. \quad (35)$$

After simplification, we get that

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le \kappa_1 |\eth(\Gamma_{2n-1},\Gamma_{2n})| + \kappa_3 \max \{ |\eth(\Gamma_{2n-1},\Gamma_{2n})|, |\eth(\Gamma_{2n},\Gamma_{2n+1})| \}. \quad (36)$$

Now, there are two possibilities:

(i) If $\eth(\Gamma_{2n-1},\Gamma_{2n})$ is a maximum term in $\{|\eth(\Gamma_{2n-1},\Gamma_{2n})|, |\eth(\Gamma_{2n},\Gamma_{2n+1})|\}$, then after simplification, (36) can be written as

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g_1 |\eth(\Gamma_{2n-1},\Gamma_{2n})|, \text{where } g_1 = \kappa_1 + \kappa_3 < 1 \quad (37)$$

(ii) If $\eth(\Gamma_{2n},\Gamma_{2n+1})$ is a maximum term in $\{\eth(\Gamma_{2n-1},\Gamma_{2n})|, |\eth(\Gamma_{2n},\Gamma_{2n+1})|\}$, then after simplification, (36) can be written as

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g_2 |\eth(\Gamma_{2n-1},\Gamma_{2n})|, \text{where } g_2 = \frac{\kappa_1}{1-\kappa_3} < 1 \quad (38)$$

Let $g \coloneqq \max\{g_1, g_2\} < 1$, then from (37) and (38), for all $n \ge 0$, we have

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g |\eth(\Gamma_{2n-1},\Gamma_{2n})|. \quad (39)$$

Similarly,

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g |\eth(\Gamma_{2n-2},\Gamma_{2n-1})|. \quad (40)$$

Now, from (40) and (39) and by induction, we have that

$$|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g |\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g^2 |\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \le \cdots \le g^{2n} |\eth(\Gamma_0,\Gamma_1)|. \quad (41)$$

Now, for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} |\eth(\Gamma_n,\Gamma_m)| &\le b|\eth(\Gamma_n,\Gamma_{n+1})| + b|\eth(\Gamma_{n+1},\Gamma_m)| \le b|\eth(\Gamma_n,\Gamma_{n+1})| + b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})| + \cdots + b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)| \\ &\le bg^n|\eth(\Gamma_0,\Gamma_1)| + b^2 g^{n+1}|\eth(\Gamma_0,\Gamma_1)| + \cdots + b^{m-n} g^{m-1}|\eth(\Gamma_0,\Gamma_1)| \\ &\le \left[bg^n + b^2 g^{n+1} + \cdots + b^{m-n} g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= \left[bg^n + b^2 g^{n+1} + \cdots + b^{m-n} g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n\left[1 + bg + b^2 g^2 \cdots + b^{m-(n+1)} g^{m-(n+1)}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n \sum_{t=0}^{m-(n+1)} b^t g^t |\eth(\Gamma_0,\Gamma_1)| \le bg^n \sum_{t=0}^{\infty} b^t g^t |\eth(\Gamma_0,\Gamma_1)| \\ &= \frac{bg^n}{1-bg}|\eth(\Gamma_0,\Gamma_1)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (42)$$

Therefore, sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s \in \Omega$ such that $\Gamma_n \longrightarrow s$, as $n \longrightarrow \infty$, or $\lim_{n\longrightarrow\infty} \Gamma_n = s$, and from (33), we have

$$\begin{aligned} \lim_{n\longrightarrow\infty} f\rho_{2n+1} &= s, \\ \lim_{n\longrightarrow\infty} f_1\rho_{2n} &= s, \\ \lim_{n\longrightarrow\infty} f_2\rho_{2n+1} &= s. \end{aligned} \quad (43)$$

As f is continuous, so

$$\begin{aligned} \lim_{n\longrightarrow\infty} f(f\rho_{2n+1}) &= fs, \\ \lim_{n\longrightarrow\infty} f(f_1\rho_{2n}) &= fs, \\ \lim_{n\longrightarrow\infty} f(f_2\rho_{2n+1}) &= fs. \end{aligned} \quad (44)$$

Since, (f, f_1) and (f, f_2) are commutable pairs, therefore, from (44), we have that

$$\begin{aligned} \lim_{n\longrightarrow\infty} f_1(f\rho_{2n}) &= fs, \\ \lim_{n\longrightarrow\infty} f_2(f\rho_{2n+1}) &= fs. \end{aligned} \quad (45)$$

Now, we have to show that $fs = s$, so by putting $\rho_1 = f\rho_{2n}$ and $\rho_2 = \rho_{2n+1}$, in (32):

$$\eth(f_1(f\rho_{2n}), f_2\rho_{2n+1}) \le \kappa_1 \eth(f(f\rho_{2n}), f\rho_{2n+1}) + \kappa_2 \frac{\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) \cdot \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n})))} + \kappa_3 \max \left\{ \begin{array}{c} \eth(f(f\rho_{2n}), f_1(f\rho_{2n})), \eth(f\rho_{2n+1}, f_2\rho_{2n+1}), \\ \dfrac{\eth(f(f\rho_{2n}), f_1(f\rho_{2n})) \cdot \eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}), f\rho_{2n+1})}, \\ \dfrac{\eth(f(f\rho_{2n}), f_1(f\rho_{2n})) \cdot \eth(f(f\rho_{2n}), f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}, \\ \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1}) \cdot \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))} \end{array} \right\}. \quad (46)$$

This implies that

$$|\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})| \le \kappa_1|\eth(f(f\rho_{2n}),f\rho_{2n+1})| + \kappa_2\frac{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|)} + \kappa_3\max\left\{\begin{array}{c}|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}),f\rho_{2n+1})|},\\ \dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}\end{array}\right\}. \quad (47)$$

Taking $\lim_{n\longrightarrow\infty}$ and using (43), (44), and (45), we get that

$$|\eth(fs,s)| \le \kappa_1|\eth(fs,s)| + \kappa_2\frac{|\eth(fs,s)|\cdot|\eth(s,fs)|}{1/2(|\eth(fs,s)|+|\eth(s,fs)|)} + \kappa_3\max\left\{\begin{array}{c}|\eth(fs,fs)|,|\eth(s,s)|,\dfrac{|\eth(fs,fs)|\cdot|\eth(s,s)|}{|1+\eth(fs,s)|},\\ \dfrac{|\eth(fs,fs)|\cdot|\eth(fs,s)|}{|\eth(fs,s)|+|\eth(s,fs)|},\dfrac{|\eth(s,s)|\cdot|\eth(s,fs)|}{|\eth(fs,s)|+|\eth(s,fs)|}\end{array}\right\}. \quad (48)$$

After simplification, we get that

$$|\eth(fs,s)| \le (\kappa_1+\kappa_2)|\eth(fs,s)| \Rightarrow (1-\kappa_1-\kappa_2)|\eth(fs,s)| \le 0. \quad (49)$$

Since $(1-\kappa_1-\kappa_2)\neq 0 \Rightarrow |\eth(fs,s)| = 0$; hence, we get that

$$fs = s. \quad (50)$$

Next, we have to show that $f_1s=s$, by the view of (32),

$$\eth(f_1s,f\rho_{2n+2}) = \eth(f_1s,f_2\rho_{2n+1}) \le \kappa_1\eth(fs,f\rho_{2n+1}) + \kappa_2\frac{\eth(fs,f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{1/2(\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s))} + \kappa_3\max\left\{\begin{array}{c}\eth(fs,f_1s),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\ \dfrac{\eth(fs,f_1s)\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(fs,f\rho_{2n+1})},\\ \dfrac{\eth(fs,f_1s)\cdot\eth(fs,f_2\rho_{2n+1})}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)},\\ \dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)}\end{array}\right\}. \quad (51)$$

This implies that

$$|\eth(f_1s,f\rho_{2n+2})| \le \kappa_1|\eth(fs,f\rho_{2n+1})| + \kappa_2\frac{|\eth(fs,f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{1/2(|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|)} + \kappa_3\max\left\{\begin{array}{c}|\eth(fs,f_1s)|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(fs,f\rho_{2n+1})|},\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(fs,f_2\rho_{2n+1})|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|}\end{array}\right\}. \quad (52)$$

Now, again applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (43) and (50), we have that

$$|\eth(f_1s,s)| \le \kappa_1|\eth(s,s)| + \kappa_2\frac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{1/2(|\eth(s,s)|+|\eth(s,f_1s)|)} + \kappa_3\max\left\{\begin{array}{c}|\eth(s,f_1s)|,|\eth(s,s)|,\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|\eth(s,s)|+|\eth(s,f_1s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{|\eth(s,s)|+|\eth(s,f_1s)|}\end{array}\right\} = \kappa_3|\eth(s,f_1s)|. \quad (53)$$

This implies that $(1-\kappa_3)\,|\,\eth(f_1s,s)\,|\le 0$. Since $(1-\kappa_3)\neq 0 \Rightarrow |\eth(f_1s,s)| = 0$. Hence,

$$f_1s = s. \quad (54)$$

Now, we have to show that $f_2s=s$, by using (32),

$$\eth(f\rho_{2n+1},f_2s) = \eth(f_1\rho_{2n},f_2s) \le \kappa_1\eth(f\rho_{2n},fs) + \kappa_2\frac{\eth(f\rho_{2n},f_2s)\cdot\eth(fs,f_1\rho_{2n})}{1/2(\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n}))} + \kappa_3\max\left\{\begin{array}{c}\eth(f\rho_{2n},f_1\rho_{2n}),\eth(fs,f_2s),\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(fs,f_2s)}{1+\eth(f\rho_{2n},fs)},\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(f\rho_{2n},f_2s)}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})},\\ \dfrac{\eth(fs,f_2s)\cdot\eth(fs,f_1\rho_{2n})}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})}\end{array}\right\}. \quad (55)$$

This implies that

$$|\eth(f\rho_{2n+1}, f_2 s)| \le \kappa_1 |\eth(f\rho_{2n}, fs)| + \kappa_2 \frac{|\eth(f\rho_{2n}, f_2 s)| \cdot |\eth(fs, f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|)} + \kappa_3 \max \left\{ \begin{array}{c} |\eth(f\rho_{2n}, f_1\rho_{2n})|, |\eth(fs, f_2 s)|, \\ \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})| \cdot |\eth(fs, f_2 s)|}{|1 + \eth(f\rho_{2n}, fs)|}, \\ \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})| \cdot |\eth(f\rho_{2n}, f_2 s)|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|}, \\ \dfrac{|\eth(fs, f_2 s)| \cdot |\eth(fs, f_1\rho_{2n})|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|} \end{array} \right\}. \tag{56}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (43) and (50), we get

$$|\eth(s, f_2 s)| \le \kappa_1 |\eth(s, s)| + \kappa_2 \frac{|\eth(s, f_2 s)| \cdot |\eth(s, s)|}{1/2(|\eth(s, f_2 s)| + |\eth(s, s)|)} + \kappa_3 \max \left\{ \begin{array}{c} |\eth(s, s)|, |\eth(s, f_2 s)|, \\ \dfrac{|\eth(s, s)| \cdot |\eth(s, f_2 s)|}{|1 + \eth(s, s)|}, \\ \dfrac{|\eth(s, s)| \cdot |\eth(s, f_2 s)|}{|\eth(s, f_2 s)| + |\eth(s, s)|}, \\ \dfrac{|\eth(s, f_2 s)| \cdot |\eth(s, s)|}{|\eth(s, f_2 s)| + |\eth(s, s)|} \end{array} \right\} = \kappa_3 |\eth(s, f_2 s)|. \tag{57}$$

This implies that $(1 - \kappa_3) \mid \eth(s, f_2 s) \mid \le 0$. Since $(1 - \kappa_3) \ne 0 \Rightarrow |\eth(s, f_2 s) \mid = 0$. Hence,

$$f_2 s = s. \tag{58}$$

Thus, from (50), (54), and (58), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs = f_1 s = f_2 s = s. \tag{59}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{60}$$

Then, from (32), we have that

$$\eth(s, s^*) = \eth(f_1 s, f_2 s^*) \le \kappa_1 \eth(fs, fs^*) + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} + \kappa_3 \max \left\{ \begin{array}{c} \eth(fs, f_1 s), \eth(fs^*, f_2 s^*), \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)}, \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)}, \\ \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right\} = (\kappa_1 + \kappa_2) \eth(s, s^*). \tag{61}$$

This implies that $|\eth(s, s^*) \mid \le (\kappa_1 + \kappa_2) |\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2) |\eth(s, s^*)| \le 0$. Since $(1 - \kappa_1 - \kappa_2) \ne 0 \Rightarrow |\eth(s, s^*) \mid = 0 \Rightarrow s = s^*$, hence proving that f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_2 = 0$ in Theorem 12, we can get the following corollary.

Corollary 13. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2) + \kappa_3 \max \left\{ \begin{array}{c} \eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right\}, \tag{62}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $\kappa_1/(1 - \kappa_3) < 1$, where $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Corollary 14. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2) + \kappa_3 \max \left\{ \begin{array}{c} \eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_2, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right\}, \tag{63}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $\kappa_1/(1 - \kappa_3) < 1$, where $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Example 15. Let $(\Omega, \eth)$ be a complex-valued b-metric space, where $\Omega = [0, 1]$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ with $\eth(\rho_1, \rho_2) = 4 |\rho_1 - \rho_2|^2/9 + i(4|\rho_1 - \rho_2|^2/9)$, for all $\rho_1, \rho_2 \in \Omega$. Now, we find b,

$$\begin{aligned}
\eth(\rho_1,\rho_2) &= \frac{4|\rho_1-\rho_2|^2}{9} + i\frac{4|\rho_1-\rho_2|^2}{9} \le \frac{4|(\rho_1-\rho_3)+(\rho_3-\rho_2)|^2}{9} \\
&\quad + i\frac{4|(\rho_1-\rho_3)+(\rho_3-\rho_2)|^2}{9} \\
&\le \left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4}{9}(2|\rho_1-\rho_3||\rho_3-\rho_2|)\right) \\
&\quad + i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4}{9}(2|\rho_1-\rho_3||\rho_3-\rho_2|)\right) \\
&\le \left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&\quad + i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) + 2i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2\left(\frac{4|\rho_1-\rho_3|^2}{9} + i\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + i\frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2[\eth(\rho_1,\rho_3) + \eth(\rho_3,\rho_2)].
\end{aligned} \tag{64}$$

That is $\eth(\rho_1,\rho_2) \le b[\eth(\rho_1,\rho_3) + \eth(\rho_3,\rho_2)]$, where $b=2$. Now, define $f_1, f_2, f : \Omega \longrightarrow \Omega$ as

$$\begin{aligned}
f_1\rho_1 &= f_2\rho_1 = \frac{3\rho_1}{20}, \\
f\rho_1 &= \frac{\rho_1}{4} \text{ for } \rho_1 \in \Omega.
\end{aligned} \tag{65}$$

Notice that

$$\left\{\begin{array}{c}
|\eth(f\rho_1,f\rho_2)|, \dfrac{|\eth(f\rho_1,f_2\rho_2)|\cdot|\eth(f\rho_2,f_1\rho_1)|}{1/2(|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|)}, \\
\max\left\{\begin{array}{c}
|\eth(f\rho_1,f_1\rho_1)|, |\eth(f\rho_2,f_2\rho_2)|, \\
\dfrac{|\eth(f\rho_1,f_1\rho_1)|\cdot|\eth(f\rho_2,f_2\rho_2)|}{|1+\eth(f\rho_1,f\rho_2)|}, \\
\dfrac{|\eth(f\rho_1,f_1\rho_1)|\cdot|\eth(f\rho_1,f_2\rho_2)|}{|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|}, \\
\dfrac{|\eth(f\rho_2,f_2\rho_2)|\cdot|\eth(f\rho_2,f_1\rho_1)|}{|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|}
\end{array}\right\}
\end{array}\right\} \ge 0. \tag{66}$$

In all regards, it is enough to show that $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$, for all $\rho_1,\rho_2 \in [0,1]$ and $\kappa_1,\kappa_2,\kappa_3 \in [0,1)$, such that $(\kappa_1+\kappa_2)<1$ and $(\kappa_1+\kappa_3)<1$, where $b \ge 1$, we have

$$\begin{aligned}
\eth(f_1\rho_1,f_2\rho_2) &= \left(\frac{4|f_1\rho_1-f_2\rho_2|^2}{9} + i\frac{4|f_1\rho_1-f_2\rho_2|^2}{9}\right) \\
&= \left(\frac{4|3\rho_1/20-3\rho_2/20|^2}{9} + i\frac{4|3\rho_1/20-3\rho_2/20|^2}{9}\right) \\
&= \left(\frac{3}{5}\right)^2\left(\frac{4|\rho_1/4-\rho_2/4|^2}{9} + i\frac{4|\rho_1/4-\rho_2/4|^2}{9}\right) \\
&= \frac{9}{25}\left(\frac{4|\rho_1/4-\rho_2/4|^2}{9} + i\frac{4|\rho_1/4-\rho_2/4|^2}{9}\right),
\end{aligned} \tag{67}$$

$$\begin{aligned}
\eth(f\rho_1,f\rho_2) &= \left(\frac{4|f\rho_1-f\rho_2|^2}{9} + i\frac{4|f\rho_1-f\rho_2|^2}{9}\right) \\
&= \left(\frac{4|\rho_1/4-\rho_2/4|^2}{9} + i\frac{4|\rho_1/4-\rho_2/4|^2}{9}\right).
\end{aligned} \tag{68}$$

For $\rho_1,\rho_2 \in [0,1]$, we discuss different cases with $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b=2$. Hence,

$$\begin{aligned}
\kappa_1+\kappa_2 &= \frac{2}{5}+\frac{1}{5} = \frac{3}{5} < 1, \\
\kappa_1+\kappa_3 &= \frac{2}{5}+\frac{1}{10} = \frac{1}{2} < 1.
\end{aligned} \tag{69}$$

Case 1. Let $\rho_1 = 0, \rho_2 = 0$, then from (67) and (68), directly, we get that $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$. Hence, (32) is satisfied with $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b=2$.

Case 2. Let $\rho_1 = 1, \rho_2 = 0$, then from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$, satisfied with $\kappa_1 = 2/5$, i.e.,

$$\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/4-0/4|^2}{9} + i\frac{4|1/4-0/4|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/4-0/4|^2}{9} + i\frac{4|1/4-0/4|^2}{9}\right)0.0099(1+i) \\
&\quad \le 0.0110(1+i).
\end{aligned} \tag{70}$$

Thus, (32) is true for $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b=2$.

Case 3. Let $\rho_1 = 1/2, \rho_2 = 1/4$; then, from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$ is satisfied with $\kappa_1 = 2/5$, i.e.,

$$\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/8-1/16|^2}{9} + i\frac{4|1/8-1/16|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/8-1/16|^2}{9} + i\frac{4|1/8-1/16|^2}{9}\right)0.00061(1+i) \\
&\quad \le 0.00068(1+i).
\end{aligned} \tag{71}$$

Thus, (32) is true for $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b=2$.

Case 4. Let $\rho_1 = 1/2, \rho_2 = 1$; then, from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$ is satisfied with $\kappa_1 = 2/5$, i.e.,

$$\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/8-1/4|^2}{9} + i\frac{4|1/8-1/4|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/8-1/4|^2}{9} + i\frac{4|1/8-1/4|^2}{9}\right)0.0024(1+i) \\
&\quad \le 0.0027(1+i).
\end{aligned} \tag{72}$$

Hence, (32) is satisfied with $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b = 2$. The pairs of self-mappings (f, f_1) and (f, f_2) are commutable; that is,

$$\begin{aligned} f_1(f(\rho_1)) &= f(f_1(\rho_1)) = \frac{3\rho_1}{80}, \\ f_2(f(\rho_1)) &= f(f_2(\rho_1)) = \frac{3\rho_1}{80}, \forall \rho_1 \in \Omega. \end{aligned} \tag{73}$$

Thus, all the conditions of Theorem 12 are satisfied with noticing that the point $0 \in \Omega$, which remains fixed under mappings f, f_1, and f_2, is indeed unique.

Theorem 16. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1\eth(f\rho_1, f\rho_2) + \kappa_2 \frac{\eth(f\rho_1, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1))} + \kappa_3 \left(\begin{array}{l} \eth(f\rho_1, f_1\rho_1) + \eth(f\rho_2, f_2\rho_2) \\ + \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)} \\ + \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \\ + \dfrac{\eth(f\rho_2, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right), \tag{74}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$, such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + 4\kappa_3) < 1$ and $b \ge 1$. If f is a continuous self-map and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\begin{aligned} \Gamma_{2n} &= f\rho_{2n+1} = f_1\rho_{2n}, \\ \Gamma_{2n+1} &= f\rho_{2n+2} = f_2\rho_{2n+1}, \\ &\forall n \ge 0. \end{aligned} \tag{75}$$

□

Now, by the view of (74) and (75),

$$\begin{aligned} \eth(\Gamma_{2n}, \Gamma_{2n+1}) &= \eth(f_1\rho_{2n}, f_2\rho_{2n+1}) \le \kappa_1\eth(f\rho_{2n}, f\rho_{2n+1}) \\ &+ \kappa_2 \frac{\eth(f\rho_{2n}, f_2\rho_{2n+1}) \cdot \eth(f\rho_{2n+1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \left(\begin{array}{l} \eth(f\rho_{2n}, f_1\rho_{2n}) + \eth(f\rho_{2n+1}, f_2\rho_{2n+1}) \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n}) \cdot \eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1 + \eth(f\rho_{2n}, f\rho_{2n+1})} \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n}) \cdot \eth(f\rho_{2n}, f_2\rho_{2n+1})}{\eth(f\rho_{2n}, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1\rho_{2n})} \\ + \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1}) \cdot \eth(f\rho_{2n+1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1\rho_{2n})} \end{array} \right) \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n}) + \kappa_2 \frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n}))} \\ &+ \kappa_3 \left(\begin{array}{l} \eth(\Gamma_{2n-1}, \Gamma_{2n}) + \eth(\Gamma_{2n}, \Gamma_{2n+1}) \\ + \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n+1})}{1 + \eth(\Gamma_{2n-1}, \Gamma_{2n})} \\ + \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})} \\ + \dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})} \end{array} \right). \end{aligned} \tag{76}$$

This implies that

$$\begin{aligned} |\eth(\Gamma_{2n}, \Gamma_{2n+1})| \le \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})| &+ \kappa_2 \frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})| \cdot |\eth(\Gamma_{2n}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})| + |\eth(\Gamma_{2n}, \Gamma_{2n})|)} \\ &+ \kappa_3 \Bigg(|\eth(\Gamma_{2n-1}, \Gamma_{2n})| + |\eth(\Gamma_{2n}, \Gamma_{2n+1})| \\ &+ \frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})| \cdot |\eth(\Gamma_{2n}, \Gamma_{2n+1})|}{|1 + \eth(\Gamma_{2n-1}, \Gamma_{2n})|} \\ &+ \frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})| \cdot |\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})| + |\eth(\Gamma_{2n}, \Gamma_{2n})|} \\ &+ \frac{|\eth(\Gamma_{2n}, \Gamma_{2n+1})| \cdot |\eth(\Gamma_{2n}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})| + |\eth(\Gamma_{2n}, \Gamma_{2n})|} \Bigg). \end{aligned} \tag{77}$$

After simplification, we get that

$$|\eth(\Gamma_{2n}, \Gamma_{2n+1})| \le g|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, \text{where } g = \frac{\kappa_1 + 2\kappa_3}{1 - 2\kappa_3} < 1. \tag{78}$$

Again, by the view of (74) and (75),

$$\begin{aligned} \eth(\Gamma_{2n-1}, \Gamma_{2n}) &= \eth(f_2\rho_{2n-1}, f_1\rho_{2n}) = \eth(f_1\rho_{2n}, f_2\rho_{2n-1}) \le \kappa_1\eth(f\rho_{2n}, f\rho_{2n-1}) \\ &+ \kappa_2 \frac{\eth(f\rho_{2n}, f_2\rho_{2n-1}) \cdot \eth(f\rho_{2n-1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n-1}) + \eth(f\rho_{2n-1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \left(\begin{array}{l} \eth(f\rho_{2n}, f_1\rho_{2n}) + \eth(f\rho_{2n-1}, f_2\rho_{2n-1}) \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n}) \cdot \eth(f\rho_{2n-1}, f_2\rho_{2n-1})}{1 + \eth(f\rho_{2n}, f\rho_{2n-1})} \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n}) \cdot \eth(f\rho_{2n}, f_2\rho_{2n-1})}{\eth(f\rho_{2n}, f_2\rho_{2n-1}) + \eth(f\rho_{2n-1}, f_1\rho_{2n})} \\ + \dfrac{\eth(f\rho_{2n-1}, f_2\rho_{2n-1}) \cdot \eth(f\rho_{2n-1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n-1}) + \eth(f\rho_{2n-1}, f_1\rho_{2n})} \end{array} \right) \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n-2}) + \kappa_2 \frac{\eth(\Gamma_{2n-1}, \Gamma_{2n-1}) \cdot \eth(\Gamma_{2n-2}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n-1}) + \eth(\Gamma_{2n-2}, \Gamma_{2n}))} \\ &+ \kappa_3 \left(\begin{array}{l} \eth(\Gamma_{2n-1}, \Gamma_{2n}) + \eth(\Gamma_{2n-2}, \Gamma_{2n-1}) \\ + \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n-2}, \Gamma_{2n-1})}{1 + \eth(\Gamma_{2n-1}, \Gamma_{2n-2})} \\ + \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n-1}, \Gamma_{2n-1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1}) + \eth(\Gamma_{2n-2}, \Gamma_{2n})} \\ + \dfrac{\eth(\Gamma_{2n-2}, \Gamma_{2n-1}) \cdot \eth(\Gamma_{2n-2}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1}) + \eth(\Gamma_{2n-2}, \Gamma_{2n})} \end{array} \right). \end{aligned} \tag{79}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n-1},\Gamma_{2n})| &\leq \kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n-2})| \\ &+ \kappa_2\frac{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|)} \\ &+ \kappa_3\left(\begin{array}{c}|\eth(\Gamma_{2n-1},\Gamma_{2n})|+|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \\ +\dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|}{|1+\eth(\Gamma_{2n-1},\Gamma_{2n-2})|} \\ +\dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|} \\ +\dfrac{|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|}\end{array}\right).\end{aligned} \tag{80}$$

After simplification, we get that

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})| \leq g|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|, \text{ since } g=\frac{\kappa_1+2\kappa_3}{1-2\kappa_3}<1. \tag{81}$$

Now, from (81) and (78) and by induction, we have

$$\begin{aligned}|\eth(\Gamma_{2n},\Gamma_{2n+1})| &\leq g|\eth(\Gamma_{2n-1},\Gamma_{2n})| \leq g^2|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \\ &\leq \cdots \leq g^{2n}|\eth(\Gamma_0,\Gamma_1)|.\end{aligned} \tag{82}$$

So, for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}|\eth(\Gamma_n,\Gamma_m)| &\leq b|\eth(\Gamma_n,\Gamma_{n+1})|+b|\eth(\Gamma_{n+1},\Gamma_m)| \leq b|\eth(\Gamma_n,\Gamma_{n+1})| \\ &+ b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})|+\cdots+b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)| \\ &\leq bg^n|\eth(\Gamma_0,\Gamma_1)|+b^2g^{n+1}|\eth(\Gamma_0,\Gamma_1)| \\ &+\cdots+b^{m-n}g^{m-1}|\eth(\Gamma_0,\Gamma_1)| \\ &\leq \left[bg^n+b^2g^{n+1}+\cdots+b^{m-n}g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n\left[1+bg+b^2g^2\cdots+b^{m-(n+1)}g^{m-(n+1)}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n\sum_{t=0}^{m-(n+1)}b^tg^t|\eth(\Gamma_0,\Gamma_1)| \leq bg^n\sum_{t=0}^{\infty}b^tg^t|\eth(\Gamma_0,\Gamma_1)| \\ &= \frac{bg^n}{1-bg}|\eth(\Gamma_0,\Gamma_1)| \longrightarrow 0, \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{83}$$

Therefore, sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s \in \Omega$ such that $\Gamma_n \longrightarrow s$, as $n \longrightarrow \infty$, and from (75), we have that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} f\rho_{2n+1}=s, \\ &\lim_{n\longrightarrow\infty} f_1\rho_{2n}=s, \\ &\lim_{n\longrightarrow\infty} f_2\rho_{2n+1}=s.\end{aligned} \tag{84}$$

As f is continuous, so

$$\begin{aligned}&\lim_{n\longrightarrow\infty} f(f\rho_{2n+1})=fs, \\ &\lim_{n\longrightarrow\infty} f(f_1\rho_{2n})=fs, \\ &\lim_{n\longrightarrow\infty} f(f_2\rho_{2n+1})=fs.\end{aligned} \tag{85}$$

Since, (f,f_1) and (f,f_2) are commutable pairs, therefore, from (85), we have that

$$\begin{aligned}&\lim_{n\longrightarrow\infty} f_1(f\rho_{2n})=fs, \\ &\lim_{n\longrightarrow\infty} f_2(f\rho_{2n+1})=fs.\end{aligned} \tag{86}$$

Now, we prove $fs = s$. So, for this, we put $\rho_1 = f\rho_{2n}$ and $\rho_2 = \rho_{2n+1}$ in (74),

$$\begin{aligned}\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1}) &\leq \kappa_1\eth(f(f\rho_{2n}),f\rho_{2n+1}) \\ &+ \kappa_2\frac{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n})))} \\ &+ \kappa_3\left(\begin{array}{c}\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))+\eth(f\rho_{2n+1},f_2\rho_{2n+1}) \\ +\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}),f\rho_{2n+1})} \\ +\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f(f\rho_{2n}),f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))} \\ +\dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}\end{array}\right).\end{aligned} \tag{87}$$

This implies that

$$\begin{aligned}|\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})| &\leq \kappa_1|\eth(f(f\rho_{2n}),f\rho_{2n+1})| \\ &+ \kappa_2\frac{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|)} \\ &+ \kappa_3\left(\begin{array}{c}|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|+|\eth(f\rho_{2n+1},f_2\rho_{2n+1})| \\ +\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}),f\rho_{2n+1})|} \\ +\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|} \\ +\dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}\end{array}\right).\end{aligned} \tag{88}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84), (85), and (86), we get that

$$\begin{aligned}|\eth(fs,s)| &\leq \kappa_1|\eth(fs,s)|+\kappa_2\frac{|\eth(fs,s)|\cdot|\eth(s,fs)|}{1/2(|\eth(fs,s)|+|\eth(s,fs)|)} \\ &+ \kappa_3\left(\begin{array}{c}|\eth(fs,fs)|+|\eth(s,s)| \\ +\dfrac{|\eth(fs,fs)|\cdot|\eth(s,s)|}{|1+\eth(fs,s)|} \\ +\dfrac{|\eth(fs,fs)|\cdot|\eth(fs,s)|}{|\eth(fs,s)|+|\eth(s,fs)|} \\ +\dfrac{|\eth(s,s)|\cdot|\eth(s,fs)|}{|\eth(fs,s)|+|\eth(s,fs)|}\end{array}\right) = (\kappa_1+\kappa_2)|\eth(fs,s)|.\end{aligned} \tag{89}$$

This implies that $(1-\kappa_1-\kappa_2)\,|\,\eth(fs,s)\,|\leq 0$. Since, $(1-\kappa_1-\kappa_2)\neq 0 \Rightarrow |\eth(fs,s)\,|=0$. Hence,

$$fs=s. \tag{90}$$

Next, we have to show that $f_1s=s$, by using (74),

$$\eth(f_1s, f\rho_{2n+2}) = \eth(f_1s, f_2\rho_{2n+1}) \leq \kappa_1\eth(fs, f\rho_{2n+1}) + \kappa_2\frac{\eth(fs, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1s)}{1/2(\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1s))} + \kappa_3\left(\begin{array}{c}\eth(fs, f_1s) + \eth(f\rho_{2n+1}, f_2\rho_{2n+1}) \\ +\dfrac{\eth(fs, f_1s)\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(fs, f\rho_{2n+1})} \\ +\dfrac{\eth(fs, f_1s)\cdot\eth(fs, f_2\rho_{2n+1})}{\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1s)} \\ +\dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1s)}{\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1s)}\end{array}\right). \tag{91}$$

This implies that

$$|\eth(f_1s, f\rho_{2n+2})| \leq \kappa_1|\eth(fs, f\rho_{2n+1})| + \kappa_2\frac{|\eth(fs, f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1s)|}{1/2(|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1s)|)} + \kappa_3\left(\begin{array}{c}|\eth(fs, f_1s)| + |\eth(f\rho_{2n+1}, f_2\rho_{2n+1})| \\ +\dfrac{|\eth(fs, f_1s)|\cdot|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|}{|1+\eth(fs, f\rho_{2n+1})|} \\ +\dfrac{|\eth(fs, f_1s)|\cdot|\eth(fs, f_2\rho_{2n+1})|}{|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1s)|} \\ +\dfrac{|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1s)|}{|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1s)|}\end{array}\right). \tag{92}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84) and (90), we get

$$|\eth(f_1s, s)| \leq \kappa_1|\eth(s,s)| + \kappa_2\frac{|\eth(s,s)|\cdot|\eth(s, f_1s)|}{1/2(|\eth(s,s)| + |\eth(s, f_1s)|)} + \kappa_3\left(\begin{array}{c}|\eth(s, f_1s)| + |\eth(s,s)| \\ +\dfrac{|\eth(s, f_1s)\,|\cdot|\,\eth(s,s)\,|}{|1+\eth(s,s)\,|} \\ +\dfrac{|\eth(s, f_1s)\,|\cdot|\,\eth(s,s)\,|}{|\eth(s,s)|+|\eth(s, f_1s)\,|} \\ +\dfrac{|\eth(s,s)|\cdot|\eth(s, f_1s)|}{|\eth(s,s)| + |\eth(s, f_1s)|}\end{array}\right). \tag{93}$$

Thus, we get that $|\eth(f_1s,s)\,|\leq\kappa_3|\eth(s,f_1s)| \Rightarrow (1-\kappa_3)|\eth(f_1s,s)|\leq 0$. Since $(1-\kappa_3)\neq 0$, therefore, $|\eth(f_1s,s)\,|=0$. Hence,

$$f_1s=s. \tag{94}$$

Now, we have to show that $f_2s=s$, by using (74),

$$\eth(f\rho_{2n+1}, f_2s) = \eth(f_1\rho_{2n}, f_2s) \leq \kappa_1\eth(f\rho_{2n}, fs) + \kappa_2\frac{\eth(f\rho_{2n}, f_2s)\cdot\eth(fs, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2s) + \eth(fs, f_1\rho_{2n}))} + \kappa_3\left(\begin{array}{c}\eth(f\rho_{2n}, f_1\rho_{2n}) + \eth(fs, f_2s) \\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(fs, f_2s)}{1+\eth(f\rho_{2n}, fs)} \\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2s)}{\eth(f\rho_{2n}, f_2s) + \eth(fs, f_1\rho_{2n})} \\ +\dfrac{\eth(fs, f_2s)\cdot\eth(fs, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2s) + \eth(fs, f_1\rho_{2n})}\end{array}\right). \tag{95}$$

This implies that

$$|\eth(f\rho_{2n+1}, f_2s)| \leq \kappa_1|\eth(f\rho_{2n}, fs)| + \kappa_2\frac{|\eth(f\rho_{2n}, f_2s)|\cdot|\eth(fs, f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n}, f_2s)| + |\eth(fs, f_1\rho_{2n})|)} + \kappa_3\left(\begin{array}{c}|\eth(f\rho_{2n}, f_1\rho_{2n})| + |\eth(fs, f_2s)| \\ +\dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})|\cdot|\eth(fs, f_2s)|}{|1+\eth(f\rho_{2n}, fs)|} \\ +\dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})|\cdot|\eth(f\rho_{2n}, f_2s)|}{|\eth(f\rho_{2n}, f_2s)| + |\eth(fs, f_1\rho_{2n})|} \\ +\dfrac{|\eth(fs, f_2s)|\cdot|\eth(fs, f_1\rho_{2n})|}{|\eth(f\rho_{2n}, f_2s)| + |\eth(fs, f_1\rho_{2n})|}\end{array}\right). \tag{96}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84) and (90), we get

$$|\eth(s, f_2s)| \leq \kappa_1|\eth(s,s)| + \kappa_2\frac{|\eth(s, f_2s)|\cdot|\eth(s,s)|}{1/2(|\eth(s, f_2s)| + |\eth(s,s)|)} + \kappa_3\left(\begin{array}{c}|\eth(s,s)| + |\eth(s, f_2s)| \\ +\dfrac{|\eth(s,s)\,|\cdot|\,\eth(s, f_2s)\,|}{|1+\eth(s,s)\,|} \\ +\dfrac{|\eth(s,s)\,|\cdot|\,\eth(s, f_2s)\,|}{|\eth(s, f_2s)|+|\eth(s,s)\,|} \\ +\dfrac{|\eth(s, f_2s)\,|\cdot|\,\eth(s,s)\,|}{|\eth(s, f_2s)|+|\eth(s,s)\,|}\end{array}\right). \tag{97}$$

So, we get that $|\eth(s,f_2s)\,|\leq\kappa_3|\eth(s,f_2s)| \Rightarrow (1-\kappa_3)|\eth(s,f_2s)|\leq 0$. Since $(1-\kappa_3)\neq 0$, therefore, $|\eth(s,f_2s)\,|=0$. Hence,

$$f_2s=s. \tag{98}$$

Thus, from (90), (94), and (98), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs = f_1 s = f_2 s = s. \tag{99}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{100}$$

Then, from (74), we have that

$$\begin{aligned} \eth(s, s^*) &= \eth(f_1 s, f_2 s^*) \le \kappa_1 \eth(fs, fs^*) \\ &\quad + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} \\ &\quad + \kappa_3 \left(\begin{array}{c} \eth(fs, f_1 s) + \eth(fs^*, f_2 s^*) \\ + \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)} \\ + \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \\ + \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right) \\ &= (\kappa_1 + \kappa_2) \eth(s, s^*). \end{aligned} \tag{101}$$

This implies that $|\eth(s, s^*)| \le (\kappa_1 + \kappa_2)|\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2)|\eth(s, s^*)| \le 0$. Since $(1 - \kappa_1 - \kappa_2) \ne 0$, therefore, $|\eth(s, s^*)| = 0 \Rightarrow s = s^*$, hence proving that f, f_1, and f_2 have a unique CFP in Ω.

Corollary 17. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\begin{aligned} \eth(f_1\rho_1, f_2\rho_2) &\le \kappa_1 \eth(f\rho_1, f\rho_2) \\ &\quad + \kappa_2 \frac{\eth(f\rho_1, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1))} \\ &\quad + \kappa_3 \left(\begin{array}{c} \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)} \\ + \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \\ + \dfrac{\eth(f\rho_2, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right), \end{aligned} \tag{102}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$, such that $(\kappa_1 + \kappa_2) < 1$ and $(\kappa_1 + \kappa_3)/(1 - \kappa_3) < 1$, with $b \ge 1$. If f is a continuous self-mapping and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Example 18. Let $\Omega = [0, \infty)$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ be defined as $\eth(\rho_1, \rho_2) = 3|\rho_1 - \rho_2|^2/13 + i3|\rho_1 - \rho_2|^2/13$ for all $\rho_1, \rho_2 \in \Omega$. Then, $(\Omega, \eth)$ is a complex-valued b-metric space. Now, we find b:

$$\begin{aligned} \eth(\rho_1, \rho_2) &= \frac{3|\rho_1 - \rho_2|^2}{13} + i\frac{3|\rho_1 - \rho_2|^2}{13} \le \frac{3|(\rho_1 - \rho_3) + (\rho_3 - \rho_2)|^2}{13} \\ &\quad + i\frac{3|(\rho_1 - \rho_3) + (\rho_3 - \rho_2)|^2}{13} \\ &\le \left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3}{13}(2|\rho_1 - \rho_3||\rho_3 - \rho_2|) \right) \\ &\quad + i\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3}{13}(2|\rho_1 - \rho_3||\rho_3 - \rho_2|) \right) \\ &\le \left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &\quad + i\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) + i2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + i\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + i\frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]. \end{aligned} \tag{103}$$

That is $\eth(\rho_1, \rho_2) \le b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$, where $b = 2$. Now, we define $f_1, f_2, f : \Omega \longrightarrow \Omega$ by

$$\begin{aligned} f_1\rho_1 &= f_2\rho_1 = \ln\left(1 + \frac{\rho_1}{4 + \rho_1}\right), \\ f\rho_1 &= e^{4\rho_1} - 1, \\ &\text{for all } \rho_1 \in \Omega. \end{aligned} \tag{104}$$

Notice that

$$\left\{ |\eth(f\rho_1, f\rho_2)|, \frac{|\eth(f\rho_1, f_2\rho_2)| \cdot |\eth(f\rho_2, f_1\rho_1)|}{1/2(|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|)}, \left(\begin{array}{c} |\eth(f\rho_1, f_1\rho_1)| + |\eth(f\rho_2, f_2\rho_2)| \\ + \dfrac{|\eth(f\rho_1, f_1\rho_1)| \cdot |\eth(f\rho_2, f_2\rho_2)|}{|1 + \eth(f\rho_1, f\rho_2)|} \\ + \dfrac{|\eth(f\rho_1, f_1\rho_1)| \cdot |\eth(f\rho_1, f_2\rho_2)|}{|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|} \\ + \dfrac{|\eth(f\rho_2, f_2\rho_2)| \cdot |\eth(f\rho_2, f_1\rho_1)|}{|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|} \end{array} \right) \right\} \ge 0. \tag{105}$$

In all regards, it is enough to show that $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$, for all $\rho_1, \rho_2 \in [0, \infty)$ and $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$, such that $(\kappa_1 + \kappa_2) < 1$ and $(\kappa_1 + 4\kappa_3) < 1$, where $b \ge 1$, we have

$$\begin{aligned}\eth(f_1\rho_1, f_2\rho_2) &= \left(\frac{3|f_1\rho_1 - f_2\rho_2|^2}{13} + i\frac{3|f_1\rho_1 - f_2\rho_2|^2}{13}\right)\\ &= \left(\frac{3|\ln(1+\rho_1/(4+\rho_1)) - \ln(1+\rho_2/(4+\rho_2))|^2}{13}\right.\\ &\quad\left. + i\frac{3|\ln(1+\rho_1/(4+\rho_1)) - \ln(1+\rho_2/(4+\rho_2))|^2}{13}\right)\\ &\le \left(\frac{3|\rho_1/(4+\rho_1) - \rho_2/(4+\rho_2)|^2}{13}\right.\\ &\quad\left. + i\frac{3|\rho_1/(4+\rho_1) - \rho_2/(4+\rho_2)|^2}{13}\right)\\ &\le \left(\frac{3|(4\rho_1 - 4\rho_2)/16|^2}{13} + i\frac{3|(4\rho_1 - 4\rho_2)/16|^2}{13}\right)\\ &= \frac{1}{16^2}\left(\frac{3|4\rho_1 - 4\rho_2|^2}{13} + i\frac{3|4\rho_1 - 4\rho_2|^2}{13}\right)\\ &\le \frac{1}{256}\left(\frac{3|e^{4\rho_1} - e^{4\rho_2}|^2}{13} + i\frac{3|e^{4\rho_1} - e^{4\rho_2}|^2}{13}\right),\end{aligned} \tag{106}$$

$$\begin{aligned}\eth(f\rho_1, f\rho_2) &= \left(\frac{3|f\rho_1 - f\rho_2|^2}{13} + i\frac{3|f\rho_1 - f\rho_2|^2}{13}\right)\\ &= \left(\frac{3|(e^{4\rho_1} - 1) - (e^{4\rho_2} - 1)|^2}{13} + i\frac{3|(e^{4\rho_1} - 1) - (e^{4\rho_2} - 1)|^2}{13}\right)\\ &= \left(\frac{3|e^{4\rho_1} - e^{4\rho_2}|^2}{13} + i\frac{3|e^{4\rho_1} - e^{4\rho_2}|^2}{13}\right).\end{aligned} \tag{107}$$

For $\rho_1, \rho_2 \in [0,\infty)$, we discuss different cases with $\kappa_1 = 1/5$, $\kappa_2 = 1/4$, and $\kappa_3 = 1/10$, where $b = 2$. Hence,

$$\begin{aligned}\kappa_1 + \kappa_2 &= \frac{1}{5} + \frac{1}{4} = \frac{9}{20} < 1,\\ \kappa_1 + 4\kappa_3 &= \frac{1}{5} + \frac{2}{5} = \frac{3}{5} < 1.\end{aligned} \tag{108}$$

Case 1. Let $\rho_1 = 0, \rho_2 = 0$. Then, from (106) and (107), directly, we get that $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$. Hence, (74) is satisfied with $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 2. Let $\rho_1 = 0$, $\rho_2 = 1$, then from (106) and (107), we find $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$ is satisfied with $\kappa_1 = 1/5$, as

$$\begin{aligned}&\frac{1}{256}\left(\frac{3|e^0 - e^4|^2}{13} + i\frac{3|e^0 - e^4|^2}{13}\right)\\ &\quad\le \kappa_1\left(\frac{3|e^0 - e^4|^2}{13} + i\frac{3|e^0 - e^4|^2}{13}\right).\end{aligned} \tag{109}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{256}\left(\frac{3|-53.5981|^2}{13} + i\frac{3|-53.5981|^2}{13}\right)\\ &\quad\le \frac{1}{5}\left(\frac{3|-53.5981|^2}{13} + i\frac{3|-53.5981|^2}{13}\right)2.5896(1+i)\\ &\quad\le 132.5887(1+i).\end{aligned} \tag{110}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 3. Let $\rho_1 = 1/2$, $\rho_2 = 1/4$, then from (106) and (107), we find $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$ is true for $\kappa_1 = 1/5$, as

$$\frac{1}{256}\left(\frac{3|e^2 - e^1|^2}{13} + i\frac{3|e^2 - e^1|^2}{13}\right) \le \kappa_1\left(\frac{3|e^2 - e^1|^2}{13} + i\frac{3|e^2 - e^1|^2}{13}\right). \tag{111}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{256}\left(\frac{3|4.6708|^2}{13} + i\frac{3|4.6708|^2}{13}\right) \le \frac{1}{5}\left(\frac{3|4.6708|^2}{13} + i\frac{3|4.6708|^2}{13}\right)\\ &\quad\cdot 0.0196(1+i) \le 1.0069(1+i).\end{aligned} \tag{112}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 4. Let $\rho_1 = 1/2, \rho_2 = 1$, then from (106) and (107), we get that $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$ is true for $\kappa_1 = 1/5$, as

$$\frac{1}{256}\left(\frac{3|e^2 - e^4|^2}{13} + i\frac{3|e^2 - e^4|^2}{13}\right) \le \kappa_1\left(\frac{3|e^2 - e^4|^2}{13} + i\frac{3|e^2 - e^4|^2}{13}\right). \tag{113}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}&\frac{1}{256}\left(\frac{3|-51.8799|^2}{13} + i\frac{3|-51.8799|^2}{13}\right)\\ &\quad\le \frac{1}{5}\left(\frac{3|-51.8799|^2}{13} + i\frac{3|-51.8799|^2}{13}\right)2.4262(1+i)\\ &\quad\le 124.2241(1+i).\end{aligned} \tag{114}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

So, all conditions of Theorem 16 are satisfied to get a unique CFP, that is "0" of the mappings f, f_1, and f_2.

4. Applications

Here, we provide an application to support our main result. To do this, we take a couple of UITEs to obtain the existing result of a common solution to check the effectiveness of our result. Let the set $\Omega = C([k_1, k_2], \mathbb{R})$ contain real-valued continuous functions defined on $[k_1, k_2]$. In the following, we use Theorem 9 to obtain the existing result of a common solution. This enables us to establish a theorem based on UITEs to attain the existing result of a common solution.

Theorem 19 (see [28]). *Let $\Omega = C([k_1, k_2], \mathbb{R})$, where $[k_1, k_2] \subseteq \mathbb{R}$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ is defined as*

$$\eth(\rho_1, \rho_2) = \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \quad (115)$$

for all $\rho_1, \rho_2 \in \Omega$ and $y \in [k_1, k_2]$. Consider that the UITEs are

$$\begin{aligned} \rho_1(y) &= \int_{k_1}^{k_2} Q_1(y, r, \rho_1(r))dr + \Gamma_1(y), \\ \rho_2(y) &= \int_{k_1}^{k_2} Q_2(y, r, \rho_2(r))dr + \Gamma_2(y), \end{aligned} \quad (116)$$

where $r \in [k_1, k_2]$. Let $Q_1, Q_2 : [k_1, k_2] \times [k_1, k_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that $D_{\rho_1}, E_{\rho_2} \in \Omega$ for every $\rho_1, \rho_2 \in \Omega$, we have that

$$\begin{aligned} D_{\rho_1}(y) &= \int_{k_1}^{k_2} Q_1(y, r, \rho_1(r))dr, \\ E_{\rho_2}(y) &= \int_{k_1}^{k_2} Q_2(y, r, \rho_2(r))dr. \end{aligned} \quad (117)$$

If there exists $\mu \in (0, 1)$ such that, for all $\rho_1, \rho_2 \in \Omega$,

$$\begin{aligned} &\left\| D_{\rho_1}(y) - E_{\rho_2}(y) + \Gamma_1(y) - \Gamma_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \\ &\leq \mu M(\rho_1, \rho_2), \end{aligned} \quad (118)$$

where

$$M(\rho_1, \rho_2) = \max \{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}, \quad (119)$$

with

$$A_1(\rho_1, \rho_2)(y) = \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1},$$

$$A_2(\rho_1, \rho_2)(y) = \frac{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \left(\sqrt{1 + k_1^2} e^{i \cot k_1} \right)}{1/2 \left(\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \right)}, \quad (120)$$

$$\begin{aligned} A_3(\rho_1, \rho_2)(y) = \min \{&a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), \\ &a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}, \end{aligned} \quad (121)$$

where

$$a_1(\rho_1, \rho_2)(y) = \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1},$$

$$a_2(\rho_1, \rho_2)(y) = \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1},$$

$$a_3(\rho_1, \rho_2)(y) = \frac{\left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \left(\sqrt{1 + k_1^2} e^{i \cot k_1} \right)^2}{1 + \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}},$$

$$a_4(\rho_1, \rho_2)(y) = \frac{\left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2},$$

$$a_5(\rho_1, \rho_2)(y) = \frac{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2}. \quad (122)$$

Then, the two UITEs, i.e., (41), have a unique common solution.

Proof. Define $f_1, f_2, f : \Omega \longrightarrow \Omega$ as

$$\begin{aligned} f_1\rho_1 &= f_1\rho_1(y) = D_{\rho_1}(y) + \Gamma_1(y) = D_{\rho_1} + \Gamma_1, \\ f\rho_1 &= f\rho_1(y) = \rho_1(y) = \rho_1, \\ f_2\rho_2 &= f_2\rho_2(y) = E_{\rho_2}(y) + \Gamma_2(y) = E_{\rho_2} + \Gamma_2, \\ f\rho_2 &= f\rho_2(y) = \rho_2(y) = \rho_2. \end{aligned} \quad (123)$$

□

Then, we have the following three cases:

(1) If $A_1(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (118), (119), and (123), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \leq \mu \|\rho_1 - \rho_2\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (124)$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, f_1, f_2, and f satisfy all conditions of Theorem 9 with $\mu = \kappa_1$ and $\kappa_2 = \kappa_3 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

(2) If $A_2(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (118), (119), and (123), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right|^2\right\| \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2 \left(\sqrt{1+k_1^2} e^{i \cot k_1}\right)}{1/2\left(\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2\right)}, \quad (125)$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, f_1, f_2, and f satisfy all conditions of Theorem 9 with $\mu = \kappa_2$ and $\kappa_1 = \kappa_3 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

(3) If $A_3(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (119), we have that

$$M(\rho_1, \rho_2) = A_3(\rho_1, \rho_2)(y). \quad (126)$$

Then, there are furthermore five subcases arising:

(i) If $a_1(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \left\|\left|D_{\rho_1} + \Gamma_1 - \rho_1\right|\right\|^2 \sqrt{1+k_1^2} e^{i \cot k_1}, \quad (127)$$

for all $\rho_1, \rho_2 \in \Omega$

(ii) If $a_2(\rho_1, \rho_2)(y)$is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \left\|\left|E_{\rho_2} + \Gamma_2 - \rho_2\right|\right\|^2 \sqrt{1+k_1^2} e^{i \cot k_1}, \quad (128)$$

for all $\rho_1, \rho_2 \in \Omega$

(iii) If $a_3(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|D_{\rho_1} + \Gamma_1 - \rho_1\right|^2\right\| \left\|E_{\rho_2} + \Gamma_2 - \rho_2\right\|^2 \left(\sqrt{1+k_1^2} e^{i \cot k_1}\right)^2}{1 + \|\rho_1 - \rho_2\|^2 \sqrt{1+k_1^2} e^{i \cot k_1}}, \quad (129)$$

for all $\rho_1, \rho_2 \in \Omega$

(iv) If $a_4(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|D_{\rho_1} + \Gamma_1 - \rho_1\right|^2\right\| \left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 \sqrt{1+k_1^2} e^{i \cot k_1}}{\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2}, \quad (130)$$

for all $\rho_1, \rho_2 \in \Omega$

(v) If $a_5(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|E_{\rho_2} + \Gamma_2 - \rho_2\right|^2\right\| \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2 \sqrt{1+k_1^2} e^{i \cot k_1}}{\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2}, \quad (131)$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, the subcases of Case 3 (Case (i-v)) for the mappings f_1, f_2, and f satisfy all the conditions of Theorem 9 with $\mu = \kappa_3$ and $\kappa_1 = \kappa_2 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

5. Conclusions

We have established some unique CFP-results in complex-valued b-metric space by using rational contraction conditions for three self-mappings in which one self-map is continuous and commutable with the other two self-mappings. In our main work, we have generalized the results (e.g., see [28, 37, 38]). To show the authenticity of our results, we have given some useful examples in the main section. We have also provided an application for our main result to indicate its utility. In this direction, many results can be contributed to the said space by applying different contractions with different types of integral operators.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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