

Research Article

A Note on Orthogonal Fuzzy Metric Space, Its Properties, and Fixed Point Theorems

Khalil Javed ¹, Awais Asif ², and Ekrem Savas ³

¹Department of Mathematics and Statistics, International Islamic University Islamabad, 44000, Pakistan

²Department of Mathematics, Kateb University, 1007, Afghanistan

³Department of Mathematics, Usak University, Turkey

Correspondence should be addressed to Awais Asif; awais.asif@kateb.edu.af

Received 8 January 2022; Revised 3 March 2022; Accepted 11 April 2022; Published 11 May 2022

Academic Editor: Anita Tomar

Copyright © 2022 Khalil Javed et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The article generalizes the notion of orthogonal fuzzy metric space into a broader term, named as orthogonal picture fuzzy metric space. The obtained results improve and extend the idea of the orthogonal fuzzy metric space and its related results. However, this article outstretches the above-mentioned notion further into a newly defined concept, named as orthogonal picture fuzzy metric space. A detailed insight is given into the topic by presenting some fixed point results in the frame of the newly defined structure. To elaborate the results more precisely, some concrete examples are given.

1. Introduction

In 2013, Cuong [2] proposed a new concept named picture fuzzy sets (PFS), which is an extension of fuzzy sets and intuitionistic fuzzy sets. In a picture fuzzy set, each element is specified by the degree of membership, the degree of non-membership, and degree of neutrality together with the condition that the sum of these grades should be less or equal to 1.

In this regard, Phong et al. [7] studied some compositions of picture fuzzy relations. Cuong and Hai [8] investigated main fuzzy logic operators: negations, conjunctions, disjunctions, and implications on picture fuzzy sets, and constructed the main operations for fuzzy inference processes in picture fuzzy systems. Singh [9] studied the correlation coefficients of picture fuzzy sets. Cuong et al. [10] then investigated the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets.

Eshaghi et al. [4] presented a new generalization of the Banach fixed point theorem (BFPT) by defining the notion of orthogonal sets. The orthogonal set is a non-empty set equipped with a binary relation (called orthogonal relation) having a special structure (see [4]). The metric defined on the orthogonal set is called orthogonal metric space. The orthogonal metric space contains partially ordered metric space and graphical metric

space. Hezarjaribi [5] further extended the results of [4] to orthogonal fuzzy metric space. Also, Ishtiaq et al. [6] extended the results of [4] to orthogonal neutrosophic metric space. Some more details about generalized orthogonal metric spaces have been provided by Javed et al. [11], Uddin et al. [12, 13], and Senapati et al. [14].

In this paper, we introduce orthogonal picture fuzzy metric space which generalize picture fuzzy metric space and orthogonal fuzzy metric spaces. We show that every picture fuzzy metric space is an orthogonal picture fuzzy metric space but not conversely. We investigate different conditions on the picture fuzzy to show the existence of fixed points in various types of contractions. We also present some examples in support of the obtained results. The authors intend to further widen the interesting idea of orthogonality to the intuitionistic fuzzy metric space and spherical fuzzy metric spaces. Some interesting results on the same two topics can be read in the articles [15, 16] and [17], respectively.

2. Preliminaries

Definition 1 (see [1]). A fuzzy set is a pair (W, f) , where W is a non-empty set, $f : W \rightarrow [0, 1]$ is a membership

function and for each $\mathfrak{F} \in W$, $f(\mathfrak{F})$ is called the grade of membership of \mathfrak{F} in (W, f) .

Definition 2 (see [2]). A picture fuzzy set A on the universe set W is an object of the form where $Y(\partial) \in [0, 1]$ is called the degree of positive membership of ∂ in A , $\mathcal{M}(\partial) \in [0, 1]$ is called the “degree of neutral membership of ∂ in A ,” and $D(\partial) \in [0, 1]$ is called the degree of negative membership of ∂ in A , and $Y(\partial), \mathcal{M}(\partial), D(\partial)$ satisfy

$$Y(\partial) + \mathcal{M}(\partial) + D(\partial) \leq 1, \quad (1)$$

for all $\partial \in A$. Then,

$$\text{for all } \partial \in A, 1 - (Y(\partial) + \mathcal{M}(\partial) + D(\partial)), \quad (2)$$

is called the degree of refusal membership of ∂ in A .

Definition 3 (see [3]). Suppose $W \neq \emptyset$ is an arbitrary set, assume a five tuple $(W, Y, \mathcal{M}, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, and Y, \mathcal{M} are FSSs on $W \times W \times (0, \infty)$. If $(W, Y, \mathcal{M}, *, \Delta)$ meet the following circumstances for all $\mathfrak{F}, \mathfrak{h}, \partial \in W$ and $\pi, \wp > 0$:

- (B1) $Y(\mathfrak{F}, \mathfrak{h}, \wp) + \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1$,
- (B2) $Y(\mathfrak{F}, \mathfrak{h}, \wp) > 0$,
- (B3) $Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1 \iff \mathfrak{F} = \mathfrak{h}$,
- (B4) $Y(\mathfrak{F}, \mathfrak{h}, \wp) = Y(\mathfrak{h}, \mathfrak{F}, \wp)$,
- (B5) $Y(\mathfrak{F}, \partial, (\wp + \pi)) \geq Y(\mathfrak{F}, \mathfrak{h}, \wp) * Y(\mathfrak{h}, \partial, \pi)$,
- (B6) $Y(\mathfrak{F}, \mathfrak{h}, \bullet)$ is non decreasing (ND) function of \mathbb{R}^+ and $\wp Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1$,
- (B7) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) > 0$,
- (B8) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0 \iff \mathfrak{F} = \mathfrak{h}$,
- (B9) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = \mathcal{M}(\mathfrak{h}, \mathfrak{F}, \wp)$,
- (B10) $\mathcal{M}(\mathfrak{F}, \partial, (\wp + \pi)) \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \Delta \mathcal{M}(\mathfrak{h}, \partial, \pi)$,
- (B11) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \bullet)$ is non increasing (NI) function of \mathbb{R}^+ and $\lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0$.

Then, $(W, Y, \mathcal{M}, *, \Delta)$ is an IFMS.

Definition 4. Suppose $W \neq \emptyset$, assume five tuples $(W, Y, \mathcal{M}, D, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, and Y, \mathcal{M}, D are picture fuzzy set on $W \times W \times \mathbb{R}^+$. If $(W, Y, \mathcal{M}, D, *, \Delta)$ meet the following circumstances for all $\mathfrak{F}, \mathfrak{h}, \partial \in W$ and $\pi, \wp > 0$:

- (P1) $Y(\mathfrak{F}, \mathfrak{h}, \wp) + \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) + D(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1$,
- (P2) $0 \leq Y(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1$,
- (P3) $Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1 \iff \mathfrak{F} = \mathfrak{h}$,
- (P4) $Y(\mathfrak{F}, \mathfrak{h}, \wp) = Y(\mathfrak{h}, \mathfrak{F}, \wp)$,
- (P5) $Y(\mathfrak{F}, \partial, (\wp + \pi)) \geq Y(\mathfrak{F}, \mathfrak{h}, \wp) * Y(\mathfrak{h}, \partial, \pi)$,
- (P6) $Y(\mathfrak{F}, \mathfrak{h}, \bullet)$ is non decreasing (ND) function of \mathbb{R}^+ and $\lim_{\wp \rightarrow \infty} Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1$,
- (P7) $0 \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1$,
- (P8) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0 \iff \mathfrak{F} = \mathfrak{h}$,
- (P9) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = \mathcal{M}(\mathfrak{h}, \mathfrak{F}, \wp)$,
- (P10) $\mathcal{M}(\mathfrak{F}, \partial, (\wp + \pi)) \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \Delta \mathcal{M}(\mathfrak{h}, \partial, \pi)$,

(P11) $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \bullet)$ is non increasing (NI) function of \mathbb{R}^+ and $\lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0$,

$$(P12) 0 \leq D(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1,$$

$$(P13) D(\mathfrak{F}, \mathfrak{h}, \wp) = 0 \iff \mathfrak{F} = \mathfrak{h},$$

$$(P14) D(\mathfrak{F}, \mathfrak{h}, \wp) = D(\mathfrak{h}, \mathfrak{F}, \wp),$$

$$(P15) D(\mathfrak{F}, \partial, (\wp + \pi)) \leq D(\mathfrak{F}, \mathfrak{h}, \wp) \Delta v(\mathfrak{h}, \partial, \pi),$$

(P16) $D(\mathfrak{F}, \mathfrak{h}, \bullet)$ is non increasing (NI) function of \mathbb{R}^+ and $\lim_{\wp \rightarrow \infty} D(\mathfrak{F}, \mathfrak{h}, \wp) = 0$,

(P17) If $\wp \leq 0$, then $Y(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 1$ and $D(\mathfrak{F}, \mathfrak{h}, \wp) = 1$.

Then, $(W, Y, \mathcal{M}, D, *, \Delta)$ is a PFMS.

Definition 5 (see [4]). Assume $W \neq \emptyset$ and $\vdash \in W \times W$ is a binary relation. Assume there exists $\mathfrak{F}_0 \in W$ such that $\mathfrak{F}_0 \vdash \mathfrak{F}$ or $\mathfrak{F} \vdash \mathfrak{F}_0$ for all $\mathfrak{F} \in W$. Thus, W is said to be an OS. Furthermore, we denote OS by (W, \vdash) .

Definition 6 (see [4]). Suppose that (W, \vdash) is an OS. A sequence $\{\mathfrak{F}_n\}$ for $n \in \mathbb{N}$ is called an (OS) if for all n , $\mathfrak{F}_n \vdash \mathfrak{F}_{n+1}$ or for all n , $\mathfrak{F}_{n+1} \vdash \mathfrak{F}_n$.

2.1. Orthogonal Picture Fuzzy Metric Space

Definition 7. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be called an OPFMS if W is a non-empty OS, $*$ is a CTN, Δ is a CTCN, and Y, \mathcal{M}, D are pfs on $W \times W \times \mathbb{R}^+$ with the following condition are satisfied for all $\mathfrak{F}, \mathfrak{h}, \partial \in W$ with either $(\mathfrak{F} \vdash \mathfrak{h}$ or $\mathfrak{h} \vdash \mathfrak{F})$, either $(\mathfrak{F} \vdash \mathfrak{h}$ or $\delta \vdash \mathfrak{F})$, and either $(\mathfrak{F} \vdash \mathfrak{h} \vee \mathfrak{h} \vdash \mathfrak{F})$:

$$P_1) Y(\mathfrak{F}, \mathfrak{h}, \wp) + \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) + D(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1,$$

$$P_2) 0 \leq Y(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1,$$

$$P_3) Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1 \text{ if and only if } \mathfrak{F} = \mathfrak{h},$$

$$P_4) Y(\mathfrak{F}, \mathfrak{h}, \wp) = Y(\mathfrak{h}, \mathfrak{F}, \wp),$$

$$P_5) Y(\mathfrak{F}, \partial, (\wp + \pi)) \geq Y(\mathfrak{F}, \mathfrak{h}, \lambda) * Y(\mathfrak{h}, e, \pi),$$

$$P_6) Y(\mathfrak{F}, \mathfrak{h}, \bullet): (0, \infty) \longrightarrow [0, 1] \text{ is continuous,}$$

$$P_7) 0 \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1,$$

$$P_8) \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 1 \text{ if and only if } \mathfrak{F} = \mathfrak{h},$$

$$P_9) \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = \mathcal{M}(\mathfrak{h}, \mathfrak{F}, \wp),$$

$$P_{10}) \mathcal{M}(\mathfrak{F}, \partial, (\wp + \pi)) \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \Delta \mathcal{M}(\mathfrak{h}, \partial, \pi),$$

$$P_{11}) \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \bullet): (0, \infty) \longrightarrow [0, 1] \text{ is continuous,}$$

$$P_{12}) 0 \leq D(\mathfrak{F}, \mathfrak{h}, \wp) \leq 1,$$

$$P_{13}) D(\mathfrak{F}, \mathfrak{h}, \wp) = 1 \text{ if and only if } \mathfrak{F} = \mathfrak{h},$$

$$P_{14}) D(\mathfrak{F}, \mathfrak{h}, \wp) = D(\mathfrak{h}, \mathfrak{F}, \wp),$$

$$P_{15}) D(\mathfrak{F}, \partial, (\wp + \pi)) \leq D(\mathfrak{F}, \mathfrak{h}, \wp) \Delta D(\mathfrak{h}, \partial, \pi),$$

$$P_{16}) D(\mathfrak{F}, \mathfrak{h}, \bullet): (0, \infty) \longrightarrow [0, 1] \text{ is continuous,}$$

(P17) If $\wp \leq 0$ then $Y(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, $\mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 1$ and $D(\mathfrak{F}, \mathfrak{h}, \wp) = 1$,

Then, $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ is called OPFMS.

Remark 8. Every PFMS is an OPFMS but the converse is not true.

Example 1. Let $W = [-7, 7]$ and define a CTN as $a * b = ab$, CTCN as $a \Delta b = \max\{a, b\}$ and define a binary relation \vdash by $\mathfrak{F} \vdash \mathfrak{h}$ iff $\mathfrak{F} + \mathfrak{h} \geq 0$. Take

$$\begin{aligned}
Y(\mathfrak{F}, \hbar, \wp) &= \begin{cases} 1 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}, \hbar\}} & \text{if otherwise,} \end{cases} \\
\mathcal{M}(\mathfrak{F}, \hbar, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\max\{\mathfrak{F}, \hbar\}}{\wp + \max\{\mathfrak{F}, \hbar\}} & \text{if otherwise,} \end{cases} \\
D(\mathfrak{F}, \hbar, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\max\{\mathfrak{F}, \hbar\}}{\wp} & \text{if otherwise,} \end{cases}
\end{aligned} \tag{3}$$

for all $\mathfrak{F}, \hbar \in W, \wp > 0$, then it is OPFS, but not an PFMS.

It is easy to see that for $\pi = \wp = 1, \mathfrak{F} = -1, \hbar = -1/2, \partial = -2$. $(P_+5), (P_+10)$, and (P_+15) fails.

Remark 9. The above example is also OPFMS if we take

$$\mathcal{M}(\mathfrak{F}, \hbar, \wp) = \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar, \\ 1 - \frac{\wp}{\wp + \max\{\mathfrak{F}, \hbar\}} & \text{if otherwise.} \end{cases} \tag{4}$$

Definition 10. An OS $\{\mathfrak{F}_n\}$ in an OPFMS $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ is said to be orthogonal convergent (O-C) to $\mathfrak{F} \in W$ if

$$\begin{aligned}
\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}, \wp) &= 1, \forall \wp > 0, \\
\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}, \wp) &= 0, \forall \wp > 0, \\
\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}, \wp) &= 0, \forall \wp > 0.
\end{aligned} \tag{5}$$

Definition 11. An OS $\{\mathfrak{F}_n\}$ in an OPFMS $(W, Y, \mathcal{M}, *, \Delta, \vdash)$ is said to be Orthogonal Cauchy (O-CS) if there exists $n \in \mathbb{N}$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= 1, \\
\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= 0, \\
\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= 0,
\end{aligned} \tag{6}$$

for all $\wp \geq 0, p \geq 1$.

Definition 12. $\Omega : W \rightarrow W$ is OC at $\mathfrak{F} \in W$ in an OPFMS $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$, whenever for each OS $\{\mathfrak{F}_n\}$ for all $n \in \mathbb{N}$ in W if $\lim_{n \rightarrow \infty} R(\mathfrak{F}_n, \mathfrak{F}, \wp) = 1, \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}, \wp) = 0$ for all $\wp > 0$, then $\lim_{n \rightarrow \infty} Y(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}, \wp) = 1, \lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}, \wp) = 0$ for all $\wp > 0$.

Definition 13. An OPFMS $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ is said to be orthogonally complete (O-complete) if every O-CS is convergent.

Example 2. Assume OPFMS as given in Example 1 and define a sequence $\{\mathfrak{F}_n\}$ in W by $\mathfrak{F}_n = 1 - 1/n, \forall n \in \mathbb{N}$ such that $(\forall n; \mathfrak{F}_n \vdash \mathfrak{F}_{n+1})$ or $(\forall n; \mathfrak{F}_{n+1} \vdash \mathfrak{F}_n)$. Define a CTN as $a * b = ab$, CTCN as $a \Delta b = \max\{a, b\}$, and define a binary relation \vdash by $\mathfrak{F} \vdash \hbar$ iff $\mathfrak{F} + \hbar \geq 0$. Take

$$\begin{aligned}
\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 1 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}_n, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} = \begin{cases} 1 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} \\
\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\max\{\mathfrak{F}_n, \mathfrak{F}\}}{\wp + \max\{\mathfrak{F}_n, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} = \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar \\ \frac{\max\{\mathfrak{F}, \mathfrak{F}\}}{\wp + \max\{\mathfrak{F}, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} \\
\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar \\ \frac{\max\{\mathfrak{F}_j, \mathfrak{F}\}}{\wp} & \text{if otherwise} \end{cases} = \begin{cases} 0 & \text{if } \mathfrak{F} = \hbar, \\ \frac{\max\{\mathfrak{F}, \mathfrak{F}\}}{\wp} & \text{if otherwise.} \end{cases}
\end{aligned} \tag{7}$$

Example 3. From proof of Example 2, $\mathfrak{F}_n = 1 - 1/n, \forall n \in \mathbb{N}$ is a O-CS in an OPFMS.

$$\begin{aligned} \lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 1 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}_n, \mathfrak{F}_{n+p}\}} & \text{if otherwise,} \end{cases} = \begin{cases} 1 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} \\ \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{F}_n, \mathfrak{F}_{n+p}\}}{\wp + \max\{\mathfrak{F}_n, \mathfrak{F}_{n+p}\}} & \text{if otherwise,} \end{cases} = \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{F}, \mathfrak{F}\}}{\wp + \max\{\mathfrak{F}, \mathfrak{F}\}} & \text{if otherwise,} \end{cases} \\ \lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{F}_n, \mathfrak{F}_{n+p}\}}{\wp} & \text{if otherwise} \end{cases} = \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{F}, \mathfrak{F}\}}{\wp} & \text{if otherwise,} \end{cases} \end{aligned} \quad (8)$$

for all $\wp \geq 0, p \geq 1$.

Lemma 14. *If for some $v \in (0, 1)$ and $\mathfrak{F}, \mathfrak{h} \in W$,*

$$\begin{aligned} Y(\mathfrak{F}, \mathfrak{h}, \wp) &\geq Y\left(\mathfrak{F}, \mathfrak{h}, \frac{\wp}{v}\right), \wp > 0, \\ \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) &\leq \mathcal{M}\left(\mathfrak{F}, \mathfrak{h}, \frac{\wp}{v}\right), \wp > 0 \\ D(\mathfrak{F}, \mathfrak{h}, \wp) &\leq D\left(\mathfrak{F}, \mathfrak{h}, \frac{\wp}{v}\right), \wp > 0 \end{aligned} \quad (9)$$

then $\mathfrak{F} = \mathfrak{h}$.

Definition 15. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an OPFMS. A map $\Omega : W \rightarrow W$ is an orthogonal contraction if there exists $\theta \in (0, 1)$ such that for every $\wp > 0$ and $\mathfrak{F}, \delta \in W$ with $\mathfrak{F} \vdash \mathfrak{h}$, we have

$$Y(\psi\mathfrak{F}, \Omega\mathfrak{h}, \theta\lambda) \geq Y(\mathfrak{F}, \mathfrak{h}, \wp), \quad (10)$$

$$\mathcal{M}(\Omega\mathfrak{F}, \Omega\mathfrak{h}, \theta\wp) \leq \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp), \quad (11)$$

$$D(\Omega\mathfrak{F}, \Omega\mathfrak{h}, \theta\wp) \leq D(\mathfrak{F}, \mathfrak{h}, \wp). \quad (12)$$

Theorem 16. *Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS such that*

$$\lim_{\wp \rightarrow \infty} Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1, \lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0, \text{ and } \lim_{\wp \rightarrow \infty} D(\mathfrak{F}, \mathfrak{h}, \wp) = 0, \forall \mathfrak{F}, \mathfrak{h} \in W. \quad (13)$$

Let $\Omega : W \rightarrow W$ be an OC, O-CON and OPR. Thus, Ω has a unique FP, say $\mathfrak{F}_* \in W$. Furthermore,

$$\begin{aligned} \lim_{n \rightarrow \infty} Y(\Omega^n \mathfrak{F}, \mathfrak{F}_*, \wp) &= 1, \lim_{n \rightarrow \infty} \mathcal{M}(\Omega^n \mathfrak{F}, \mathfrak{F}_*, \wp) = 0, \\ \text{and } \lim_{n \rightarrow \infty} D(\Omega^n \mathfrak{F}, \mathfrak{F}_*, \wp) &= 0 \forall \mathfrak{F} \in W \text{ and } \wp > 0. \end{aligned} \quad (14)$$

Proof. Since $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ is an O-complete PFMS, there exists $\mathfrak{F}_0 \in W$ such that

$$\mathfrak{F}_0 \vdash \mathfrak{h} \text{ for all } \mathfrak{h} \in W. \quad (15)$$

That is, $\mathfrak{F}_0 \vdash \Omega \mathfrak{F}_0$. Take

$$\mathfrak{F}_n = \Omega^n \mathfrak{F}_0 = \Omega \mathfrak{F}_{n-1} \text{ for all } n \in \mathcal{M}. \quad (16)$$

Since Ω is OPR, $\{\mathfrak{F}_n\}$ is an OS. Now, since Ω is an O-CON, we get

$$Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) = Y(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \geq Y(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp), \quad (17)$$

for all $n \in \mathcal{M}$ and $\wp > 0$. Note that Y is nondecreasing on $(0, \infty)$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned} Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\geq Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) = Y(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \\ &\geq Y(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp) = Y(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_{n-2}, \wp) \\ &\geq Y\left(\mathfrak{F}_{n-1}, \mathfrak{F}_{n-2}, \frac{\wp}{\theta}\right) \geq \dots \geq Y\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{\theta^n}\right), \end{aligned} \quad (18)$$

for all $n \in \mathcal{M}$ and $\wp > 0$. Thus, from (15) and (P₅), we have

$$\begin{aligned}
 Y(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) &\geq Y\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2}\right) \geq Y\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+3}, \frac{\wp}{2}\right) * \\
 &\vdots \\
 &* Y\left(\mathfrak{F}_{n+\alpha-1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2^{n+\alpha}}\right) \geq Y\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2\theta^n}\right) * Y\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^2\theta^n}\right) * \\
 &\vdots \\
 &* Y\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^{n+\alpha}\theta^n}\right).
 \end{aligned} \tag{19}$$

We know that $\lim_{\wp \rightarrow \infty} Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1$, for all $\mathfrak{F}, \mathfrak{h} \in W$ and $\wp > 0$. So, from (19) we get,

$$\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) \geq 1 * 1 * \dots * 1 = 1.$$

$$\mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) = \mathcal{M}(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \leq \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp), \tag{20}$$

for all $n \in \mathfrak{h}$ and $\wp > 0$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned}
 \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) = \mathcal{M}(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \\
 &\leq \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp) = \mathcal{M}(\Omega\mathfrak{F}_{n-1}, \Psi\mathfrak{F}_{n-2}, \wp) \\
 &\leq \mathcal{M}\left(\mathfrak{F}_{n-1}, \mathfrak{F}_{n-2}, \frac{\wp}{\theta}\right) \leq \dots \leq \mathcal{M}\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{\theta^n}\right),
 \end{aligned} \tag{21}$$

for all $n \in \mathfrak{h}$ and $\wp > 0$. Thus, from (21) and $(P_{\perp}10)$, we have

$$\begin{aligned}
 \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) &\leq \mathcal{M}\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2}\right) \\
 &\leq \mathcal{M}\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \\
 &\quad \Delta \mathcal{M}\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+3}, \frac{\wp}{2}\right) \Delta \dots \Delta \mathcal{M}\left(\mathfrak{F}_{n+\alpha-1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2^{n+\alpha}}\right) \\
 &\leq \mathcal{M}\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2\theta^n}\right) \Delta \mathcal{M}\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^2\theta^n}\right) \Delta \dots \\
 &\quad \Delta \mathcal{M}\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^{n+\alpha}\theta^n}\right).
 \end{aligned} \tag{22}$$

We know that $\lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, for all $\mathfrak{F}, \mathfrak{h} \in W$ and $\wp > 0$. So, from (22) we get,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) &\leq 0 \Delta 0 \Delta \dots \Delta 0 = 0, \\
 D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) &= D(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \\
 &\leq D(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp),
 \end{aligned} \tag{23}$$

for all $n \in \mathfrak{h}$ and $\wp > 0$. Therefore, by applying the above

expression, we can deduce

$$\begin{aligned}
 D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) = D(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_{n-1}, \theta\wp) \\
 &\leq D(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp) = D(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_{n-2}, \wp) \\
 &\leq D\left(\mathfrak{F}_{n-1}, \mathfrak{F}_{n-2}, \frac{\wp}{\theta}\right) \leq \dots \leq D\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{\theta^n}\right),
 \end{aligned} \tag{24}$$

for all $n \in \mathfrak{h}$ and $\wp > 0$. Thus, from (24) and $(P_{\perp}15)$, we have

$$\begin{aligned}
 D(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) &\leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2}\right) \\
 &\leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \\
 &\quad \Delta D\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+3}, \frac{\wp}{2}\right) \Delta \dots \Delta D\left(\mathfrak{F}_{n+\alpha-1}, \mathfrak{F}_{n+\alpha}, \frac{\wp}{2^{n+\alpha}}\right) \\
 &\leq D\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2\theta^n}\right) \Delta D\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^2\theta^n}\right) \Delta \dots \Delta D\left(\mathfrak{F}_1, \mathfrak{F}_0, \frac{\wp}{2^{n+\alpha}\theta^n}\right),
 \end{aligned} \tag{25}$$

We know that $\lim_{\wp \rightarrow \infty} D(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, for all $\mathfrak{F}, \mathfrak{h} \in W$ and $\wp > 0$. So, from (25) we get,

$$\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_{n+\alpha}, \wp) \leq 0 \Delta 0 \Delta \dots \Delta 0 = 0. \tag{26}$$

So, $\{\mathfrak{F}_n\}$ is a O-CS. The O-completeness of the PFMS $(W, Y, \mathcal{M}, D, *, \Delta, \dagger)$ ensures that there exist $\mathfrak{F}_* \in W$ such that $Y(\mathfrak{F}_n, \mathfrak{F}_*, \wp) \rightarrow 1, \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_*, \wp) \rightarrow 0$, and $D(\mathfrak{F}_n, \mathfrak{F}_*, \wp) \rightarrow 0$ as $n \rightarrow +\infty$ for all $\wp > 0$. Now, since Ω is an OC, $Y(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) = Y(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) \rightarrow 1, \mathcal{M}(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) = \mathcal{M}(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) \rightarrow 0$, and $D(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) = D(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) \rightarrow 0$ as $n \rightarrow +\infty$. Now, we have

$$\begin{aligned}
 Y(\mathfrak{F}_*, \Psi\mathfrak{F}_*, \wp) &\geq Y\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \frac{\wp}{2}\right), \\
 \mathcal{M}(\mathfrak{F}_*, \Omega\mathfrak{F}_*, \wp) &\leq \mathcal{M}\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \frac{\wp}{2}\right), \\
 D(\mathfrak{F}_*, \Omega\mathfrak{F}_*, \wp) &\leq D\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \frac{\wp}{2}\right).
 \end{aligned} \tag{27}$$

Taking limit as $n \rightarrow +\infty$, we get $Y(\mathfrak{F}_*, \Omega\mathfrak{F}_*, \wp) = 1 * 1 = 1, \mathcal{M}(\mathfrak{F}_*, \Omega\mathfrak{F}_*, \wp) = 0 \Delta 0 = 0$, and $D(\mathfrak{F}_*, \Omega\mathfrak{F}_*, \wp) = 0 \Delta 0 = 0$ and hence $\Omega\mathfrak{F}_* = \mathfrak{F}_*$.

Now, we show the uniqueness of the FP of the mapping Ω . Assume that \mathfrak{F}_* and \mathfrak{h}_* are two FPs of Ω such that $\mathfrak{F}_* \neq \mathfrak{h}_*$. We can get

$$\mathfrak{F}_0 \dagger \mathfrak{F}_* \text{ and } \mathfrak{F}_0 \dagger \mathfrak{h}_*. \tag{28}$$

Since D is OPR, one writes

$$\Omega^n \mathfrak{F}_0 \vdash \Omega^n \mathfrak{F}_* \text{ and } \Omega^n \mathfrak{F}_0 \vdash \Omega^n \mathfrak{h}_*, \quad (29)$$

for all $n \in \mathcal{M}$. So from (10), we can derive

$$\begin{aligned} Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) &\geq Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \geq Y\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{\theta^n}\right), \\ Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \wp) &\geq Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \theta \wp) \geq Y\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{\theta^n}\right). \end{aligned} \quad (30)$$

Therefore,

$$\begin{aligned} Y(\mathfrak{F}_*, \mathfrak{h}_*, \wp) &= Y(\Omega^n \mathfrak{F}_*, \Omega^n \mathfrak{h}_*, \wp) \geq Y\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \frac{\wp}{2}\right) \\ &\quad * Y\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \frac{\wp}{2}\right) \geq Y\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{2\theta^n}\right) \\ &\quad * Y\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (31)$$

So from (11), we can derive

$$\begin{aligned} \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) &\leq \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \leq \mathcal{M}\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{\theta^n}\right), \\ \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \wp) &\leq \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \theta \wp) \leq \mathcal{M}\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{\theta^n}\right). \end{aligned} \quad (32)$$

Therefore,

$$\begin{aligned} \mathcal{M}(\mathfrak{F}_*, \mathfrak{h}_*, \wp) &= \mathcal{M}(\Omega^n \mathfrak{F}_*, \Omega^n \mathfrak{h}_*, \wp) \leq \mathcal{M}\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \frac{\wp}{2}\right) \\ &\leq \mathcal{M}\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{2\theta^n}\right) \Delta \mathcal{M}\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (33)$$

Similarly, from (12), we can derive

$$\begin{aligned} D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) &\leq D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \leq D\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{\theta^n}\right), \\ D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \wp) &\leq D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \theta \wp) \leq D\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{\theta^n}\right). \end{aligned} \quad (34)$$

Therefore,

$$\begin{aligned} D(\mathfrak{F}_*, \mathfrak{h}_*, \wp) &= D(\Omega^n \mathfrak{F}_*, \Omega^n \mathfrak{h}_*, \wp) \\ &\leq D\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \frac{\wp}{2}\right) \Delta D\left(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{h}_*, \frac{\wp}{2}\right) \\ &\leq D\left(\mathfrak{F}_0, \mathfrak{F}_*, \frac{\wp}{2\theta^n}\right) \Delta D\left(\mathfrak{F}_0, \mathfrak{h}_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (35)$$

So, $\mathfrak{F}_* = \mathfrak{h}$; hence, \mathfrak{F}_* is the unique FP. \square

Corollary 17. Assume $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O -complete PFMS. Assume $\Omega : W \longrightarrow W$ be O -CON and OPR and if $\{\mathfrak{F}_n\}$ is an OS with $\mathfrak{F}_n \longrightarrow \mathfrak{F} \in W$, then $\mathfrak{F} \vdash \mathfrak{F}_n$ for all $n \in \mathbb{N}$. Then, Ω has a unique FP, say $\mathfrak{F}_* \in W$.

Proof. We can similarly derive as in the proof of Theorem 16 that $\{\mathfrak{F}_n\}$ is a O -CS and so it converges to $\mathfrak{F}_* \in W$. Hence, $\mathfrak{F}_* \vdash \mathfrak{F}_n$ for all $n \in \mathbb{N}$. from (10), we can get

$$\begin{aligned} Y(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= Y(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp) \geq Y(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp \theta) \geq Y(\mathfrak{F}_*, \mathfrak{F}_n, \wp), \\ \lim_{n \rightarrow \infty} Y(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= 1. \end{aligned} \quad (36)$$

Then, we can write

$$Y(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) \geq Y\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \Omega \mathfrak{F}_*, \frac{\wp}{2}\right). \quad (37)$$

Taking limit as $n \longrightarrow +\infty$, we get $Y(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) = 1 * 1 = 1$ and from (11), we can get

$$\begin{aligned} \mathcal{M}(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= \mathcal{M}(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp) \leq \mathcal{M}(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp \theta) \leq \mathcal{M}(\mathfrak{F}_*, \mathfrak{F}_n, \wp), \\ \lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= 0. \end{aligned} \quad (38)$$

Then, we can write

$$\mathcal{M}(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) \leq \mathcal{M}\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \Omega \mathfrak{F}_*, \frac{\wp}{2}\right). \quad (39)$$

Taking limit as $n \longrightarrow +\infty$, we get

$$\mathcal{M}(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) = 0 \Delta 0 = 0, \quad (40)$$

and from (12), we can get

$$\begin{aligned} D(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= D(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp) \leq D(\Omega \mathfrak{F}_*, \Omega \mathfrak{F}_n, \wp \theta) \leq D(\mathfrak{F}_*, \mathfrak{F}_n, \wp), \\ \lim_{n \rightarrow \infty} D(\Omega \mathfrak{F}_*, \mathfrak{F}_{n+1}, \wp) &= 0. \end{aligned} \quad (41)$$

Then, we can write

$$D(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) \leq D\left(\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \Omega \mathfrak{F}_*, \frac{\wp}{2}\right). \quad (42)$$

Taking limit as $n \longrightarrow +\infty$, we get

$$D(\mathfrak{F}_*, \Omega \mathfrak{F}_*, \wp) = 0 \Delta 0 = 0, \quad (43)$$

so $\Omega \mathfrak{F}_* = \mathfrak{F}_*$. Next proof is similar as in Theorem 16. \square

Example 4. Let $W = [-3, 3]$. We define a binary relation \vdash by $\mathfrak{F} \vdash \mathfrak{h} \iff \mathfrak{F} + \mathfrak{h} \geq 0$.

Define an OPFMS as in Example 1 by

$$\begin{aligned}
 Y(\mathfrak{S}, \hbar, \wp) &= \begin{cases} 1 & \text{if } \mathfrak{S} = \hbar, \\ \frac{\wp}{\wp + \max\{\mathfrak{S}, \hbar\}} & \text{if otherwise,} \end{cases} \\
 \mathcal{M}(\mathfrak{S}, \hbar, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{S} = \hbar, \\ \frac{\max\{\mathfrak{S}, \hbar\}}{\wp + \max\{\mathfrak{S}, \hbar\}} & \text{if otherwise,} \end{cases} \\
 D(\mathfrak{S}, \hbar, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{S} = \hbar, \\ \frac{\max\{\mathfrak{S}, \hbar\}}{\wp} & \text{if otherwise,} \end{cases}
 \end{aligned} \tag{44}$$

for all $\mathfrak{S}, \delta \in W, \wp > 0$, with the CTN $a * b = a \cdot b$ and CTCN $a \Delta b = \max\{a, b\}$. Then, $(W, Y, \mathcal{M}, D, *, \Delta, +)$ is an O-complete PFMS. Define $\Omega : W \rightarrow W$ by

$$\Omega \mathfrak{S} = \begin{cases} \frac{\mathfrak{S}}{4}, & \mathfrak{S} \in [-3, 0] \\ 0, & \mathfrak{S} \in (0, 3] \end{cases} \tag{45}$$

Then, the following cases are satisfied:

- (1) If $\mathfrak{S} \in [-3, 0]$ and $\hbar \in (0, 3]$, then $\Omega \mathfrak{S} = \mathfrak{S}/4$ and $\psi \hbar = 0$
- (2) If $\mathfrak{S}, \hbar \in [-3, 0]$, then $\Omega \mathfrak{S} = \mathfrak{S}/4$ and $\Omega \hbar = \hbar/4$
- (3) If $\mathfrak{S}, \hbar \in (0, 3]$, then $\Omega \mathfrak{S} = 0$ and $\Omega \hbar = 0$
- (4) If $\mathfrak{S} \in (0, 3]$ and $\hbar \in [-3, 0]$, then $\Omega \mathfrak{S} = 0$ and $\Omega \hbar = \hbar/4$

This clearly implies that $\Omega \mathfrak{S} + \Omega \hbar \geq 0$. Hence, Ω is OPR. We can easily see that if $\lim_{n \rightarrow \infty} Y(\mathfrak{S}_n, \mathfrak{S}, \wp) = 1$, then $\lim_{n \rightarrow \infty} Y(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{S}_n, \mathfrak{S}, \wp) = 0$, then $\lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 0$ and $\lim_{n \rightarrow \infty} D(\mathfrak{S}_n, \mathfrak{S}, \wp) = 0$, then $\lim_{n \rightarrow \infty} D(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 0$ for all $\mathfrak{S} \in W$ and $\wp > 0$. Hence, Ω is OC.

The above four cases for $\theta \in [1/2, 1)$ satisfies the below contractive conditions:

$$\begin{aligned}
 Y(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\geq Y(\mathfrak{S}, \hbar, \wp), \\
 \mathcal{M}(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\leq \mathcal{M}(\mathfrak{S}, \hbar, \wp), \\
 D(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\leq D(\mathfrak{S}, \hbar, \wp).
 \end{aligned} \tag{46}$$

All conditions of Theorem 16 are satisfied. Also, 0 is FP of Ω .

Theorem 18. Let $(W, Y, \mathcal{M}, D, *, \Delta, +)$ be an O-complete PFMS such that

$$\lim_{\wp \rightarrow \infty} Y(\mathfrak{S}, \hbar, \wp) = 1 \text{ and } \lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{S}, \hbar, \wp) = 0, \forall \mathfrak{S}, \hbar \in W \text{ and } \wp > 0. \tag{47}$$

Let $\Omega : W \rightarrow W$ be OC, O-CON and OPR. Assume that there exist $\theta \in (0, 1)$ and $\wp > 0$ such that

$$\begin{aligned}
 Y(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\geq \min\{Y(\Omega \mathfrak{S}, \mathfrak{S}, \wp), Y(\Omega \hbar, \hbar, \wp)\}, \\
 \mathcal{M}(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\leq \min\{\mathcal{M}(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \mathcal{M}(\Omega \hbar, \hbar, \wp)\}, \\
 D(\Omega \mathfrak{S}, \Omega \hbar, \theta \wp) &\leq \min\{D(\Omega \mathfrak{S}, \mathfrak{S}, \wp), D(\Omega \hbar, \hbar, \wp)\},
 \end{aligned} \tag{48}$$

for all $\mathfrak{S}, \hbar \in W, \wp > 0$. Then, Ω has a unique FP, so $\mathfrak{S}_* \in W$. Furthermore, $\lim_{n \rightarrow \infty} Y(\Omega^n \mathfrak{S}, \mathfrak{S}_*, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\Omega^n \mathfrak{S}, \mathfrak{S}_*, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\Omega^n \mathfrak{S}, \mathfrak{S}_*, \wp) = 0$ for all $\mathfrak{S} \in W$ and $\wp > 0$.

Proof. Since $(W, Y, \mathcal{M}, D, *, \Delta, +)$ is an O-complete PFMS, there exists $\mathfrak{S}_0 \in W$ such that

$$\mathfrak{S}_0 + \hbar, \forall \delta \in W. \tag{49}$$

Thus, $\mathfrak{S} + \hbar$. Consider

$$\mathfrak{S}_n = \Omega^n \mathfrak{S}_0 = \Omega \mathfrak{S}_{n-1}, \forall n \in \mathbb{S}. \tag{50}$$

Since Ω is OPR, $\{\mathfrak{S}_n\}$ is an OS. We can get

$$\begin{aligned}
 Y(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \wp) &\geq Y(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \theta \wp) = Y(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}_{n-1}, \theta \wp) \geq \min\{Y(\Omega \mathfrak{S}_n, \mathfrak{S}_n, \wp), Y(\Omega \mathfrak{S}_{n-1}, \mathfrak{S}_{n-1}, \wp)\}, \\
 \mathcal{M}(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \wp) &\leq \mathcal{M}(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \theta \wp) = \mathcal{M}(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}_{n-1}, \theta \wp) \leq \min\{\mathcal{M}(\Omega \mathfrak{S}_n, \mathfrak{S}_n, \wp), \mathcal{M}(\Omega \mathfrak{S}_{n-1}, \mathfrak{S}_{n-1}, \wp)\}, \\
 D(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \wp) &\leq D(\mathfrak{S}_{n+1}, \mathfrak{S}_n, \theta \wp) = D(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}_{n-1}, \theta \wp) \leq \min\{D(\Omega \mathfrak{S}_n, \mathfrak{S}_n, \wp), D(\Omega \mathfrak{S}_{n-1}, \mathfrak{S}_{n-1}, \wp)\}.
 \end{aligned} \tag{51}$$

Two cases arise:

Case1: If $Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) \geq Y(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp)$, then

$$\begin{aligned} Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\geq Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \geq Y(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp) = Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp), \\ \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq \mathcal{M}(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp). \end{aligned} \quad (52)$$

Then,

$$\begin{aligned} \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \leq \mathcal{M}(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp) = \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp), \\ D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq D(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp). \end{aligned} \quad (53)$$

Then,

$$\begin{aligned} D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \leq D(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \wp) \\ &= D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp). \end{aligned} \quad (54)$$

Then by Lemma 14, $\mathfrak{F}_n = \mathfrak{F}_{n+1}$ for all $n \in \mathbb{N}$.

Case2: If $Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) \geq Y(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp)$, then

$$\begin{aligned} Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\geq Y(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \geq Y(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp) \geq Y(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp), \\ \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq \mathcal{M}(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp). \end{aligned} \quad (55)$$

Then,

$$\mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) \leq \mathcal{M}(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \leq \mathcal{M}(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp) \leq \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp), \quad (56)$$

and

$$D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) \leq D(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp). \quad (57)$$

Then,

$$\begin{aligned} D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \wp) &\leq D(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \theta\wp) \leq D(\Omega\mathfrak{F}_{n-1}, \mathfrak{F}_{n-1}, \wp) \\ &\leq D(\mathfrak{F}_n, \mathfrak{F}_{n-1}, \wp), \end{aligned} \quad (58)$$

for all $n \in \mathbb{N}$ and $\wp > 0$. Then by Theorem 16, we have a OCS. By completeness of $(W, Y, \mathcal{M}, D, *, \Delta, \dagger)$, there exists \mathfrak{F}_* in W such that $\lim_{n \rightarrow \infty} R(\mathfrak{F}_n, \mathfrak{F}_*, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_*, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_*, \wp) = 0$, for all $\wp > 0$.

We know that Ω is an OC, then

$$\begin{aligned} \lim_{n \rightarrow \infty} Y(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) &= \lim_{n \rightarrow \infty} Y(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) = 1, \\ \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) &= \lim_{n \rightarrow \infty} \mathcal{M}(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) = 0, \\ \lim_{n \rightarrow \infty} D(\mathfrak{F}_{n+1}, \Omega\mathfrak{F}_*, \wp) &= \lim_{n \rightarrow \infty} D(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_*, \wp) = 0. \end{aligned} \quad (59)$$

Now, we prove that \mathfrak{F}_* is a FP of Ω . Let $\wp_1 \in (\theta, 1)$ and $\wp_2 = 1 - \wp_1$. Then,

$$\begin{aligned} Y(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\geq Y\left(\Omega\mathfrak{F}_*, \mathfrak{F}_{n+1}, \frac{\wp\wp_1}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_*, \frac{\wp\wp_2}{2}\right) \\ &= Y\left(\Omega\mathfrak{F}_*, \Omega\mathfrak{F}_n, \frac{\wp\wp_1}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_*, \frac{\wp\wp_2}{2}\right) \\ &\geq \min\left\{Y\left(\Omega\mathfrak{F}_*, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_*, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{Y\left(\Omega\mathfrak{F}_*, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_*, \frac{\wp\wp_2}{2}\right). \end{aligned} \quad (60)$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} Y(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\geq \min\left\{Y\left(\Omega\mathfrak{F}_*, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), 1\right\} * 1, \\ Y(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\geq Y\left(\Omega\mathfrak{F}_*, \mathfrak{F}_n, \frac{\wp}{v}\right), \wp > 0, \\ \mathcal{M}(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\leq \mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp\wp_1}{2}\right) \Delta \mathcal{M} \\ &\left(\mathfrak{F}_{n+1}, \mathfrak{F}_*, \frac{\wp\wp_2}{2}\right) = \mathcal{M}\left(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_n, \frac{\wp\wp_1}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right) \\ &\leq \min\left\{\mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), \mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{\mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right). \end{aligned} \quad (61)$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} \mathcal{M}(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\leq \min\left\{\mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), 0\right\} \Delta 0, \\ \mathcal{M}(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\leq \mathcal{M}\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp}{v}\right), \wp > 0, \\ D(\Omega\mathfrak{F}_*, \mathfrak{F}_*, \wp) &\leq D\left(\Omega\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp\wp_1}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right) = D\left(\Omega\mathfrak{F}_n, \Omega\mathfrak{F}_n, \frac{\wp\wp_1}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right) \\ &\leq \min\left\{D\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), D\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{D\left(\Omega\mathfrak{F}_n, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right), D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_n, \frac{\wp\wp_2}{2}\right). \end{aligned} \quad (62)$$

Taking $n \rightarrow \infty$, we get

$$D(\Omega \mathfrak{F}_*, \mathfrak{F}_*, \wp) \leq \min \left\{ D\left(\Omega \mathfrak{F}_*, \mathfrak{F}_*, \frac{\wp \wp_1}{2\theta}\right), 0 \right\} \Delta 0,$$

$$D(\Omega \mathfrak{F}_*, \mathfrak{F}_*, \wp) \leq D\left(\Omega \mathfrak{F}_*, \mathfrak{F}_*, \frac{\wp}{\nu}\right), \wp > 0. \tag{63}$$

Here, $\nu = 2\theta/\wp_1 \in (0, 1)$, from Lemma 14, we

have $\Omega \mathfrak{F}_* = \mathfrak{F}_*$. Suppose ν_* and \hbar_* the FPs of Ω . We have

$$\mathfrak{F}_0 \dagger \mathfrak{F}_* \text{ and } \mathfrak{F}_0 \dagger \mathfrak{F}_*. \tag{64}$$

Because Ω is an OPR, so we can write

$$\Omega^n \mathfrak{F}_0 \dagger \Omega^n \mathfrak{F}_* \text{ and } \Omega^n \mathfrak{F}_0 \dagger \Omega^n \mathfrak{F}_* \text{ for all } n \in \mathbb{N}. \tag{65}$$

We can write

$$Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) \geq Y(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \geq \min \{ Y(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), Y(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \wp) \},$$

$$Y(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \wp) \geq Y(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \theta \wp) \geq \min \{ Y(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), Y(\Omega^n \hbar_*, \hbar_*, \wp) \}. \tag{66}$$

Hence, we write that

$$Y(\mathfrak{F}_*, \hbar_*, \wp) = Y(\Omega^n \mathfrak{F}_*, \Omega^n \hbar_*, \wp) \geq \min \left\{ Y\left(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \frac{\wp}{\theta}\right), Y\left(\Omega^n \hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{ 1, 1 \} = 1,$$

$$\mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) \leq \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \leq \min \{ \mathcal{M}(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), \mathcal{M}(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \wp) \},$$

$$\mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \wp) \leq \mathcal{M}(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \theta \wp) \leq \min \{ \mathcal{M}(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), \mathcal{M}(\Omega^n \hbar_*, \hbar_*, \wp) \}.$$

Hence, we write that

$$\mathcal{M}(\mathfrak{F}_*, \hbar_*, \wp) = \mathcal{M}(\Omega^n \mathfrak{F}_*, \Omega^n \hbar_*, \wp) \leq \min \left\{ \mathcal{M}\left(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \frac{\wp}{\theta}\right), \mathcal{M}\left(\Omega^n \hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{ 0, 0 \} = 0,$$

$$D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \wp) \leq D(\Omega^n \mathfrak{F}_0, \Omega^n \mathfrak{F}_*, \theta \wp) \leq \min \{ D(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), D(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \wp) \},$$

$$D(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \wp) \leq D(\Omega^n \mathfrak{F}_0, \Omega^n \hbar_*, \theta \wp) \leq \min \{ D(\Omega^n \mathfrak{F}_0, \mathfrak{F}_0, \wp), D(\Omega^n \hbar_*, \hbar_*, \wp) \}.$$

Hence, we write that

$$D(\mathfrak{F}_*, \hbar_*, \wp) = D(\Omega^n \mathfrak{F}_*, \Omega^n \hbar_*, \wp) \leq \min \left\{ D\left(\Omega^n \mathfrak{F}_*, \mathfrak{F}_*, \frac{\wp}{\theta}\right), D\left(\Omega^n \hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{ 0, 0 \} = 0, \tag{69}$$

for all $\wp > 0$. Hence, $\mathfrak{F}_* = \hbar_*$. □

Corollary 19. Let $(W, Y, \mathcal{M}, *, \Delta, \dagger)$ be an O-complete PFMS and $\Omega : W \rightarrow W$ be an OC and OPR. Then $\theta \in (0, 1)$, we get $\wp > 0$,

$$Y(\Omega \mathfrak{F}, \Omega \hbar, \theta \wp) \geq \min \{ Y(\Omega \mathfrak{F}, \mathfrak{F}, \wp), Y(\Omega \hbar, \hbar, \wp), Y(\mathfrak{F}, \hbar, \wp) \},$$

$$\mathcal{M}(\Omega \mathfrak{F}, \Omega \hbar, \theta \wp) \leq \min \{ \mathcal{M}(\Omega \mathfrak{F}, \mathfrak{F}, \wp), \mathcal{M}(\Omega \hbar, \hbar, \wp), \mathcal{M}(\mathfrak{F}, \hbar, \wp) \},$$

$$D(\Omega \mathfrak{F}, \Omega \hbar, \theta \wp) \leq \min \{ D(\Omega \mathfrak{F}, \mathfrak{F}, \wp), D(\Omega \hbar, \hbar, \wp), D(\mathfrak{F}, \hbar, \wp) \}, \tag{70}$$

Then, Ω has a unique FP.

Proof. It follows from Theorem 16 and Theorem 18. \square

Example 5. Let $W = [-2, 2]$ and define a binary relation \perp by

$$\mathfrak{S} \perp \mathfrak{h} \iff \mathfrak{S} + \mathfrak{h} \geq 0. \quad (71)$$

Define Y and \mathcal{M} by

$$\begin{aligned} Y(\mathfrak{S}, \mathfrak{h}, \wp) &= \begin{cases} 1 & \text{if } \mathfrak{S} = \mathfrak{h} \\ \frac{\wp}{\wp + \max\{\mathfrak{S}, \mathfrak{h}\}} & \text{otherwise,} \end{cases} \\ \mathcal{M}(\mathfrak{S}, \mathfrak{h}, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{S} = \mathfrak{h} \\ \frac{\max\{\mathfrak{S}, \mathfrak{h}\}}{\lambda + \max\{\mathfrak{S}, \mathfrak{h}\}} & \text{otherwise,} \end{cases} \\ D(\mathfrak{S}, \mathfrak{h}, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{S} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{S}, \mathfrak{h}\}}{\lambda} & \text{otherwise,} \end{cases} \end{aligned} \quad (72)$$

for all $\mathfrak{S}, \mathfrak{h} \in W$ and $\wp > 0$, with the CTN and CTCN, respectively; $a * b = a.b$, $a \Delta b = \max\{a, b\}$ then $(W, Y, \mathcal{M}, D, *, \Delta, \perp)$ is an O-complete PFMS. Note that $\lim_{\wp \rightarrow \infty} Y(\mathfrak{S}, \mathfrak{h}, \wp) = 1$, $\lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{S}, \mathfrak{h}, \wp) = 0$, and $\lim_{\wp \rightarrow \infty} D(\mathfrak{S}, \mathfrak{h}, \wp) = 0 \forall \mathfrak{S}, \mathfrak{h} \in E$. Define $\Omega : W \rightarrow W$ by

$$\Omega \mathfrak{S} = \begin{cases} \frac{\mathfrak{S}}{4}, & \mathfrak{S} \in \left[-2, \frac{2}{3}\right] \\ 1 - \mathfrak{S}, & \mathfrak{S} \in \left(\frac{2}{3}, 1\right] \\ \mathfrak{S} - \frac{1}{2}, & \mathfrak{S} \in (1, 2] \end{cases}. \quad (73)$$

We have the following cases:

- (1) If $\mathfrak{S}, \mathfrak{h} \in [-2, 2/3]$, then $\Omega \mathfrak{S} = \mathfrak{S}/4$ and $\Omega \mathfrak{h} = \mathfrak{h}/4$
- (2) If $\mathfrak{S}, \mathfrak{h} \in (2/3, 1]$, then $\Omega \mathfrak{S} = 1 - \mathfrak{S}$ and $\Omega \mathfrak{h} = 1 - \mathfrak{h}$
- (3) If $\mathfrak{S}, \mathfrak{h} \in (1, 2]$, then $\Omega \mathfrak{S} = \mathfrak{S} - 1/2$ and $\Omega \mathfrak{h} = \mathfrak{h} - 1/2$
- (4) If $\mathfrak{S} \in [-2, 2/3]$ and $\mathfrak{h} \in (2/3, 1]$, then $\Omega \mathfrak{S} = \mathfrak{S}/4$ and $\Omega \mathfrak{h} = 1 - \mathfrak{h}$
- (5) If $\mathfrak{S} \in [-2, 2/3]$ and $\mathfrak{h} \in (1, 2]$, then $\Omega \mathfrak{S} = \mathfrak{S}/4$ and $\Omega \mathfrak{h} = \mathfrak{h} - 1/2$
- (6) If $\mathfrak{S} \in (2/3, 1]$ and $\mathfrak{h} \in (1, 2]$, then $\Omega \mathfrak{S} = 1 - \mathfrak{S}$ and $\Omega \mathfrak{h} = \mathfrak{h} - 1/2$
- (7) If $\mathfrak{S} \in (1, 2]$ and $\mathfrak{h} \in (2/3, 1]$, then $\Omega \mathfrak{S} = \mathfrak{S} - 1/2$ and $\Omega \mathfrak{h} = 1 - \mathfrak{h}$
- (8) If $\mathfrak{S} \in (1, 2]$ and $\mathfrak{h} \in [-2, 2/3]$, then $\Omega \mathfrak{S} = \mathfrak{S} - 1/2$ and $\Omega \mathfrak{h} = \mathfrak{h}/4$

- (9) If $\mathfrak{S} \in (2/3, 1]$ and $\mathfrak{h} \in [-2, 2/3]$, then $\Omega \mathfrak{S} = 1 - \mathfrak{S}$ and $\Omega \mathfrak{h} = \mathfrak{h}/4$

Because $\mathfrak{S} \perp \mathfrak{h} \iff \mathfrak{S} + \mathfrak{h} \geq 0$, it is clearly implying that $\Omega \mathfrak{S} + \Omega \mathfrak{h} \geq 0$. Hence, Ω is OPR. Let $\{\mathfrak{S}_n\}$ be an arbitrary OS in W that converges to $\mathfrak{S} \in W$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} Y(\mathfrak{S}_n, \mathfrak{S}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 1 & \text{if } \mathfrak{S} = \mathfrak{h}, \\ \frac{\wp}{\wp + \max\{\mathfrak{S}_n, \mathfrak{S}\}} & \text{otherwise} = 1, \end{cases} \\ \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{S}_n, \mathfrak{S}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 1 & \text{if } \mathfrak{S} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{S}_n, \mathfrak{S}\}}{\wp + \max\{\mathfrak{S}_n, \mathfrak{S}\}} & \text{otherwise} = 0, \end{cases} \\ \lim_{n \rightarrow \infty} D(\mathfrak{S}_n, \mathfrak{S}, \wp) &= \lim_{n \rightarrow \infty} \begin{cases} 1 & \text{if } \mathfrak{S} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{S}_n, \mathfrak{S}\}}{\wp} & \text{otherwise} = 0. \end{cases} \end{aligned} \quad (74)$$

Note that if $\lim_{n \rightarrow \infty} Y(\mathfrak{S}_n, \mathfrak{S}, \lambda) = 1$, $\lim_{n \rightarrow \infty} S(\mathfrak{S}_n, \mathfrak{S}, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\mathfrak{S}_n, \mathfrak{S}, \wp) = 0$, then $\lim_{n \rightarrow \infty} Y(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\Omega \mathfrak{S}_n, \Omega \mathfrak{S}, \wp) = 0$ for all $\mathfrak{S} \in W$ and $\wp > 0$. Hence, Ω is OC. The case $\mathfrak{S} = \mathfrak{h}$ is clear. Let $\mathfrak{S} \neq \mathfrak{h}$. We have

$$\begin{aligned} Y(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &\geq \min\{Y(\Omega \mathfrak{S}, \mathfrak{S}, \wp), Y(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\}, \\ \mathcal{M}(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &\leq \min\{\mathcal{M}(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \mathcal{M}(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\}, \\ D(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &\leq \min\{D(\Omega \mathfrak{S}, \mathfrak{S}, \wp), D(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\}. \end{aligned} \quad (75)$$

Indeed, it is satisfied for all above 9 cases. But, Ω is not a contraction. Assume

$$\begin{aligned} \min\{Y(\Omega \mathfrak{S}, \mathfrak{S}, \wp), Y(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\} &= Y(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \\ \min\{\mathcal{M}(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \mathcal{M}(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\} &= \mathcal{M}(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \\ \min\{D(\Omega \mathfrak{S}, \mathfrak{S}, \wp), D(\Omega \mathfrak{h}, \mathfrak{h}, \wp)\} &= D(\Omega \mathfrak{S}, \mathfrak{S}, \wp), \end{aligned} \quad (76)$$

then for $\mathfrak{S} = -1, \mathfrak{h} = -2$, we have

$$\begin{aligned} Y(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &= \frac{\theta \wp}{\theta \wp + \max\{(\mathfrak{S}/4), (\mathfrak{h}/4)\}} = \frac{4\theta \wp}{4\theta \wp - 1} \geq 1, \\ \mathcal{M}(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &= \frac{\max\{(\mathfrak{S}/4), (\mathfrak{h}/4)\}}{\theta \lambda + \max\{(\mathfrak{S}/4), (\mathfrak{h}/4)\}} = \frac{-1}{4\theta \wp - 1} \leq 0, \\ \mathcal{M}(\Omega \mathfrak{S}, \Omega \mathfrak{h}, \theta \wp) &= \frac{\max\{(\mathfrak{S}/4), (\mathfrak{h}/4)\}}{\theta \lambda} = \frac{-1}{4\theta \wp} \leq 0. \end{aligned} \quad (77)$$

It is a contradiction. Hence, all the conditions of Theorem 18 are satisfied and 0 is the unique FP of ψ .

Definition 20. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an OPFMS. A mapping $\Omega : W \rightarrow W$ is named to be an PF \vdash -contractive if $\exists \theta \in (0, 1)$ so that

$$\frac{1}{Y(\Omega\mathfrak{F}, \Omega\mathfrak{h}, \lambda)} - 1 \leq \theta \left[\frac{1}{Y(\mathfrak{F}, \mathfrak{h}, \lambda)} - 1 \right], \tag{78}$$

$$\mathcal{M}(\Omega\mathfrak{F}, \Omega\mathfrak{h}, \wp) \leq \theta \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \text{ and } D(\Omega\mathfrak{F}, \Omega\mathfrak{h}, \wp) \leq \theta D(\mathfrak{F}, \mathfrak{h}, \wp), \tag{79}$$

for all $\mathfrak{F}, \mathfrak{h} \in W$ and $\wp > 0$. Here, θ is called the PFS \vdash -contractive constant of Ω .

Theorem 21. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS such that

$$\begin{aligned} \lim_{\wp \rightarrow \infty} Y(\mathfrak{F}, \mathfrak{h}, \wp) &= 1, \lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) \\ &= 0, \text{ and } \lim_{\wp \rightarrow \infty} D(\mathfrak{F}, \mathfrak{h}, \wp) = 0, \forall \mathfrak{F}, \mathfrak{h} \in W. \end{aligned} \tag{80}$$

Let $\Omega : W \rightarrow W$ be an OC, \vdash -contraction and OPR. Thus, Ω has a FP, say $v \in W$, $Y(v, v, \wp) = 1$, $\mathcal{M}(v, v, \wp) = 0$ and $D(v, v, \wp) = 0$ for all $\wp > 0$.

Proof. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS. For an arbitrary $\mathfrak{F}_0 \in W$,

$$\mathfrak{F}_0 \vdash \mathfrak{h}, \forall \mathfrak{h} \in W. \tag{81}$$

That is, $\mathfrak{F}_0 \vdash \Omega\mathfrak{F}_0$. Consider

$$\mathfrak{F}_n = \Omega^n \mathfrak{F}_0 = \Omega \mathfrak{F}_{n-1} \text{ for all } n \in \mathbb{N}. \tag{82}$$

Since Ω is OPR, $\{\mathfrak{F}_n\}$ is an OS. If $\mathfrak{F}_n = \mathfrak{F}_{n-1}$ for some $n \in \mathbb{N}$, then \mathfrak{F}_n is a FP of Ω . We assume that $\mathfrak{F}_n \neq \mathfrak{F}_{n-1}$ for all $n \in \mathbb{N}$. For all $\wp > 0$ and $n \in \mathbb{N}$, we get from (12),

$$\begin{aligned} \frac{1}{Y(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp)} - 1 &= \frac{1}{Y(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_n, \wp)} - 1 \leq \theta \left[\frac{1}{Y(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp)} - 1 \right], \\ \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp) &= \mathcal{M}(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_n, \wp) \leq \theta \mathcal{M}(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp), \\ D(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp) &= D(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_n, \wp) \leq \theta D(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp). \end{aligned} \tag{83}$$

We have $\forall \wp > 0$

$$\frac{1}{Y(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp)} \leq \frac{\theta}{Y(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp)} + (1 - \theta), \tag{84}$$

Implying that

$$\frac{\theta}{Y(\Omega\mathfrak{F}_{n-2}, \Omega\mathfrak{F}_{n-1}, \wp)} + (1 - \theta) \leq \frac{\theta^2}{Y(\mathfrak{F}_{n-2}, \mathfrak{F}_{n-1}, \wp)} + \theta(1 - \theta) + (1 - \theta). \tag{85}$$

Continuing in this way, we get

$$\begin{aligned} \frac{1}{Y(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp)} &\leq \frac{\theta^n}{Y(\mathfrak{F}_0, \mathfrak{F}_1, \wp)} + \theta^{n-1}(1 - \theta) + \theta^{n-2}(1 - \theta) \\ &+ \dots + \theta(1 - \theta) + (1 - \theta) \leq \frac{\theta^n}{Y(\mathfrak{F}_0, \mathfrak{F}_1, \wp)} \\ &+ (\theta^{n-1} + \theta^{n-2} + \dots + 1)(1 - \theta) \leq \frac{\theta^n}{Y(\mathfrak{F}_0, \mathfrak{F}_1, \wp)} + (1 - \theta^n). \end{aligned} \tag{86}$$

We have

$$\frac{1}{(\theta^n / Y(\mathfrak{F}_0, \mathfrak{F}_1, \wp)) + (1 - \theta^n)} \leq Y(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp), \forall \wp > 0, n \in \mathbb{N}, \tag{87}$$

$$\begin{aligned} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp) &= \mathcal{M}(\Omega\mathfrak{F}_{n-1}, \Omega\mathfrak{F}_n, \wp) \leq \theta \mathcal{M}(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp) \\ &= \theta \mathcal{M}(\Omega\mathfrak{F}_{n-2}, \Omega\mathfrak{F}_{n-1}, \wp) \leq \theta^2 \mathcal{M}(\mathfrak{F}_{n-2}, \mathfrak{F}_{n-1}, \wp) \\ &\leq \dots \leq \theta^n \mathcal{M}(\mathfrak{F}_0, \mathfrak{F}_1, \wp). \forall \wp > 0, n \in \mathbb{N}, \end{aligned} \tag{88}$$

$$\begin{aligned} D(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \wp) &= D(\psi\mathfrak{F}_{n-1}, \psi\mathfrak{F}_n, \wp) \leq \theta D(\mathfrak{F}_{n-1}, \mathfrak{F}_n, \wp) \\ &= \theta D(\psi\mathfrak{F}_{n-2}, \psi\mathfrak{F}_{n-1}, \wp) \leq \theta^2 D(\mathfrak{F}_{n-2}, \mathfrak{F}_{n-1}, \wp) \\ &\leq \dots \leq \theta^n D(\mathfrak{F}_0, \mathfrak{F}_1, \wp). \forall \wp > 0, n \in \mathbb{N}. \end{aligned} \tag{89}$$

Now, for $m \geq 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} Y(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) &\geq Y\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+m}, \frac{\wp}{2}\right) \\ &\geq Y\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \\ &\quad * Y\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+m}, \frac{\wp}{2}\right). \end{aligned} \tag{90}$$

Again, continuing in this way, we get

$$\begin{aligned} Y(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) &\geq Y\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) * \dots * Y\left(\mathfrak{F}_{n+m-1}, \mathfrak{F}_{n+m}, \frac{\wp}{2}\right), \\ \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) &\leq \mathcal{M}\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \leq \mathcal{M}\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+m}, \frac{\wp}{2}\right). \end{aligned} \tag{91}$$

Continuing in this way, we get

$$\begin{aligned} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &\leq \mathcal{M}\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2^2}\right) \Delta \cdots \Delta \mathcal{M}\left(\mathfrak{F}_{n+p-1}, \mathfrak{F}_{n+p}, \frac{\wp}{2^{p-1}}\right), \\ D(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &\leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2}\right) \leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2^2}\right) \Delta D\left(\mathfrak{F}_{n+2}, \mathfrak{F}_{n+p}, \frac{\wp}{2^2}\right). \end{aligned} \tag{92}$$

Continuing in this way, we get

$$D(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) \leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2^2}\right) \Delta \cdots \Delta D\left(\mathfrak{F}_{n+p-1}, \mathfrak{F}_{n+p}, \frac{\wp}{2^{p-1}}\right). \tag{93}$$

By using (87) in the above inequality, we have

$$\begin{aligned} Y(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) &\geq \frac{1}{(\theta^n/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2))) + (1 - \theta^n)} * \frac{1}{(\theta^{n+1}/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2^2))) + (1 - \theta^{n+1})} * \cdots * \\ &\cdot \frac{1}{(\theta^{n+m-1}/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2^{m-1}))) + (1 - \theta^{n+m-1})} \geq \frac{1}{(\theta^n/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2))) + 1} \\ &* \frac{1}{(\theta^{n+1}/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2^2))) + 1} * \cdots * \frac{1}{(\theta^{n+m-1}/Y(\mathfrak{F}_0, \mathfrak{F}_1, (\wp/2^{m-1}))) + 1}, \end{aligned} \tag{94}$$

using (88),

$$\begin{aligned} S(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &\leq S\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta S\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2^2}\right) \\ &\Delta \cdots \Delta S\left(\mathfrak{F}_{n+p-1}, \mathfrak{F}_{n+p}, \frac{\wp}{2^{p-1}}\right), \end{aligned} \tag{95}$$

and using (89)

$$\begin{aligned} D(\mathfrak{F}_n, \mathfrak{F}_{n+p}, \wp) &\leq D\left(\mathfrak{F}_n, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \frac{\wp}{2^2}\right) \\ &\Delta \cdots \Delta D\left(\mathfrak{F}_{n+p-1}, \mathfrak{F}_{n+p}, \frac{\wp}{2^{p-1}}\right), \end{aligned} \tag{96}$$

$\theta \in (0, 1)$ we deduce from the above expression that $\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_{n+m}, \wp) = 0$ for all $\wp > 0, m \geq 1$.

Therefore, $\{\mathfrak{F}_n\}$ is a O-CS in $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$. By the completeness of $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$, we know that Ω is an OC and there exists $v \in W$ such that

$$\lim_{n \rightarrow \infty} Y(\mathfrak{F}_{n+1}, v, \wp) = \lim_{n \rightarrow \infty} Y(\Omega \mathfrak{F}_n, \Omega v, \wp) = 1, \forall \wp > 0, \tag{97}$$

$$\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_{n+1}, v, \wp) = \lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{F}_n, \Omega v, \wp) = 0 \forall \wp > 0, \tag{98}$$

$$\lim_{n \rightarrow \infty} D(\mathfrak{F}_{n+1}, v, \wp) = \lim_{n \rightarrow \infty} D(\Omega \mathfrak{F}_n, \Omega v, \wp) = 0 \forall \wp > 0. \tag{99}$$

Now, we prove that v is a FP of Ω . For this, we obtain from (78) that

$$\frac{1}{Y(\Omega \mathfrak{F}_n, \Omega v, \wp)} - 1 \leq \theta \left[\frac{1}{Y(\mathfrak{F}_n, v, \wp)} - 1 \right] = \frac{\theta}{Y(\mathfrak{F}_n, v, \wp)} - \theta. \tag{100}$$

That is,

$$\frac{1}{(\theta/Y(\mathfrak{F}_n, v, \wp)) + 1 - \theta} \leq Y(\Omega \mathfrak{F}_n, \Omega v, \wp). \tag{101}$$

Using the above inequality, we obtain

$$\begin{aligned}
 Y(v, \Omega v, \wp) &\geq Y\left(v, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{F}_{n+1}, \Omega v, \frac{\wp}{2}\right) = Y\left(v, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * Y(\Omega \mathfrak{F}_n, \Omega v, \wp) \geq Y\left(v, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) * \frac{1}{(\theta/Y(\mathfrak{F}_n, v, (\wp/2))) + 1 - \theta}, \\
 \mathcal{M}(w, v, \wp) &= \mathcal{M}(\Omega w, \Omega v, \wp) \leq \theta \mathcal{M}(w, v, \wp) < \mathcal{M}(w, v, \wp), = \mathcal{M}\left(w, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}(\Omega \mathfrak{F}_n, \Omega w, \wp) \leq \mathcal{M}\left(w, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \theta \mathcal{M}\left(\mathfrak{F}_n, w, \frac{\wp}{2}\right), \\
 D(w, v, \wp) &= D(\Omega w, \Omega v, \wp) \leq \theta D(w, v, \wp) < D(w, v, \wp), = D\left(w, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta D\left(\Omega \mathfrak{F}_n, \Omega w, \frac{\wp}{2}\right) \leq D\left(w, \mathfrak{F}_{n+1}, \frac{\wp}{2}\right) \Delta \theta D\left(\mathfrak{F}_n, w, \frac{\wp}{2}\right).
 \end{aligned} \tag{102}$$

Taking limit as $n \rightarrow \infty$ and using (97), (98), and (99) in the above expression, we get that $Y(v, \Omega v, \wp) = 1$, $\mathcal{M}(v, \Omega v, \wp) = 0$, and $D(v, \Omega v, \wp) = 0$, that is, $\Omega v = v$. Therefore, v is a FP of Ω , $Y(v, v, \wp) = 1$, $\mathcal{M}(v, v, \wp) = 0$, and $D(v, v, \wp) = 0$ for all $\wp > 0$. \square

Corollary 22. Let $(W, Y, \mathcal{M}, D, *, \Delta, \dagger)$ be a O-complete PFMS and $\Omega : W \rightarrow W$ satisfy

$$\begin{aligned}
 \frac{1}{Y(\Omega^n \mathfrak{F}, \Omega^n \mathfrak{h}, \wp)} - 1 &\leq \theta \left[\frac{1}{Y(\mathfrak{F}, \mathfrak{h}, \wp)} - 1 \right], \\
 \mathcal{M}(\Omega^n \mathfrak{F}, \Omega^n \mathfrak{h}, \wp) &\leq \theta \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp), \\
 D(\Omega^n \mathfrak{F}, \Omega^n \mathfrak{h}, \wp) &\leq \theta D(\mathfrak{F}, \mathfrak{h}, \wp),
 \end{aligned} \tag{103}$$

for all $n \in \mathbb{N}$, $\mathfrak{F}, \mathfrak{h} \in W$, $\wp > 0$, where $0 < \theta < 1$. Then, Ω has a FP.

Proof. $v \in W$ is the unique FP of Ω^n by using Theorem 21, and $Y(v, v, \wp) = 1$, $\mathcal{M}(v, v, \wp) = 0$, $D(v, v, \wp) = 0$, $\forall \wp > 0$. Ωv is also a FP of Ω^n as $\Omega^n(\Omega v) = \Omega v$. From Theorem 21, $\Omega v = v$, v is a FP since the FP of Ω is also a FP of Ω^n . \square

Example 6. Let $W = [-1, 2]$ and define a binary relation \dagger by

$$\mathfrak{F} \dagger \mathfrak{h} \iff \mathfrak{F} + \mathfrak{h} \geq 0. \tag{104}$$

Define Y, \mathcal{M}, D by

$$\begin{aligned}
 Y(\mathfrak{F}, \mathfrak{h}, \wp) &= \begin{cases} 1 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\wp}{\wp + \max\{\mathfrak{F}, \mathfrak{h}\}} & \text{if otherwise,} \end{cases} \\
 \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ 1 - \frac{\wp}{\wp + \max\{\mathfrak{F}, \mathfrak{h}\}} & \text{if otherwise,} \end{cases} \\
 D(\mathfrak{F}, \mathfrak{h}, \wp) &= \begin{cases} 0 & \text{if } \mathfrak{F} = \mathfrak{h}, \\ \frac{\max\{\mathfrak{F}, \mathfrak{h}\}}{\wp} & \text{if otherwise.} \end{cases}
 \end{aligned} \tag{105}$$

With CTN $a * b = a.b$ and CTCN $a \Delta b = \max\{a, b\}$ then $(W, Y, \mathcal{M}, D, *, \Delta, \dagger)$ is an O-complete PFMS. Also observe that $\lim_{\wp \rightarrow \infty} Y(\mathfrak{F}, \mathfrak{h}, \wp) = 1$, $\lim_{\wp \rightarrow \infty} \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, and $\lim_{\wp \rightarrow \infty} D(\mathfrak{F}, \mathfrak{h}, \wp) = 0$, $\forall \mathfrak{F}, \mathfrak{h} \in W$.

Define $\Omega : W \rightarrow W$ by

$$\Omega \mathfrak{F} = \begin{cases} 2 - \mathfrak{F}, & \mathfrak{F} \in [-1, 1) \\ 1, & \mathfrak{F} \in [1, 2] \end{cases}. \tag{106}$$

Therefore, it will satisfy the following cases:

- (1) If $\mathfrak{F}, \mathfrak{h} \in [-1, 1)$, then $\Omega \mathfrak{F} = 2 - \mathfrak{F}$ and $\Omega \mathfrak{h} = 2 - \mathfrak{h}$
- (2) If $\mathfrak{F}, \mathfrak{h} \in [1, 2]$, then $\Omega \mathfrak{F} = \Omega \mathfrak{h} = 1$
- (3) If $\mathfrak{F} \in [-1, 1)$ and $\mathfrak{h} \in [1, 2]$, then $\Omega \mathfrak{F} = 2 - \mathfrak{F}$ and $\mathfrak{h} = 1$
- (4) If $\mathfrak{F} \in [1, 2]$ and $\mathfrak{h} \in [-1, 1)$, then $\Omega \mathfrak{F} = 1$ and $\Omega \mathfrak{h} = 2 - \mathfrak{h}$

Because $\mathfrak{F} \dagger \mathfrak{h} \iff \mathfrak{F} + \mathfrak{h} \geq 0$, it is clearly implying that $\Omega \mathfrak{F} + \Omega \mathfrak{h} \geq 0$. Hence, Ω is OPR. Let $\{\mathfrak{F}_n\}$ be an arbitrary OS in W that $\{\mathfrak{F}_n\}$ converges to $\mathfrak{F} \in W$.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_n, \wp) &= 1, \\
 \lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_n, \wp) &= 0, \\
 \lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_n, \wp) &= 0,
 \end{aligned} \tag{107}$$

as $\{\mathfrak{F}_n\}$ converges to \mathfrak{F} . We can easily see that if $\lim_{n \rightarrow \infty} Y(\mathfrak{F}_n, \mathfrak{F}_n, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\mathfrak{F}_n, \mathfrak{F}_n, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\mathfrak{F}_n, \mathfrak{F}_n, \wp) = 0$, then clearly $\lim_{n \rightarrow \infty} Y(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}_n, \wp) = 1$, $\lim_{n \rightarrow \infty} \mathcal{M}(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}_n, \wp) = 0$, and $\lim_{n \rightarrow \infty} D(\Omega \mathfrak{F}_n, \Omega \mathfrak{F}_n, \wp) = 0$ for all $\mathfrak{F} \in W$ and $\wp > 0$. Hence, Ω is OC. Also above all cases satisfied PFS \dagger -contractive mapping

$$\begin{aligned}
 \frac{1}{Y(\Omega \mathfrak{F}, \Omega \mathfrak{h}, \wp)} - 1 &\leq \theta \left[\frac{1}{Y(\mathfrak{F}, \mathfrak{h}, \wp)} - 1 \right], \\
 \mathcal{M}(\Omega \mathfrak{F}, \Omega \mathfrak{h}, \wp) &\leq \theta \mathcal{M}(\mathfrak{F}, \mathfrak{h}, \wp), \\
 D(\Omega \mathfrak{F}, \Omega \mathfrak{h}, \wp) &\leq \theta D(\mathfrak{F}, \mathfrak{h}, \wp).
 \end{aligned} \tag{108}$$

All conditions of Theorem 21 are satisfied and 1 is a FP of Ω .

3. Conclusions

A picture fuzzy set is more proficient and more capable than an intuitionistic fuzzy set and fuzzy to cope with uncertain

and unpredictable information in realistic issues. Herein, we have introduced the notion of orthogonal picture fuzzy metric space and investigated some new type of fixed point theorems in this new setting. Moreover, we have provided non-trivial examples to demonstrate the viability of the proposed results. Since our structure is more general than the class of picture fuzzy metric spaces, our results and notions expand and generalize several previous results. This work can be easily extended in the structure of orthogonal picture fuzzy cone metric spaces, and orthogonal picture fuzzy bipolar metric spaces.

Data Availability

No data was used during this research

Conflicts of Interest

The authors declare that they have no competing interests.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] B. C. Cuong, *Picture Fuzzy Sets-First Results. Part 1, Seminar Neuro-Fuzzy Systems with Applications*, Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi-Vietnam, 2013.
- [3] J. H. Park, "Intuitionistic fuzzy metric spaces," *Chaos, Solitons and Fractals*, vol. 22, no. 5, pp. 1039–1046, 2004.
- [4] M. Eshaghi, M. Ramezani, M. D. L. Sen, and Y. J. Cho, "On orthogonal sets and Banach fixed point theorem," *Fixed Point Theory*, vol. 18, no. 2, pp. 569–578, 2017.
- [5] M. Hezarjaribi, "Fixed point result in orthogonal fuzzy metric space," *Jordan Journal of Mathematics and Statistics*, vol. 11, no. 4, pp. 295–308, 2018.
- [6] U. Ishtiaq, K. Javed, F. Uddin, M. de la Sen, K. Ahmed, and M. U. Ali, "Fixed point results in orthogonal neutrosophic metric spaces," *Complexity*, vol. 2021, Article ID 2809657, 18 pages, 2021.
- [7] P. H. Phong, D. T. Hieu, R. T. H. Ngan, and P. T. Them, "Some compositions of picture fuzzy relations," in *Proceedings of the 7th National Conference on Fundamental and Applied Information Technology Research, FAIR'7*, pp. 19–20, Thai Nguyen, 2014.
- [8] B. C. Cuong and P. V. Hai, "Some fuzzy logic operators for picture fuzzy sets," in *2015 Seventh International Conference on Knowledge and Systems Engineering (KSE)*, pp. 132–137, Ho Chi Minh City, Vietnam, October 2015.
- [9] P. Singh, "Correlation coefficients for picture fuzzy sets," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 2, pp. 591–604, 2015.
- [10] B. C. Cuong, V. Kreinovich, and R. T. Ngan, "A classification of representable t-norm operators for picture fuzzy sets," in *2016 Eighth International Conference on Knowledge and Systems Engineering (KSE)*, Hanoi, Vietnam, October 2016.
- [11] K. Javed, H. Aydi, F. Uddin, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.
- [12] F. Uddin, C. Park, K. Javed, M. Arshad, and J. R. Lee, "Orthogonal m-metric spaces and an application to solve integral equations," *Advances in Difference Equations*, vol. 2021, no. 1, 15 pages, 2021.
- [13] F. Uddin, K. Javed, H. Aydi, U. Ishtiaq, and M. Arshad, "Control fuzzy metric spaces via orthogonality with an application," *Journal of Mathematics*, vol. 2021, Article ID 5551833, 12 pages, 2021.
- [14] T. Senapati, L. K. Dey, B. Damjanovic, and A. Chanda, "New fixed point results in orthogonal metric spaces with an application," *Kragujevac Journal of Mathematics*, vol. 42, no. 4, pp. 505–516, 2018.
- [15] W. Shatanawi, V. Gupta, and A. Kanwar, "New results on modified intuitionistic generalized fuzzy metric spaces by employing E.A property and common E.A property for coupled maps," *Journal of Intelligent and Fuzzy Systems*, vol. 38, no. 3, pp. 3003–3010, 2020.
- [16] V. Gupta, R. K. Saini, and A. Kanwar, "Some coupled fixed point results on modified intuitionistic fuzzy metric spaces and application to integral type contraction," *Iranian Journal of Fuzzy Systems*, vol. 14, no. 5, pp. 123–137, 2017.
- [17] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.