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Research Article

Kadec-Klee Property in Orlicz Function Spaces Equipped with S-Norms

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Using some new techniques, the necessary and sufficient conditions for Kadec-Klee property of Orlicz function spaces equipped with s-norms are presented. An original method that was used in the process of inquiry and the obtained results also systematically complete and broaden the characterization of Kadec-Klee property of Orlicz spaces.

1. Introduction and Preliminaries

Orlicz spaces, introduced by W. Orlicz in 1932, form a wide class of Banach spaces of measurable functions (in the case of atomless measure) or sequences (in the case of counting measure) (see [1]). On Orlicz spaces of measurable functions, the classical Orlicz and Luxemburg norm can be defined by use of the Amemiya formula: $\|x\|_\Phi^o = \inf_{k>0}(1/k)(1 + I_\Phi(kx))$ and $\|x\|_\Phi = \inf_{k>0}(1/k)\max\{1, I_\Phi(kx)\}$, respectively, where Φ is an Orlicz function and $I_\Phi(x) = \int_G \Phi(x(t))\,dt$. Based on these statements, H. Hudzik and L. Maligrada introduced Orlicz spaces equipped with the p-Amemiya norms, where $p \in [1, \infty]$ and defined $\|x\|_\Phi^p = \inf_{k>0}(1/k)(1 + I_\Phi^p(kx))^{(1/p)}$ in 2000 (see [2, 3]). M. Wisła presented a universal and general method of introducing norms (s-norms) in Orlicz spaces in 2019, and the introduction of Orlicz spaces equipped with s-norms covers all the cases mentioned above.

1.1. Introduction. In the following, by N we will denote the sets of natural numbers and R and R^+ the sets of real and nonnegative real numbers, respectively. By $S(X)$ and $B(X)$, we will denote the unit sphere and the unit ball of the Banach space X, respectively.

Let (G, Σ, μ) be a finite nonatomic measure space, and $L^0 = L^0(\mu)$ be the set of all (μ-equivalence) classes of Σ-measurable real functions defined on G.

Definition 1. A continuous function $\Phi: R \longrightarrow R^+$ is called convex if

$$\Phi\left(\frac{u+v}{2}\right) \le \frac{\Phi(u) + \Phi(v)}{2}, \quad (1)$$

for all $u, v \in R$. If, in addition, the two sides of formula (1) are not equal for all $u \ne v$, then we say Φ is strictly convex.

Definition 2. A function $\Phi: R \longrightarrow R^+$ goes by name of an Orlicz function if Φ is nonnegative, even, convex satisfying $\Phi(0) = 0$ and $\lim_{u \longrightarrow \infty}(\Phi(u)/u) = \infty$.

$\Psi(v) = \sup\{u|v| - \Phi(u): u \ge 0\}$ for all $v \in R$ is called the complementary function of $\Phi(u)$ in the sense of Young. Obviously, Ψ is also an Orlicz function.

Definition 3. I_Φ is introduced as a modular of Φ by

$$I_\Phi(u) = \int_G \Phi(u(t))\,dt, \quad (2)$$

for all $u \in L^0(\mu)$.

Then, the Orlicz space L_Φ and its subspace E_Φ are generated by an Orlicz function and Φ are linear spaces of measurable functions defined by the following formulas:

$$\begin{aligned} L_\Phi &= \{u\colon I_\Phi(\lambda u) < \infty \text{ for some } \lambda > 0\}; \\ E_\Phi &= \{u\colon I_\Phi(\lambda u) < \infty \text{ for any } \lambda > 0\}. \end{aligned} \quad (3)$$

For each $u \in L_\Phi$, let Luxemburg norm be defined by

$$\|x\|_\Phi = \inf\left\{k > 0, I_\Phi\left(\frac{x}{k}\right) \le 1\right\}, \quad (4)$$

and Orlicz norm by

$$\|x\|_\Phi^o = \inf_{k>0}\frac{1}{k}\left(1 + I_\Phi(kx)\right). \quad (5)$$

Definition 4 (see [4]). A function s: $[0,\infty) \longrightarrow [1,\infty)$ will be called an outer function, if it is convex and

$$\operatorname{Max}\{u,1\} \le s(u) \le u + 1 \text{ forall } u \ge 0. \quad (6)$$

Definition 5 (see [4]). Let s be an outer function and Φ be an Orlicz function. Then, the function

$$\|x\|_{\Phi,s} = \inf_{k>0}\frac{1}{k}s\left(I_\Phi(kx)\right), \quad (7)$$

is a norm that will be called s-norm on the Orlicz space L_Φ.

Combing the definitions of various norms, the inequality

$$\|x\|_\Phi \le \|x\|_{\Phi,s} \le \|x\|_\Phi^o, \quad (8)$$

holds. Denote

$$L_{\Phi,s} = \left(L_\Phi, \|\cdot\|_{\Phi,s}\right), E_{\Phi,s} = \left(E_\Phi, \|\cdot\|_{\Phi,s}\right). \quad (9)$$

Definition 6 (see [4]). s^* is an outer function that is conjugate to s in the Hölder sense.

For any $x \in L_{\Phi,s}$ and $y \in L_{\Psi,s^*}$ the following Hölder inequality holds.

$$\left|\int_G x(t)y(t)\,dt\right| \le \|x\|_{\Phi,s}\|y\|_{\Psi,s^*}. \quad (10)$$

Definition 7 (see [4]). For an outer function s and its right-hand derivative s'_+, define

$$\omega(v) = \int_0^v {s'}_+^{-1}(t)\,dt, \quad (11)$$

whenever $v \in [0,1]$.

For all $u \in [0,\infty)$, $v \in [0,\infty)$, define

$$\beta_s(u,v) = 1 - \omega\left(s'_+(u)\right) - v s'_+(u), \quad (12)$$

and for all $x \in L_{\Phi,s}/\{(0)\}$,

$$\begin{aligned} k^*(x) &= \inf\{k > 0\colon \beta_s\left(I_\Phi(kx), I_\Psi\left(p_+(k|x|)\right)\right) \le 0\}, \\ k^{**}(x) &= \sup\{k > 0\colon \beta_s\left(I_\Phi(kx), I_\Psi\left(p_+(k|x|)\right)\right) \ge 0\}. \end{aligned} \quad (13)$$

Lemma 1 (see [4]). *For any $x \in L_{\Phi,s}/\{0\}$, if and only if $k \in (0,\infty) \cap k(x)$, where $k(x) = [k^*(x), k^{**}(x)]$ for short, we have*

$$\|x\|_{\Phi,s} = \frac{1}{k}s\left(I_\Phi(kx)\right). \quad (14)$$

Definition 8. We say that an Orlicz function Φ satisfies condition Δ_2 ($\Phi \in \Delta_2$, for short) if there exists $K > 0$ and $u_0 > 0$, such that

$$\Phi(2u) \le K\Phi(u), \quad |u| \ge u_0. \quad (15)$$

Definition 9. Let X be a Banach space. If $x_n \in S(X)$, $x \in S(X)$, and $x_n \longrightarrow^w x$ imply $x_n \longrightarrow x\,(n \longrightarrow \infty)$, then we say that X has the Kadec-Klee property (see [5–7]).

Lemma 2. *For any $f \in (L_{\Phi,s})^*$ has a unique decomposition,*

$$f = \varphi + v\left(\varphi \in F, v \in \left(E_{\Phi,s}\right)^*\right), \quad (16)$$

where F is the set of all singular functionals on $E_{\Phi,s}$.

Lemma 3 (see [8]). *If Φ is strictly convex, then for any $[a,b] \subset (0,1)$ and $D, \varepsilon > 0$, there exists $\delta > 0$, such that*

$$\Phi[\lambda u + (1-\lambda)v] \le (1-\delta)[\lambda\Phi(u) + (1-\lambda)\Phi(v)], \quad (17)$$

whenever $\lambda \in [a,b]$, $|u| \le D$, $|v| \le D$, $|u - v| \ge \varepsilon$.

Lemma 4 (see [8]). *A subset $A = \{x_n\} \subset L_{\Phi,s}$ is L_{Ψ,s^*} weakly compact (i.e., there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $x \in L_{\Phi,s}$, such that $v(x_{n_i}) \longrightarrow v(x)$ for each $v \in L_{\Psi,s^*}$) if*

$$\lim_{\lambda\longrightarrow 0}\sup\frac{I_\Phi(\lambda x_n)}{\lambda} = 0. \quad (18)$$

Lemma 5 (see [8]). *For any $x_n, x \in L_{\Phi,s}$, assume that $I_\Phi(k_n x_n) \longrightarrow I_\Phi(k_0 x)$ and $k_n x_n(t) \longrightarrow^\mu k_0 x(t)$; if in addition, $\Phi \in \Delta_2$, then*

$$\|x_n - x\|_{\Phi,s} \longrightarrow 0. \quad (19)$$

2. Main Results

Theorem 1. *Orlicz space $L_{\Phi,s}$ has the Kadec-Klee property if and only if*

(1) $\Phi \in \Delta_2$

(2) Φ is strictly convex on R

Proof. Necessity. (i) If $\Phi \notin \Delta_2$, there exists $x \in S(L_{\Phi,s}/E_{\Phi,s})$, such that $\lim_{n\longrightarrow\infty}\|x - x_n\|_{\Phi,s} = \theta(x) > 0$, where $\theta(x) = \inf\{\lambda > 0, I_\Phi(x/\lambda) < \infty\}$ and $I_\Phi(x - x_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

For any $n, m > 0$, put

$$G_n = \{t \in G\colon |x(t)| \le n\}, G_{n,m} = \{t \in G\colon n \le |x(t)| \le m\}. \quad (20)$$

Using $\lim_{m\longrightarrow\infty}\|x\chi_{G_{n,m}}\|_{\Phi,s}$, we have

$$\lim_{m\longrightarrow\infty}\left\|x\chi_{G_{n,m}}\right\|_{\Phi,s}\geq\theta(x). \tag{21}$$

Hence, for each $n\in N$, there exist $m_n>n>0$ satisfying $\|x\chi_{G_{n,m_n}}\|_{\Phi,s}\geq(\theta(x)/2)$. Define $\overline{x_n}=x\chi_{G_{n,m_n}}$. Then,

$$\left\|\overline{x_n}\right\|_{\Phi,s}\geq\frac{\theta(x)}{2}. \tag{22}$$

Let us prove that $\overline{x_n}\longrightarrow^{w}0$. For any $f=\nu+\varphi\in(L_{\Phi,s})^*$, we have $\varphi(\overline{x_n})=0$, thanks to $\overline{x_n}\in E_{\Phi,s}$ and

$$f\left(\overline{x_n}\right)=\nu\left(\overline{x_n}\right)=\int_G\nu(t)\overline{x_n}(t)\,\mathrm{d}t=\int_{G_{n,m_n}}\nu(t)x(t)\,\mathrm{d}t. \tag{23}$$

Using $\int_G\nu(t)x(t)\,\mathrm{d}t<\infty$ and $\lim_{n\longrightarrow\infty}m(G_{n,m_n})=0$, the quality

$$\lim_{n\longrightarrow\infty}\int_{G_{n,m_n}}\nu(t)x(t)=0, \tag{24}$$

holds.

Put $y_n(t)=x(t)-\overline{x_n}(t)$. Then, $y_n\longrightarrow^{w}x$, and

$$1\geq\left\|y_n\right\|_{\Phi,s}\geq\left\|x\chi_{G|G_n}\right\|_{\Phi,s}\longrightarrow\|x\|_{\Phi,s}=1, \tag{25}$$

that is $\|y_n\|_{\Phi,s}=1$. But,

$$\left\|y_n-x\right\|_{\Phi,s}=\left\|\overline{x_n}\right\|_{\Phi,s}\geq\frac{\theta(x)}{2}, \tag{26}$$

this contract with that $L_{\Phi,s}$ has the Kadec-Klee property.

(ii) If Φ is not strictly convex, there exists $x\in S(L_{\Phi,s})$, such that x is not an extreme point, i.e., there exists $k_0\in[k^*(x),k^{**}(x)]$, such that $m(\{t\in G\colon|k_0x(t)|\notin S_\Phi\})>0$. Therefore, there exist $0<a<b$, such that

$$\Phi(u)=Au+B, \tag{27}$$

for some $A,B\in R^+$ when $u\in(a,b)$.

Put

$$\begin{aligned}G_0&=\left\{t\in G\colon\left|k_0x(t)\right|\in(a,b)\right\},\\G_n&=\left\{t\in G_0\colon\left|k_0x(t)\right|\in\left(a+\frac{1}{n},b-\frac{1}{n}\right)\right\}.\end{aligned} \tag{28}$$

Then, $\lim_{n\longrightarrow\infty}m(G_n)=m(G_0)$. Without loss of generality, we may assume that $x(t)\geqslant0$ for each $t\in G_0$.

Therefore, there exists a $n_0\in N$, such that $m(G_{n_0})>0$. Let $c=(1/m(G_{n_0}))\int_{G_{n_0}}x(t)\,\mathrm{d}t$. Then, $(a/k_0)<c<(b/k_0)$. Put $x_0(t)=c\chi_{G_{n_0}}+x(t)\chi_{G|G_{n_0}}$ and $\delta=(k_0/n_0b)$. Then, we have $kx(t)\in(a,b)$ for any $k\in(k_0-\delta,k_0+\delta)$ and $t\in G_{n_0}$. Hence,

$$\begin{aligned}I_\Phi(kx(t))&=\int_{G_{n_0}}\Phi(kx(t))\,\mathrm{d}t+\int_{G|G_{n_0}}\Phi(kx(t))\,\mathrm{d}t\\&=\int_{G_{n_0}}[A(kx(t))+B]\,\mathrm{d}t+\int_{G|G_{n_0}}\Phi(kx(t))\,\mathrm{d}t\\&=Akcm\left(G_{n_0}\right)+Bm\left(G_{n_0}\right)+\int_{G|G_{n_0}}\Phi(kx(t))\,\mathrm{d}t\\&=\Phi(kc)m\left(G_{n_0}\right)+\int_{G|G_{n_0}}\Phi(kx(t))\,\mathrm{d}t\\&=I_\Phi(kx_0(t)),\end{aligned} \tag{29}$$

i.e., $I_\Phi(kx_0(t))=I_\Phi(kx(t))$. Then, we have that $\beta_s(I_\Phi(kx_0),I_\Psi(p_+(k|x_0|)))\leq0$ when $\beta_s(I_\Phi(kx),I_\Psi(p_+(k|x|)))\leq0$ and $\beta_s(I_\Phi(kx_0),I_\Psi(p_+(k|x_0|)))\geq0$ when $\beta_s(I_\Phi(kx),I_\Psi(p_+(k|x|)))\geq0$. By formula (27), we have

$$\|x\|_{\Phi,s}=\frac{1}{k_0}s\left(I_\Phi\left(k_0x(t)\right)\right)=\frac{1}{k_0}s\left(I_\Phi\left(k_0x_0(t)\right)\right)=\left\|x_0\right\|_{\Phi,s}. \tag{30}$$

Divide G_0 into two disjoint sets $G_1^{(1)}$ and $G_1^{(2)}$, such that

$$m\left(G_1^{(1)}\right)=m\left(G_1^{(2)}\right)=\frac{1}{2}m\left(G_{n_0}\right), \tag{31}$$

and divide $G_1^{(1)}$ and $G_1^{(2)}$ into two disjoint sets $G_2^{(1)}$, $G_2^{(2)}$ and $G_2^{(3)}$, $G_2^{(4)}$, respectively, such that

$$m\left(G_2^{(1)}\right)=m\left(G_2^{(2)}\right)=m\left(G_2^{(3)}\right)=m\left(G_2^{(4)}\right)=\frac{1}{2^2}m\left(G_{n_0}\right). \tag{32}$$

By induction as above, we obtain a sequence $\{G_n^{(i)}\}$, such that

$$m\left(G_n^{(2i-1)}\right)=m\left(G_n^{(2i)}\right)=\frac{1}{2^n}m\left(G_0\right), \tag{33}$$

for every $n\in N$ and $i=1,2,\ldots,2^n$. Define

$$x_n(t)=(c+\varepsilon)\chi_{\cup_{i=1}^{2^{n-1}}G_n^{(2i-1)}}+(c-\varepsilon)\chi_{\cup_{i=1}^{2^{n-1}}G_n^{(2i)}}+x(t)\chi_{G|G_{n_0}}. \tag{34}$$

Since, for $k_0\in[k^*(x),k^{**}(x)]$, the following equalities hold:

$$\begin{aligned}I_\Phi\left(k_0x_n(t)\right)&=\int_{G_1^{(1)}}\Phi\left(k_0(c+\varepsilon)\right)\mathrm{d}t+\int_{G_1^{(2)}}\Phi\left(k_0(c-\varepsilon)\right)\mathrm{d}t\\&\quad+\int_{G|G_{n_0}}\Phi\left(k_0x(t)\right)\mathrm{d}t\\&=\int_{G_{n_0}}\Phi\left(k_0c\right)\mathrm{d}t+\int_{G|G_{n_0}}\Phi\left(k_0x(t)\right)\mathrm{d}t=I_\Phi\left(k_0x_0(t)\right),\end{aligned} \tag{35}$$

whence $\|x\|_{\Phi,s} = \|x_0\|_{\Phi,} \quad s = (1/k_0)s(I_\Phi(k_0x_0(t))) = (1/k_0)s(I_\Phi(k_0x_n(t))) = \|x_n\|_{\Phi,s}$.

Let $\overline{x_n}(t) = x_n(t) - x(t)\chi_{G|G_{n_0}}$; we get that $\lim_{\lambda\longrightarrow 0}\sup(I_\Phi(\lambda\overline{x_n})/\lambda) = 0$, without loss of generality, using Lemma 4, we may assume that $\overline{x_n} \longrightarrow^{L_\Psi,s^*} \overline{x}$ for some $\overline{x} \in L_{\Phi,s}$; using $\overline{x_n} \in E_{\Phi,s}$, we have that $\overline{x_n} \longrightarrow^{w} \overline{x}$ holds. Put $\widetilde{x} = \overline{x} + x(t)\chi_{G|G_{n_0}}$. Then, $x_n \longrightarrow^{w} \widetilde{x}$.

Let $y(t) \in S(L_{\Psi,s^*})$ be a support functional of $x_0(t)$. Denote $d = (1/m(G_{n_0}))\int_{G_{n_0}} y(t)\,\mathrm{d}t$ and $y_0(t) = d\chi_{G_{n_0}} + y(t)\chi_{G|G_{n_0}}$. We get the following:

$$\begin{aligned}
y_0(x_0) &= \int_G y_0(t)x_0(t)\,\mathrm{d}t \\
&= \int_{G_{n_0}} y_0(t)x_0(t)\,\mathrm{d}t + \int_{G|G_{n_0}} y(t)x(t)\,\mathrm{d}t \\
&= \int_{G_{n_0}} cdd\ t + \int_{G|G_{n_0}} y(t)x(t)\,\mathrm{d}t \\
&= c\int_{G_{n_0}} y(t)\,\mathrm{d}t + \int_{G|G_{n_0}} y(t)x(t)\,\mathrm{d}t \\
&= y(x_0) = \|x_0\|_{\Phi,s} = 1,
\end{aligned} \tag{36}$$

whence $\|y_0\|_{\Psi,s^*} = \|y_0\|_{\Psi,s^*} \cdot \|x_0\|_{\Phi,s} \geq y_0(x_0) = y(x_0) = \|x\|_{\Phi,s} = 1$.

On the other hand, take $k > 0$, satisfying $\|y\|_{\Psi,s^*} = (1/k)s^*(I_\Psi(ky))$. Then,

$$\begin{aligned}
\|y_0\|_{\Psi,s^*} &\leq \frac{1}{k}s^*(I_\Psi(ky_0(t))) = \frac{1}{k}s^*\left(\int_G \Psi(ky_0(t))\,\mathrm{d}t\right) \\
&= \frac{1}{k}s^*\left(\int_{G_{n_0}} \Psi(k\ d)\,\mathrm{d}t + \int_{G|G_{n_0}} \Psi(ky(t))\,\mathrm{d}t\right) \\
&= \frac{1}{k}s^*\left(\int_{G_{n_0}}\left(\frac{1}{m(G_{n_0})}\int_{G_{n_0}} \Psi(ky(t))\,\mathrm{d}t\right)\mathrm{d}t + \int_{G|G_{n_0}} \Psi(ky(t))\,\mathrm{d}t\right) \\
&= \frac{1}{k}s^*(I_\Psi(ky(t))) = 1.
\end{aligned} \tag{37}$$

Hence, we have that $\|y_0\|_{\Psi,s^*} = 1$ holds.

Since $x_n \longrightarrow^{w} \widetilde{x}$, $\|\widetilde{x}\|_{\Phi,s} \leq \liminf_{n\longrightarrow\infty} \|x_n\|_{\Phi,s} = 1$, and $\|\widetilde{x}\|_{\Phi,s} \geq y_0(\widetilde{x}) = 1$, so $\|\widetilde{x}\|_{\Phi,s} = 1$.

Furthermore,

$$\|x_n - x_m\|_{\Phi,s} = 2\varepsilon\left\|\chi_{G_1^{(1)}}\right\|_{\Phi,s}. \tag{38}$$

It follows that x_n is not a Cauchy sequence, a contradiction.

Sufficiency. Assume that s is an outer function, and Orlicz function Φ is strictly convex satisfying the Δ_2 condition. Take $x_n, x \in S(L_{\Phi,s})$ satisfying $\|x_n + x\|_{\Phi,s} \longrightarrow 2$ and $x_n \longrightarrow^{w} x$.

For any $x \in L_{\Phi,s}$, let $f(k) = (1/k)s(I_\Phi(kx))$ and $G_a = \{t\colon |x(t)| > a\}$ for some $a > 0$ with $m(G_a) > 0$. By the definition of s, we have

$$f(k) = \frac{1}{k}s(I_\Phi(kx)) \geq \frac{I_\Phi(kx)}{k} \geq \int_{G_a} \frac{\Phi(kx)}{k}\,\mathrm{d}t \geq \int_{G_a} \frac{\Phi(ka)}{k}\,\mathrm{d}t \geq \frac{\Phi(ka)}{k}m(G_a), \tag{39}$$

and

$$f(k) = \frac{1}{k}s(I_\Phi(kx)) \geq \frac{1}{k}. \tag{40}$$

Since $\lim_{u\longrightarrow\infty}(\Phi(u)/u) = \infty$, we obtain $\lim_{k\longrightarrow\infty} f(k) = \infty$ and $\lim_{k\longrightarrow 0} f(k) = \infty$, i.e., $k(x) \neq \varnothing$. Then, there exist $k_n \in [k^*(x_n), k^{**}(x_n)]$, such that

$$\|x_n\|_{\Phi,s}=\frac{1}{k_n}s\left(I_\Phi\left(k_nx_n\right)\right)=1. \tag{41}$$

First, we will prove that $x_n \longrightarrow x$ in measure. Suppose that $k_nx_n \overset{\mu}{\nrightarrow} k_0x$ as $n\longrightarrow\infty$, there exist $\varepsilon_0>0,\delta_0>0$, such that $m(\{t\in G\colon |k_nx_n-k_0x|\geq\varepsilon_0\})\geq\delta_0$ for any $n\in N$.

(i) Assume that k_n is bounded, and we have $\sup_{n>0}k_n=\overline{k}<\infty$.
Hence,

$$0<\frac{k_0}{k_0+\overline{k}}\leq\frac{k_0}{k_0+k_n};\frac{k_n}{k_0+k_n}\leq\frac{\overline{k}}{k_0+\overline{k}}<1. \tag{42}$$

Using $k_n=s(I_\Phi(k_nx_n))$, we have

$$\begin{aligned}\overline{k}&\geq k_n=s\left(I_\Phi\left(k_nx_n\right)\right)\\&\geq\int_G\Phi\left(k_nx_n(t)\right)\mathrm{d}t\\&\geq\int_{\{t\in G\colon |k_nx_n(t)|>k\}}\Phi\left(k_nx_n(t)\right)\mathrm{d}t\\&\geq\Phi(k)m\left(\left\{t\in G\colon \left|k_nx_n(t)\right|>k\right\}\right).\end{aligned} \tag{43}$$

Then, there exists $k\in N$, such that $m(\{t\in G\colon |k_nx_n(t)|>k\})\leq(\overline{k}/\Phi(k))<(\delta_0/3)$. Set

$$G_n=\left\{t\in G\colon \begin{array}{c}\left|k_nx_n(t)\right|\leq k\\ \left|k_0x(t)\right|\leq k\\ \left|k_nx_n(t)-k_0x(t)\right|\geq\varepsilon_0\end{array}\right\}. \tag{44}$$

Then, $m(G_n)<(\delta_0/3)$. In virtue of Lemma 3, there exists $\delta_1>0$, such that

$$\Phi\left(\frac{k_0k_n}{k_0+k_n}\left(x_n(t)+x(t)\right)\right)\leq\left(1-\delta_1\right)\left[\frac{k_0}{k_0+k_n}\Phi\left(k_nx_n(t)\right)+\frac{k_n}{k_0+k_n}\Phi\left(k_0x(t)\right)\right]. \tag{45}$$

Whenever $t\in G_n$. Combining with the definition of the functions s and Φ as well as with formulas (42) and (45), we obtain

$$\begin{aligned}2\leftarrow\left\|x_n+x\right\|_{\Phi,s}&\leq\frac{k_0+k_n}{k_0k_n}s\left(I_\Phi\left(\frac{k_0k_n}{k_0+k_n}\left(x_n(t)+x(t)\right)\right)\right)\\&\leq\frac{k_0+k_n}{k_0k_n}s\left(\int_{G_n}\Phi\left(\frac{k_0k_n}{k_0+k_n}\left(x_n(t)+x(t)\right)\right)\mathrm{d}t+\int_{G|G_n}\Phi\left(\frac{k_0+k_n}{k_0k_n}\left(x_n(t)+x(t)\right)\right)\mathrm{d}t\right)\\&\leq\frac{k_0+k_n}{k_0k_n}s\left(\left(1-\delta_1\right)\left(\frac{k_0}{k_0+k_n}\int_{G_n}\Phi\left(k_nx_n(t)\right)\mathrm{d}t+\frac{k_n}{k_0+k_n}\int_{G_n}\Phi\left(k_0x(t)\right)\mathrm{d}t\right)\right.\\&\quad\left.+\frac{k_0}{k_0+k_n}\int_{G|G_n}\Phi\left(k_nx_n(t)\right)\mathrm{d}t+\frac{k_n}{k_0+k_n}\int_{G|G_n}\Phi\left(k_0x(t)\right)\mathrm{d}t\right.\\&\leq\frac{k_0+k_n}{k_0k_n}\left(s\left(\frac{k_0}{k_0+k_n}\int_G\Phi\left(k_nx_n(t)\right)dt+\frac{k_n}{k_0+k_0}\int_G\Phi\left(k_0x(t)\right)dt\right.\right.\\&\quad\left.\left.-s\left(\delta_1\left(\frac{k_0}{k_0+k_n}\int_{G_n}\Phi\left(k_nx_n(t)\right)dt+\frac{k_n}{k_0+k_n}\int_{G_n}\Phi\left(k_0x(t)\right)dt\right)\right)\right]\\&\leq 2-\frac{k_0+k_n}{k_0k_n}s\left(\delta_1\left(\frac{k_0}{k_0+k_n}\int_{G_n}\Phi\left(k_nx_n(t)\right)dt+\frac{k_n}{k_0+k_n}\int_{G_n}\Phi\left(k_0x(t)\right)dt\right)\right)\\&\leq 2-\delta_1\left(\frac{1}{k_n}\int_{G_n}\Phi\left(k_nx_n(t)\right)\mathrm{d}t+\frac{1}{k_0}\int_{G_n}\Phi\left(k_0x(t)\right)\mathrm{d}t\right)\\&\leq 2-\frac{2\delta_1}{\overline{k}}\frac{\int_{G_n}\left(\Phi\left(k_nx_n(t)\right)+\Phi\left(k_0x(t)\right)\right)\mathrm{d}t}{2}\leq 2-\frac{2\delta_1}{\overline{k}}\int_{G_n}\Phi\left(\frac{k_nx_n(t)-k_0x(t)}{2}\right)\mathrm{d}t<2-\frac{2\delta_1}{\overline{k}}\Phi\left(\frac{\varepsilon_0}{2}\right)\frac{\delta_0}{3},\end{aligned} \tag{46}$$

A contradiction. Consequently, $k_n x_n \longrightarrow^{\mu} k_0 x$ as $n \longrightarrow \infty$.

(ii) If the sequence (k_n) is not bounded and without loss of generality, we may assume that $\lim_{n\longrightarrow+\infty} k_n = +\infty$ and set $y_n = (x_n + x/2)$, $l_n = (k_n k_0/k_n + k_0)$; then, $y_n \longrightarrow^{w} x$, $\lim_{n\longrightarrow\infty}\|y_n\|_{\Phi,s} = 1$, and the following inequalities hold.

$$\begin{aligned}2\lim_{n\longrightarrow\infty}\|y_n\|_{\Phi,s} &\leqslant \lim_{n\longrightarrow\infty}\frac{1}{l_n}s\left(I_\Phi\left(l_n y_n\right)\right)\\ &= \lim_{n\longrightarrow\infty}\frac{k_n+k_0}{k_n k_0}s\left(I_\Phi\left(\frac{k_0}{k_n+k_0}k_n x_n + \frac{k_n}{k_n+k_0}k_0 x\right)\right)\\ &\leqslant \lim_{n\longrightarrow\infty}\frac{1}{k_n}\left(1+I_\Phi\left(k_n x_n\right)\right) + \frac{1}{k_0}\left(1+I_\Phi\left(k_0 x\right)\right)\\ &= 2,\end{aligned} \tag{47}$$

It is clear that the sequence (l_n) is bounded, and

$$\lim_{n\longrightarrow\infty} l_n = \lim_{n\longrightarrow\infty}\frac{k_n k_0}{k_n+k_0} = \lim_{n\longrightarrow\infty}\frac{k_0}{1+(k_0/k_n)} = k_0. \tag{48}$$

Similar as the proof above, we have $l_n x_n \longrightarrow^{\mu} k_0 x$.

Second, we will show that $\lim_{n\longrightarrow\infty} k_n = k_0$. We only need to prove that $k_n x_n(t)$ converges $k_0 x(t)$ weakly star.

For any $\varepsilon > 0$ and $v \in E_{\Psi,s^*}$, using that v has absolutely continuous norm, there exists $\delta > 0$, $E \subset G$, and $m(E) < \delta$, such that

$$\|v\chi_E\|_{\Psi,s^*} < \frac{\varepsilon}{3\bar{k}}. \tag{49}$$

By Yegolov's theorem, there exists $e_0 \subset G$ with $m(e_0) < \delta$ for which $k_n x_n(t) \longrightarrow k_0 x(t)$ uniformly $t \in G|e_0$.

Hence, there is $n_0 \in N$ satisfying

$$\left\|(k_n x_n - k_0 x)\chi_{G|e_0}\right\|_{\Phi,s} \leqslant \frac{\varepsilon}{3\|v\|_{\Psi,s^*}}, \tag{50}$$

Whenever $n \geq n_0$ and $t \in G|e_0$.

So,

$$\begin{aligned}&\left|\int_G k_n x_n(t)v(t)\,\mathrm{d}t - \int_G k_0 x(t)v(t)\,\mathrm{d}t\right|\\ &\leq \int_{G|e_0}\left|k_n x_n(t) - k_0 x(t)\|v(t)\right|\mathrm{d}t + \int_{e_0}\left|k_n x_n(t)v(t)\right|\mathrm{d}t + \int_{e_0}\left|k_0 x(t)v(t)\right|\mathrm{d}t\\ &\leq \left\|(k_n x_n - k_0 x)\chi_{G|e_0}\right\|_{\Phi,s}\|v\|_{\Psi,s^*} + \bar{k}\|x_n\|_{\Phi,s}\left\|v\chi_{e_0}\right\|_{\Psi,s^*} + \bar{k}\|x\|_{\Phi,s}\left\|v\chi_{e_0}\right\|_{\Psi,s^*}\\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\ &= \varepsilon,\end{aligned} \tag{51}$$

i.e., $k_n x_n \longrightarrow^{w^*} k_0 x$. By $x_n \longrightarrow^{w} x$, we can get that $k_n \longrightarrow k_0$ holds.

Finally, by formula (41), we have $s^{-1}(k_n) = I_\Phi(k_n x_n)$ and $s^{-1}(k_0) = I_\Phi(k_0 x)$. Hence, we get that $I_\Phi(k_n x_n) \longrightarrow I_\Phi(k_0 x)$ holds. In virtue of Lemma 5, we obtain $\|x_n - x\|_{\Phi,s} \longrightarrow 0$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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