Hindawi
Journal of Function Spaces
Volume 2023, Article ID 9762590, 1 page
https://doi.org/10.1155/2023/9762590

Retraction

Retracted: Characterization of Bipolar Vague Soft *S*-Open Sets

Journal of Function Spaces

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] F. Afzal, A. Mehmood, S. Al Ghour, M. Zafar, H. Sakidin, and S. Gul, "Characterization of Bipolar Vague Soft *S*-Open Sets," *Journal of Function Spaces*, vol. 2022, Article ID 5964872, 13 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5964872, 13 pages
https://doi.org/10.1155/2022/5964872

Research Article

Characterization of Bipolar Vague Soft S-Open Sets

Farkhanda Afzal,[1] **Arif Mehmood**,[2] **Samer Al Ghour**,[3] **Mudasar Zafar**,[4] **Hamzah Sakidin**,[4] **and Saeed Gul**[5]

[1]*Department of Humanities and Basic Sciences, MCS, National University of Sciences and Technology, Islamabad, Pakistan*
[2]*Department of Mathematics, University of Science and Technology Bannu, Pakistan*
[3]*Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan*
[4]*Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, 32610 Seri Iskandar Perak, Malaysia*
[5]*Faculty of Economics, Kardan University, Parwan-e-Du Square, Kabul, Afghanistan*

Correspondence should be addressed to Saeed Gul; s.gul@kardan.edu.af

Received 20 January 2022; Accepted 15 April 2022; Published 5 May 2022

Academic Editor: Hanan Alolaiyan

This paper concerns the study of the concept of bipolar vague soft s-open set, bipolar vague soft s-interior, bipolar vague soft s-closer, and bipolar vague soft s-exterior in bipolar vague soft topological spaces. By using such concepts, some results are addressed in bipolar vague soft topological spaces. The engagements among these results are also addressed by using bipolar vague soft s-open sets.

1. Introduction

Zadeh [1] familiarized the notion of fuzzy set theory. Pawlak [2] initiated rough set theory. Molodtsov [3] inaugurated the soft set theory. Maji et al. [4] made soft set theory more powerful by using it in different practical problems. Maji et al. [5] filled up the gaps that exist in [2]. Since the concept of soft set theory was too young, so work was continued on the journey. Pei and Miao [6] and Chen [7] improved the work of Maji et al. Broumi et al. [8] became the founder of the idea interval valued neutrosophic soft relation which is improvement over relations including soft, fuzzy soft, and intuitionistic fuzzy soft relations. Çağman et al. [9] inaugurated the most valuable structure of mathematics known as soft topology. Shabir and Naz [10] also worked on the same structure. Bayramov and Gunduz [11] have driven soft topology with new points known as soft points. Atanassov [12] founded the notion of intuitionistic fuzzy set theory. Some untouched results were left. Bayramov and Gunduz [13] touched the fundamentals and inaugurated the idea of intuitionistic fuzzy topology. Hayat et al. [14] discussed complex structure known as type 2 soft sets. Hayat et al. [15] traditionalized correspondence between a vertex and its neighbors. Hayat et al. [16] ushered in TOPSIS and the Shannon entropy on the idea of soft set. Shabir and Naz [17] made the concept of bipolar soft sets and its fundamentals. Karaaslan and Karatas [18] developed a new access to bipolar soft set. Ozturk [19] organized bipolar soft topology. Al-shami et al. [20] addressed several journaled type of soft semicompact spaces. Al-shami et al. [20] addressed soft compact and Lindelof spaces via soft preopen set.

In our study, vague soft set, bipolar vague soft set, bipolar vague soft complements, bipolar vague null set, soft set, absolute bipolar vague soft absolute set, bipolar vague soft subset, bipolar vague soft equal sets, bipolar vague soft union, and bipolar vague soft intersection, some fundamental results are based on these operations. Bipolar vague soft topology $\langle\langle \mathrm{BVST}\rangle\rangle$ is defined, and related structures are discussed with respect to s-open. This s-open is chosen among eight new definitions which are introduced in bipolar vague soft topology. Examples are given for supporting some results. The concept of interior and closure is inaugurated.

On the basis of these concepts, related results are addressed. The results that engage the interior with closure are addressed.

2. New Concepts

This section is devoted to the basic notions which are necessary for the upcoming section of this particular piece of work.

Definition 1. Let $\langle\langle M\rangle\rangle$ be master set, and $\mathscr{L}$ be a set of parameters. Let $P\langle\langle\langle M\rangle\rangle\rangle$ signify power of vague set $\langle\langle \mathrm{VS}\rangle\rangle$ of $\langle\langle M\rangle\rangle$; then a vague soft set $\langle\langle \mathfrak{f}, \mathscr{L}\rangle\rangle$ over $\langle\langle M\rangle\rangle$ is a set given by $\mathfrak{f} : \mathscr{L} \longrightarrow P\langle\langle M\rangle\rangle$. In other words, $\langle\langle \mathfrak{f}, \mathscr{L}\rangle\rangle = [(\nu, \langle \mathfrak{x}, \gamma_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \delta_{\mathfrak{f}^{\sim}(\nu)\langle \mathfrak{x}\rangle\rangle : \mathfrak{x}\in^{\sim}\langle\langle \mathrm{M}\rangle\rangle): \nu\in^{\sim}\mathscr{L}]$ where $\delta_{\mathfrak{f}\langle \nu\rangle}\langle \mathfrak{x}\rangle \in^{\sim} [0, 1]$ and $\gamma_{\mathfrak{f}\langle \nu\rangle}\langle \mathfrak{x}\rangle \in^{\sim} [0, 1]$ with $0 \leq^{\sim} \gamma_{\mathfrak{f}\langle \nu\rangle}\langle \mathfrak{x}\rangle + \delta_{\mathfrak{f}\langle \nu\rangle}\langle \mathfrak{x}\rangle \leq^{\sim} 2$.

Definition 2. Let $\langle M\rangle$ be master set, and $\mathscr{L}$ be a set of parameters. A bipolar vague soft set $\langle\langle \mathrm{BVSS}\rangle\rangle$

$$\left\langle\left\langle \tilde{\mathfrak{f}}, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \gamma^{\oplus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \delta^{\oplus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \\ \gamma^{\ominus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \delta^{\ominus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \end{pmatrix} \right\rangle : \mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle\right) : \nu\in^{\sim}\mathscr{L}\right], \quad (1)$$

where $\gamma^{\oplus}_{\mathfrak{f}}, \delta^{\oplus}_{\mathfrak{f}} \longrightarrow [0, 1], \gamma^{\ominus}_{\mathfrak{f}}, \delta^{\ominus}_{\mathfrak{f}} \longrightarrow [-1, 0]$.

Definition 3. Let $\langle\langle \tilde{\mathfrak{f}}, \mathscr{L}\rangle\rangle$ be a $\langle\langle \mathrm{BVS}\rangle\rangle$ set over $\langle M\rangle$, and then, complement of a $\langle\langle \mathrm{BVS}\rangle\rangle$ set $\langle\langle \tilde{\mathfrak{f}}, \mathscr{L}\rangle\rangle$ is signified by $\langle\langle \tilde{\mathfrak{f}}, \mathscr{L}\rangle\rangle^{C}$ and given as

$$\left\langle\left\langle \tilde{\mathfrak{f}}, \mathscr{L}\right\rangle\right\rangle^{C} = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \delta^{\oplus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \gamma^{\oplus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \\ \delta^{\ominus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}(\nu)}\langle \mathfrak{x}\rangle \end{pmatrix} \right\rangle : \mathfrak{x}\in^{\sim}\langle M\rangle\right) : \nu\in^{\sim}\mathscr{L}\right]. \quad (2)$$

Definition 4. An empty $\langle\langle \mathrm{BVS}\rangle\rangle$ set $\langle\langle \tilde{\mathfrak{f}}_{\mathrm{null}}, \mathscr{L}\rangle\rangle$ over $\langle M\rangle$ is defined by

$$\left\langle\left\langle \tilde{\mathfrak{f}}_{\mathrm{null}}, \mathscr{L}\right\rangle\right\rangle = [(e, \langle \mathfrak{x}, (0, 0, -1, 0)\rangle : \mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle) : \nu\in^{\sim}\mathscr{L}]. \quad (3)$$

An absolute $\langle\langle \mathrm{BVS}\rangle\rangle$ set $\langle\langle\langle M\rangle_{\mathrm{aboslute}}, \mathscr{L}\rangle\rangle$ over $\langle M\rangle$ is defined by

$$\left\langle\left\langle \langle \tilde{M}\rangle_{\mathrm{aboslute}}, \mathscr{L}\right\rangle\right\rangle = [(\nu, \langle \mathfrak{x}, (1, 1, 0, -1)\rangle : \mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle) : \nu\in^{\sim}\mathscr{L}]. \quad (4)$$

Definition 5. Let $\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle$ and $\langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$ be two $\langle\langle \mathrm{BVS}\rangle\rangle$ sets over $\langle M\rangle$. $\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle$ is said to be $\langle\langle \mathrm{BVS}\rangle\rangle$ subset of $\langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$ if

$$\begin{gathered} \gamma^{\oplus}{}_{\mathfrak{f}_1{}^{\sim}(\nu)\langle \mathfrak{x}\rangle \preccurlyeq^{\sim} \gamma^{\oplus}\underset{\mathfrak{f}_2(\nu)}{\sim}}\langle \mathfrak{x}\rangle, \\ \delta^{\oplus}{}_{\mathfrak{f}_1{}^{\sim}(\nu)\langle \mathfrak{x}\rangle \succcurlyeq^{\sim} \delta^{\oplus}\underset{\mathfrak{f}_2(\nu)}{\sim}}\langle \mathfrak{x}\rangle, \\ \gamma^{\ominus}{}_{\mathfrak{f}_1{}^{\sim}(\nu)\langle \mathfrak{x}\rangle \preccurlyeq^{\sim} \gamma^{\ominus}\underset{\mathfrak{f}_2(\nu)}{\sim}}\langle \mathfrak{x}\rangle, \\ \delta^{+}{}_{\mathfrak{f}_1{}^{\sim}(\nu)\langle \mathfrak{x}\rangle \succcurlyeq^{\sim} \delta^{+}\underset{\mathfrak{f}_2(\nu)}{\sim}}\langle \mathfrak{x}\rangle \forall \langle \nu, \mathfrak{x}\rangle \in^{\sim} \mathscr{L} \times \langle M\rangle. \end{gathered} \quad (5)$$

It is denoted by $\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle \Subset^{\sim} \langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$.

$\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle$ is said to be $\langle\langle \mathrm{BVS}\rangle\rangle$ equal to $\langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$ if $\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle$ is $\langle\langle \mathrm{BVS}\rangle\rangle$ subset of $\langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$ and $\langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$ is $\langle\langle \mathrm{BVS}\rangle\rangle$subset of $\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle$ and is signified by$\langle\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\rangle\rangle = \langle\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\rangle\rangle$.

Example 1. Let $\langle M\rangle = \{\mathfrak{x}_1, \mathfrak{x}_2\}$ and $\mathscr{L} = \{\nu_1, \nu_2\}$, if

$$\begin{aligned} \left\langle\left\langle \widetilde{\mathfrak{f}_1}, \mathscr{L}\right\rangle\right\rangle &= \begin{pmatrix} \left(\nu_1, \left\langle \mathfrak{x}_1, \left(06\times 10^{-1}, 05\times 10^{-1}, -08\times 10^{-1}, -04\times 10^{-1}\right)\right\rangle, \\ \left\langle \mathfrak{x}_2, \left(05\times 10^{-1}, 04\times 10^{-1}, -06\times 10^{-1}, -03\times 10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left(\left\langle\left(05\times 10^{-1}, 07\times 10^{-1}, -06\times 10^{-1}, -05\times 10^{-1}\right)\right\rangle\right), \\ \left\langle \mathfrak{x}_2, \left(03\times 10^{-1}, 05\times 10^{-1}, -04\times 10^{-1}, -02\times 10^{-1}\right)\right\rangle\right) \end{pmatrix}, \\ \left\langle\left\langle \widetilde{\mathfrak{f}_2}, \mathscr{L}\right\rangle\right\rangle &= \begin{pmatrix} \left(\nu_1, \left\langle \mathfrak{x}_1, \left(07\times 10^{-1}, 08\times 10^{-1}, -05\times 10^{-1}, -06\times 10^{-1}\right)\right\rangle, \\ \left\langle \mathfrak{x}_2, \left(06\times 10^{-1}, 06\times 10^{-1}, -05\times 10^{-1}, -07\times 10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle \mathfrak{x}_1, \left(06\times 10^{-1}, 09\times 10^{-1}, -04\times 10^{-1}, -07\times 10^{-1}\right)\right\rangle, \\ \left\langle \mathfrak{x}_2, \left(04\times 10^{-1}, 07\times 10^{-1}, -03\times 10^{-1}, -06\times 10^{-1}\right)\right\rangle\right) \end{pmatrix}. \end{aligned} \quad (6)$$

Then, $\langle\langle \widetilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle \Subset^{\sim} \langle\langle \widetilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle$.

Definition 6. Let

$$\left\langle\left\langle \widetilde{\mathfrak{f}}_1, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \\ \gamma^{\ominus}_{\widetilde{B_i}(e)}(x), \delta^{\ominus}_{B_i^{\sim}(e)(x)} \end{pmatrix}\right\rangle : \mathfrak{x} \in^{\sim} \langle \mathrm{M}\rangle\right) : \nu \in^{\sim} \mathscr{L}\right], \tag{7}$$

for $i = 1, 2$ be two $\langle\langle \mathrm{BVS}\rangle\rangle$ sets over $\langle M\rangle$. Then, their union is signified by $\langle\langle \widetilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle \Cup^{\sim} \langle\langle \widetilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle$ and it is given as

$$\coprod_{i=1}^{2}\left\langle\left\langle \widetilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \langle \mathfrak{x}, \begin{pmatrix} max\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ max\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix} \rangle : \mathfrak{x} \in^{\sim} \langle \mathrm{M}\rangle\right) : \nu \in^{\sim} \mathscr{L}\right]. \tag{8}$$

Definition 7. Let

$$\left\langle\left\langle \widetilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \\ \gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle} \end{pmatrix}\right\rangle : \mathfrak{x} \in^{\sim} \langle \mathrm{M}\rangle\right) : \nu \in^{\sim} \mathscr{L}\right], \tag{9}$$

for $i = 1, 2$ be two $\langle\langle \mathrm{BVS}\rangle\rangle$ sets over $\langle M\rangle$.

Then, their intersection is signified by $\langle\langle \widetilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle \Cap^{\sim} \langle\langle \widetilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle$ and it is given as

$$\prod_{i=1}^{2}\left\langle\left\langle \widetilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} min\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ min\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix}\right\rangle : \mathfrak{x} \in^{\sim} \langle \mathrm{M}\rangle\right) : e \in^{\sim} \mathscr{L}\right]. \tag{10}$$

Definition 8. Let

$$\left\langle\left\langle \widetilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \\ \gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle} \end{pmatrix}\right\rangle : \mathfrak{x} \in^{\sim} \langle \mathrm{M}\rangle\right) : \nu \in^{\sim} \mathscr{L}\right], \tag{11}$$

for $i \in^{\sim} I$ be a family of $\langle\langle \mathrm{BVS}\rangle\rangle$ sets over $\langle M\rangle$. Then,

$$\coprod_{i\in I}\left\langle\left\langle\widetilde{\mathfrak{f}}_i,\mathscr{L}\right\rangle\right\rangle=\left[\left(\nu,\left\langle\mathfrak{x},\begin{pmatrix}\sup\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\},\inf\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\},\\ \sup\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle\right\},\inf\left\{\delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle}\right\}\end{pmatrix}\right\rangle:\mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle\right):\nu\in^{\sim}\mathscr{L}\right],$$
$$\prod_{i\in I}\left\langle\left\langle\widetilde{\mathfrak{f}}_i,\mathscr{L}\right\rangle\right\rangle=\left[\left(\nu,\left\langle\mathfrak{x},\begin{pmatrix}\inf\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\},\sup\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle\right\},\\ \inf\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle\right\},\sup\left\{\delta^{\ominus}_{\mathfrak{f}_i^{\sim}(\nu)\langle\mathfrak{x}\rangle}\right\}\end{pmatrix}\right\rangle:\mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle\right):\nu\in^{\sim}\mathscr{L}\right].\tag{12}$$

Proposition 9. *Let $\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle$ and $\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$ be the empty $\langle\langle BVS\rangle\rangle$ set and absolute $\langle\langle BVS\rangle\rangle$ set over $\langle M\rangle$, respectfully. Then,*

(1) $\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle\Subset\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$

(2) $\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle=\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$

(3) $\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle=\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle$

Proof. Straightforward. □

Definition 10. Let $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$ and $\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$ be two $\langle\langle$BVS$\rangle\rangle$ sets over $\langle M\rangle$. Then, $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$ difference $\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$ operation on them is given by $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\setminus\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle=\langle\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle$ and is signified by $\langle\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle=\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^{c}$ as follows:

$$\left\langle\left\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\right\rangle\right\rangle=\left[\left(\nu,\left\langle\mathfrak{x},\begin{pmatrix}\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle,\\ \gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\mathfrak{f}_3^{\sim}(\nu)\langle\mathfrak{x}\rangle}\end{pmatrix}\right\rangle:\mathfrak{x}\in^{\sim}\langle \mathrm{M}\rangle\right):\nu\in^{\sim}\mathscr{L}\right],\tag{13}$$

where

$$\begin{aligned}\delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle&=\left[min\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\},\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle\right.\\&\left.=min\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}\right],\\ \delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle&=\left[max\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\},\delta^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu)}\langle\mathfrak{x}\rangle\right.\\&\left.=max\left\{\delta^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}\right].\end{aligned}\tag{14}$$

Definition 11. Let $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$ and $\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$ be two $\langle\langle$BVS$\rangle\rangle$ sets over $\langle M\rangle$. Then, "AND" operation on them is given by $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\wedge^{\sim}\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle=\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\times\mathscr{L}\rangle$ and is signified by

$$\begin{aligned}&\left(\widetilde{B}_3,\mathscr{L}\times\mathscr{L}\right)\\&=\left[\left(\left(\nu_1,\nu_2\right),\left\langle\mathfrak{x},\begin{pmatrix}\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle\\ \gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle\end{pmatrix}\right\rangle:x\in^{\sim}\langle \mathrm{M}\rangle\right):\langle\nu_1,\nu_2\rangle\in^{\sim}\mathscr{L}\times\mathscr{L}\right],\end{aligned}\tag{15}$$

where

$$\begin{aligned}\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle&=\left[min\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle\right.\\&\left.=min\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\right],\\ \gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle&=\left[max\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\},F^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle\right.\\&\left.=max\left\{\delta^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\right].\end{aligned}\tag{16}$$

Definition 12. Let $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$ and $\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$ be two $\langle\langle$BVS$\rangle\rangle$ sets over $\langle M\rangle$. Then, "OR" operation on them is signified by $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\vee^{\sim}\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle=(\widetilde{\mathfrak{f}}_3,\mathscr{L}\times\mathscr{L})$ and is given by

$$\begin{aligned}&\left(\widetilde{\mathfrak{f}}_3,\mathscr{L}\times\mathscr{L}\right)\\&=\left[\left(\left(\nu_1,\nu_2\right),\left\langle\mathfrak{x},\begin{pmatrix}\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle,\\ \gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle\end{pmatrix}\right\rangle:x\in^{\sim}\langle \mathrm{M}\rangle\right):(\nu_1,\nu_2)\in^{\sim}\mathscr{L}\times\mathscr{L}\right],\end{aligned}\tag{17}$$

where

$$\begin{aligned}&\left[\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle=max\left\{\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},\right.\\&\left.\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle=max\left\{\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\right],\\&\left[\delta^{\oplus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle=min\left\{\delta^{\oplus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},\right.\\&\left.\gamma^{\ominus}_{\widetilde{\mathfrak{f}}_3(\nu_1,\nu_2)}\langle\mathfrak{x}\rangle=min\left\{\delta^{\ominus}_{\widetilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\widetilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\right].\end{aligned}\tag{18}$$

Example 2. Let $\langle M\rangle=\{\mathfrak{x}_1,\mathfrak{x}_2\}$ and $\mathscr{L}=\{\nu_1,\nu_2\}$, if

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left(\nu_1,\left\langle\mathfrak{x}_1,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-07\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-08\times10^{-1}\right)\right\rangle\right\rangle),\\ \left(\nu_2,\left\langle\mathfrak{x}_1,\left(04\times10^{-1},04\times10^{-1},-04\times10^{-1},-03\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(05\times10^{-1},08\times10^{-1},-09\times10^{-1},-07\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix},$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left(\nu_1,\left\langle\mathfrak{x}_1,\left(4\times10^{-1},06\times10^{-1},-03\times10^{-1},-09\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(04\times10^{-1},06\times10^{-1},-02\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left(\nu_2,\left\langle\mathfrak{x}_1,\left(03\times10^{-1},03\times10^{-1},-06\times10^{-1},-08\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(04\times10^{-1},05\times10^{-1},-01\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix}. \tag{19}$$

Then,

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle\Cup^{\sim}\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left(\nu_1,\left\langle\mathfrak{x}_1,\left(04\times10^{-1},06\times10^{-1},-03\times10^{-1},-09\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(4\times10^{-1},06\times10^{-1},-02\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left(\nu_2,\left\langle\mathfrak{x}_1,\left(04\times10^{-1},04\times10^{-1},-04\times10^{-1},-08\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2\left(05\times10^{-1},08\times10^{-1},-01\times10^{-1},-07\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix},$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle\Cap^{\sim}\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left(\nu_1,\left\langle\mathfrak{x}_1,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-07\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left(\nu_2,\left\langle\mathfrak{x}_1,\left(03\times10^{-1},03\times10^{-1},-06\times10^{-1},-03\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(05\times10^{-1},05\times10^{-1},-09\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix},$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle\setminus\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left(\nu_1,\left\langle\mathfrak{x}_1,\left(03\times10^{-1},04\times10^{-1},-07\times10^{-1},-05\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(02\times10^{-1},04\times10^{-1},-08\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left(\nu_2,\left\langle\mathfrak{x}_1,\left(04\times10^{-1},04\times10^{-1},-04\times10^{-1},-03\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(03\times10^{-1},05\times10^{-1},-09\times10^{-1},-06\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix},$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle\wedge^{\sim}\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-07\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(03\times10^{-1},03\times10^{-1},-06\times10^{-1},-07\times10^{-1}\right)\right\rangle\right.\\ \left.,\left\langle\mathfrak{x}_2\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1\left(04\times10^{-1},04\times10^{-1},-04\times10^{-1},-03\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(04\times10^{-1},06\times10^{-1},-02\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(03\times10^{-1},03\times10^{-1},-06\times10^{-1},-03\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(04\times10^{-1},05\times10^{-1},-09\times10^{-1},-03\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix},$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle\vee^{\sim}\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle=\begin{pmatrix}\left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(04\times10^{-1},06\times10^{-1},-03\times10^{-1},-09\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(04\times10^{-1},06\times10^{-1},-02\times10^{-1},-08\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(03\times10^{-1},05\times10^{-1},-05\times10^{-1},-08\times10^{-1}\right)\right\rangle\right.\\ \left.,\left\langle\mathfrak{x}_2,\left(04\times10^{-1},05\times10^{-1},-01\times10^{-1},-08\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(04\times10^{-1},06\times10^{-1},-03\times10^{-1},-09\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(05\times10^{-1},08\times10^{-1},-02\times10^{-1},-07\times10^{-1}\right)\right\rangle\right\rangle),\\ \left((\nu_1,\nu_2),\left\langle\mathfrak{x}_1,\left(04\times10^{-1},04\times10^{-1},-04\times10^{-1},-08\times10^{-1}\right)\right\rangle,\right.\\ \left.\left\langle\mathfrak{x}_2,\left(05\times10^{-1},08\times10^{-1},-01\times10^{-1},-07\times10^{-1}\right)\right\rangle\right\rangle)\end{pmatrix}. \tag{20}$$

Proposition 13. *Let $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$, $\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$, and $\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle$ be $\langle\langle BVS\rangle\rangle$ sets over $\langle M\rangle$. Then,*

(1) $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}[\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]=[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle$ and $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}[\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]=[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle$

(2) $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}[\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]=[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]\Cup^{\sim}[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]$, $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}[\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]=[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]\Cup^{\sim}[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle]$

(3) $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle=\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$, $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle=\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle$

(4) $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle=\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$ and $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle=\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$

(5) $\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle\setminus\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle=\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle$ and $\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle\setminus\langle\langle\tilde{\mathfrak{f}}_{null},\mathscr{L}\rangle\rangle=\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$

Proof. Straightforward. □

Proposition 14. *Let $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$ and $\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$ be two $\langle\langle BVS\rangle\rangle$ sets over $\langle M\rangle$. Then,*

(1) $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cup^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]^C=[\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle]^C\Cap^{\sim}[\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle]^C$

(2) $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\Cap^{\sim}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]^C=[\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle]^C\Cup^{\sim}[\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle]^C$

Proof.

(1) For all $\nu\in^{\sim}\mathscr{L}\&\mathfrak{x}\in^{\sim}\langle M\rangle$,

$$\coprod_{i=1}^{2}\left\langle\left\langle \tilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle = \left\{\left(\nu, \langle \mathfrak{x}, \begin{pmatrix} max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix} \rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\},$$

$$\left[\coprod_{i=1}^{2}\left\langle\left\langle \tilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle\right]^c = \left\{\left(\nu, \left\langle \nu, \begin{pmatrix} min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix}\right\rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\}. \tag{21}$$

Now,

$$\left\langle\left\langle \tilde{\mathfrak{f}}_1, \mathscr{L}\right\rangle\right\rangle^c = \left\{\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} \delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \\ \delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle} \end{pmatrix}\right\rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\},$$

$$\left\langle\left\langle \tilde{\mathfrak{f}}_2, \mathscr{L}\right\rangle\right\rangle^c = \left\{\left(\nu, \left\langle : \mathfrak{x}, \begin{pmatrix} \delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle, \\ \delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle} \end{pmatrix}\right\rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\}. \tag{22}$$

Then,

$$\begin{aligned}\prod_{i=1}^{2}\left\langle\left\langle \tilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle^c &= \left\{\left(\nu, \left\langle : \mathfrak{x}, \begin{pmatrix} min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix}\right\rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\} \\ &= \left\{\left(\nu, \left\langle : \mathfrak{x}, \begin{pmatrix} min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, \\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\mathfrak{f}_2\tilde{}(\nu)\langle\mathfrak{x}\rangle}\right\} \end{pmatrix}\right\rangle : \mathfrak{x}\tilde{\in}\langle M\rangle\right) : \nu\tilde{\in}\mathscr{L}\right\}.\end{aligned} \tag{23}$$

Thus, $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \tilde{\Cup} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle]^c = \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^c \tilde{\Cap} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle^c$.

(2) Obvious

□

Proposition 15. *Let $\langle\langle\tilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle$ and $\langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle$ be two $\langle\langle BVS\rangle\rangle$ sets over $\langle M\rangle$. Then,*

(1) $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \tilde{\vee} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle]^c = \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C \tilde{\wedge} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle^c$

(2) $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \tilde{\wedge} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle]^c = \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C \tilde{\vee} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle^c$

Proof.

(1) For all $(\nu_1,\nu_2)\tilde{\in}\mathscr{L}\times\mathscr{L}\&\mathfrak{x}\tilde{\in}\langle M\rangle$,

$$\begin{aligned}\bigvee_{i=1}^{2}\left\langle\left\langle \tilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle &= \left[\left((\nu_1,\nu_2), \left\langle \mathfrak{x}, \begin{matrix} max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, \\ max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\} \end{matrix}\right\rangle\right)\right], \\ \left[\bigvee_{i=1}^{2}\left\langle\left\langle \tilde{\mathfrak{f}}_i, \mathscr{L}\right\rangle\right\rangle\right]^c &= \left[(\nu_1,\nu_2), \left\langle x, \begin{matrix} min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, \\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}, max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle, \gamma^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\} \end{matrix}\right\rangle\right].\end{aligned} \tag{24}$$

Now,

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle^c=\left[\nu_1,\left\langle \mathfrak{x},\begin{matrix}\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle,\\ \delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\mathfrak{f}_2^{\sim}(\nu_2)\langle\mathfrak{x}\rangle}\end{matrix}\right\rangle : \nu\tilde{\in}\mathscr{L}\right],$$

$$\left\langle\left\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle^c=\left[\nu_2,\left\langle \mathfrak{x},\begin{matrix}\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle,\\ \delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\mathfrak{f}_2^{\sim}(\nu_2)\langle\mathfrak{x}\rangle}\end{matrix}\right\rangle : \nu\tilde{\in}\mathscr{L}\right]. \tag{25}$$

Then,

$$\begin{aligned}\bigwedge_{i=1}^{2}\left\langle\left\langle\tilde{\mathfrak{f}}_i,\mathscr{L}\right\rangle\right\rangle^c&=\left[(\nu_1,\nu_2),\left\langle\mathfrak{x},\begin{matrix}min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},\\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\end{matrix}\right\rangle\right]\\ &=\left[(\nu_1,\nu_2),\left\langle\mathfrak{x},\begin{matrix}min\left\{\delta^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},max\left\{\gamma^{\oplus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\oplus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},\\ min\left\{\delta^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\delta^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\},max\left\{\gamma^{\ominus}_{\tilde{\mathfrak{f}}_1(\nu_1)}\langle\mathfrak{x}\rangle,\gamma^{\ominus}_{\tilde{\mathfrak{f}}_2(\nu_2)}\langle\mathfrak{x}\rangle\right\}\end{matrix}\right\rangle\right].\end{aligned} \tag{26}$$

Thus, $[\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle\tilde{\vee}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]^c=\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C\tilde{\wedge}\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^C$.

(2) Following the steps of (1) of this theorem

□

3. Bipolar Vague Soft Structure

This is the most important section. This section has been devoted to few new sets included bipolar vague soft toplogy ⟨⟨BVST⟩⟩ and eight new definitions. The concept of interior and closure is inaugurated. On the basis of these concepts, related results are addressed. The results that engage the interior with closure are also given a touch.

Definition 16. Let BVSS $\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$ be family of all ⟨⟨BVS⟩⟩ sets over $\langle\langle M\rangle\rangle$ and $\mathscr{T}^{\mathrm{BVSS}}\Subset$ BNS set $\langle\langle\langle\tilde{M}\rangle_{\mathrm{absolute}},\mathscr{L}\rangle\rangle$, then $\mathscr{T}^{\mathrm{BVSS}}$ is said to be a bipolar vague soft topology ⟨⟨BVST⟩⟩ on $\langle M\rangle$. If

(1) $\langle\langle\tilde{\mathfrak{f}}_{\mathrm{null}},\mathscr{L}\rangle\rangle$ and $\langle\langle\langle\tilde{M}\rangle_{\mathrm{absolute}},\mathscr{L}\rangle\rangle\tilde{\in}\mathscr{T}^{\mathrm{BVSS}}$`

(2) Union of any number of ⟨⟨BVS⟩⟩ sets in $\mathscr{T}^{\mathrm{BVSS}}\tilde{\in}\mathscr{T}^{\mathrm{BVSS}}$

(3) Intersection of finite number of ⟨⟨BVS⟩⟩ soft sets in $\mathscr{T}^{\mathrm{BVSS}}\tilde{\in}\mathscr{T}^{\mathrm{BVSS}}$

Then, $\langle\langle\langle\tilde{M}\rangle,\mathscr{T}^{\mathrm{BVSS}},\mathscr{L}\rangle\rangle$ is said to be a ⟨⟨BVSTS⟩⟩ over $\langle\tilde{M}\rangle$.

Definition 17. Let $\langle\langle\langle\tilde{M}\rangle,\mathscr{T}^{\mathrm{BVSS}},\mathscr{L}\rangle\rangle$ be a ⟨⟨BVSTS⟩⟩ over $\langle\tilde{M}\rangle$, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ be a ⟨⟨BVS⟩⟩ set over $\langle\tilde{M}\rangle$. Then, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ is said to be ⟨⟨BVS⟩⟩ closed set iff its complement is a ⟨⟨BVS⟩⟩ open set.

Definition 18. Let $\langle\langle\langle\tilde{M}\rangle,\mathscr{T}^{\mathrm{BVSS}},\mathscr{L}\rangle\rangle$ be a ⟨⟨BVSTS⟩⟩ and $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ be a ⟨⟨BVS⟩⟩ set over $\langle\tilde{M}\rangle$, then $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ is called BVS.

(1) Semiopen if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) and BVS semiclose if $(\tilde{\mathfrak{f}},\theta)\tilde{\Supset}$BVS int (BVScl$(\tilde{\mathfrak{f}},\theta)$)

(2) Preopen if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) and BVS preclose if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Supset}$BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))

(3) α-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVSint(BVScl(BVSint ($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))) and BVS$\alpha$-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Supset}$BVScl(BVSint (BVScl ($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)))

(4) β-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVScl(BVSint(BVScl ($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))) and BVS$\beta$-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Supset}$BVSint(BVScl (BVSint $(\tilde{\mathfrak{f}},\theta)$))

(5) b-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ $\tilde{\Subset}$ BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) ∪ BVSint(NBVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) and BVS$b$-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Supset}$BVSint(BVScl$(\tilde{\mathfrak{f}},\theta)$)$\tilde{\Cap}$BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))

(6) $*_b$-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVScl (BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) $\tilde{\Cap}$BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)) and BVS$*_b$-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Supset}$

BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))
$\tilde{\Cup}$BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))

(7) b^{**}-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVSint(BVScl (BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)))$\tilde{\Cup}$BVScl(BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))) and BVS b^{**}-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ $\tilde{\Supset}$BVScl(BVSint(BVScl ($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))) $\tilde{\Cap}$BVSint(BVScl(BVSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)))

(8) $**_b$-open if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle\tilde{\Subset}$BVSint (BVScl(BVSint$(\tilde{\mathfrak{f}},\theta)$)) $\tilde{\Cap}$NScl(BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))) and BVS soft

$**_b$-close if $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \ni^{\sim}$ BVScl(BVSint(BVScl($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$)))$\Cup^{\sim}$BVSint(BVScl(NSint($\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$))))

Proposition 19. *Let $\langle\langle\langle\tilde{M}\rangle, \mathscr{T}^{BVSS}, \mathscr{L}\rangle\rangle$ be a $\langle\langle BVSTS\rangle\rangle$ over $\langle\tilde{M}\rangle$. Then,*

(1) *$\langle\langle\tilde{\mathfrak{f}}_{null}, \mathscr{L}\rangle\rangle$, $\langle\langle\langle\tilde{M}\rangle_{absolute}, \mathscr{L}\rangle\rangle$ are $\langle\langle BVS\rangle\rangle$s-closed sets over $\langle\tilde{M}\rangle$*

(2) *$\tilde{\Cap}$ of any number of $\langle\langle BVS\rangle\rangle$s-closed sets $\langle\langle CS\rangle\rangle$ is a $\langle\langle BVS\rangle\rangle$s-closed sets $\langle\langle CS\rangle\rangle$ over $\langle\tilde{M}\rangle$*

(3) *$\tilde{\Cup}$ of finite number of $\langle\langle BVS\rangle\rangle$s-closed sets $\langle\langle CS\rangle\rangle$ is a $\langle\langle BVS\rangle\rangle$s-closed sets $\langle\langle CS\rangle\rangle$ over $\langle\tilde{M}\rangle$*

Proof. Obvious. □

Definition 20. Let BNSS$\langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle$ be the family of all $\langle\langle$BVS$\rangle\rangle$ sets over $\langle\tilde{M}\rangle$.

(1) If $\mathscr{T}^{\text{BVSS}} = \{\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle\}$, then $\mathscr{T}^{\text{BVSS}}$ is said to be $\langle\langle$BVS$\rangle\rangle$ indiscrete topology, $(\langle\tilde{M}\rangle, \mathscr{T}^{\text{BVSS}}, E)$ is said to be a $\langle\langle$BVS$\rangle\rangle$ indiscrete topological space over $\langle\tilde{M}\rangle$

(2) If $\mathscr{T}^{\text{BVSS}} = \text{BVSS}\langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle$, then $\mathscr{T}^{\text{BVSS}}$ is said to be $\langle\langle$BVS$\rangle\rangle$ discrete topology, $\langle\langle\langle\tilde{M}\rangle, \mathscr{T}^{\text{BVSS}}, \mathscr{L}\rangle\rangle$ is said to be a $\langle\langle$BVS$\rangle\rangle$ discrete topological space over $\langle\tilde{M}\rangle$

Proposition 21. *Let $\langle\langle\langle\tilde{M}\rangle, \mathscr{T}^{BVSS}{}_1, \mathscr{L}\rangle\rangle$ and $\langle\langle X, \mathscr{T}^{BVSS}{}_2, \mathscr{L}\rangle\rangle$ be two $\langle\langle BVSTS\rangle\rangle$ over $\langle\tilde{M}\rangle$. Then, $\langle\langle\langle\tilde{M}\rangle, \mathscr{T}^{BVSS}{}_1 \Cap^{\sim} \mathscr{T}^{BVSS}{}_2, \mathscr{L}\rangle\rangle$ is $\langle\langle BVSTS\rangle\rangle$ over $\langle\tilde{M}\rangle$.*

Proof.

(1) Since $\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1$ and $\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$, then $\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1 \Cap^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$

(2) Let $\{\langle\langle\tilde{\mathfrak{f}}, \mathscr{L}\rangle\rangle : i \in I\}$ be a family of $\langle\langle$BVS$\rangle\rangle$ sets in $\mathscr{T}^{\text{BVSS}}{}_1 \tilde{\Cap} \mathscr{T}^{\text{BVSS}}{}_2$. Then, $\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1$, $\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_2 \forall \text{i} \in \text{I}$, so $\coprod_{i\in I}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1 \& \coprod_{i\in I}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in \tau_2{}^{\text{BN}}$. Thus, $\coprod_{i\in I}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1 \Cap^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$

(3) Let $\{\langle\langle\tilde{\mathfrak{f}}, \mathscr{L}\rangle\rangle : i = \overline{1, n}\}$ be a family of finite number of $\langle\langle$BVS$\rangle\rangle$ sets in $\mathscr{T}^{\text{BVSS}}{}_1 \tilde{\Cap} \mathscr{T}^{\text{BVSS}}{}_2$. Then, $\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1$, $\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$ for $i = \overline{1, n}$, so $\prod_{i=1}^{n}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in \mathscr{T}^{\text{BVSS}}{}_1$ and $\prod_{i=1}^{n}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$. Thus, $\prod_{i=1}^{n}\langle\langle\tilde{\mathfrak{f}}_i, \mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}}{}_1 \Cap^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$

□

Remark 22. The union of two $\langle\langle$BVSTS$\rangle\rangle$ over $\langle M\rangle$ may not be a $\langle\langle$BVSTS$\rangle\rangle$ on $\langle M\rangle$.

Example 3. Let $\langle M\rangle = \{\mathfrak{x}_1, \mathfrak{x}_2\}$, $\mathscr{L} = \{\nu_1, \nu_2\}$ be a set of parameters, $\mathscr{T}^{\text{BVSS}}{}_1 = \{\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_3, \mathscr{L}\rangle\rangle\}$, $\mathscr{T}^{\text{BVSS}}{}_2 = \{\langle\langle\tilde{\mathfrak{f}}_{\text{null}}, \mathscr{L}\rangle\rangle, \langle\langle\langle\tilde{M}\rangle_{\text{absolute}}, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_4, \mathscr{L}\rangle\rangle\}$ be two $\langle\langle$BVSTS$\rangle\rangle$ over $\langle M\rangle$. Here, $\langle\langle$BVS$\rangle\rangle$ sets $\langle\langle\tilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_3, \mathscr{L}\rangle\rangle$, and $\langle\langle\tilde{\mathfrak{f}}_4, \mathscr{L}\rangle\rangle$ over $\langle M\rangle$ are as succeeding:

$$\left\langle\left\langle\tilde{\mathfrak{f}}_1, \mathscr{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(09\times10^{-1}, 04\times10^{-1}, -03\times10^{-1}, -07\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(05\times10^{-1}, 06\times10^{-1}, -02\times10^{-1}, -08\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(07\times10^{-1}, 03\times10^{-1}, -05\times10^{-1}, -04\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(06\times10^{-1}, 06\times10^{-1}, -07\times10^{-1}, -05\times10^{-1}\right)\right\rangle\right) \end{pmatrix},$$
$$\left\langle\left\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(07\times10^{-1}, 04\times10^{-1}, -04\times10^{-1}, -06\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 05\times10^{-1}, -03\times10^{-1}, -07\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(06\times10^{-1}, 02\times10^{-1}, -06\times10^{-1}, -03\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(05\times10^{-1}, 04\times10^{-1}, -08\times10^{-1}, -04\times10^{-1}\right)\right\rangle\right) \end{pmatrix},$$
$$\left\langle\left\langle\tilde{\mathfrak{f}}_3, \mathscr{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(05\times10^{-1}, 03\times10^{-1}, -05\times10^{-1}, -05\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(03\times10^{-1}, 04\times10^{-1}, -04\times10^{-1}, -06\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left\langle\left(04\times10^{-1}, 01\times10^{-1}, -07\times10^{-1}, -02\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 03\times10^{-1}, -09\times10^{-1}, -03\times10^{-1}\right)\right\rangle\right) \end{pmatrix},$$
$$\left\langle\left\langle\tilde{\mathfrak{f}}_4, \mathscr{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(08\times10^{-1}, 05\times10^{-1}, -02\times10^{-1}, -08\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(05\times10^{-1}, 06\times10^{-1}, -01\times10^{-1}, -09\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(07\times10^{-1}, 03\times10^{-1}, -04\times10^{-1}, -05\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(06\times10^{-1}, 05\times10^{-1}, -06\times10^{-1}, -06\times10^{-1}\right)\right\rangle\right) \end{pmatrix}. \tag{27}$$

Since $\langle\langle\tilde{\mathfrak{f}}_1, \mathscr{L}\rangle\rangle \Cup^{\sim} \langle\langle\tilde{\mathfrak{f}}_2, \mathscr{L}\rangle\rangle = \mathscr{T}^{\text{BVSS}}{}_1 \Cup^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$, then $\mathscr{T}^{\text{BVSS}}{}_1 \Cup^{\sim} \mathscr{T}^{\text{BVSS}}{}_2$ is not a $\langle\langle$BVVS$\rangle\rangle$ over $\langle M\rangle$.

Proposition 23. *Let* $\langle\langle\langle M\rangle, \mathcal{T}^{BVSS}, \mathcal{L}\rangle\rangle$ *be a* $\langle\langle BVSTS\rangle\rangle$ *over* $\langle M\rangle$, $\mathcal{T}^{BVSS} = \{\langle\langle\tilde{\mathfrak{f}}_i, \mathcal{L}\rangle\rangle: \langle\langle\tilde{\mathfrak{f}}_i, \mathcal{L}\rangle\rangle \tilde{\in} BVSS(\langle M\rangle, \mathcal{L})\}$ *where*

$$\left\langle\left\langle\tilde{\mathfrak{f}}_i, \mathcal{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle \mathfrak{x}, \begin{pmatrix} T^{\oplus}_{\tilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, F^{\oplus}_{\tilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, \\ T^{\ominus}_{\tilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, F^{\ominus}_{\tilde{\mathfrak{f}}_i(\nu)\langle\mathfrak{x}\rangle} \end{pmatrix}\right\rangle: \mathfrak{x} \tilde{\in} \langle \mathrm{M}\rangle\right): \nu \tilde{\in} \mathcal{L}\right], \tag{28}$$

for $i \tilde{\in} I$.

Then,

$$\mathcal{T}^{\mathrm{BVSS}} = \left[\left\langle\left\langle\widetilde{\mathfrak{f}^{\oplus}}_i, \mathcal{L}\right\rangle\right\rangle = \left[\left(\nu, \left\langle\mathfrak{x}, \left(T^{\oplus}_{\tilde{\mathfrak{f}}_i(\nu)}\langle\mathfrak{x}\rangle, F^{\oplus}_{\tilde{\mathfrak{f}}_i(\nu)\langle\mathfrak{x}\rangle}\right)\right\rangle: \mathfrak{x}\tilde{\in}\langle \mathrm{M}\rangle\right): \nu \in \mathcal{L}\right]: \left\langle\left\langle\widetilde{\mathfrak{f}^{\oplus}}_i, \mathcal{L}\right\rangle\right\rangle \tilde{\in} VSS\left\langle\left\langle\langle\tilde{\mathrm{M}}\rangle, \mathcal{L}\right\rangle\right\rangle\right] \tag{29}$$

define $\langle\langle BVST\rangle\rangle$ *on* $\langle M\rangle$.

Proof. Obvious. □

Definition 24. Let $\langle\langle\langle M\rangle, \mathcal{T}^{\mathrm{BVSS}}, \mathcal{L}\rangle\rangle$ be a $\langle\langle\mathrm{BVSTS}\rangle\rangle$ over $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle \in \mathrm{BNSS}\langle\langle\langle\tilde{M}\rangle, \mathcal{L}\rangle\rangle$ be a $\langle\langle\mathrm{BVS}\rangle\rangle$ set. Then, $\langle\langle$ BVS$\rangle\rangle$ interior of $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$, denoted $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$, is defined as $\langle\langle\mathrm{BVS}\rangle\rangle$ union of all $\langle\langle\mathrm{BVS}\rangle\rangle s$-open subsets of $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$.

Clearly, $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$ is the biggest $\langle\langle\mathrm{BVS}\rangle\rangle s$-open set contained by $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$.

Example 4. Let us consider $\langle\langle\mathrm{BVSTS}\rangle\rangle$, i.e., $\mathcal{T}^{\mathrm{BVSS}}{}_1$ given in Example 3. Let $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle \tilde{\in} \mathrm{BVSS}\langle\langle\langle\tilde{M}\rangle, \mathcal{L}\rangle\rangle$ be defined as succeeding:

$$\left\langle\left\langle\tilde{\mathfrak{f}}, \mathcal{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(08\times10^{-1}, 04\times10^{-1}, -02\times10^{-1}, -06\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 07\times10^{-1}, -01\times10^{-1}, -09\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(09\times10^{-1}, 02\times10^{-1}, -06\times10^{-1}, -05\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(07\times10^{-1}, 05\times10^{-1}, -06\times10^{-1}, -06\times10^{-1}\right)\right\rangle\right) \end{pmatrix}. \tag{30}$$

Then, $\langle\langle\tilde{\mathfrak{f}}_{\mathrm{null}}, \mathcal{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_2, \mathcal{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_3, \mathcal{L}\rangle\rangle \Subset \langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$; therefore, $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ} = \langle\langle\tilde{\mathfrak{f}}_{\mathrm{null}}, \mathcal{L}\rangle\rangle \tilde{\Cup} \langle\langle\tilde{\mathfrak{f}}_2, \mathcal{L}\rangle\rangle \tilde{\Cup} \langle\langle\tilde{\mathfrak{f}}_3, \mathcal{L}\rangle\rangle = \langle\langle\tilde{\mathfrak{f}}_2, \mathcal{L}\rangle\rangle$.

Theorem 25. *Let* $\langle\langle\langle M\rangle, \mathcal{T}^{BVSS}, \mathcal{L}\rangle\rangle$ *be a* $\langle\langle BVSTS\rangle\rangle$ *over* $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle \in BVSS\langle\langle\langle\tilde{M}\rangle, \mathcal{L}\rangle\rangle$. $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$ *is a* $\langle\langle BVS\rangle\rangle s$ *-open set iff* $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle = \langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$.

Proof. Let $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$ be a $\langle\langle\mathrm{BVS}\rangle\rangle s$-open set. Then, the biggest $\langle\langle\mathrm{BVS}\rangle\rangle s$-open set that is contained by $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$ is equal to $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$. Hence, $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle = \langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$.

Contrariwise, it is known that $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$ is a $\langle\langle\mathrm{BVS}\rangle\rangle s$ -open set; if $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle = \langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle^{\circ}$, then $\langle\langle\tilde{\mathfrak{f}}, \mathcal{L}\rangle\rangle$ is a $\langle\langle\mathrm{BVS}\rangle\rangle$ set. □

Theorem 26. *Let* $\langle\langle\langle M\rangle, \mathcal{T}^{BVSS}, \mathcal{L}\rangle\rangle$ *be a* $\langle\langle BVSTS\rangle\rangle$ *over* $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}}_1, \mathcal{L}\rangle\rangle, \langle\langle\tilde{\mathfrak{f}}_2, \mathcal{L}\rangle\rangle \tilde{\in} BVSS\langle\langle\langle\tilde{M}\rangle, \mathcal{L}\rangle\rangle$. *Then,*

(1) $[\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ}]^{\circ} = \langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ}$

(2) $\langle\langle(\tilde{\mathfrak{f}}_{null}, \mathcal{L})\rangle\rangle^{\circ} = \langle\langle(\tilde{\mathfrak{f}}_{null}, \mathcal{L})\rangle\rangle \& \langle\langle\langle\tilde{M}\rangle_{absolute}, \mathcal{L}\rangle\rangle^{\circ} = \langle\langle\langle\tilde{M}\rangle_{absolute}, \mathcal{L}\rangle\rangle$

(3) $\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle \Subset \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle \Rightarrow \langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ} \Subset \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle^{\circ}$

(4) $[\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle]^{\circ} = \langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ} \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle^{\circ}$

(5) $\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ} \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle^{\circ} \Subset [\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle]^{\circ}$

Proof.

(1) Let $\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ} = \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle$. Then, $\langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle \in \tau^{\mathrm{BN}}$ iff $\langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle = \langle\langle(\tilde{\mathfrak{f}}_2, \mathcal{L})\rangle\rangle^{\circ}$. So, $[\langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ}]^{\circ} = \langle\langle(\tilde{\mathfrak{f}}_1, \mathcal{L})\rangle\rangle^{\circ}$

(2) Obvious

(3) Let $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, $\langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$. Since $\langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ}$ is the biggest $\langle\langle\text{BVS}\rangle\rangle s$-open set covered in $\langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, so $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ}$

(4) Since $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \& \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle$, then $[\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ}$, $[\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ}$ and so $[\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ}$. On the other hand, since $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle$ and $\langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, then $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$; besides, $[\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ} \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$ and it is the biggest $\langle\langle\text{BVS}\rangle\rangle s$-open set. Therefore, $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^{\circ} \Cap^{\sim} \langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^{\circ} \Subset [\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \Cap^{\sim} \langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]^{\circ}$. Thus, $[\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \Cap^{\sim} \langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle]^{\circ} = \langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^{\circ} \Cap^{\sim} \langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^{\circ}$

(5) Since $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle (\Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \& \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)$, then $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Subset [\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ}$ and $\langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ} \Subset [\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ}$. Therefore, $\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^{\circ} \Cap^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^{\circ} \Subset [\langle\langle(\widetilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\widetilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]^{\circ}$

□

Definition 27. Let $\langle\langle\langle M\rangle, \mathscr{T}^{\text{BVSS}}, \mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVSTS}\rangle\rangle$ over $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \in \text{BVSS}\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVS}\rangle\rangle$ set. Then, $\langle\langle\text{BVS}\rangle\rangle s$-closure of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$,denoted $\overline{\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle}$, is defined as $\langle\langle\text{BVS}\rangle\rangle$ soft intersection of all $\langle\langle\text{BVS}\rangle\rangle s$-closed supersets of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$.

Clearly, $\overline{\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle}$ is the smallest $\langle\langle\text{BVS}\rangle\rangle s$-closed set covering by $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$.

Definition 28. Let $\langle\langle\langle M\rangle, \mathscr{T}^{\text{BVSS}}, \mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVSTS}\rangle\rangle$ over $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \in \text{BVSS}\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVS}\rangle\rangle$ set then the boundary of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ is dented by Fr. $(\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle)$ is defined as a $\langle\langle\text{BVS}\rangle\rangle$ point. ${\mathfrak{t}_1}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)}$ is called boundary of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ if every $\langle\langle\text{BVS}\rangle\rangle s$-open set containing ${\mathfrak{t}_1}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)}$ contains at least one point of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ and least one $\langle\langle\text{BVS}\rangle\rangle$ point of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle^{c}$.

Definition 29. Let $\langle\langle\langle M\rangle, \mathscr{T}^{\text{BVSS}}, \mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVSTS}\rangle\rangle$ over $\langle M\rangle$, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \in \text{BVSS}\langle\langle\langle\tilde{\text{M}}\rangle,\mathscr{L}\rangle\rangle$ be a $\langle\langle\text{BVS}\rangle\rangle$ set then $\langle\langle\text{BVS}\rangle\rangle$ exterior of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ is dented by Ext. $(\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle)$ is defined as a $\langle\langle\text{BVS}\rangle\rangle$ point. $\mathfrak{t}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)}$ is called exterior of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ if $\mathfrak{t}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)}$ if $\langle\langle\text{BVS}\rangle\rangle$ point $\mathfrak{t}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)}$ is $\langle\langle\text{BVS}\rangle\rangle$ interior of $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle^{c}$, that is, there exists $\langle\langle\text{BVS}\rangle\rangle s$-open set $\langle\langle\tilde{g},\mathscr{L}\rangle\rangle$ such that $\mathfrak{t}^{\mathfrak{m}}{}_{(\mathfrak{p}_1,\mathfrak{p}_2)} \in \langle\langle\tilde{g},\mathscr{L}\rangle\rangle \subseteq \langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle^{c}$.

Example 5. Let $\langle\langle\text{BVS}\rangle\rangle$ topology $\mathscr{T}^{\text{BVSS}}$ given in Example 3. Suppose $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \in^{\sim} \text{BVSS}\langle\langle\langle\tilde{\text{M}}\rangle,\mathscr{L}\rangle\rangle$ is defined as succeeding:

$$\left\langle\left\langle\tilde{\mathfrak{f}},\mathscr{L}\right\rangle\right\rangle = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(01\times10^{-1}, 04\times10^{-1}, -09\times10^{-1}, -01\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 02\times10^{-1}, -08\times10^{-1}, -01\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(02\times10^{-1}, 03\times10^{-1}, -07\times10^{-1}, -02\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(01\times10^{-1}, 02\times10^{-1}, -07\times10^{-1}, -04\times10^{-1}\right)\right\rangle\right) \end{pmatrix}. \tag{31}$$

Obviously, $\langle\langle\tilde{\mathfrak{f}}_{\text{null}},\mathscr{L}\rangle\rangle$, $\langle\langle\langle\tilde{M}\rangle_{\text{absolute}},\mathscr{L}\rangle\rangle$, $\langle\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^{C}$, $\langle\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^{C}$ and $\langle\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle^{C}$ are all $\langle\langle\text{BVS}\rangle\rangle s$-closed sets over $\langle\langle\langle M\rangle, \mathscr{T}^{\text{BVSS}}, \mathscr{L}\rangle\rangle$. They are given as follows:

$$\begin{aligned}
&\left\langle\left\langle\tilde{\mathfrak{f}}_{\text{null}},\mathscr{L}\right\rangle\right\rangle^{C} = \left\langle\left\langle\langle\tilde{M}\rangle_{\text{absolute}},\mathscr{L}\right\rangle\right\rangle, \left\langle\left\langle\langle\tilde{M}\rangle_{\text{absolute}},\mathscr{L}\right\rangle\right\rangle^{C} = \left\langle\left\langle\tilde{\mathfrak{f}}_{\text{null}},\mathscr{L}\right\rangle\right\rangle, \\
&\left\langle\left\langle\widetilde{\mathfrak{f}}_1,\mathscr{L}\right\rangle\right\rangle^{C} = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(03\times10^{-1}, 06\times10^{-1}, -07\times10^{-1}, -02\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(05\times10^{-1}, 04\times10^{-1}, -08\times10^{-1}, -01\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(04\times10^{-1}, 07\times10^{-1}, -05\times10^{-1}, -04\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(02\times10^{-1}, 04\times10^{-1}, -03\times10^{-1}, -06\times10^{-1}\right)\right\rangle\right) \end{pmatrix}, \\
&\left\langle\left\langle\widetilde{\mathfrak{f}}_2,\mathscr{L}\right\rangle\right\rangle^{C} = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(05\times10^{-1}, 06\times10^{-1}, -07\times10^{-1}, -03\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(05\times10^{-1}, 05\times10^{-1}, -07\times10^{-1}, -02\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(04\times10^{-1}, 08\times10^{-1}, -04\times10^{-1}, -05\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 07\times10^{-1}, -02\times10^{-1}, -07\times10^{-1}\right)\right\rangle\right) \end{pmatrix}, \\
&\left\langle\left\langle\widetilde{\mathfrak{f}}_3,\mathscr{L}\right\rangle\right\rangle^{C} = \begin{pmatrix} \left(\nu_1, \left\langle\mathfrak{x}_1, \left(06\times10^{-1}, 07\times10^{-1}, -05\times10^{-1}, -03\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(07\times10^{-1}, 06\times10^{-1}, -06\times10^{-1}, -03\times10^{-1}\right)\right\rangle\right), \\ \left(\nu_2, \left\langle\mathfrak{x}_1, \left(05\times10^{-1}, 09\times10^{-1}, -03\times10^{-1}, -06\times10^{-1}\right)\right\rangle, \left\langle\mathfrak{x}_2, \left(04\times10^{-1}, 07\times10^{-1}, -02\times10^{-1}, -07\times10^{-1}\right)\right\rangle\right) \end{pmatrix}.
\end{aligned} \tag{32}$$

Then, $\langle\langle\langle\tilde{M}\rangle_{\text{absolute}},\mathscr{L}\rangle\rangle^C, \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C, \langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^C, \langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle^C \ni^{\sim} \langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$; therefore, $\overline{\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle} = \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C \Cap^{\sim} \langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle^C \Cap^{\sim} \langle\langle\tilde{\mathfrak{f}}_3,\mathscr{L}\rangle\rangle^C = \langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle^C$.

Theorem 30. *Let $\langle\langle\langle M\rangle,\mathscr{T}^{BVSS},\mathscr{L}\rangle\rangle$ be a $\langle\langle BVSTS\rangle\rangle$ over $\langle\tilde{M}\rangle$, $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle \in BNSS\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$. $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$ is a $\langle\langle BVS\rangle\rangle s$-open set iff $\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle = \langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle$.*

Proof. Obvious. □

Theorem 31. *Let $\langle\langle\langle M\rangle,\mathscr{T}^{BVSS},\mathscr{L}\rangle\rangle$ be a $\langle\langle BVSTS\rangle\rangle$ over $\langle M\rangle$, $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle, \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \in BVSS\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$. Then,*

(1) $\overline{[\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}]} = \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$

(2) $\overline{\langle\langle(\tilde{\mathfrak{f}}_{null},\mathscr{L})\rangle\rangle} = \langle\langle(\tilde{\mathfrak{f}}_{null},\mathscr{L})\rangle\rangle$ *and* $\overline{\langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle} = \langle\langle\langle\tilde{M}\rangle_{absolute},\mathscr{L}\rangle\rangle$

(3) $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Rightarrow \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

(4) $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cap^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

(5) $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]} = \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cup^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

Proof.

(1) Let $\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} = \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, then $\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$ is a $\langle\langle BVS\rangle\rangle$ s-closed set. Hence, $\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle} = \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$. So, $\overline{[\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}]} = \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$

(2) Obvious

(3) It is known that $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$, $\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$. Since $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$. Since $\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$ is the smallest $\langle\langle BVS\rangle\rangle$ s-closed set covering $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle$, then so $\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

(4) Since $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, $\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, then $\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Subset \overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]}$, $\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle} \Subset \overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]}$ and so $\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cup^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle} \Subset \overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]}$. Contrariwise, since $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}$, $\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$, then $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cup^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$. Besides, $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]}$ is the smallest $\langle\langle BVS\rangle\rangle s$-closed set covering $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$. Therefore, $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cup^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$. Thus, $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cup^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

(5) Since $\langle\langle\tilde{\mathfrak{f}}_1,\mathscr{L}\rangle\rangle \Cap^{\sim} \langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle \Subset^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cap^{\sim} \overline{\langle\langle\tilde{\mathfrak{f}}_2,\mathscr{L}\rangle\rangle}$ and $[\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cap^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}]$ is then smallest $\langle\langle BVS\rangle\rangle$ s-closed set covering $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$, then $\overline{[\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle]} \Subset \overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \Cap^{\sim} \overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}$

□

Theorem 32. *Let $\langle\langle\langle M\rangle,\mathscr{T}^{BVSS},\mathscr{L}\rangle\rangle$ be a $\langle\langle BVSTS\rangle\rangle$ over $\langle M\rangle$, $\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle \in BVSS\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$.*

(1) $[\overline{\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle}]^c = [\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^c]^\circ$

(2) $[\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^\circ]^c = [\overline{\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^C}]$

Proof.

(1) $\overline{\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle} = \prod_{i\in I}\{\langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \in^{\sim} (\mathscr{T}^{\text{BVSS}})^C : \langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \ni^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle\} \Rightarrow [\overline{\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle}]^c = [\prod_{i\in I}\{\langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \in^{\sim} (\mathscr{T}^{\text{BVSS}})^c : \langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \ni^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle \forall i \in^{\sim} I\}]^c = \coprod_{i\in I}\{\langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle^c \in^{\sim} (\mathscr{T}^{\text{BVSS}}) : \langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle^c \Subset^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^c\} = [\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^c]^\circ$

(2) $\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^\circ = \coprod_{i\in I}\{\langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \in \mathscr{T}^{\text{BVSS}} : \langle\langle\tilde{\mathfrak{f}}_i,\mathscr{L}\rangle\rangle \Subset^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle\} \Rightarrow [\langle\langle\tilde{\mathfrak{f}},\mathscr{L}\rangle\rangle^\circ]^C = [\coprod_{i\in I}\{\langle\langle\tilde{\mathfrak{f}}_i,\mathscr{L}\rangle\rangle \in^{\sim} \mathscr{T}^{\text{BVSS}} : \langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle \Subset^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle\}]^c = \prod_{i\in I}\{\langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle^c \in^{\sim} (\mathscr{T}^{\text{BVSS}})^c : \langle\langle(\tilde{\mathfrak{f}}_i,\mathscr{L})\rangle\rangle^c \ni^{\sim} \langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^c\} = [\overline{\langle\langle(\tilde{\mathfrak{f}},\mathscr{L})\rangle\rangle^c}]$

□

Theorem 33. *Let $\langle\langle\langle M\rangle,\mathscr{T}^{BVSS},\mathscr{L}\rangle\rangle$ be a $\langle\langle BVSTS\rangle\rangle$ over $\langle M\rangle$, $\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle, \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle \in BVSS\langle\langle\langle\tilde{M}\rangle,\mathscr{L}\rangle\rangle$, then*

(1) $Ext(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) = Ext(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \Cup^{\sim} Ext(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)$

(2) $Ext(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) \ni^{\sim} Ext(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \Cup^{\sim} Ext(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)$

(3) $Fr(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} \langle g,\ell\rangle) \Subset^{\sim} Fr\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} Fr\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$

(4) $Fr(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cap^{\sim} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) \Subset^{\sim} Fr\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \Cup^{\sim} Fr\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$

Proof.

(1) Since $\mathrm{Ext}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) = ((\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c)^\circ = (\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^c \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^c)^\circ = (\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle^c)^\circ \tilde{\Cap} (\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle^c)^\circ = \mathrm{Ext}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \tilde{\Cap} \mathrm{Ext}(\langle g,\ell\rangle)$

(2) $\mathrm{Ext}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) = ((\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c)^\circ = ((\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c \tilde{\Cup} (\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c)^\circ \tilde{\Supset} ((\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c)^\circ \tilde{\Cup} ((\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c)^\circ = \mathrm{Ext}\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \mathrm{Ext}\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$ that is $\mathrm{Ext}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) \tilde{\Supset} Ext(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \tilde{\Cup} \mathrm{Ext}(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)$

(3) $\mathrm{Fr}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) = \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c} = (\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle} \tilde{\Cup} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)}) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c} \tilde{\Subset} (\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cup} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c} \{(\overline{\langle f,\ell\rangle}) \tilde{\Cup} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c}\} \tilde{\Cap} \overline{(\langle g,\ell\rangle)^c} = \{((\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c}) \tilde{\Cup} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle}) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c}\} \tilde{\Cup} \{\overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c}\} = \{\mathrm{Fr}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c}\} \tilde{\Cup} \{\mathrm{Fr}(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c}\} \tilde{\Subset} \mathrm{Fr}\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \mathrm{Fr}\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$

(4) $\mathrm{Fr}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle) = \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)} \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cap} \langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c} \tilde{\Subset} ((\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cap} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle})) \tilde{\Cap} (\overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c} \tilde{\Cup} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c}) = \{((\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cap} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle})) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle)^c}\} \tilde{\Cup} \{((\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cap} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle})) \tilde{\Cap} \overline{(\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle)^c}\} = \{\mathrm{Fr}(\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle) \tilde{\Cap} (\overline{\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle})\} \tilde{\Cup} \{(\overline{\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle}) \tilde{\Cap} \mathrm{Fr}(\langle g,\ell\rangle)\} \tilde{\Subset} \mathrm{Fr}\langle\langle(\tilde{\mathfrak{f}}_1,\mathscr{L})\rangle\rangle \tilde{\Cup} \mathrm{Fr}\langle\langle(\tilde{\mathfrak{f}}_2,\mathscr{L})\rangle\rangle$

□

4. Conclusions

During the study, we have gone into detail about defining and finding out the characteristics of bipolar vague soft sets and fundamental operations in a new way. These operations are discussed with examples. On the basis of these operations, vague soft topology is defined. Some structures are discussed with respect to bipolar vague soft s-open sets. These bipolar vague soft s-open sets are chosen among eight new definitions which are introduced in bipolar vague soft topology. Concepts of interior and closure with respect to s-open set are inaugurated. On the basis of these concepts, related results are addressed. The results that engage the interior with closure are also addressed. In the future, we will extend the study to bipolar vague soft bitopology with respect to bipolar vague soft s-open sets and bipolar vague soft β -open sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

Authors' Contributions

All authors read and approved the final manuscript.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] Z. Pawlak, "Rough sets," *International Journal of Computer Science Issues*, vol. 11, pp. 341–356, 1982.

[3] D. Molodtsov, "Soft set theory–first results," *Computers & Mathematcs with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[4] P. K. Maji, R. Biswas, and R. Roy, "An application of soft sets in a decision making problem," *Computers & Mathematcs with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[5] P. K. Maji, R. Biswas, and R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[6] D. Pie and D. Miao, "From soft sets to information systems," in *2005 IEEE International Conference on Granular Computing*, vol. 2, pp. 617–621, Beijing, China, 2005.

[7] D. Chen, E. C. C. Tsang, D. S. Yeung, and X. Wang, "The parameterization reduction of soft sets and its applications," *Computers & Mathematics with Applications*, vol. 49, no. 5-6, pp. 757–763, 2005.

[8] S. Broumi, I. Deli, and F. Smarandache, "Neutrosophic parametrized soft set theory and its decision making," *International Frontier Science Letters*, vol. 1, pp. 1–10, 2014.

[9] N. Çağman, S. Karatas, and S. Enginoglu, "Soft topology," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 351–358, 2011.

[10] M. Shabir and M. Naz, "On soft topological spaces," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1786–1799, 2011.

[11] S. Bayramov and C. Gunduz, "A new approach to separability and compactness in soft topological spaces," *TWMS Journal of Pure*, vol. 9, no. 1, pp. 82–93, 2018.

[12] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[13] S. Bayramov and C. Gunduz, "On intuitionistic fuzzy soft topological spaces," *Applied Mathematics*, vol. 5, no. 1, pp. 66–79, 2014.

[14] K. Hayat, M. I. Ali, B. Y. Cao, and F. Karaaslan, "New results on type-2 soft sets," *Hacettepe Journal of Mathematics*, vol. 5, no. 46, pp. 855–876, 2017.

[15] K. Hayat, M. I. Ali, B. Y. Cao, F. Karaaslan, and X. P. Yang, "A new type-2 soft set: type-2 soft graphs and their applications," *Advances in Fuzzy Systems*, vol. 2017, Article ID 6162753, 17 pages, 2017.

[16] K. Hayat, M. I. Ali, B. Y. Cao, F. Karaaslan, X. P. Yang, and M. H. Shah, "Design concept evaluation using soft sets based on acceptable and satisfactory levels: an integrated TOPSIS and Shannon entropy," *Soft Computing*, vol. 24, no. 3, pp. 2229–2263, 2020.

[17] M. Shabir and M. Naz, "On bipolar soft sets," 2013, https://arxiv.org/abs/1303.1344.

[18] F. Karaaslan and S. Karatas, "A new approach to bipolar soft sets and its applications," *Discrete Mathematics, Algorithms and Applications*, vol. 7, no. 4, article 1550054, 2015.

[19] T. Y. Ozturk, "On bipolar soft topological spaces," *Journal of New Theory*, vol. 20, pp. 64–75, 2018.

[20] T. M. Al-shami, M. E. El-Shafei, and M. Abo-Elhamayel, "Seven generalized types of soft semi-compact spaces," *The Korean Journal of Mathematics*, vol. 27, no. 3, pp. 661–690, 2019.